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Summary

A theoretical analysis is presented for the determination of the fluctuating lift generated by a moving blade-row interacting with the potential flow disturbances of an upstream blade-row. This analysis is an extension of the existing theories, Refs. 1 and 2, for an isolated airfoil moving through transverse and chord-wise gusts to include the effects due to a cascade of airfoils. When the case of infinite spacing between the airfoils of the cascade is considered the present analysis gives the same results as the isolated airfoil theory.

The mathematical representation of the cascade of airfoils is by a continuous distribution of vorticity on a reference blade and in the wakes of all the other blades. The effect of the bound vorticity of the neighbouring blades is simulated by a vortex at their quarter chord points. This representation is suggested by the steady state cascade analysis of Tanabe and Horlock[11] and the present analysis gives the same results as Reference 11 for steady flow. Using thin airfoil theory an expression for the unsteady lift acting on a two-dimensional cascade of thin, slightly cambered airfoils moving through a sinusoidal disturbance in through flow velocity, is derived. This expression can be expressed as a function of a reduced frequency (as for an isolated airfoil Ref. 7) and other parameters representing the effects of the neighbouring blades.

The resulting expression for the fluctuating lift for a cascade of airfoils with circular arc camber lines at small angle of incidence is presented.

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* Replaces A.R.C. 32 368.

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1. *Introduction.*

Methods for the prediction of the fluctuating lift on an isolated airfoil, moving through gusts normal and parallel to its chord, are presented in Refs. 1 and 2. The basis of these methods is an unsteady thin airfoil theory in which the airfoil is represented by a continuous distribution of vorticity on its chord and wake. The assumption of a sinusoidally varying gust velocity, allows the strength of this vorticity and hence the unsteady lift on the airfoil to be determined.

The extension of these methods to predict the fluctuating lift of a cascade of airfoils has been attempted by several authors. For a cascade of rigid (not vibrating) airfoils moving through either the potential flow disturbances of the adjacent blade rows or the viscous wakes shed from the upstream blades, the work of Refs. 3 and 4 is best known. However this work only includes the effect of the unsteady part of the circulation of the reference blade and hence neglects the effect of all vortex wakes except that shed by the reference blade. The isolated airfoil theory is used to include the effect of the unsteady part of circulation of the reference blade. For a cascade of vibrating airfoils the unsteady representation has been more general and includes the effect of the unsteady part of the circulation of all the airfoils in the cascade. Ref. 5 presents such an analysis for a cascade of flat plate airfoils in which each airfoil is replaced by a finite number of vortices and its wake is represented by a continuous distribution of vorticity. Ref. 6 is an extension of this work to thin, cambered airfoils.

Now, in the analysis of an isolated airfoil Refs. 1 and 2 the fluctuating lift is expressed as a function of the reduced frequency, which is the ratio of the airfoil chord length to the wave length of the disturbance. Such a relationship is invaluable to the designer since for a given disturbance the chord length can be chosen to minimize the lift fluctuations. The effect of camber can similarly be included. However similar functions, accounting for the full effect of all the blades in unsteady flow, have not been developed for a cascade of airfoils. The approaches to the solution of blade vibration Refs. 5 and 6 include these unsteady effects, but the solutions are by numerical methods and as such are best suited for the analysis of a given blade geometry and disturbance characteristics.

The present method gives a functional relationship for the fluctuating lift on a cascade of airfoils similar to that for an isolated airfoil. This analysis is limited to the interaction of a single rotor with potential flow disturbances, Figure 1 and assumes two-dimensional, incompressible, inviscid flow.

2. *Representation of the Cascade and the Formulation of Induced Velocities.*

The approach used is that of unsteady thin airfoil theory. In this theory the airfoil is replaced by a continuous distribution of vorticity γ along its chord line and it is assumed to have zero thickness. The resulting induced velocities are calculated on the chord line in terms of the vorticity γ. Further it is assumed that the flow is two-dimensional, inviscid and incompressible.

From the cascade geometry shown in Figure 1, the normal and chordwise induced velocities dv and du at a point X_p on the reference blade, due to an element of circulation $(\gamma_n\, dx)$ on the nth blade of the cascade, can be determined. These are

$$du_0\,(x_p) = -\frac{1}{2\pi}\frac{\gamma_n(x_n)\, ns \cos\xi\, dx_n}{(ns\cos\xi)^2+(x_n-x_p+ns\sin\xi)^2}$$

$$dv_0\,(x_p) = \frac{1}{2\pi}\frac{\gamma_n(x_n)\,[x_n-x_p+ns\sin\xi]\, dx_n}{(ns\cos\xi)^2+(x_n-x_p+ns\sin\xi)^2}. \qquad (1)$$

It is assumed that the circulation on each of the blades in the cascade is of the same amplitude but there exists a constant phase difference from blade to blade. Therefore,

$$\gamma_n(x_n) = \gamma_0(x)e^{in\tau}$$

where $\tau = {}^{2\Pi s}/l$. As in Refs. 1 and 2 it is assumed that all changes with time are sinusoidal. Hence

$$\gamma_0 = \bar{\gamma}_0 \, e^{i\nu t}$$
$$v_0 = \bar{v}_0 \, e^{i\nu t}$$
$$\text{and } u_0 = \bar{u}_0 \, e^{i\nu t} \tag{2}$$

Substitution of these relations into equation (1) gives

$$d\bar{u}_0(x_p) = -\frac{1}{2\pi} \frac{\bar{\gamma}_0(x)e^{in\tau}\, ns \cos \xi \, dx_n}{(ns \cos \xi)^2 + (x_n - x_p + ns \sin \xi)^2},$$

$$d\bar{v}_0(x_p) = \frac{1}{2\pi} \frac{\bar{\gamma}_0(x)e^{in\tau}[x_n - x_p + ns \sin \xi]dx_n}{(ns \cos \xi)^2 + (x_n - x_p + ns \sin \xi)^2}. \tag{3}$$

These expressions give the velocity induced at a point x_p on the reference blade by an element of circulation associated with the nth blade and located at a point x_n. These expressions can be integrated from the leading edge and along the wake of the nth blade to obtain the total velocity induced at x_p. Before this can be done it is necessary to relate the wake vorticity to that on the blade.

Consider an element of wake vorticity of the reference blade of strength $\gamma_{0_w} = \bar{\gamma}_{0_w} e^{i\nu t}$ which is transported away from the blade with velocity W_2 in a direction parallel to the chord. Then at any location $(\lambda, 0)$ downstream of the trailing edge.

$$\bar{\gamma}_{0_w} e^{i\nu t} = \text{constant } e^{i\nu\left(t - \frac{\lambda}{W_2}\right)} \tag{4}$$

During an interval of time, δt, the bound circulation on the blade changes its strength by

$$\frac{d\Gamma_0}{dt}\delta t = \frac{d(\bar{\Gamma}_0 \, e^{i\nu t})}{dt}\delta t = i\nu\bar{\Gamma}_0 \, e^{i\nu t} \, \delta t$$

where $\Gamma_0 \equiv \int_0^c \gamma_0 \, dx$. This change in bound circulation results in an element of wake circulation which is shed at $\lambda = c$ and moves downstream a distance $W_2 \, \delta t$ in time δt. This wake circulation is of opposite sign to the change in circulation on the blade. Therefore from equation (4)

$$-\frac{i\nu\bar{\Gamma}_0 \, e^{i\nu t} \, \delta t}{W_2 \, \delta t} = \text{constant } e^{i\nu\left(t - \frac{c}{W_2}\right)}$$

thus determining the constant in equation (4) and leading to an expression for the strength of the wake vorticity,

$$\bar{\gamma}_{0w} = -\frac{i\nu}{W_2}\bar{\Gamma}_0 \, e^{\, i\nu\left(\frac{c-\lambda}{W_2}\right)} \tag{5}$$

Since this analysis is restricted to lightly loaded thin airfoils it is assumed that the relative velocity at the exit W_2 can be replaced by the mean relative velocity $W_m = (W_1 + W_2)/2$. This assumption is employed in the remainder of this Report.

