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Aircraft Centre of Gravity Response to Two-Dimensional Spectra of Turbulence

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Summary.

The energy spectrum of atmospheric turbulence is used to predict the normal acceleration response of a rigid aircraft. Particular reference is made to spanwise variations of gust velocity and this effect is described in terms of a general input spectrum which is applicable to any wing with a particular spanwise loading distribution operating under a wide range of conditions. Tables of the gust response factor and the number of zero crossings are presented for a range of parameters.

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1. *Introduction.*

The loads imposed on an aircraft by atmospheric turbulence are often predicted by comparison with the loads measured on another aircraft. This procedure requires that it must be possible to derive from the measurements of an aircraft response, such as the normal acceleration at the centre of gravity, the characteristics of the assumed turbulence model producing it and then to determine the response of the second aircraft to the same disturbances.

When comparing aircraft of a generally similar type under the same conditions adequate results may be obtained with simple representations of the atmospheric turbulence and the aircraft dynamics. For many years the discrete gust method[1] has been used, which assumes that observed loads may be referred to isolated gusts, and more recently spectral techniques have become established. These require that the turbulence is continuous and possesses a mathematically defined uniformity. When an aircraft is within a particular patch of turbulence it is commonly encountering disturbances that satisfy these conditions.

The continuous approach has been described in many papers, for example Refs. 2–7, which have made similar assumptions to those used here for the dynamics (basically that the aircraft and its motion form a linear system). In this study the spanwise variations of gust velocities are incorporated in the atmospheric turbulence model and this has an appreciable effect on the response at high frequencies. The number of times that a value of a response is crossed per unit distance is sensitive to the high frequency part of the spectrum and so the inclusion of spanwise variations facilitates the study of this concept. The crossings of a particular value may be described by reference to the parameter (zero crossings) defined by the number of times that the mean value is crossed in each direction per unit distance.

The model proposed by Von Kármán is considered to give a good general description of atmospheric turbulence, and various functions associated with it are given in Section 2, together with the corresponding expressions for the Dryden model which has often been used. In Section 3 forms of the transfer function are developed to relate the turbulence input to the aircraft acceleration response. The gust response and zero crossings factors may then be found and Section 4 contains a description of the methods used to calculate the results presented in the tables.

2. *Mathematical Model of Turbulence.*

In order to derive the responses of an aircraft to atmospheric turbulence it is necessary to make some basic assumptions on the nature of the atmospheric motions to be considered in the analysis. These are that the turbulence is a stationary random process which is both homogeneous in the horizontal plane containing the flight path and symmetrical about a vertical axis. These conditions imply that the statistical characteristics of the turbulence are invariant under a rotation of co-ordinates about a vertical axis and a horizontal translation of the origin. A time displacement t is equivalent to a distance displacement x in the direction of motion

$$x = Ut \tag{1}$$

subject to the restrictions that the aircraft speed remains approximately constant with mean U during this time and is much larger than the magnitude of the gust velocity component along the x axis.

It is also assumed that the forces produced by the motion and the disturbances are linear and hence superposable.

2.1. *Statistical Functions in One-Dimension.*

Functions are defined by reference to co-ordinate axes $Oxyz$ moving with the aircraft, the Ox axis being directed along the mean flight path, the Oz axis along the upward vertical and the origin O on the aircraft centreline at the position of the leading edge of the wing at its root. Perturbation velocities in the x, y, z directions are denoted by u, v, w respectively.

Relationships for a one-dimensional random process are demonstrated here in terms of the vertical component $w(x)$ of gust velocity along any horizontal axis Ox.

The correlation function is defined by

$$f_w(x) = \overline{w(r)\, w(x+r)} \tag{2}$$

where the mean is defined by the space average of this product over all positions r on the x axis, i.e.

$$\overline{w(r)\, w(x+r)} = \lim_{R\to\infty} \frac{1}{2R} \int_{-R}^{R} w(r)\, w(x+r)\, dr\,.$$

In particular $f_w(0)$ equals σ_w^2, the mean square of w.

The Fourier transform of the autocorrelation function is given by

$$S_w(k) = \int_{-\infty}^{\infty} f_w(x)\, e^{-ikx}\, dx \tag{3}$$

with the corresponding inverse

$$f_w(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_w(k)\, e^{ikx}\, dk\,, \tag{4}$$

this arrangement of the constant being used for all one-dimensional Fourier transforms in this paper. $S_w(k)$ is the one-dimensional spectrum of lateral turbulence energy per unit mass. The spectrum is real, positive and an even function of the wave number k. The total turbulence energy per unit mass ($= \frac{1}{2}\sigma_w^2$) is given by

$$\tfrac{1}{2}\sigma_w^2 = \frac{1}{2\pi} \int_{0}^{\infty} S_w(k)\, dk\,. \tag{5}$$

The spectrum has been defined in terms of the wave number k rather than inverse wavelength ($= k/(2\pi)$) or circular frequency ($= k\,U$) or frequency ($= k\,U/(2\pi)$) for convenience in the analysis.

An analytical measure of the scale of the turbulence, L, is derived from the correlation function:

$$L = \frac{2}{\sigma_w^2} \int_{0}^{\infty} f_w(r)\, dr\,. \tag{6}$$

The spectrum defines the energy density at all wave numbers but this is not directly related to the frequency with which a particular velocity will occur. If it is assumed that the distribution of velocities is a normal distribution then Rice[8] has shown that the number of times that different velocities are crossed has also a normal distribution, which is given by

$$N_w = N_0 \exp(-w^2/2\sigma_w^2) \tag{7}$$

where N_w is the number per unit distance of crossings in each direction of velocity w and N_0 is the number of times the zero value is crossed in each direction in unit distance. N_0 is related to the energy spectrum by

$$N_0 = \frac{1}{2\pi}\left[\frac{\int_0^\infty k^2 S_w(k)\,dk}{\int_0^\infty S_w(k)\,dk}\right]^{\frac{1}{2}} \tag{8}$$

This result may be applied to any random process for which the random variable and its first derivative are independent and each have a Gaussian distribution. However, its usefulness is dependent upon the high frequency behaviour of the spectrum being such that the integral converges. While N_0 has been defined using the turbulence spectrum as an example zero crossings are only evaluated for responses, since for most mathematical forms of atmospheric turbulence spectra the integral does not converge. This is because the expressions used do not adequately describe the behaviour of the spectrum in the viscous range. Aircraft responses to these small eddies with low wavelengths are negligible and so equation (8) can be used to evaluate the response number of crossings.

2.2. *Two-Dimensional Spectra.*

If the aircraft being considered was small compared with the scale of the turbulence in which it was flying the gust velocity at all points on its surface at any instant would be approximately the same. The one-dimensional spectrum $S_w(k)$ describes the properties of the vertical component of this velocity.

The effect of variations of gust velocity over the aircraft surface is now studied, this being required for all aircraft of a size that is not negligible compared with the turbulence scale. Since the largest dimensions ofn aircraft are in the horizontal plane and the only deviations from the flight path to be examined here are comparatively small perturbations in the vertical direction, the turbulence spectrum is required in a form relating vertical components of gusts at different points in a horizontal plane. The spectrum is expressed in terms of a two-dimensional wave number vector with components k_1 and k_2 in the x and y directions respectively. Under the assumption of axisymmetry the correlation function $f_w(x,y)$ associated with this spectrum is the same as that for one dimension along a horizontal axis in the appropriate direction, i.e. $f_w(\sqrt{x^2+y^2})$.

The spectrum and correlation function are related by a Fourier transform in two dimensions corresponding to equations (3) and (4):

$$S_w(k_1, k_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1 x+k_2 y)} f_w(x, y)\,dx\,dy \tag{9}$$

$$f_w(x, y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1 x+k_2 y)} S_w(k_1, k_2)\,dk_1\,dk_2 . \tag{10}$$

2.3. *Specific Turbulence Energy Spectra.*

Taylor[5] gives expressions for the energy distributions and correlation functions in one and two dimensions in terms of a general longitudinal correlation function involving shape and scale parameters. By choosing particular values of these the atmospheric turbulence models proposed by Dryden and Von Kármán are obtained.

For the Von Kármán model the lateral correlation function is

$$f_w(r) = (4r/L_1)^{1/3}\,\sigma_w^2\,[K_{1/3}(r/L_1) - \tfrac{1}{2}(r/L_1)\,K_{-2/3}(r/L_1)]/\Gamma\left(\frac{1}{3}\right) \tag{11}$$

and the lateral spectra are

$$S_w(k) = \sigma_w^2 L\left[1 + \left(\frac{8}{3}\right)(k\,L_1)^2\right]\Big/[1 + (k\,L_1)^2]^{11/6} \tag{12}$$

$$S_w(k_1, k_2) = \frac{16}{9}\sigma_w^2\,\pi\,L_1^4\,(k_1^2 + k_2^2)\Big/[1 + L_1^2\,(k_1^2 + k_2^2)]^{7/3}\,, \tag{13}$$

where $L_1 = 1{\cdot}339L$.

The corresponding functions for the Dryden model are

$$f_w(r) = \sigma_w^2\,(1 - \tfrac{1}{2}\,|\,r\,|\,/L)\,exp\,(-\,|\,r\,|\,/L)$$

$$S_w(k) = \sigma_w^2\,L[1 + 3\,(k\,L)^2]\,/\,[1 + (k\,L)^2]^2$$

$$S_w(k_1, k_2) = 3\,\sigma_w^2\,\pi\,L^4\,(k_1^2 + k_2^2)\,/\,[1 + L^2\,(k_1^2 + k_2^2)]^{5/2}\,.$$

The energy density given by the Von Kármán model varies with the $-5/3$ power of frequency at high frequencies while the corresponding power for the Dryden model is -2. Practical and theoretical studies, as summarised by Taylor[5] and Tatarski[9] for example, indicate that the $-5/3$ power law at high frequencies is a close representation of atmospheric turbulence. In analytic studies the simpler form of the Dryden model favours the use of these functions but the more accurate model is used here in conjunction with numerical methods.

3. *Calculation of Frequency Response Function.*

3.1. *Response Factors.*

The aircraft response with which this Report is directly concerned is the normal acceleration at the centre of gravity, which is studied throughout under the assumptions that the aircraft is rigid and responds only in heave. A statistical description of this acceleration is given by its root mean square σ_a and the number of zero crossings per unit distance, which values may be obtained by integration of the normal acceleration spectrum $S_a(k)$. It is convenient to refer to these quantities in terms of the gust response factor K and a zero crossings factor M_0 defined by

$$K = (\mu\,\bar{c}/U)\,\sigma_a/\sigma_w \tag{14}$$

$$M_0 = \bar{c}\,K\,N_0 \tag{15}$$

and by analogy with equations (5) and (8) the factors are related to the spectrum $S_a(k)$ by

$$K^2 = \left(\frac{\mu\bar{c}}{U\sigma_w}\right)^2 \frac{1}{\pi}\int_0^\infty S_a(k)\,dk \tag{16}$$

$$M_0^2 = \left(\frac{\bar{c}}{2\pi}\frac{\mu\bar{c}}{U\sigma_w}\right)^2 \frac{1}{\pi}\int_0^\infty k^2 S_a(k)\, dk \tag{17}$$

$$\text{where } \mu = \frac{2W}{S\rho g \bar{c} a}.$$

Under this definition M_0 divided by K equals the number of zero crossings in each direction per mean chord length. By studying the parameter M_0 instead of N_0 it is possible to separate the effect on K and M_0 of different terms in the acceler ˙ɔn spectrum. For example, while the gust response factor is found to vary greatly with the mass parameter the zero crossings factor is almost independent of it, and so the effects of changes in this parameter need only be studied closely in the former case.

3.2. *Simple Response Function.*

The normal acceleration at any instant is related to the gust velocity through the equation of motion. From this equation the frequency response, or transfer function must be derived in order to relate the spectra of these random variables. Generally, if $r_1(x)$ and $r_2(x)$ are the input and output functions of some linear system then the output spectrum $S_2(k)$ is connected to the input spectrum $S_1(k)$ by

$$S_2(k) = |\, T_{12}(k)\,|^2\, S_1(k) \tag{18}$$

where the transfer function $T_{12}(k)$ is the response of the system to a unit sinusoidal input and is the ratio of the Fourier transforms of the appropriate variables,

$$T_{12}(k) = F\{r_2(x)\}/F\{r_1(x)\}\,. \tag{19}$$

The transfer function for the simplest form of the equation of motion is found in this Section and in the following Sections it is extended to more precise representations of the physical system.

In addition to the assumptions made throughout this Report the following simplifications are made in this section. The aircraft is considered to be sufficiently small compared with the scale of turbulence for the gust velocity at any instant to be the same at all points on the surface and it is assumed that this gust velocity produces corresponding lift changes instantaneously. For this system the lift of the wing at any instant is $\frac{1}{2}\rho\, U\, S\, a\,(w-w_r)$ where w, w_r are the instantaneous values of the vertical gust velocity and the vertical response velocity of the aircraft. The equation of motion is thus

$$\left(\frac{d}{dx}+\frac{1}{\mu\bar{c}}\right) w_r = l \tag{20}$$

where μ is the mass parameter and the lift parameter l is a function of x describing the variation of that part of the lift due to the vertical component of gust velocity. This lift force equals $(UW/g)l$, where the factor UW/g is taken to be constant for the duration of a turbulence encounter. The transformation (1) is used to replace time by distance as the variable of the differential in this equation.

Application of equation (19) yields the transfer function from the lift parameter to normal acceleration:

$$T_{la}(k) = \frac{i\,k\,U}{i\,k + 1/(\mu\bar{c})} \tag{21}$$

and by (18)

$$S_a(k) = \frac{k^2 U^2}{k^2 + 1/(\mu\bar{c})^2} S_l(k). \tag{22}$$

Under the approximations of this section the lift parameter is directly proportional to the vertical component of gust velocity, $l = w/(\mu\bar{c})$ and so

$$S_l(k) = S_w(k)/(\mu\bar{c})^2 \tag{23}$$

giving the transfer function from gust velocity to normal acceleration and the required spectrum as:

$$T_{wa}(k) = \frac{i\,k\,U}{i\,k\,\mu\bar{c} + 1} \tag{24}$$

$$S_a(k) = \frac{k^2 U^2}{(k\,\mu\bar{c})^2 + 1} S_w(k). \tag{25}$$

The factors that have been neglected in deriving this result all tend to reduce the predicted acceleration response to the high frequency part of the input spectrum and, since this part dominates the integral for the zero crossings factor, no useful information can be obtained about this quantity. It is possible to calculate an approximate gust response factor and for some turbulence models this may be carried out analytically by contour integration or other methods, such as that given by Huntley[6].

3.3. *Response Function Including Unsteady Lift.*

The transfer function represents the physical system more closely, particularly for high frequencies, when unsteady lift is included in the equation of motion. These aerodynamic effects occur in two forms, the Küssner function describing the variation in lift following a change in vertical gust velocity and the Wagner effect representing that variation after a change in aircraft vertical velocity.

These functions have been approximated by exponential terms for various aspect ratios and planforms. For example, Zbrozek[1] has tabulated them for aspect ratios of 3 and 6 and for the two-dimensional case, with compressibility included in the latter case for Mach numbers up to 0·7. In the form given these functions describe the transient response to unit step change in the corresponding vertical velocity so that a transformation is required for use in this analysis. The Küssner function $\psi(x)$ and the impulse response function $h(x)$ represent the lift when the leading edge has moved a distance x after encountering, respectively, a step and an impulsive increase in vertical gust velocity. Since $H(k)$, the Fourier transform of $h(x)$, is given by

$$H(k) = 1 + i\,k \int_0^\infty e^{-ikx} (\psi(x) - 1)\, dx$$

and the unsteady lift functions have been approximated in the form

$$\psi(x) = 1 - \sum_j A_j\, e^{-B_j x/\bar{c}} \tag{26}$$

it follows that the corresponding $H(k)$ takes the form

$$H(k) = 1 - i\,k \sum_j \frac{A_j}{i\,k + B_j/\bar{c}} \tag{27}$$

$|H(k)|^2$ tends to zero like k^{-2} for high wave numbers if the constants A_j are such that $\Sigma A_j = 1$. This implies that $\psi(0) = 0$, which is in fact satisfied by the exact Küssner function; for some approximations, such as those given by Jones[10] and by Zbrozek for finite aspect ratios the curves are not fitted through the origin (since this was not important in the uses for which they were derived). It is found here that sets of A_j values which do not satisfy this condition produce considerable error in determining the zero crossings factor but have less effect on the gust response factor. For the evaluations in this Report the function quoted by Zbrozek with two terms and applicable to the two-dimensional case has been used, this being

$$\psi(x) = 1 - 0{\cdot}5\, e^{-0{\cdot}26x/\bar{c}} - 0{\cdot}5\, e^{-2x/\bar{c}}$$

the Wagner function

$$\phi(x) = 1 - 0{\cdot}458\, e^{-0{\cdot}265x/\bar{c}}$$

The effect of compressibility has been studied using the constants given in Table 1 for a maximum Mach number of 0·7. The three term function for incompressible two-dimensional flow does not satisfy the required condition for $x = 0$ but the gust response factors obtained with it are found to be very close to those for the function quoted above. Drischler[11] summarises these functions and also some that have been obtained for supersonic flow over various wing planforms.