It is next assumed that the vorticity on the nth blade is concentrated at a chordwise point x_{c_n} while the vorticity in its wake is continuously distributed. The circulation on the nth blade is related to that on the reference blade by $\Gamma_n = \Gamma_0 \, e^{in\tau}$. Also the cordinate axes are transformed to the mid-chord of the reference blade, i.e. $x^+ = \frac{2}{c}x - 1$ and the chord length of the blades is assumed equal to 2. Substitution of these assumptions into equation (3) gives the total velocity induced by the nth blade at the point x_p^+ as

$$\bar{u}_0(x_p^+) = -\frac{\Gamma_0}{4\pi s}\Theta(\chi_c^+)+\frac{i\omega\Delta}{4\pi}\frac{c}{s}\int_1^\infty e^{-i\omega\lambda^+}\,\Theta(\chi_\lambda^+)d\lambda^+$$

$$\text{and } \bar{v}_0(x_p^+) = -\frac{\Gamma_0}{4\pi i s}\Phi(\chi_c^+)+\frac{\omega\Delta}{4\pi}\frac{c}{s}\int_1^\infty e^{-i\omega\lambda^+}\,\Phi(\chi_\lambda^+)d\lambda^+. \tag{6}$$

The total velocity induced at the point x_p^+ by all of the blades can be found by taking the summation of equation (6) from $n = -\infty$ to $n = +\infty$. In addition, it is assumed that the vorticity on the reference blade is continuously distributed rather than being a concentrated vortex of strength Γ_0. If x_p^+ is written as x^+ then equation (6) becomes

$$\bar{u}_0(x^+) = -\frac{\Gamma_0}{4\pi s}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\Theta(\chi_c^+)+\frac{i\omega\Delta}{4\pi}\frac{c}{s}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\int_1^{x} e^{-i\omega\lambda^+}\,\Theta(\chi_\lambda^+)\,d\lambda^+$$

$$\bar{v}_0(x^+) = -\frac{\Gamma_0}{4\pi i s}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\Phi(\chi_c^+)+\frac{1}{2\pi}\int_{-1}^{1}\frac{\bar{\gamma}_0(x_1^+)dx_1^+}{(x_1^+-x^+)}+$$

$$+\frac{\omega\Delta}{4\pi}\frac{c}{s}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\int_1^\infty e^{-i\omega\lambda^+}\,\Phi(\chi_\lambda^+)d\lambda^+ +\frac{\omega\Delta}{i\pi}\int_1^\infty\frac{e^{-i\omega\lambda^+}\,d\lambda^+}{(\lambda^+-x^+)}. \tag{7}$$

Within the assumptions of thin airfoil theory the reference blade induces no chordwise velocity u_0 on itself whereas the second and fourth integrals in the expression for $\bar{v}_0(x^+)$ represent the effect of the reference blade. If only the effects of the reference blade are considered, equation (7) reduces to that of Ref. 9, equation 5-328, which considers the unsteady lift on an isolated airfoil.

3. *Determination of Vorticity Distribution on the Reference Blade.*

The determination of the unsteady lift generated in a cascade of rigid airfoils experiencing a sinusoidal perturbation in its inlet velocity requires that the vorticity $\gamma_0(x^+)$ on the reference blade be known. To determine this vorticity the approach outlined in Ref. 9 for an isolated airfoil is followed.

The basis of this solution is the use of the Sohngen Inversion Formula i.e.,

$$\text{if } g(\sigma) = -\frac{1}{2\pi}\int_{-1}^{1}\frac{f(\xi)\,d\xi}{\xi-\sigma}$$

$$\text{then } f(\sigma) = -\frac{2}{\pi}\sqrt{\frac{1-\sigma}{1+\sigma}}\int_{-1}^{1}\sqrt{\frac{1+\xi}{1-\xi}}\,\frac{g(\xi)}{(\sigma-\xi)}\,d\xi \tag{8}$$

where $f(\sigma)$ is the desired unknown function. Comparison of this inversion formula and equation (7) leads to the following expression for $\bar{\gamma}_0(x^+)$

$$\bar{\gamma}_0(x^+) = \frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{\bar{v}_0(x_1^+)dx_1^+}{(x^+-x_1^+)} - \frac{2\omega\Delta}{i\pi^2}\sqrt{\frac{1-x^+}{1+x^+}}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\int_{1}^{\infty}\frac{e^{-i\omega\lambda^+}\,d\lambda^+\,dx_1^+}{(\lambda^+-x_1^+)(x^+-x_1^+)} +$$

$$+\frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{\Gamma_0}{4\pi i s}\Phi(\chi_c^+)\frac{dx_1^+}{(x^+-x_1^+)} -$$

$$-\frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{\omega\Delta}{4\pi}\frac{c}{s}\int_{1}^{\infty}e^{-i\omega\lambda^+}\,\Phi(\chi_\lambda^+)\frac{d\lambda^+\,dx_1^+}{(x^+-x_1^+)} \qquad (9)$$

where the $\left(\sum_{\infty}^{-1}+\sum_{1}^{\infty}\right)$ has been removed from the integral since the same result can be obtained by first considering only the reference blade and the nth blade and then performing the summation.

Only the induced velocity $\bar{v}_0(x_1^+)$ appears in the expression for $\bar{\gamma}_0(x^+)$. $\bar{v}_0(x_1^+)$ will eventually be specified through the boundary conditions on the airfoil; at the same time the influence of $\bar{u}_0(x_1^+)$ will be discussed.

By interchanging the order of integration in equation (9) and introducing the expression for $\Phi(\chi_c^+)$ given by equation (A.5) Appendix I, $\bar{\gamma}_0(x^+)$ becomes

$$\bar{\gamma}_0(x^+) = \frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{\bar{v}_0(x_1^+)dx_1^+}{(x^+-x_1^+)} - \frac{2\omega\Delta}{i\pi^2}\sqrt{\frac{1-x^+}{1+x^+}}\int_{1}^{\infty}e^{-i\omega\lambda^+}\cdot\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{dx_1^+}{(\lambda^+-x_1^+)(x^+-x_1^+)} -$$

$$-\frac{\Gamma_0}{\pi^2 c}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\left\{\frac{1}{g_c-x_1^+}+\frac{1}{h_c-x_1^+}\right\}\frac{dx_1^+}{(x^+-x_1^+)} - \qquad (10)$$

$$-\frac{\omega\Delta}{i\pi^2}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\int_{1}^{\infty}e^{-i\omega\lambda^+}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\left\{\frac{1}{g_\lambda-x_1^+}+\frac{1}{h_\lambda-x_1^+}\right\}\frac{dx_1^+}{(x^+-x_1^+)}$$

where $g_c = x_c^+ + n/ia$, $g_\lambda = \lambda^+ + n/ia$, $h_c = x_c^+ - n/ib$ and $h_\lambda = \lambda^+ - n/ib$.

The products of the form $(s'-x_1^+)^{-1}(x^+-x_1^+)^{-1}$ which occur in equation (10) can be rewritten as

$$\frac{1}{(s'-x_1^+)}\frac{1}{(x^+-x_1^+)} = \frac{1}{s'-x^+}\left\{\frac{1}{x^+-x_1^+}-\frac{1}{s'-x_1^+}\right\}.$$

Substitution of this relation into equation (10) allows the integrals with respect to x_1^+ to be evaluated using relations 1 and 2 of Appendix II. However the quantities x^+, λ^+, g_c, h_c, g_λ and h_λ must be examined to assure that they fulfil the conditions for using these integrals, i.e. $x^{+2} \leqslant 1$ and $\lambda^{+2} > 1$. If p represents x_c^+ or λ^+ in the general case then,

$$g_p^2 = h_p^2 = p^2 + \frac{4ns}{c}\sin\xi + \frac{4n^2s^2}{c^2}$$

Thus for application of relation 2, Appendix II, it is necessary that

$$\frac{4ns}{c}\sin\xi + \frac{4n^2s^2}{c^2} > 1 - p^2.$$

For practical reasons both s/c and $\sin(\xi)$ are positive. The condition that the inequality hold for $p^2 = 1$ is $s/c > \sin\xi$ for all values of n except $n = 0$ which is excluded from the summation. With the left hand side of the inequality positive, the most severe restriction on s/c occurs when $p^2 = 0$. For this condition the inequality is true if $s/c > \frac{1}{2}$. Therefore the use of relation 2, Appendix II, is restricted to values of $s/c > \frac{1}{2}$ or $\sin\xi$ depending upon which is larger. These restrictions are similar to those derived in the analysis of the steady lift of a cascade whose blades are represented by concentrated vortices (Ref. 11).

With the above conditions imposed equation (10) becomes

$$\bar{\gamma}_0(x^+) = \frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\int_{-1}^{1}\sqrt{\frac{1+x^+}{1-x^+}}\frac{\bar{v}_0(x_1^+)dx_1^+}{(x^+-x_1^+)} - \frac{2i\omega\Delta}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\int_{1}^{\infty}\sqrt{\frac{\lambda^++1}{\lambda^+-1}}\frac{e^{-i\omega\lambda^+}}{(\lambda^+-x^+)}d\lambda^+ +$$

$$+\frac{\bar{\Gamma}_0}{\pi c}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\frac{1}{(g_c-x^+)}\sqrt{\frac{g_c+1}{g_c-1}}+\frac{1}{(h_c-x^+)}\sqrt{\frac{h_c+1}{h_c-1}}\right\}-$$

$$-\frac{i\omega\Delta}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\int_{1}^{\infty}\sqrt{\frac{g_\lambda+1}{g_\lambda-1}}\frac{e^{-i\omega\lambda^+}}{(g_\lambda-x^+)}d\lambda^+ + \int_{1}^{\infty}\sqrt{\frac{h_\lambda+1}{h_\lambda-1}}\frac{e^{-i\omega\lambda^+}}{(h_\lambda-x^+)}d\lambda^+\right\}. \quad (11)$$