The value of the lift parameter is a function of the gust velocities at all previous points on the flight path:

$$l(x) = \frac{1}{\mu\bar{c}} \int_{-\infty}^{x} w(x_1)\, h(x - x_1)\, dx_1$$

which with a change of origin becomes

$$l(x) = \frac{1}{\mu\bar{c}} \int_{0}^{\infty} w(x - x_1)\, h(x_1)\, dx_1 \tag{28}$$

and so

$$F\{l(x)\} = H(k)\, F\{w(x)/(\mu\bar{c})\}\,.$$

Thus the relation between the lift and gust spectra (i.e. the unsteady lift form of equation (23)) is

$$S_l(k) = S_w(k)\, |H(k)|^2/(\mu\bar{c})^2\,. \tag{29}$$

Both velocity terms in the equation of motion are modified in this way and the improved form of equation (24) is

$$T_{wa}(k) = \frac{i\,k\,U\,H_1(k)}{i\,k\,\mu\bar{c} + H_2(k)} \tag{30}$$

$H_1(k)$ and $H_2(k)$ are the Fourier transforms corresponding to the Küssner and Wagner functions and may be evaluated using (27) for appropriate values of the summation constants A_j and B_j.

3.4. *Spanwise Variation of Gust Velocity.*

In this Section a transfer function is derived that is applicable to an aircraft flying through turbulence of a scale such that the spanwise variations of the gust velocity may not be neglected. The general assumptions that have been used above are retained but the vertical component of turbulence energy is now represented by its two-dimensional spectrum..

The gust velocity varies with both the spanwise position y and the distance x along the flight path and is thus denoted by $w(x, y)$. Neglecting unsteady aerodynamic effects the instantaneous lift due to this gust velocity acting on an elementary strip (width σy) of the wing is

$$\tfrac{1}{2}\rho\, U\, a\, \bar{c}\, \gamma(y)\, w(x, y)\, \delta y$$

and with unsteady effects included the lift force when the leading edge of the wing is at x is

$$\tfrac{1}{2}\rho\, U\, a\, \bar{c}\, \gamma(y)\, \delta y \int_0^\infty h(x_1, y)\, w(x-x_1, y)\, dx_1 \tag{31}$$

where $h(x, y)$ is the local unsteady lift function giving the fractional value of the lift of this element when its leading edge has moved a distance x after encountering a gust velocity impulse. $\gamma(y)$ is the spanwise loading coefficient, defined as the ratio of the local value of chord times lift coefficient to the mean value for the whole wing.

If the wing has negligible sweep the local unsteady lift functions at all points across the span may be referred to the same origin on the x-axis and the lift of the whole wing is given by

$$\frac{U\, W}{g} l(x) = \int_{-b/2}^{b/2} \tfrac{1}{2}\rho\, U\, a\, \bar{c}\, \gamma(y) \int_0^\infty h(x_1, y)\, w(x-x_1, y)\, dx_1\, dy\,. \tag{32}$$

This defines the lift parameter of equation (20) as a function of position along the flight path; it is an extended form of (28). To obtain the statistical description of the response of the aircraft the spectrum of $l(x)$ must be related to the two-dimensional lateral turbulence energy spectrum.

The correlation function is obtained by the definition given in equation (2),

$$f_l(x) = \frac{1}{(\mu S)^2} \int_{-b/2}^{b/2} \int_{-b/2}^{b/2} \gamma(y_1)\, \gamma(y_2) \int_0^\infty \int_0^\infty h(x_1, y_1) \times$$

$$\times\ h(x_2, y_2)\, \overline{w(r-x_1, y_1)\, w(r+x-x_2, y_2)}\, dx_1\, dx_2\, dy_1\, dy_2$$

If the turbulence is a stationary random process the correlation function depends only upon the difference of the space co-ordinates and so

$$\overline{w(r-x_1, y_1)\, w(r+x-x_2, y_2)} = f_w(x+x_1-x_2, y_2-y_1)$$

and by using equation (10)

$$f_l(x) = \frac{1}{(2\pi\, \mu\, \bar{c})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{ik_1 x}\, S_w(k_1, k_2)\, |F_2(k_1, k_2)|^2\, dk_1\, dk_2$$

where

$$F_2(k_1, k_2) = \frac{1}{b}\int_{-b/2}^{b/2}\int_{-\infty}^{\infty} \gamma(y)\, h(x, y) e^{-i(k_1 x + k_2 y)}\, dx\, dy\,. \tag{33}$$

Hence the spectrum of the lift parameter is

$$S_l(k) = \frac{1}{\pi(\mu\bar{c})^2}\int_0^{\infty} S_w(k, k_2)\,|F_2(k, k_2)|^2\, dk_2 \tag{34}$$

this being the generalisation of (29) to include spanwise variation of gust velocity.

Drischler[12] has shown that for a wing experiencing a sudden change in vertical velocity (corresponding to the Wagner effect) the spanwise loading distribution remains almost constant, except possibly for a short distance after the change. It seems reasonable to assume that the same applies for the Küssner effect, that is, the spanwise lift distribution remains constant after an unswept wing has entered a vertical gust of constant velocity over the entire span. This implies that the local unsteady lift function introduced in equation (31) is independent of spanwise position. Denoting the Fourier transform of this function of x by $H(k)$, equation (33) may be rewritten as

$$F_2(k_1, k_2) = H(k_1)\, F_1(k_2)$$

where

$$F_1(k_2) = \frac{1}{b}\int_{-b/2}^{b/2} \gamma(y)\, e^{-ik_2 y}\, dy$$

and for the usual case of a symmetric steady lift distribution, $F_1(k_2)$ is a real function:

$$F_1(k_2) = \frac{2}{b}\int_0^{b/2} \gamma(y) \cos k_2 y\, dy\,. \tag{36}$$

With this form of $F_2(k_1, k_2)$ equation (34) is now of a similar form to (29), the one-dimensional spectrum $S_w(k)$ in the latter being replaced by an effective spectrum $\hat{S}_w(k)$ of one-dimension, related to $S_w(k_1, k_2)$ by

$$\hat{S}_w(k) = \frac{1}{\pi}\int_0^{\infty} S_w(k, k_2)\,[F_1(k_2)]^2\, dk_2\,. \tag{37}$$

Since the aircraft is assumed to be rigid the effect of unsteady aerodynamics on the part of the lift fluctuations due to aircraft heaving is the same as that in the case neglecting spanwise gust variations. Hence the normal acceleration spectrum is

$$S_a(k) = \frac{k^2\, U^2 |H_1(k)|^2}{|i\, k\, \mu\bar{c} + H_2(k)|^2}\, \hat{S}_w(k)\,. \tag{38}$$

3.5. *Effect of Sweep.*

For an unswept aircraft flying through a patch of turbulence giving a significant spanwise variation the mean square normal acceleration is lower than that with negligible variation across the span. For a swept wing aircraft of the same span there is a larger reduction in the magnitude of the response since the two-dimensional variations are now 'averaged' both laterally and longitudinally. The averaging of variation along the flight path occurs even when the one-dimensional spectrum is assumed and for aircraft of very low aspect ratio can be more significant than the spanwise variations. Little information is available in general form for the unsteady lift of a swept wing and so a transfer function is derived under the assumptions of the previous Section; in particular it is assumed that the local unsteady lift function of each chordwise element of the wing is independent of other elements and the same at all span positions (referred to the leading-edge position of the appropriate element).