From this expression for $\bar{\gamma}_0(x^+)$, the total circulation on the reference blade, $\bar{\Gamma}_0$, can be obtained by integration of equation (11). Again bringing the summations outside of the integrals and employing relations 1 and 2 of Appendix II,

$$\frac{2\bar{\Gamma}_0}{c} = -2\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\bar{v}_0(x_1^+)dx_1^+ + \frac{\bar{\Gamma}_0}{c}\left\{\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left[\sqrt{\frac{g_c+1}{g_c-1}}+\sqrt{\frac{h_c+1}{h_c-1}}-2\right]-\right.$$

$$-2i\omega e^{i\omega}\int_{1}^{\infty}\left[\sqrt{\frac{\lambda^++1}{\lambda^+-1}}-1\right]e^{-i\omega\lambda^+}d\lambda^+ - i\omega e^{i\omega}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\int_{1}^{\infty}\left[\sqrt{\frac{g_\lambda+1}{g_\lambda-1}}-1\right]e^{-i\omega\lambda^+}d\lambda^+ +\right.$$

$$\left.\left.+\int_{1}^{\infty}\left[\sqrt{\frac{h_\lambda+1}{h_\lambda-1}}-1\right]e^{-i\omega\lambda^+}d\lambda^+\right\}\right\} \quad (12)$$

Using the definition of terms in Appendix III and relation 3 of Appendix II, the total circulation $\bar{\Gamma}_0$ on the blade becomes

$$-\frac{i\omega\Delta}{\pi}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\left\{\int_{1}^{\infty}\left\{\sqrt{\frac{g_\lambda+1}{g_\lambda-1}}\left[\frac{\pi}{2}+\sin^{-1}\sigma^+\right]+\Lambda(\sigma^+,g_\lambda)\right\}e^{-i\omega\lambda^+}d\lambda^+ +\right.\right.$$

$$\left.\left.+\int_{1}^{\infty}\left\{\sqrt{\frac{h_\lambda+1}{h_\lambda-1}}\left[\frac{\pi}{2}+\sin^{-1}\sigma^+\right]+\Lambda(\sigma^+,h_\lambda)\right\}e^{-i\omega\lambda^+}d\lambda^+\right\}\right\}. \tag{16}$$

Following the steps outlined in Ref. 9, this expression can be simplified by subtracting from it the product of equation (12) and the quantity $\left(\frac{1}{2}+\frac{1}{\pi}\sin^{-1}\sigma^+\right)$. This leads to,

$$\int_{-1}^{\sigma^+}\bar{\gamma}_0(x_1^+)dx_1^+ = -\frac{2}{\pi}\int_{-1}^{1}\bar{v}_0(x_1^+)\Omega(\sigma^+,x_1^+)dx_1^+ - \frac{2i\omega\Delta}{\pi}\int_{1}^{\infty}e^{-i\omega\lambda^+}\Lambda(\sigma^+,\lambda^+)\,d\lambda^+ +$$

$$+\frac{\Delta e^{-i\omega}}{\pi}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left[\Lambda(\sigma^+,g_c)+\Lambda(\sigma^+,h_c)\right]-$$

$$-\frac{i\omega\Delta}{\pi}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\int_{1}^{\infty}e^{-i\omega\lambda^+}\Lambda(\sigma^+,g_\lambda)\,d\lambda^+ + \int_{1}^{\infty}e^{-i\omega\lambda^+}\Lambda(\sigma^+,h_\lambda)\,d\lambda^+\right\}. \tag{17}$$

Substitution of equations (11) and (17) into equation (15) gives the unsteady pressure differential Δp at a point σ^+ on the blade exclusive of the effect of u_d. Using 6 and 7 of Appendix II and the definition of terms in Appendix III $\overline{\Delta p}(\sigma^+)$ can be expressed as

$$-\frac{\overline{\Delta p}(\sigma^+)}{\rho W_m} = \frac{2}{\pi}\sqrt{\frac{1-\sigma^+}{1+\sigma^+}}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{\bar{v}_0(x_1^+)dx_1^+}{(\sigma^+-x_1^+)} - \frac{2i\omega}{\pi}\int_{-1}^{1}\bar{v}_0(x_1^+)\Omega(\sigma^+,x_1^+)dx_1^+ -$$

$$-\omega\Delta H_0^{(2)}(\omega)\sqrt{\frac{1-\sigma^+}{1+\sigma^+}} - \frac{2i\Delta\omega e^{-i\omega}}{\pi}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left[\frac{\partial A_1}{\partial\sigma^+}+\frac{\partial A_2}{\partial\sigma^+}-\frac{\partial A_3}{\partial\sigma^+}-\frac{\partial A_4}{\partial\sigma^+}\right]+$$

$$+\frac{\Delta e^{-i\omega}}{\pi}\sqrt{\frac{1-\sigma^+}{1+\sigma^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\left\{\frac{1}{(g_c-\sigma^+)}\sqrt{\frac{g_c+1}{g_c-1}}+\right.\right.$$

$$\left.\left.+\frac{1}{(h_c-\sigma^+)}\sqrt{\frac{h_c+1}{h_c-1}} - i\omega e^{i\omega}\left[B_1+B_2\right]\right\}\right\} \tag{18}$$

5. *Determination of Unsteady Lift for the Case of Zero Chordwise Disturbance Velocity.*

The lift on a blade in a cascade can now be determined in terms of the velocity $\bar{v}_0(x_1^+)$ induced normal to its chord. The lift is given as

$$L = \frac{c}{2}\int_{-1}^{1} -\Delta p'(\sigma^+)d\sigma^+ = -e^{ivt}\int_{-1}^{1}\overline{\Delta p}(\sigma^+)d\sigma^+ \tag{19}$$

since $c = 2$. From equation (18) and relations 1, 8 and 9 of Appendix II

$$\frac{L}{\rho W_m e^{ivt}} = -2\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\,\bar{v}_0(x_1^+)dx_1^+ - 2i\omega\int_{-1}^{1}\sqrt{1-x_1^{+2}}\,\bar{v}_0(x_1^+)dx_1^+ -$$

$$-\pi\omega\Delta H_0^{(2)}(\omega) - \frac{2i\omega\Delta e^{-i\omega}}{\pi}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left[A_1+A_2-A_3-A_4\right] -$$

$$-\Delta e^{-i\omega}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left[i\omega e^{i\omega}\{B_1+B_2\}-\{C_1+C_2-2\}\right]. \tag{20}$$

This expression for the lift can be written in a form similar to that for an isolated airfoil by substitution of equation (13) into equation (20).

$$\frac{L}{\rho W_m e^{ivt}} = -2G(\omega, s/c, \xi)\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\,\bar{v}_0(x_1^+)dx_1^+ - 2i\omega\int_{-1}^{1}\sqrt{1-x_1^{+2}}\,\bar{v}_0(x_1^+)dx_1^+ \tag{21}$$

where

$$G(\omega, s/c, \xi) = 1 + \frac{\pi\omega\, H_0^{(2)}(\omega) + \left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{\frac{2i\omega e^{-i\omega}}{\pi}\left[A_1+A_2-A_3-A_4\right]+i\omega\left[B_1+B_2\right]-e^{-i\omega}\left[C_1+C_2-2\right]\right\}}{i\omega\pi\left[H_1^{(2)}(\omega)+iH_0^{(2)}(\omega)\right]+\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)e^{in\tau}\left\{e^{-i\omega}\left[C_1+C_2-2\right]-i\omega\left[D_1+D_2\right]\right\}}. \tag{22}$$

6. *General Expression for Unsteady Lift in a Cascade with a Normal and Chordwise Disturbance.*

As shown in equation (14) an additional term $\rho u_d \gamma_{o_s}(\sigma^+)$ contributes to the pressure difference across the blade when a chordwise gust u_d is present. Since equation (21) was derived assuming $u_d = 0$ and the subject analysis is linear, the term

$$\int_{-1}^{1}\frac{u_d\gamma_{o_s}(x_1^+)dx_1^+}{W_m e^{ivt}} \tag{23}$$

can be added to equation (21) to include this effect. Thus, the total lift becomes,

$$\frac{L}{\rho W_m e^{ivt}} = \int_{-1}^{1} \frac{\bar{u}_d \gamma_{0_s}(x_1^+) dx_1^+}{W_m} - 2G(\omega, s/c, \xi) \int_{-1}^{1} \sqrt{\frac{1+x_1^+}{1-x_1^+}}\, \bar{v}_0(x_1^+) dx_1^+ - 2i\omega \int_{-1}^{1} \sqrt{1-x_1^{+2}}\; \bar{v}_0(x_1^+) dx_1^+$$

From equations (11), (12) and (22), with $\omega = 0$, the steady circulation $\gamma_{0_s}(x_1^+)$ can be expressed as,

$$\gamma_{0_s}(x_1^+) = \frac{2}{\pi} \sqrt{\frac{1-x_1^+}{1+x_1^+}} \int_{-1}^{1} \sqrt{\frac{1+k}{1-k}} \left\{ \frac{1}{(x_1^+ - k)} - \frac{G(0, s/c, \xi)}{2} \left(\sum_{-\infty}^{-1} + \sum_{1}^{\infty} \right) \left[\frac{C_1}{(g_c - x_1^+)} + \frac{C_2}{(h_c - x_1^+)} \right] \right\} v_{o_s}(k) dk \quad (25)$$

where $G(o, s/c, \xi)$ is the modified Theodorsens function with $\omega = 0$ (as defined after equation (30)).