The expression corresponding to (32) for the lift of a wing with a straight leading edge swept at an angle Λ is

$$\frac{U\,W}{g}\,l(x) = \int_{-b/2}^{b/2} \tfrac{1}{2}\rho\, U\, a\, \bar{c}\, \gamma(y) \int_0^\infty h(x_1)\, w(x - x_1 - |y| \tan\Lambda, y)\, dx_1\, dy\,. \tag{39}$$

Proceeding as in Section 3.4 leads to an equation similar to equation (34) but with the function $F_2(k_1, k_2)$ now defined by

$$F_2(k_1, k_2) = \frac{1}{b}\int_{-b/2}^{b/2}\int_{-\infty}^{\infty} \gamma(y)\, h(x)\, e^{-i(k_1 x + k_1 |y| \tan\Lambda + k_2 y)}\, dx\, dy$$

which for a symmetrical loading function reduces to

$$F_2(k_1, k_2) = H(k_1)\frac{2}{b}\int_0^{b/2} \gamma(y) \cos(k_1 y \tan\Lambda) \cos k_2 y\, dy\,. \tag{40}$$

For the particular case of constant loading this gives

$$F_2(k_1, k_2) = H(k_1)\frac{p \sin p \cos q - q \cos p \sin q}{p^2 - q^2}$$

where

$$p = \frac{k_1 b}{2}\tan\Lambda\,, \quad q = \frac{k_2 b}{2}\,.$$

Equation (40) has replaced the function $F_1(k_2)$, defined by (36), by an expression involving both k_1 and k_2. This is in addition to the common factor $H(k_1)$ and thus represents the additional reduction in the response to a swept wing.

A form of the effective spectrum $\hat{S}_w(k)$ has now been obtained that includes the effect of sweep but to obtain the transfer function to normal acceleration the Wagner function must be modified. In the absence of general functions for such a planform some indication of the effect of sweep is found by assuming a constant spanwise loading in addition to the other simplifications used in this section. The Fourier transform of the Wagner function is then equal to the transform for the corresponding unswept wing multiplied by a factor $[1-\exp(-i k \bar{c} \eta)]/(i k \bar{c} \eta)$ where $\eta = \frac{1}{2} b \tan \Lambda/\bar{c}$.

4. *Evaluation of Response Factors.*

4.1. *Calculation of Effective Spectrum.*

It was decided to evaluate numerically the integrals for gust response and zero crossings factors. For some forms of spectrum and transfer function it is possible to use analytical methods but by using one basic computer programme any combination of spectrum and transfer function may be used. The numerical calculations would take a prohibitive time if it were necessary to carry out a double integration for every condition under which the response factors of an aircraft were required. The methods by which this is avoided are outlined below, the mathematical details being given in Appendix A.

The only parameters involved in the evaluation of $\hat{S}_w(k)$ by equation (37) are the loading distribution, wing span and turbulence scale, with the last two being combined in the non-dimensional variable of span-scale ratio. In particular $\hat{S}_w(k)$ is independent of both the chord-scale ratio and the mass parameter so that the response factors can be calculated simultaneously for a range of μ and C values using only one evaluation of $\hat{S}_w(k)$ for each step of the variable k. Thus it would appear that a complete range of the integrated two-dimensional spectrum is required to be calculated for each scale of turbulence through which a particular aircraft flies (i.e. for a given span-scale ratio for a prescribed span-wise loading distribution); however a further large reduction in the numerical integration required can be made by relating the values of $\hat{S}_w(k)$ at higher frequencies to a general spectrum. The results of the integrations for the general spectrum for a particular loading are expressed in tabular form and $\hat{S}_w(k)$ is determined from these values by multiplying by a simple power of the required span-scale ratio.

This Table of values need be calculated only once for each loading distribution and thus it is feasible to use a longer, more accurate method than would be possible if it was necessary to incorporate more variables. For small k values $\hat{S}_w(k)$ must still be found by integration for each span-scale ratio but this typically represents only one fifth of the total number of k values which must be used to find the gust response factor and a still lower proportion of the total required to find the slower converging zero crossings factor. At these lower frequencies the response reduction due to span effect is less significant and all spectra then approach the one-dimensional form. Fig. 2 shows the ratio of the effective spectrum to the one-dimensional spectrum for various β values, with constant loading.

Most of the commonly occurring spanwise loading distributions fall between the constant and the triangular cases which thus correspond to the maximum and minimum reduction in response that may be expected. The difference between these two cases is much smaller than that between the spectrum for triangular loading and the one-dimensional spectrum; this is demonstrated in Fig. 3 for the value of $\beta = 0{\cdot}2$. Thus it is only necessary to use a very simple approximation to the loading function in order to obtain a spectrum of much greater accuracy than the one-dimensional one.

4.2. *Response Factors.*

Evaluation of equations (16) and (17) gives the value of the required gust response and zero crossings factors. With the effective spectrum calculated as described in the previous Section and using a particular transfer function (equation (38)) the integrals are evaluated by the numerical method described earlier.

The gust response and zero crossings factors for constant loading are given in Table 4 for aspect ratios from 2 to 16 and span-scale ratios from 0·025 to 0·4. The relationships between the factors and these parameters are illustrated in Figs. 5 to 8. The corresponding values for an elliptical loading distribution are given in Table 5 for a less extensive range of variables and show that there is very little difference between the two cases. The zero crossings factor converges rapidly to its limiting value for large μ, which can be seen in the tables of M_0. For an aircraft flying in turbulence of a given scale length it will often be sufficiently accurate to evaluate M_0 for just one μ value, the variation of the actual number of zero crossings N_0 with changing μ being entirely due to changes in K (from equation (15)).

For Tables 4 and 5 the Küssner and Wagner unsteady lift functions for two-dimensional flow were used, as given in the first line of Table 1. The magnitudes of K obtained using these values and those appropriate to finite aspect ratios of 3 and 6 are compared in Fig. 4; the curves are seen to be similar with less difference for the higher aspect ratio, as would be expected. If the Wagner effect is neglected K is underestimated and while M_0 retains the same limiting value for high μ, convergence is slower. Neglecting the effect of aspect ratio on the unsteady lift functions means that the response factors depend on this quantity only through its influence on the effective spectrum (i.e. for a given C, β is proportional to aspect ratio); since this in turn has a more powerful effect at high frequencies, M_0 has a higher percentage variation with aspect ratio than K. This slight variation of K is shown to be approximately linear in Fig. 5 whereas Fig. 6 demonstrates the nearly linear dependence of M_0 on the logarithm of aspect ratio. Figs. 7 and 8 show the same quantities plotted against chord-scale ratio. All of these illustrations are for the constant loading case.

Calculations have been made using unsteady lift functions for two-dimensional compressible flow given in Table 1. These cause only a very small fractional decrease in K but the variation of the zero crossings factor with Mach number is more pronounced, as shown in Fig. 9.

5. *Conclusions.*

The calculation of the root mean square of aircraft normal acceleration response to turbulence by an elementary spectral method is described which is similar to many simple general methods that have been used in the past.

This approach is extended to include the spanwise variations of the gust velocity when flying through turbulence that is homogenous in the horizontal plane. This reduces the predicted response at high frequencies and enables studies of the numbers of crossings of response levels to be made, which is of direct application to the interpretation of observations.

A general effective spectrum is derived for any given spanwise loading distribution that is applicable to all but the lowest frequencies; the spectrum of turbulence energy averaged across the span is found from this by inserting an appropriate scaling factor (β, the ratio of span to turbulence scale length). In this way a full double integration is avoided and only a little more numerical work is involved than in a method neglecting spanwise variations completely.

Numerical results are expressed in terms of the gust response factor and a zero crossings factor, the latter being defined such that it is almost independent of the mass parameter and eliminating the need for repeated evaluations for different weights of an aircraft. The factors are tabulated for a range of parameter values using two-dimensional unsteady lift functions which have been found to be an acceptable simplification. The effect of compressibility on these functions is illustrated. The tables are suitable for interpolating values of the response factors between the given parameters and curves are drawn to demonstrate the relationships between these.

APPENDIX

Derivation of General Spectrum

The possible simplifications in the determination of the effective one-dimensional spectrum that were described in Section 4.1 are developed here using as an example the Von Kármán spectrum and an aircraft having negligible sweep and a constant spanwise loading distribution.

The non-dimensional variables $\xi\ (= kL)$, $\beta (= b/L)$ and $C (= \bar{c}/L)$ are used.

$F_1(k_2)$ may be evaluated from (36) for a particular loading distribution and expressions are given in Table 2 for various distributions. For the example used here it is $\dfrac{\sin(\beta\,\xi/2)}{(\beta\,\xi/2)}$.

From (13) the two-dimensional spectrum is

$$S_w(\xi_1/L, \xi_2/L) = (1{\cdot}339^2 \cdot 16/9)\,\sigma_w^2\,\pi\,L^2 \frac{1{\cdot}339^2\,(\xi_1^2+\xi_2^2)}{[1+1{\cdot}339^2\,(\xi_1^2+\xi_2^2)]^{7/3}}.$$

For this example equation (37) becomes

$$\hat{S}_w(\xi/L) = 3{\cdot}1874\, \sigma_w^2\, L \int_0^\infty \frac{1{\cdot}339^2\, (\xi^2+\xi_2^2)}{[1+1{\cdot}339^2\, (\xi^2+\xi_2^2)]^{7/3}} \left[\frac{\sin(\beta\, \xi_2/2)}{\beta\, \xi_2/2}\right]^2 d\xi_2 \qquad (41)$$

For values of ξ sufficiently large that $1{\cdot}339^2\, \xi^2$ is much greater than unity this may be simplified to give

$$\hat{S}_w(\xi/L) = \beta^{5/3}\, S_G(\beta\, \xi) \qquad (42)$$

where

$$S_G(\beta\, \xi) = 3{\cdot}1874\, \sigma_w^2\, L \int_0^\infty [1{\cdot}339^2\, \{(\beta\, \xi)^2 + r^2\}]^{-4/3} \left(\frac{\sin(r/2)}{r/2}\right)^2 dr\,. \qquad (43)$$

The general spectrum ($S_G(\beta\, \xi)$ need be calculated only once for any particular loading distribution and equation (42) can then be used to find the effective spectrum for all values of β, for sufficiently large ξ. For small values of ξ it is necessary to evaluate $\hat{S}_w(\xi/L)$ directly from equation (41) for each value of β.

For use in the calculation of the gust response factors, $S_G(\beta\, \xi)$ was tabulated at values of $\beta\, \xi$ that were evenly spaced on a logarithmic scale. This spacing was chosen such that during the integration over ξ (to calculate the response factors) it did not exceed the step length that was required to give sufficient accuracy of the response factor results. The numerical scheme used to evaluate the integrals of equations (42) and (43) employed a variable interval to keep accuracy within a prescribed limit and the integration was continued until the truncation error come within another specified limit.

Table 3 gives the general spectra for rectangular, elliptical and triangular loading distributions on a coarsely spaced set of $\beta\, \xi$ values. Fig. 1 shows the constant loading averaged spectra for a number of β values, plotted against $\beta\, \xi$; the inclusion of the factor $\beta^{-5/3}$ demonstrates the approach of these spectra to the general spectrum.

TABLE 1

Unsteady Lift Functions.

Aspect ratio	Mach No.	A_1	A_2	A_3	B_1	B_2	B_3
		Küssner effect					
∞	0	0·5	0·5	—	0·26	2·0	—
6	0	0·48	0·334	—	0·588	1·93	—
3	0	0·679	0·227	—	1·16	6·4	—
∞	0	0·236	0·513	0·171	0·116	0·728	4·84
∞	0·5	0·390	0·407	0·203	0·1432	0·748	4·33
∞	0·6	0·328	0·430	0·242	0·1090	0·514	2·922
∞	0·7	0·402	0·461	0·137	0·1084	0·625	2·948
		Wagner effect					
∞	0	0·458	—	—	0·265	—	—
6	0	0·361	—	—	0·762	—	—
3	0	0·283	—	—	1·080	—	—
∞	0	0·165	0·335	—	0·090	0·600	—
∞	0·5	0·352	0·216	−0·670	0·1508	0·744	3·780
∞	0·6	0·362	0·504	−0·715	0·1292	0·962	1·916
∞	0·7	0·364	0·405	−0·419	0·1072	0·714	1·804

TABLE 2

Spanwise Loading Distributions and Associated Functions.

$\xi = k_2 L$

Loading	$\gamma(y)$	$F_1(k_2)$
Rectangular (constant)	1	$\dfrac{\sin(\beta\,\xi/2)}{\beta\,\xi/2}$
Triangular	$2(1-2y/b)$	$\dfrac{2[1-\cos(\beta\,\xi/2)]}{(\beta\,\xi/2)^2}$
General taper	$A+2By/b$	$\dfrac{(A+B)\sin(\beta\,\xi/2)}{\beta\,\xi/2}-\dfrac{B[1-\cos(\beta\,\xi/2)]}{(\beta\,\xi/2)^2}$
Elliptical	$\dfrac{4}{\pi}[1-(2y/b)^2]^{\frac{1}{2}}$	$\dfrac{2\,J_1(\beta\,\xi/2)}{\beta\,\xi/2}$

TABLE 3

General Form of Effective Von Karman Spectra.

$\log_{10}(\beta\,\xi)$	$S_G(\beta\,\xi)/(\sigma_w^2\,L)$		
	rectangular	triangular	elliptical
—0·80	0·3491 2	0·3508 2	0·3499 2
−0·64	0·1874 2	0·1889 2	0·1881 2
−0·48	0·1000 2	0·1014 2	0·1006 2
−0·32	0·5289 1	0·5408 1	0·5345 1
−0·16	0·2759 1	0·2860 1	0·2805 1
0·00	0·1410 1	0·1491 1	0·1447 1
0·16	0·6998 0	0·7617 0	0·7274 0
0·32	0·3343 0	0·3780 0	0·3532 0
0·48	0·1524 0	0·1805 0	0·1642 0
0·64	0·6618 −1	0·8235 −1	0·7263 −1
0·80	0·2747 −1	0·3572 −1	0·3058 −1
0·96	0·1101 −1	0·1478 −1	0·1235 −1
1·12	0·4313 −2	0·5883 −2	0·4838 −2
1·28	0·1664 −2	0·2280 −2	0·1859 −2
1·44	0·6359 −3	0·8696 −3	0·7062 −3
1·60	0·2414 −3	0·3287 −3	0·2665 −3
1·76	0·9124 −4	0·1236 −3	0·1002 −3
1·92	0·3439 −4	0·4639 −4	0·3761 −4
2·08	0·1291 −4	0·1739 −4	0·1410 −4
2·24	0·4883 −5	0·6514 −5	0·5287 −5

TABLE 4(i)

Response Factors for Constant Loading, Aspect Ratio 2.

β	0·025	0·05	0·10	0·20	0·40
C	0·0125	0·025	0·05	0·1	0·2
μC	Gust response factor, K				
0·0500	0·2672	0·2334	0·1936	0·1507	0·1086
0·0707	0·3141	0·2797	0·2376	0·1902	0·1413
0·1000	0·3649	0·3313	0·2872	0·2358	0·1803
0·1414	0·4201	0·3865	0·3418	0·2869	0·2252
0·2000	0·4784	0·4457	0·4003	0·3427	0·2752
0·2828	0·5386	0·5068	0·4617	0·4018	0·3289
0·4000	0·5995	0·5686	0·5238	0·4622	0·3844
0·5657	0·6589	0·6288	0·5842	0·5213	0·4392
0·8000	0·7147	0·6851	0·6406	0·5764	0·4905
1·1314	0·7650	0·7355	0·6906	0·6253	0·5361
1·6000	0·8080	0·7784	0·7331	0·6664	0·5745
2·2627	0·8432	0·8134	0·7674	0·6994	0·6051
3·2000	0·8707	0·8406	0·7939	0·7247	0·6285
	Zero crossings, M_0				
0·0500	0·0208	0·0250	0·0288	0·0313	0·0314
0·0707	0·0211	0·0257	0·0303	0·0339	0·0354
0·1000	0·0213	0·0262	0·0314	0·0362	0·0391
0·1414	0·0214	0·0266	0·0323	0·0380	0·0423
0·2000	0·0215	0·0268	0·0329	0·0393	0·0450
0·2828	0·0216	0·0270	0·0333	0·0404	0·0470
0·4000	0·0216	0·0271	0·0336	0·0411	0·0486
0·5657	0·0216	0·0271	0·0338	0·0415	0·0496
0·8000	0·0216	0·0271	0·0339	0·0418	0·0502
1·1314	0·0216	0·0271	0·0339	0·0419	0·0506
1·6000	0·0216	0·0271	0·0339	0·0419	0·0507
2·2627	0·0216	0·0271	0·0339	0·0419	0·0508
3·2000	0·0216	0·0271	0·0339	0·0419	0·0508

TABLE 4(ii)

Response Factors for Constant Loading, Aspect Ratio 4.