7. *Specification of Induced Velocity* $\bar{v}_0$.

The expression for the lift on an airfoil in a cascade, equation (24), is written in terms of the velocity $\bar{v}_0$ induced normal to the chord of the airfoil. To obtain a solution for the lift it is necessary to introduce the induced velocity $\bar{v}_0$ through the specification of the boundary conditions on the airfoil. Following thin airfoil theory, the boundary condition at any point on the airfoil requires the flow at that point to be tangential to the camber line of the airfoil, i.e.

$$\frac{W_m \sin \alpha_i + v_0 + v_d}{W_m \cos \alpha_i + u_0 + u_d} = \frac{dy_c}{dx} \quad (26)$$

where α_i is the angle of incidence of the airfoil and y_c the coordinate of the camber line. The subscripts o and d denote the induced and disturbance velocities, respectively.

The model chosen to represent the cascade of airfoils allows several simplifying assumptions to be made regarding equation (26). The major simplification concerns the induced velocity u_o. The mathematical representation of the cascade by a distributed vorticity on the reference blade and concentrated vorticity on the remaining blades is identical to that used in Ref. 11 for the steady flow. A comparison of the results obtained in Ref. 11 using this representation and a boundary condition which neglects the chordwise induced velocity with the exact inviscid analysis of Weinig is shown in Figure 2. Also shown on this figure is the range to which this unsteady analysis is restricted, i.e. $s/c > \frac{1}{2}$ or sin ξ.

Figure 2 demonstrates that for the analysis of the steady lift in a cascade the neglect of the chordwise induced velocity produces adequate results for all values of s/c if the stagger angle ξ is $\leqslant 50°$ and for $s/c > 1{\cdot}5$ if $\xi = 70°$. On the basis of these results, the chordwise induced velocity u_0 is neglected. With this assumption the boundary condition becomes

$$\frac{W_m \sin \alpha_i + v_o + v_d}{W_m \cos \alpha_i + u_d} = \frac{dy_c}{dx}. \quad (27)$$

The steady state boundary condition, i.e. $u_d = v_d = 0$, becomes

$$v_{o_s} = W_m \cos \alpha_i \frac{dy_c}{dx} - W_m \sin \alpha_i.$$

8. *Steady Lift of a Cascade.*

Using the analysis presented above the steady lift generated in a cascade can be determined. This derived lift is identical to that predicted by the analysis of Ref. 11 thus providing a check on the validity of the present analysis.

The steady lift in a cascade of airfoils can be expressed as

$$L_s = \rho W_m \Gamma_{o_s} = \sigma W_m \int_{-1}^{1} \gamma_{o_s}(x^+)dx^+ \tag{28}$$

where W_m is the mean relative velocity of the cascade. From equation (11) with $\omega = 0$ the steady circulation distribution $\gamma_{o_s}(x_1^+)$ is

$$\gamma_{o_s}(x^+) = \frac{2}{\pi}\sqrt{\frac{1-x^+}{1+x^+}}\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\frac{v_{o_s}(x_1^+)dx_1^+}{(x^+-x_1^+)} + \frac{\Gamma_{o_s}}{2\pi}\sqrt{\frac{1-x^+}{1+x^+}}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)\left[\frac{C_1}{g_c-x^+}+\frac{C_2}{h_c-x^+}\right] \tag{29}$$

With $\omega = 0$ then $\bar{\gamma}_o = \gamma_{o_s}$, $\bar{\Gamma}_o = \Gamma_{o_s}$ and $\bar{v}_o = v_{o_s}$.

Employing relations 1 and 2 of Appendix II and the definition

$$\Gamma_{o_s} = \int_{-1}^{1} \gamma_{o_s}(x^+)dx^+$$

then

$$\int_{-1}^{1} \gamma_{o_s}(x^+)dx^+ = \frac{-2\int_{-1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\,v_{o_s}(x_1^+)dx_1^+}{1-\frac{1}{2}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)[C_1+C_2-2]}. \tag{30}$$

Defining $G(o, s/c, \xi) = \left\{1-\frac{1}{2}\left(\sum_{-\infty}^{-1}+\sum_{1}^{\infty}\right)[C_1+C_2-2]\right\}^{-1}$ the steady lift becomes

$$L_s = -2\rho W_m G(o, s/c, \xi)\int_{-1}^{1}\sqrt{\frac{1+x_1^+}{1-x_1^+}}\,v_{o_s}(x_1^+)dx_1^+. \tag{31}$$

The function $G(o, s/c, \xi)$ represents the cascade lattice coefficient discussed in Ref. 11 and the quasi-steady value, (i.e. $\omega = 0$) of the modified Theodorsen function $G(\omega, s/c, \xi)$. In Figure 2 values of this function are shown for various values of s/c and ξ and the comparison is presented between Weinigs lattice coefficient and that presented in Ref. 11.

9. *Unsteady Lift of a Circular Arc Cascade with Incidence.*

The unsteady lift for a particular cascade configuration can be determined by substitution of the proper boundary conditions into equation (24). Consider the case of a cascade of circular-arc thin airfoils operating at a steady state angle of incidence α_i. The assumption is made that the angle of incidence is small so that equation (27) can be written as

$$v_o + v_d = (W_m + u_d)\frac{dy}{dx} - W_m \alpha_i .$$

The equation describing the displacement y_p^+ of a circular-arc camber line from the chord line as a function of x^+ is

$$y_p^+ = y_{\max}(1 - x_p^{+2})$$

where $y_{\max}$ is the maximum displacement of the camber line. Thus, the boundary condition becomes

$$v_0 + v_d = -2y_{\max}(W_m + u_d)x^+ - W_m \alpha_i \tag{32}$$

where $v_{o_s} = -W_m(2y_{\max} x^+ + \alpha_i)$ represents the contribution due to the steady flow. Substitution of this steady boundary condition into equation (25) and the employment of relations 13, 14 and 19 of Appendix II, the steady circulation distribution becomes

$$\gamma_{o_s}(x_1^+) = W_m\sqrt{\frac{1-x_1^+}{1+x_1^+}}\left\{4(1+x_1^+)y_{\max} + 2\alpha_i + (\alpha_i + y_{\max})\, G(o, s/c, \xi)\left(\sum_{-\infty}^{-1} + \sum_{1}^{\infty}\right)\left[\frac{C_1}{g_c - x_1^+} + \frac{C_2}{h_c - x_1^+}\right]\right\}. \tag{33}$$

This relation is used to determine the contribution of v_{o_s} to the unsteady lift.

If only the unsteady portion of the flow is considered and the disturbance velocities v_d and u_d are of the form

$$\text{constant}\; e^{ivt}\, e^{-i\omega x_1^+}$$

the unsteady boundary condition from equation (29) becomes

$$\bar{v}_0 = -2\hat{u}y_{\max}x_1^+\, e^{-i\omega x_1^+} - \hat{v}e^{-i\omega x_1^+}. \tag{34}$$

Substitution of equations (32), (33) and (34) into equation (27) with the use of relations 10, 11, 12, 15, 16, 17, 18 and 19 of Appendix II gives the following expression for the unsteady lift of a circular-arc cascade of airfoils with incidence α_i experiencing both a chordwise and normal disturbance,

$$\frac{\tilde{L}}{2\pi\rho W_m e^{ivt}} = \hat{u}_d\left\{ y_{\max}[3J_2(\omega) + J_0(\omega)] + \alpha_i[J_0(\omega) + iJ_1(\omega)] + y_{\max}G(\omega, s/c, \xi)\,[J_0(\omega) - J_2(\omega) - i2J_1(\omega)] + \right.$$

$$\left. + \frac{(y_{\max} + \alpha_i)}{2}\, G(o, s/c, \xi)\left(\sum_{-\infty}^{-1} + \sum_{1}^{\infty}\right)[(C_1 + C_2)J_o(\omega) - 2 + 2LJ_1(\omega)]\right\} +$$

$$+ \hat{v}_d\{G(\omega, s/c, \xi)\,[J_o(\omega) - iJ_1(\omega)] + iJ_1(\omega)\}. \tag{35}$$

The summations indicated in this expression represent the contribution of the neighbouring blades to the steady circulation of the reference blade. As seen from relation 18 of Appendix II this contribution is of the form of an infinite sum of Bessel functions of the first kind. Since the analysis presented here is limited to a circular-arc camber line, the contribution of those terms involving $J_k(\omega)$ for $k \geqslant 2$ are neglected by the same reasoning as used in Ref. 3.

The reduced frequency and the disturbance velocities are referred to the mean relative velocity W_m rather than the inlet or exit relative velocity. Thus the reduced frequency $\omega = \frac{\nu c}{2W_m}$. Equation (35) is expressed in terms of the maximum amplitudes $\hat{u}$ and $\hat{v}$ of the velocities u_d and v_d. Referring to Figure 1 the unsteady lift $\tilde{L}$ can be written in terms of an upstream disturbance velocity $w_d = \hat{w}e^{i\nu(t-x_i^+/W_m)}$ where $u_d = w_d \cos\xi$ and $v_d = -w_d \sin\xi$. Thus the unsteady lift becomes

$$\frac{\tilde{L}}{2\pi\rho W_m \hat{w} e^{i\nu t}} = \cos\xi \Bigg\{ y_{\max}[3J_2(\omega)+J_0(\omega)] + \alpha_i[J_0(\omega)+iJ_1(\omega)] + y_{\max}G(\omega, s/c, \xi)\,[J_0(\omega)-J_2(\omega)-i2J_1(\omega)] + \\ + \frac{(y_{\max}+\alpha_i)}{2}\, G(o, s/c, \xi)\left(\sum_{-\infty}^{-1} + \sum_{1}^{\infty}\right)[(C_1-C_2)J_o(\omega)-2+2LJ_1(\omega)] \Bigg\} \\ -\sin\xi\{G(\omega, s/c, \xi)\,[J_o(\omega)-iJ_1(\omega)]+iJ_1(\omega)\}. \tag{36}$$

10. *Unsteady Lift on an Isolated Airfoil.*

To check the validity of equation (36) consider the case of an isolated airfoil in a turbomachine which experiences an upstream disturbance w_d. This case is similar to that considered in Refs. 7 and 2. To make a direct comparison with these previous results, equation (36) for $s = \infty$ is expressed in terms of the disturbance perturbations parallel and normal to the inlet velocity direction, $\bar{u}_d'$ and $\bar{v}_d'$ respectively. Hence,

$$\hat{u} = \hat{w}\cos\xi = \bar{u}_d' + \bar{v}_d'\alpha_i$$
$$\hat{v} = -\hat{w}\sin\xi = -\bar{v}_d' + \bar{u}_d'\alpha_i$$

because the angle of incidence α_i is assumed small. Since the modified Theodorsen's function $G(\omega, s/c, \xi)$ equals $C(\omega)$ the Theodorsen function for an isolated airfoil, for $s = \infty$ *see* (Appendix V), equation (36) becomes

$$\frac{\tilde{L}}{2\pi\rho W_m e^{i\nu t}} = y_{\max}(\bar{u}_d' + \bar{v}_d'\alpha_i)\,\{3J_2(\omega)+J_o(\omega)+C(\omega)\,[J_o(\omega)- \\ -J_2(\omega)-i2J_1(\omega)]\} + (\bar{u}_d'+\bar{v}_d'\alpha_i)\alpha_i\{J_o(\omega)+iJ_1(\omega)\} + (\bar{u}_d'\alpha_i - \bar{v}_d')\,\{C(\omega)\,[J_o(\omega)-iJ_1(\omega)]+iJ_1(\omega)\}. \tag{37}$$

Following the nomenclature of Ref. 7 the term $C(\omega)$ is expressed as $C(\omega) = 1-(a'+ib')$. Substitution of this relation into the above and neglecting the higher order terms containing $y_{\max}\alpha_i$ and α_i^2 gives

$$\frac{\tilde{L}}{2\pi\rho W_m e^{i\nu t}} = \bar{u}_d' y_{\max}\{[(2+a)J_2(\omega)+(2-a)J_0(\omega)-2bJ_1(\omega)]+ \\ + i[2(a-1)J_1(\omega)-bJ_o(\omega)+bJ_2(\omega)]\} + \bar{u}_d'\alpha_i\{[(2-a)J_o(\omega)-bJ_1(\omega)]+ \\ + i[(a+1)J_1(\omega)-bJ_o(\omega)]\} - \bar{v}_d'\{[(1-a)J_o(\omega)-bJ_1(\omega)]+i[aJ_1(\omega)-bJ_0(\omega)]\}. \tag{38}$$

In terms of the functions $S(\omega)$, $T(\omega)$ and $T'(\omega)$, the Sears, Horlock and Holmes functions of Ref. 7, this can be rewritten as

$$\frac{\tilde{L}}{2\pi\rho W_m e^{i\nu t}} = \bar{u}_d'[\alpha_i T(\omega) + y_{\max}T'(\omega)] - \bar{v}_d' S(\omega). \tag{39}$$

Thus the expression derived in this report for the unsteady lift in a cascade of circular-arc airfoil at an incidence α_i, equation (36), reduces to the solution for an isolated circular-arc airfoil when the spacing of the cascade is allowed to become infinite.

11. *Discussion.*

An alaysis for the prediction of the unsteady lift in a cascade of airfoils which experiences an upstream disturbance has been presented. The essential features of this analysis are:

(1) The assumption of thin airfoil theory has been made thus restricting this analysis to thin airfoils with small camber operating in an incompressible, inviscid, two dimensional flow.

(2) The disturbances considered are perturbations to the steady flow and do not include the interference of adjacent blade rows. The analysis is therefore restricted to small disturbances in the incoming flow to a cascade of rotating blades.

(3) The representation of the neighbouring blades in the cascade by concentrated vortices restricts the use of this analysis to values of spacing to chord ratio, $s/c > \frac{1}{2}$ or $\sin \xi$, where ξ is the stagger angle. The reference blade of the cascade and the shed vorticity of all the blades are represented by continuous vorticity distributions. It is assumed that all trailing vorticity is transported downstream with a velocity equal to the steady velocity.

(4) Because of the restriction of $s/c > \frac{1}{2}$ or $\sin\xi$ the effects of chord wise induced velocity can be neglected as in the similar steady cascade analysis of Ref. 11. The effect of chord wise disturbance velocities are included however since for certain stagger angles these can be of the same order of magnitude as the transverse disturbances.

(5) An expression for the unsteady lift generated in a cascade of circular-arc airfoils with incidence α_i is presented in equation (36). The resulting unsteady lift is of a form similar to that of an isolated airfoil experiencing the same disturbance. The effect of the cascade is represented by the modified Theodorsen function $G(\omega, s/c, \xi)$, equation (22). For the case of infinite spacing the results of this analysis reduce to those presented in Ref. 7 for an isolated airfoil.

(6) When only the steady lift generated by a cascade of airfoils is considered this analysis reduces to that presented in Ref. 11. This condition together with the case of infinite cascade spacing serve as a check on the validity of the present analysis.

12. *Acknowledgment.*

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LIST OF SYMBOLS

a	$= \dfrac{ce^{-i\xi}}{2s}$
b	$= \dfrac{ce^{i\xi}}{2s}$
c	Chord length
$C(\omega)$	Theodorsen function
d	constant
g_c	Defined following equation (10) also (h_c, g_λ and h_λ)
$G(\omega, s/c, \xi)$	Cascade Theodorsen function, defined equation 22
$H_m^{(2)}(\omega)$	Hankel Function of second kind, order m argument ω
$J_q(\omega)$	Bessel function of first kind, order q argument ω
l	Wave length of upstream disturbance
L	Total lift
$\tilde{L}$	Unsteady lift
n	Index relating to blades in cascade (0-reference blade)
p'	Static pressure
s	Blade spacing
t	Time
u	Chordwise induced velocity
u_d	Chordwise disturbance velocity
v	Transverse induced velocity
v_d	Disturbance velocity normal to chord
V	Free stream velocity at $-\infty$
W	Relative velocity
x	Coordinates parallel to chord

x^+	Non dimensional coordinate $\left(=\frac{2}{c}x-1\right)$
$x_1{}^+$	Dummy variable of integration
y	Coordinate normal to chord
z	Defined in Appendix I
α	Angle of zero lift
α_i	Steady angle of incidence $(\beta_m-\xi)$
γ	Vorticity
γ_s	Steady local vorticity
Γ	Circulation
Δ	Non dimensional circulation $\left(=\frac{\overline{\Gamma}_o}{c}e^{i\omega}\right)$
Θ	Defined in Appendix I
λ	Chordwise coordinate in wake
ν	Frequency $\left(=2\pi\frac{\overline{U}}{l}\right)$
ξ	Stagger angle
ρ	Fluid density
τ	Phase relation between adjacent blades $(=2\pi s/l)$
ϕ	Velocity potential
Φ	Defined in Appendix I
χ	Defined in Appendix I
ω	Reduced frequency

Subscripts

c	Point of concentrated bound vorticity
d	Disturbance
m	Mean

o	Reference blade
p	Point on chord where induced velocity is calculated
s	Steady state
w	Wake
1	Blade leading edge
2	Blade trailing edge

REFERENCES

No.	*Author(s)*	*Title, etc.*
1	T. von Karman and W. R. Sears	Airfoil Theory for Non-uniform Motion. *Journal of Aeronautical Sciences*, Vol. 5, (1938) pp. 379–390.
2	J. H. Horlock	Fluctuating Lift Forces on Aerofoils Moving through Transverse and Chordwise Gusts. *Transactions ASME Journal of Basic Engineering* (Dec. 1968), pp. 494–500.
3	N. H. Kemp and W. R. Sears	Aerodynamic Interference Between Moving Blade Rows. *Journal of Aeronautical Sciences*, Vol. 20, (1953), pp. 585–598.
4	N. H. Kemp and W. R. Sears	The Unsteady Forces Due to Viscous Wakes in Turbomachinery. *Journal of Aeronautical Sciences*, Vol. 22, (1955), pp. 478–483.
5	D. S. Whitehead	Force and Moment Coefficients for Vibrating Aerofoils in Cascade. ARC R & M 3254, (1960).
6	Y. Hanamura .. .. .. and H. Tanaka	The Flexure-Torsions Flutter of Aerofoils in Cascade—Part I. Bulletin of the Japan Society of Mechanical Engineers. Vol. 10, No. 40, (1968).
7	J. H. Horlock	Unsteady Flow in Turbomachines. Reprint No. 2674, 3rd Australasian Conference on Hydraulics and Fluid Mechanics, Sydney, Australia (Nov. 1968).
8	D. S. Whitehead	Bending Flutter of Unstalled Cascade Blades at Finite Deflection. A.R.C. R. & M. 3386, (1962).
9	R. L. Bisplinghoff, H. Ashley and R. L. Halfman	*Aerolasticity*. Addison-Wesley, Reading, Mass., (1955).
10	M. D. Van Dyke	Second Order Subsonic Airfoil Theory Including Edge Effects. NACA Report, 1274, 1956.