β	0·025	0·05	0·10	0·20	0·40
C	0·00625	0·0125	0·025	0·05	0·1
μC	Gust response factor, K				
0·0500	0·2845	0·2543	0·2165	0·1740	0·1307
0·0707	0·3314	0·3016	0·2627	0·2166	0·1673
0·1000	0·3819	0·3537	0·3142	0·2654	0·2105
0·1414	0·4365	0·4089	0·3703	0·3196	0·2596
0·2000	0·4942	0·4681	0·4297	0·3782	0·3141
0·2828	0·5540	0·5288	0·4917	0·4399	0·3724
0·4000	0·6144	0·5903	0·5543	0·5025	0·4324
0·5657	0·6735	0·6505	0·6153	0·5635	0·4914
0·8000	0·7292	0·7068	0·6722	0·6205	0·5466
1·1314	0·7795	0·7574	0·7231	0·6711	0·5957
1·6000	0·8227	0·8008	0·7664	0·7140	0·6370
2·2627	0·8580	0·8362	0·8017	0·7487	0·6701
3·2000	0·8858	0·8639	0·8292	0·7755	0·6956
	Zero crossings factor, M_0				
0·0500	0·0142	0·0172	0·0204	0·0230	0·0244
0·0707	0·0143	0·0176	0·0211	0·0245	0·0268
0·1000	0·0144	0·0178	0·0217	0·0256	0·0289
0·1414	0·0144	0·0180	0·0221	0·0266	0·0306
0·2000	0·0144	0·0181	0·0224	0·0272	0·0320
0·2828	0·0145	0·0182	0·0226	0·0277	0·0331
0·4000	0·0145	0·0182	0·0277	0·0281	0·0339
0·5657	0·0145	0·0182	0·0228	0·0283	0·0344
0·8000	0·0145	0·0182	0·0229	0·0284	0·0347
1·1314	0·0145	0·0182	0·0229	0·0285	0·0348
1·600	0·0145	0·0182	0·0229	0·0285	0·0348
2·2627	0·0145	0·0182	0·0229	0·0285	0·0348
3·200	0·0145	0·0182	0·0229	0·0285	0·0348

TABLE 4(iii)

Response Factors for Constant Loading, Aspect Ratio 8.

β	0·025	0·05	0·10	0·20	0·04
C	0·003125	0·00625	0·0125	0·025	0·05
μC	Gust response factor, K				
0·0500	0·2951	0·2683	0·2330	0·1915	0·1479
0·0707	0·3417	0·3157	0·2802	0·2362	0·1873
0·1000	0·3919	0·3675	0·3326	0·2868	0·2333
0·1414	0·4461	0·4225	0·3887	0·3425	0·2852
0·2000	0·5031	0·4813	0·4481	0·4022	0·3424
0·2828	0·5628	0·5417	0·5101	0·4646	0·4023
0·4000	0·6228	0·6030	0·5727	0·5280	0·4656
0·5657	0·6819	0·6630	0·6337	0·5897	0·5267
0·8000	0·7377	0·7193	0·6908	0·6475	0·5840
1·1314	0·7878	0·7701	0·7421	0·6991	0·6350
1·6000	0·8310	0·8136	0·7860	0·7431	0·6783
2·2627	0·8664	0·8493	0·8218	0·7788	0·7133
3·2000	0·8942	0·8773	0·8498	0·8066	0·7403
	Zero crossings factor, M_0				
0·0500	0·0092	0·0114	0·0137	0·0159	0·0176
0·0707	0·0093	0·0116	0·0141	0·0166	0·0189
0·1000	0·0093	0·0117	0·0144	0·0173	0·0200
0·1414	0·0094	0·0118	0·0146	0·0177	0·0209
0·2000	0·0094	0·0118	0·0147	0·0181	0·0216
0·2828	0·0094	0·0118	0·0148	0·0183	0·0221
0·4000	0·0094	0·0118	0·0148	0·0185	0·0225
0·5657	0·0094	0·0119	0·0149	0·0186	0·0228
0·8000	0·0094	0·0119	0·0149	0·0186	0·0229
1·1314	0·0094	0·0119	0·0149	0·0186	0·0230
1·6000	0·0094	0·0119	0·0149	0·0186	0·0230
2·2627	0·0094	0·0119	0·0149	0·0186	0·0230
3·2000	0·0094	0·0119	0·0149	0·0186	0·0230

TABLE 4(iv)

Response Factors for Constant Loading, Aspect Ratio 16.

β	0·025	0·05	0·10	0·20	0·40
C	0·0015625	0·003125	0·00625	0·0125	0·025
μC	Gust response factor, K				
0·0500	0·3009	0·2760	0·2430	0·2032	0·1603
0·0707	0·3472	0·3234	0·2903	0·2486	0·2011
0·1000	0·3971	0·3745	0·3423	0·2998	0·2486
0·1414	0·4512	0·4299	0·3986	0·3558	0·3017
0·2000	0·5079	0·4882	0·4579	0·4156	0·3600
0·2828	0·5674	0·5485	0·5197	0·4782	0·4216
0·4000	0·6274	0·6096	0·5822	0·5416	0·4847
0·5657	0·6863	0·6695	0·6433	0·6037	0·5466
0·8000	0·7421	0·7259	0·7006	0·6619	0·6049
1·1314	0·7922	0·7767	0·7521	0·7141	0·6570
1·6000	0·8354	0·8204	0·7963	0·7587	0·7015
2·2627	0·8709	0·8562	0·8325	0·7951	0·7377
3·2000	0·8970	0·8843	0·8608	0·8236	0·7659
	Zero crossings factor, M_0				
0·0500	0·0059	0·0073	0·0089	0·0105	0·0119
0·0707	0·0059	0·0074	0·0091	0·0109	0·0126
0·1000	0·0059	0·0074	0·0092	0·0112	0·0132
0·1414	0·0059	0·0074	0·0093	0·0114	0·0136
0·2000	0·0059	0·0075	0·0093	0·0115	0·0140
0·2828	0·0059	0·0075	0·0094	0·0117	0·0143
0·4000	0·0059	0·0075	0·0094	0·0118	0·0145
0·5657	0·0059	0·0075	0·0094	0·0118	0·0146
0·8000	0·0059	0·0075	0·0094	0·0118	0·0147
1·1314	0·0059	0·0075	0·0094	0·0118	0·0147
1·6000	0·0059	0·0075	0·0094	0·0118	0·0147
2·2627	0·0059	0·0075	0·0094	0·0118	0·0147
3·2000	0·0059	0·0075	0·0094	0·0118	0·0147

TABLE 5(i)

Response Factors for Elliptical Loading, Aspect Ratio 2.

β	0·05	0·1	0·2
C	0·025	0·05	0·1
μC	Gust response factor, K		
0·05	0·2358	0·1967	0·1542
0·10	0·3333	0·2904	0·2399
0·20	0·4478	0·4035	0·3470
0·40	0·5706	0·5267	0·4665
0·80	0·6869	0·6432	0·5806
1·60	0·7800	0·7356	0·6705
3·20	0·8421	0·7964	0·7287
	Zero crossings factor, M_0		
0·05	0·0259	0·0299	0·0327
0·10	0·0271	0·0325	0·0376
0·20	0·0277	0·0340	0·0408
0·40	0·0279	0·0347	0·0424
0·80	0·0280	0·0349	0·0434
1·60	0·0280	0·0349	0·0433
3·20	0·0280	0·0349	0·0433

TABLE 5(ii)

Response Factors for Elliptical Loading, Aspect Ratio 4.

β	0·05	0·1	0·2
C	0·0125	0·025	0·05
μC	Gust response factor, K		
0·05	0·2575	0·2206	0·1786
0·10	0·3564	0·3183	0·2707
0·20	0·4707	0·4339	0·3839
0·40	0·5929	0·5581	0·5080
0·80	0·7091	0·6756	0·6258
1·60	0·8028	0·7696	0·7191
3·20	0·8658	0·8322	0·7805
	Zero crossings factor, M_0		
0·05	0·0179	0·0213	0·0241
0·10	0·0185	0·0226	0·0268
0·20	0·0188	0·0233	0·0284
0·40	0·0189	0·0236	0·0291
0·80	0·0189	0·0237	0·0294
1·60	0·0189	0·0237	0·0295
3·20	0·0189	0·0237	0·0295

TABLE 5(iii)

Response Factors for Elliptical Loading, Aspect Ratio 8.

β	0·5	0·1	0·2
C	0·00625	0·0125	0·025
μC	Gust response factor, K		
0·05	0·2723	0·2380	0·1972
0·10	0·3710	0·3374	0·2933
0·20	0·4846	0·4531	0·4091
0·40	0·6061	0·5772	0·5347
0·80	0·7221	0·6950	0·6539
1·60	0·8161	0·7898	0·7492
3·20	0·8796	0·8535	0·8126
	Zero crossings factor, M_0		
0·05	0·0119	0·0143	0·0167
0·10	0·0122	0·0150	0·0181
0·20	0·0123	0·0153	0·0189
0·40	0·0123	0·0154	0·0192
0·80	0·0123	0·0155	0·0194
1·60	0·0123	0·0155	0·0194
3·20	0·0123	0·0155	0·0194

TABLE 5(iv)

Response Factors for Elliptical Loading, Aspect Ratio 16.

β	0·05	0·1	0·2
C	0·003125	0·00625	
μC	Gust response factor, K		
0·05	0·2808	0·2488	0·2098
0·10	0·3790	0·3483	0·3073
0·20	0·4920	0·4637	0·4235
0·40	0·6132	0·5874	0·5494
0·80	0·7291	0·7053	0·6692
1·60	0·8233	0·8007	0·7657
3·20	0·8871	0·8651	0·8304
	Zero crossings factor, M_0		
0·05	0·0072	0·0093	0·0010
0·10	0·0076	0·0096	0·0117
0·20	0·0078	0·0098	0·0121
0·40	0·0078	0·0098	0·0123
0·80	0·0078	0·0098	0·0123
1·60	0·0078	0·0098	0·0123
3·20	0·0078	0·0098	0·0123

LIST OF SYMBOLS

A_j	Constants in unsteady lift function approximations
a	Aircraft lift slope
B_j	Constants in unsteady lift function approximations.
b	Wing span
C	$= \bar{c}/L$, chord-scale ratio
$\bar{c}$	Mean chord
$F_1(k), F_2(k_1, k_2)$	Functions in averaged spectrum
$f(x)$	One-dimensional correlation function
$f(x, y)$	Two-dimensional correlation function
$H(k)$	Fourier transform of $h(x)$
$h(x)$	Impulsive lift response function of wing
$h(x, y)$	Local impulsive lift response function of a section of wing at position y
K	Gust response factor, equation (14)
k, k_1, k_2	Wave numbers
L	Scale of turbulence
l	Lift parameter ($= g/WU$ times lift force due to vertical gust velocity)
M_0	Zero crossings factor, equation (15)
N_0	Number of times that zero value is crossed in each direction per unit distance
R, r	Distance along x axis
S	Wing area
$S_a(k)$	Normal acceleration spectrum
$S_G(\beta\xi)$	General spectrum, equation (43)
$S_l(k)$	Lift parameter spectrum
$S_w(k)$	One-dimensional lateral turbulence energy spectrum
$S_w(k_1, k_2)$	Two-dimensional lateral turbulence energy spectrum
$\hat{S}_w(k)$	Effective one-dimensional turbulence spectrum
$T(k)$	Transfer function
t	Time
U	Mean horizontal velocity
u, v, w	Components of gust velocity along x, y, z axes
W	Aircraft weight
x	Distance along flight path
y	Horizontal distance normal to flight path

z	Vertical distance normal to flight path
β	Span-scale ratio $= b/L$
$\gamma(y)$	Spanwise loading distribution
Λ	Angle of sweep
μ	Mass parameter $= 2W/(\rho\, S\, \bar{c}\, g\, a)$
ξ	$= kL$
ρ	Air density
σ_a	Root mean square normal acceleration
σ_w	Root mean square vertical gust velocity
$\phi(x)$	Wagner function
$\psi(x)$	Küssner function

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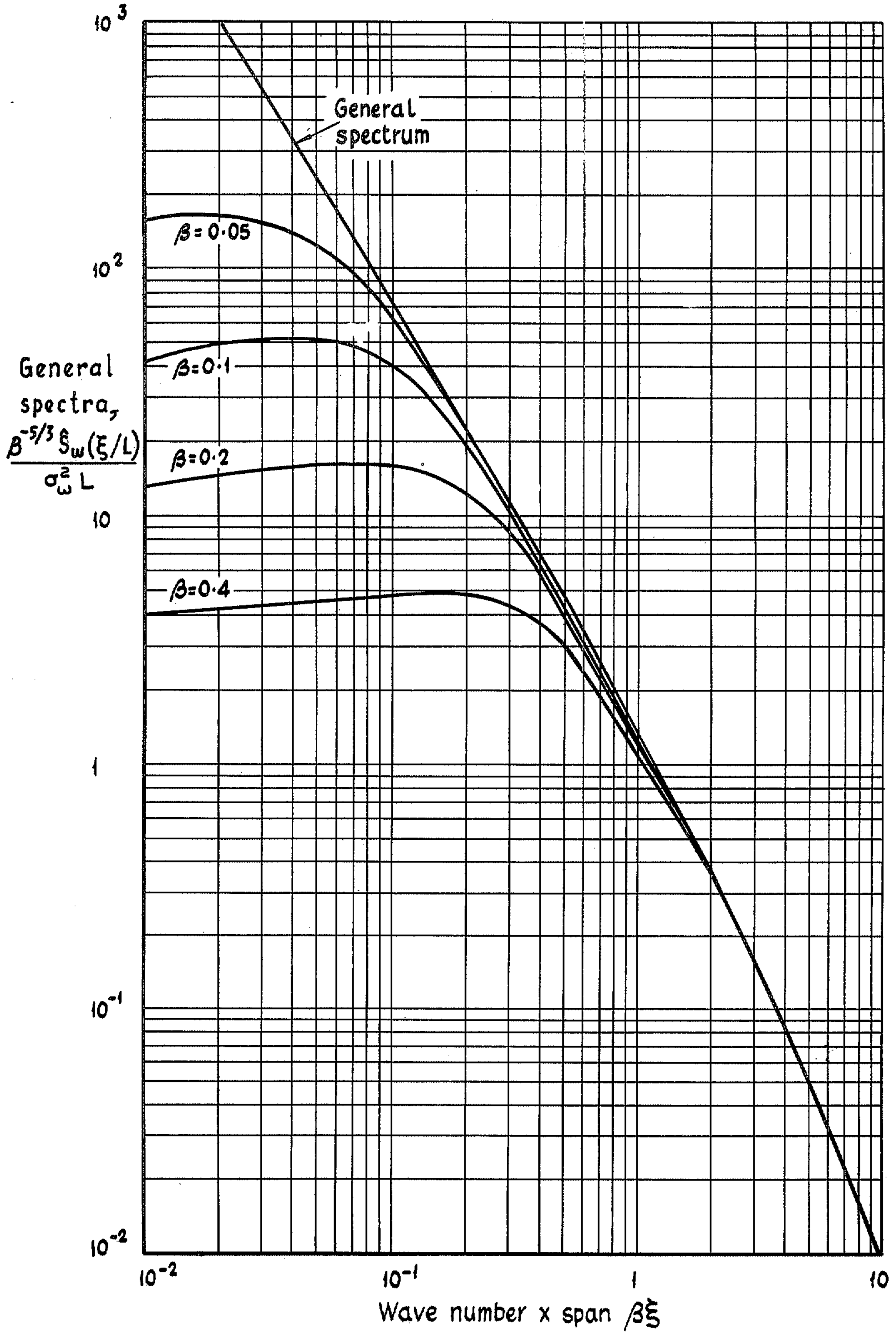

FIG. 1. Effective spectra for constant loading, showing approach to general spectrum.