REFERENCES—*continued*

11	K. Tanabe and J. H. Horlock	A Simple Method for Predicting the Performance of Cascades of Low Solidity. *Aeronautical Quarterly*, Vol. 18, Part 3, Aug. 1967.
12	D. S. Gradshteyn and D. M. Ryzhik	*Tables of Integrals Series and Products.* Academic Press, New York and London, (1965).
13	E. Jahnke and F. Emde ..	*Tables of Functions.* Dover Publications, New York, 4th Edition, (1945).
14	A. Pope	*Basic Wing and Airfoil Theory.* McGraw-Hill Co., New York (1951).
15	T. Theodorsen	General Theory of Aerodynamic Instability and the Mechanism of Flutter. NCA Report 496, (1949).

APPENDIX I

Cascade Influence Functions.

Consider the terms in equation (2) which relate the velocity induced by a vortex on the nth blade of the cascade at the point x_p i.e.

$$A = \frac{e^{in\tau}\, n\cos\xi}{\left(\frac{x_n - x_p}{s} + n\sin\xi\right)^2 + (n\cos\xi)^2}$$

$$\text{and } B = \frac{e^{in\tau}\left[\frac{x_n - x_p}{s} + n\sin\xi\right]}{\left(\frac{x_n - x_p}{s} + n\sin\xi\right)^2 + (n\cos\xi)^2} \qquad \text{(A.1)}$$

Following the approach presented in Ref. 8, these can be rewritten as

$$A = \tfrac{1}{2}\{e^{-i\xi}f(\chi) + e^{i\xi}f(\bar{\chi})\}$$

$$\text{and } B = -\frac{1}{2i}\{e^{-i\xi}f(\chi) - e^{i\xi}f(\bar{\chi})\} \qquad \text{(A.2)}$$

where $\chi = ize^{-i\xi}$, $\bar{\chi} = -ize^{i\xi}$, $z = \frac{x_n - x_p}{s}$ and $f(\chi) = \frac{e^{in\tau}}{\chi + n}$.

If the following definitions are made

$$\Phi(\chi) = e^{-i\xi}f(\chi) - e^{i\xi}f(\bar{\chi})$$

$$\text{and } \Theta(\chi) = e^{-i\xi}f(\chi) + e^{i\xi}f(\bar{\chi}) \qquad \text{(A.3)}$$

then

$$A = \tfrac{1}{2}\Theta(\chi)$$

$$\text{and } B = -\frac{1}{2i}\Phi(\chi).$$

Consider next a transformation of coordinates such that $x^+ = \frac{2}{c}x - 1$. Then $z^+ = \left(\frac{x_n^+ - x_p^+}{2}\right)\frac{c}{s}$ and $\chi^+ = iz^+e^{-i\xi}$. Therefore

$$A^+ = \tfrac{1}{2}\Theta(\chi^+)$$

$$\text{and } B^+ = -\frac{1}{2i}\Phi(\chi^+). \qquad \text{(A.4)}$$

The expression for $\Phi(\chi_n^+)$ can be written as

$$\Phi(\chi_n^+) = e^{in\tau}\left\{\frac{e^{-i\xi}}{i(x_n^+ - x^+)\frac{ce^{-i\xi}}{2s} + n} + \frac{e^{i\xi}}{i(x_n^+ - x^+)\frac{ce^{i\xi}}{2s} - n}\right\}$$

(A.5)

$$= \frac{2s}{ic} e^{in\tau} \left\{ \frac{1}{g_n - x^+} + \frac{1}{h_n - x^+} \right\}$$

where $g_n = x_n^+ - \frac{in}{a}$, $h_n = x_n^+ + \frac{in}{b}$, $a = \frac{ce^{-i\xi}}{2s}$ and $b = \frac{ce^{i\xi}}{2s}$.

Similarly,

$$\Theta(\chi_n^+) = \frac{2s}{ic} e^{in\tau} \left\{ \frac{1}{g_n - x^+} - \frac{1}{h_n - x^+} \right\} \tag{A.6}$$

APPENDIX II

Tabulation of Integrals.

The following is a tabulation of those integrals which are used repeatedly in this report. Also indicated are the restrictions which are placed upon their use and where appropriate the reference from which they were obtained.

1. $$\int_{-1}^{1} \sqrt{\frac{1+s'}{1-s'}} \frac{ds'}{(t-s')} = -\pi \text{ for } t^2 \leqslant 1$$ (Ref. 10)

2. $$\int_{-1}^{1} \sqrt{\frac{1+s'}{1-s'}} \frac{ds'}{(r-s')} = -\pi \left\{ 1 - \sqrt{\frac{r+1}{r-1}} \right\} \text{ for } r^2 > 1$$

3. $$\int_{1}^{\infty} \left\{ \sqrt{\frac{s'+1}{s'-1}} - 1 \right\} e^{-iks'} ds' = -\frac{\pi}{2} [H_1^{(2)}(k) + iH_0^{(2)}(k)] - \frac{e^{-ik}}{ik}$$ (Ref. 12)

4. $$\int_{-1}^{\sigma} \sqrt{\frac{1-s'}{1+s'}} \frac{ds'}{(r-s')} = \frac{\pi}{2} + \sin^{-1}\sigma + \sqrt{\frac{r-1}{r+1}} \Lambda(\sigma, r) \text{ for } r^2 > 1$$

   where $\Lambda(\sigma, r) = 2\tan^{-1}\left\{ \sqrt{\left(\frac{1-\sigma}{1+\sigma}\right)\left(\frac{r+1}{r-1}\right)} \right\} - \pi$ (Ref. 12)

5. $$\int_{-1}^{\sigma} \sqrt{\frac{1-s'}{1+s'}} \frac{ds'}{(r-s')} = \frac{\pi}{2} + \sin^{-1}\sigma + \sqrt{\frac{1-r}{1+r}} \Omega(\sigma, r) \text{ for } r^2 < 1$$

   where $\Omega(\sigma, r) = \frac{1}{2} \ln \left\{ \frac{1 - \sigma r + \sqrt{1-r^2}\sqrt{1-\sigma^2}}{1 - \sigma r - \sqrt{1-r^2}\sqrt{1-\sigma^2}} \right\}$. (Ref. 12)

6. $$\int_{1}^{\infty} \Lambda(\sigma, r) e^{-iks'} ds' = \frac{e^{-ik}}{ik} \left\{ 2\tan^{-1}\left[\sqrt{\left(\frac{1-\sigma}{1+\sigma}\right)\left(\frac{\bar{r}+1}{\bar{r}-1}\right)} \right] - \pi \right.$$

   $$\left. + \frac{1}{ik} \int_{1}^{\infty} \sqrt{\frac{1-\sigma}{1+\sigma}} \left[\frac{1}{\sqrt{r^2-1}} + \sqrt{\frac{r+1}{r-1}} \frac{1}{(\sigma - r)} \right] e^{-iks'} ds' \right.$$ where $r = s' + d$, $\bar{r} = 1 + d$.

7. $$\int_{-1}^{1} \sqrt{\frac{1-s'}{1+s'}}\, ds' = \pi.$$

8. $$\int_{-1}^{1} \Omega(s', r) ds' = \pi\sqrt{1-r^2} \text{ where } r^2 \leqslant 1.$$

9. $$\int_{1}^{\infty} \frac{e^{-iks'}}{\sqrt{s'^2-1}}\, ds' = -i\frac{\pi}{2} H_0^{(2)}(k).$$ (Ref. 12)

10. $$\int_{-1}^{1} \sqrt{\frac{1+s'}{1-s'}}\, e^{-iks'} ds' = \pi\left[J_o(k) - J_1(k)\right].$$ (Ref. 12)

11. $$\int_{-1}^{1} \sqrt{1-s'^2}\, e^{-iks'} ds' = \frac{\pi}{k} J_1(k).$$

12. $$\int_{-1}^{1} \sqrt{\frac{1-s'}{1+s'}}\, e^{-iks'} ds' = \pi\left[J_o(k) + iJ_1(k)\right].$$ (Ref. 12)

13. $$\int_{-1}^{1} \frac{s'^2 ds'}{\sqrt{1-s'^2}\,(r-s')} = -\pi.$$ (Ref. 10)

14. $$\int_{-1}^{1} \frac{s'^2 ds'}{\sqrt{1-s'^2}\,(r-s')} = -\pi r.$$ (Ref. 10)

15. $$\int_{-1}^{1} \sqrt{\frac{1-s'}{1+s'}}\, s' e^{-iks'} ds' = \frac{\pi}{2}\left[J_2(k) - J_o(k) - i2J_1(k)\right].$$

16. $$\int_{-1}^{1} \sqrt{\frac{1+s'}{1-s'}}\, s' e^{-iks'} ds' = \frac{\pi}{2}\left[J_o(k) - J_2(k) - i2J_1(k)\right].$$

17. $$\int_{-1}^{1} s'\sqrt{1-s'^2}\, e^{-iks'} ds' = -\frac{i\pi}{4}\left[J_1(k) + J_3(k)\right].$$

18. $$\int_{-1}^{1} \sqrt{\frac{1-s'}{1+s'}}\, \frac{e^{-iks'}}{(r-s')}\, ds' = \pi\left\{ J_o(k) - \sqrt{\frac{r-1}{r+1}} - 2r^2\sqrt{\frac{r-1}{r+1}} \sum_{1}^{\infty} (-1)^t\, J_t(k)\{\sqrt{r^2-1} - r\}^{t+} \right\}$$

19. $\displaystyle\int_{-1}^{1}\sqrt{\frac{1+s'}{1-s'}}\,s'ds' = \frac{\pi}{2}.$

APPENDIX III

Cascade Functions.

The following is a tabulation of those functions which arise in the solution of the unsteady lift in a cascade and are not represented by a common mathematical function.

1. $C_1 = \sqrt{\dfrac{g_c+1}{g_c-1}}.$

2. $C_2 = \sqrt{\dfrac{h_c+1}{h_c-1}}.$

3. $D_1 = \displaystyle\int_{1}^{\infty}\left\{\sqrt{\frac{g_\lambda+1}{g_\lambda-1}}-1\right\} e^{-i\omega\lambda^+}\,d\lambda^+ .$

4. $D_2 = \displaystyle\int_{1}^{\infty}\left\{\sqrt{\frac{h_\lambda+1}{h_\lambda-1}}-1\right\} e^{-i\omega\lambda^+}\,d\lambda^+ .$

5. $A_1 = \displaystyle\int_{-1}^{1}\tan^{-1}\left\{\sqrt{\left(\frac{1-\sigma^+}{1+\sigma^+}\right)\left(\frac{\dfrac{n}{ib}-2}{\dfrac{n}{ib}}\right)}\right\} d\sigma^+ .$

6. $A_2 = \displaystyle\int_{-1}^{1}\tan^{-1}\left\{\sqrt{\left(\frac{1-\sigma^+}{1+\sigma^+}\right)\left(\frac{\dfrac{n}{ia}+2}{\dfrac{n}{ia}}\right)}\right\} d\sigma^+ .$

7. $A_3 = \displaystyle\int_{-1}^{1}\tan^{-1}\left\{\sqrt{\left(\frac{1-\sigma^+}{1+\sigma^+}\right)\frac{\left(x_c^+ +1-\dfrac{n}{ib}\right)}{\left(x_c^+ -1-\dfrac{n}{ib}\right)}}\right\} d\sigma^+ .$

8. $A_4 = \displaystyle\int_{-1}^{1}\tan^{-1}\left\{\sqrt{\left(\frac{1-\sigma^+}{1+\sigma^+}\right)\frac{\left(x_c^+ +1+\dfrac{n}{ia}\right)}{\left(x_c^+ -1+\dfrac{n}{ia}\right)}}\right\} d\sigma^+ .$

9. $B_1 = \int_1^\infty \frac{e^{-i\omega\lambda^+}}{\sqrt{g_\lambda^2-1}} d\lambda^+ .$

10. $B_2 = \int_1^\infty \frac{e^{-i\omega\lambda^+}}{\sqrt{h_\lambda^2-1}} d\lambda^+ .$

11. $L = g_c^2\{\sqrt{g_c^2-1}-g_c\}+h_c^2\{\sqrt{h_c^2-1}-h_c\} .$

APPENDIX IV

Pressure Difference Across Airfoil.

To obtain the lift on an airfoil it is necessary to calculate the difference in pressure across the airfoil. This is done using the unsteady Bernoulli equation for a flow in which a velocity potential ϕ is assumed to exist. This can be written as

$$\frac{\partial \phi}{\partial t}+\tfrac{1}{2}q^2+\frac{p'}{\rho} = f(t) \tag{D.1}$$

where q is the total velocity along the airfoil. The pressure difference $\Delta p' = p'_{upper}-p'_{lower}$, then becomes

$$\Delta p' = -\rho\left\{ \frac{\partial}{\partial t}(\phi_u-\phi_l)+\tfrac{1}{2}(q_u^2-q_l^2) \right\} \tag{D.2}$$

In general, $q^2=(V+u)^2+v^2$ where V is the free stream velocity along the chord and u and v the perturbation velocities of order ε parallel and normal to the chord. Therefore,

$$q_u^2-q_l^2 = 2V(u_u-u_l)+(u_u^2-u_l^2)+(v_u^2-v_l^2).$$

Now, $u_u=u_d+\gamma/2$ and $u_l=u_d-\gamma/2$ where u_d is the disturbance velocity carried with the free stream velocity and γ the local vorticity. Since the airfoil is a solid boundary, $v_u = -v_l$. Thus neglecting velocities of order ε^2,

$$q_u^2-q_l^2 = 2V(u_u-u_l)+2u_d\gamma = 2(V+u_d)\gamma \tag{D.3}$$

since $u_u-u_l = \gamma$. From this relation, $\phi_u-\phi_l = \frac{c}{2}\int_{-1}^{\sigma^+}\gamma dx_1^+$. Therefore the pressure difference at a point σ^+ is

$$-\Delta p'(\sigma^+) = \rho(V+u_d)\,\gamma(\sigma^+)+\rho\frac{c}{2}\frac{\partial}{\partial t}\int_{-1}^{\sigma^+}\gamma(x_1^+)\,dx_1^+ .$$

The vorticity $\gamma(\sigma^+)$ is made up of a steady and unsteady part, i.e. $\gamma(\sigma^+) = \gamma_s(\sigma^+)+\gamma_{u.s.}(\sigma^+)$. As with the usual linearized unsteady flow theories $\gamma_s = 0(1)$ and $\gamma_{u.s.} = 0(\varepsilon)$. Since $u_d = 0(\varepsilon)$ the neglect of terms $0(\varepsilon^2)$ gives

$$-\Delta' p(\sigma^+) = \rho V\gamma(\sigma^+)+\rho u_d\gamma_s(\sigma^+)+\rho\frac{c}{2}\frac{\partial}{\partial t}\int_{-1}^{\sigma^+}\gamma(x_1^+)dx_1^+. \tag{D.4}$$

APPENDIX V

Modified Theodorsen Function for the Case of Infinite Spacing.

The case of infinite spacing, $s = \infty$, represents a unique condition for the modified Theodorsen Function, $G(\omega, s/c, \xi)$ equation (22). At this condition, $G(\omega, s/c, \xi)$ can be shown to reduce to the familiar Theodorsen Function $C(\omega)$ associated with an isolated airfoil.

To examine the value of $G(\omega, s/c, \xi)$ it is necessary to consider the limits of the cascade functions, Appendix III, as $s \to \infty$. This requires the determinate of $\lim_{s\to\infty}\sqrt{\frac{g_\lambda+1}{g_\lambda-1}}$ and $\lim_{s\to\infty}\sqrt{\frac{h_\lambda+1}{h_\lambda-1}}$. The first of the limits can be written as

$$\lim_{s\to\infty}\sqrt{\frac{g_\lambda+1}{g_\lambda-1}} = \lim_{s\to\infty}\left\{\frac{\lambda^+ +1+\frac{2ns}{c}\sin\xi - i\frac{2ns}{c}\cos\xi}{\lambda^+ -1+\frac{2ns}{c}\sin\xi - i\frac{2ns}{c}\cos\xi}\right\}^{\frac{1}{2}}.$$

Dividing both the numerator and denominator of this expression by s gives

$$\lim_{s\to\infty}\sqrt{\frac{g_\lambda+1}{g_\lambda-1}} = \lim_{s\to\infty}\left\{\frac{\frac{\lambda^+}{s}+\frac{1}{s}+\frac{2n}{c}\sin\xi - i\frac{2n}{c}\cos\xi}{\frac{\lambda^+}{s}-\frac{1}{s}+\frac{2n}{c}\sin\xi - i\frac{2n}{c}\cos\xi}\right\}^{\frac{1}{2}} = 1.$$

Similarly,

$$\lim_{s\to\infty}\sqrt{\frac{h_\lambda+1}{h_\lambda-1}} = 1, \quad \lim_{s\to\infty}\sqrt{\frac{g_c+1}{g_c-1}} = 1 \text{ and } \lim_{s\to\infty}\sqrt{\frac{h_c+1}{h_c-1}} = 1.$$

From Appendix III the following values of the cascade functions are obtained as

$$\lim_{s\to\infty} C_1 = 1 \qquad \lim_{s\to\infty} A_1 = \int_{-1}^{1}\tan^{-1}\sqrt{\frac{1-\sigma^+}{1+\sigma^+}}d\sigma^+,$$

$$\lim_{s\to\infty} C_1 = 1 \qquad \lim_{s\to\infty} A_2 = \int_{-1}^{1}\tan^{-1}\sqrt{\frac{1-\sigma^+}{1+\sigma^+}}d\sigma^+,$$

$$\lim_{s \to \infty} D_1 = 0 \qquad \lim_{s \to \infty} A_3 = \int_{-1}^{1} \tan^{-1} \sqrt{\frac{1-\sigma^+}{1+\sigma^+}}\, d\sigma^+ ,$$

$$\lim_{s \to \infty} D_2 = 0 \qquad \lim_{s \to \infty} A_4 = \int_{-1}^{1} \tan^{-1} \sqrt{\frac{1-\sigma^+}{1+\sigma^+}}\, d\sigma^+ .$$

The remaining cascade functions B_1 and B_2 involve a term of the form $\{\sqrt{g_\lambda^2 - 1}\}^{-1}$ which becomes zero when $s \to \infty$. Hence

$$\lim_{s \to \infty} B_1 = 0 \text{ and } \lim_{s \to \infty} B_2 = 0.$$

Substitution of the above limits into equation (22) gives the following expression for $G(\omega, s/c, \xi)$,

$$G(\omega, \infty, \xi) = 1 + \frac{\omega\pi\, H_0^{(2)}(\omega)}{i\omega\pi \left[H_1^{(2)}(\omega) + iH_0^{(2)}(\omega)\right]}$$

$$= \frac{H_1^{(2)}(\omega)}{H_1^{(2)}(\omega) + iH_0^{(2)}(\omega)} .$$

This latter expression is identically equal to the Theodorsen function $C(\omega)$. It should be noted that the term $e^{in\tau}$ involves the spacing s and is undefined as $s \to \infty$ since it represents the summation of sines and cosines. However its value does remain finite and drops out through the multiplication by zero.

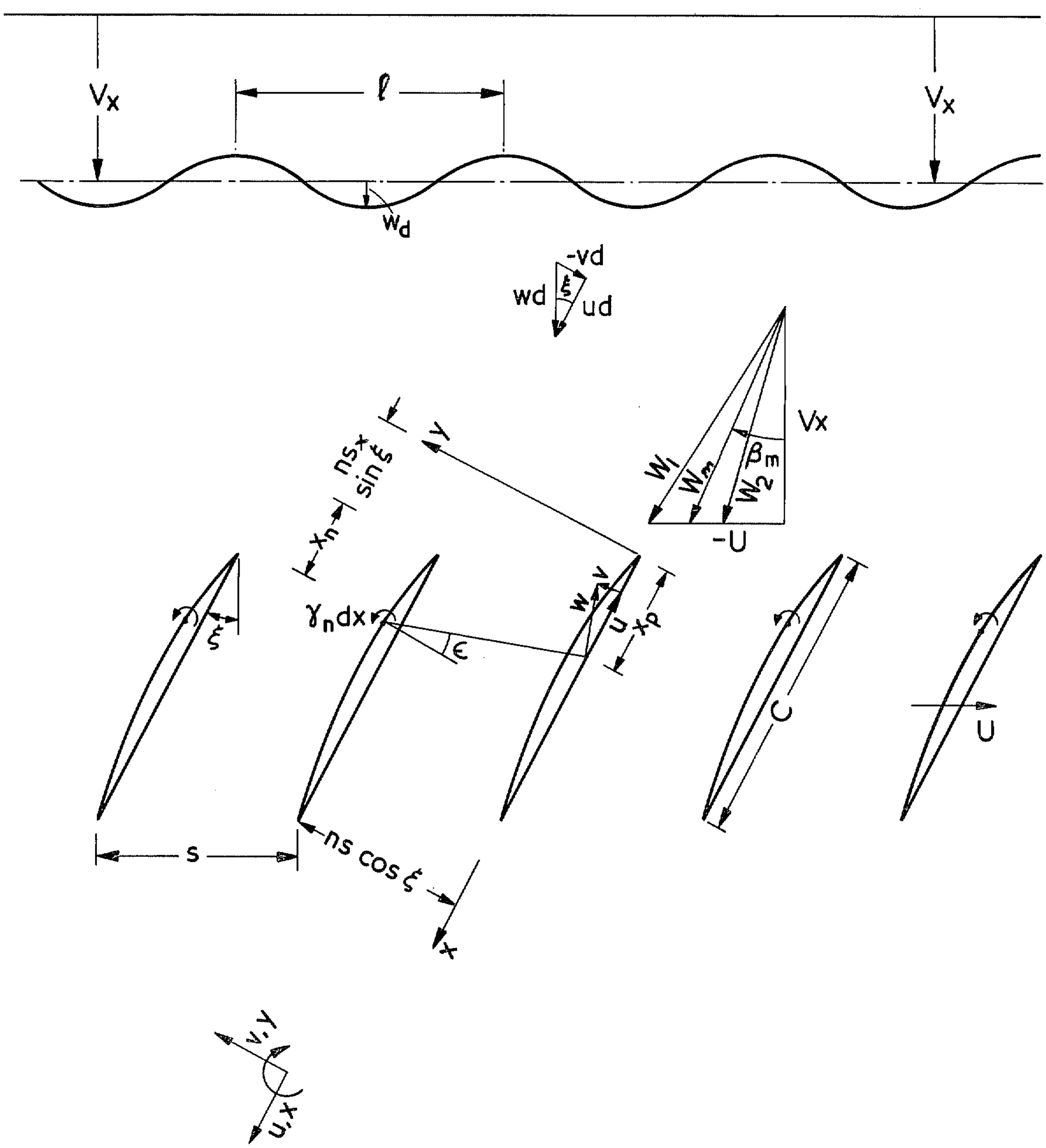

FIG. 1. Cascade of blades moving through a disturbance in the flow.

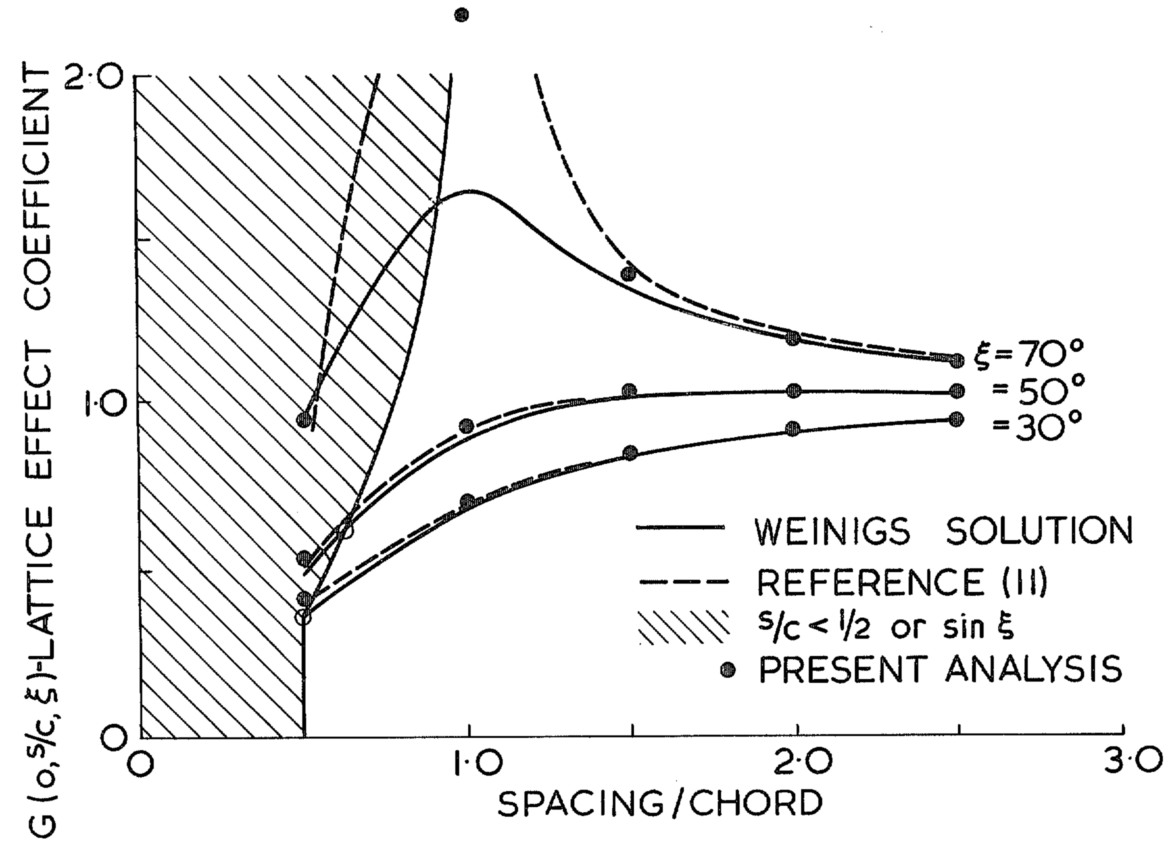

FIG. 2. Comparison of the exact and approximate solutions for steady flow.

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