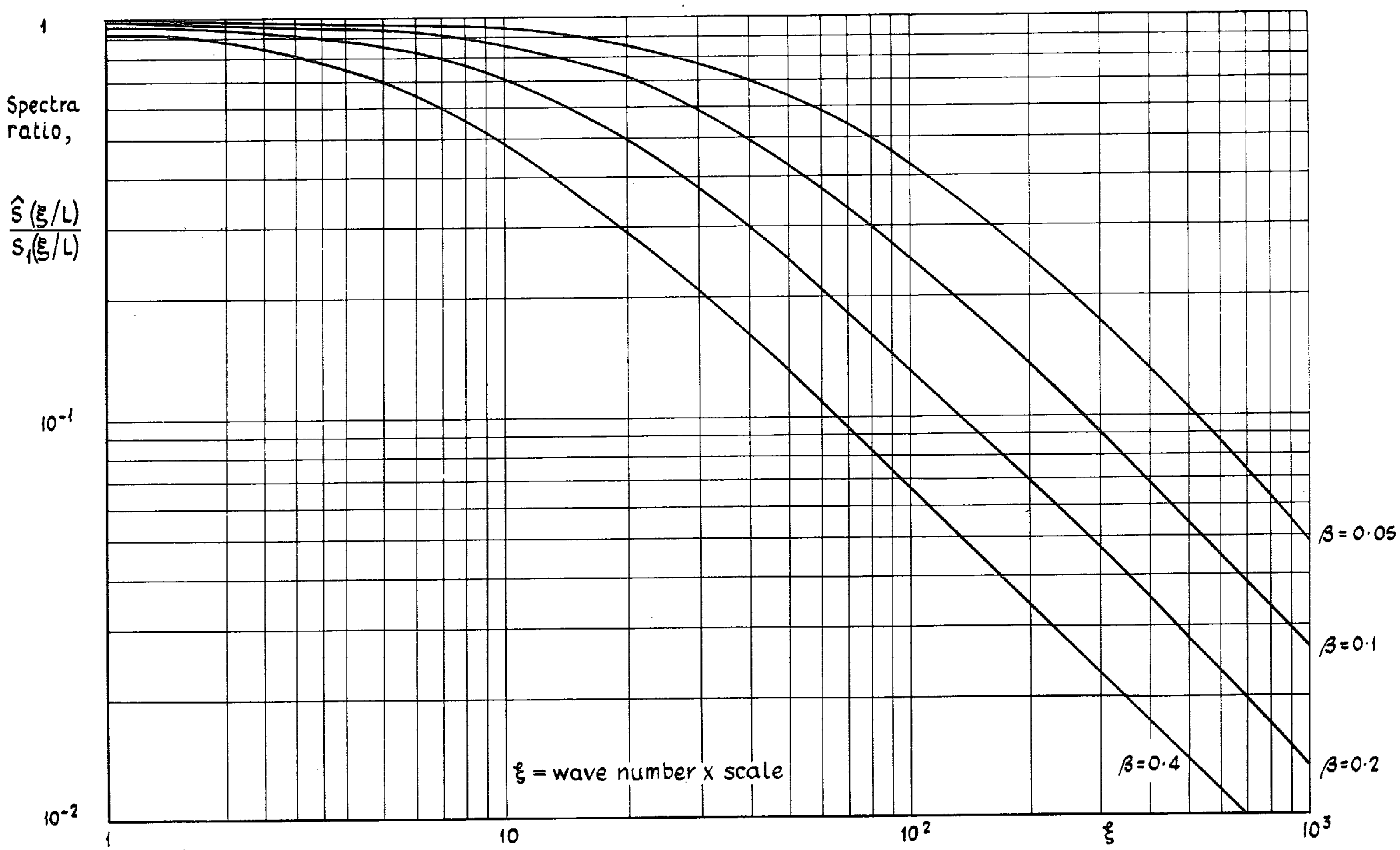

FIG. 2. Ratio of Von Karman spectra with and without span effect for constant loading.

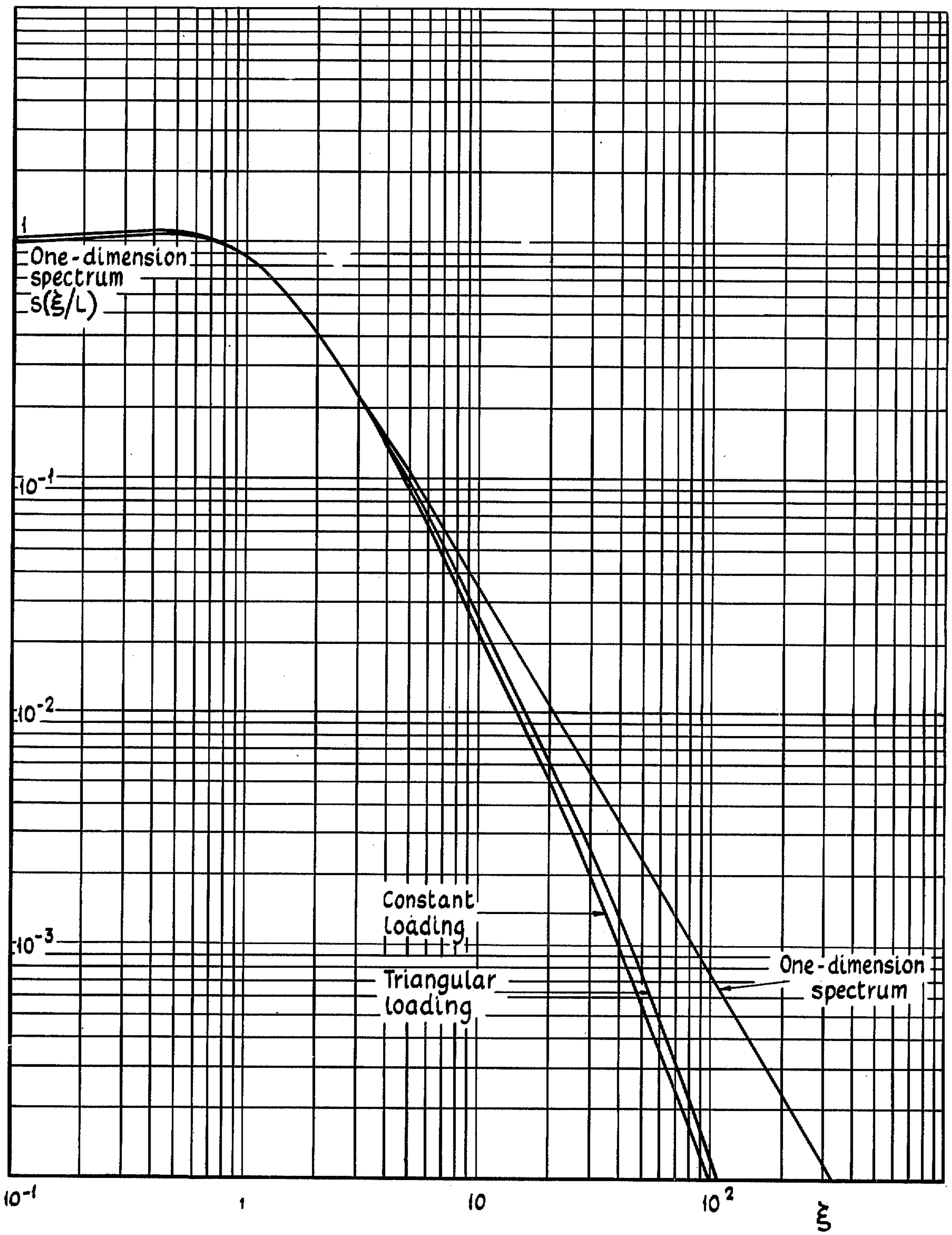

FIG. 3. Effective and one dimension Von Karman spectra for $\beta = 0{\cdot}2$, ξ = wave number x scale.

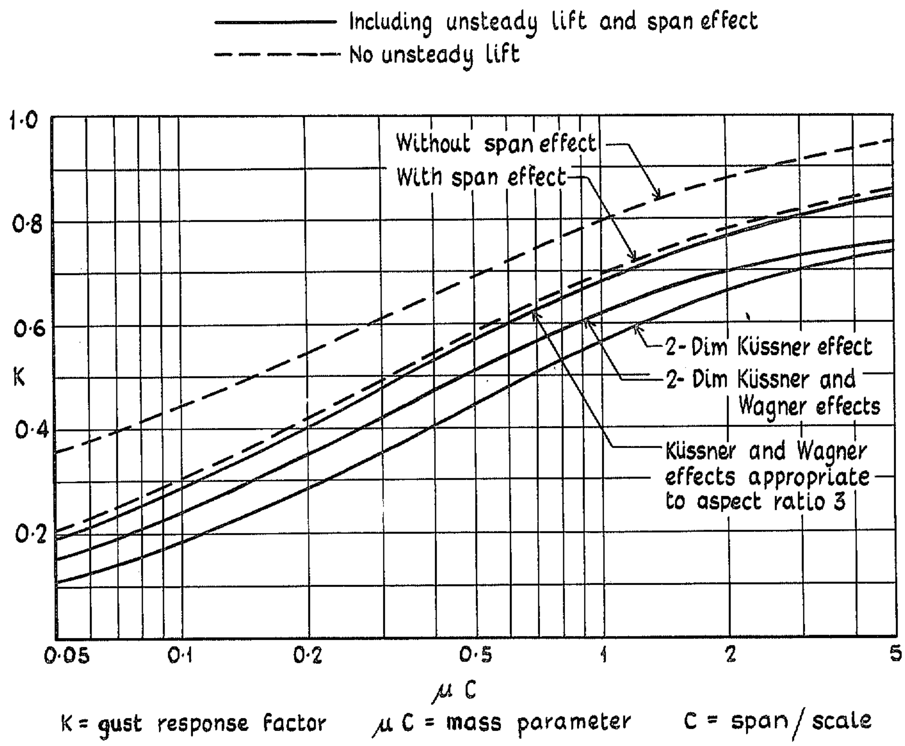

FIG. 4a. Gust response factor aspect ratio 3. $C = 0{\cdot}08$.

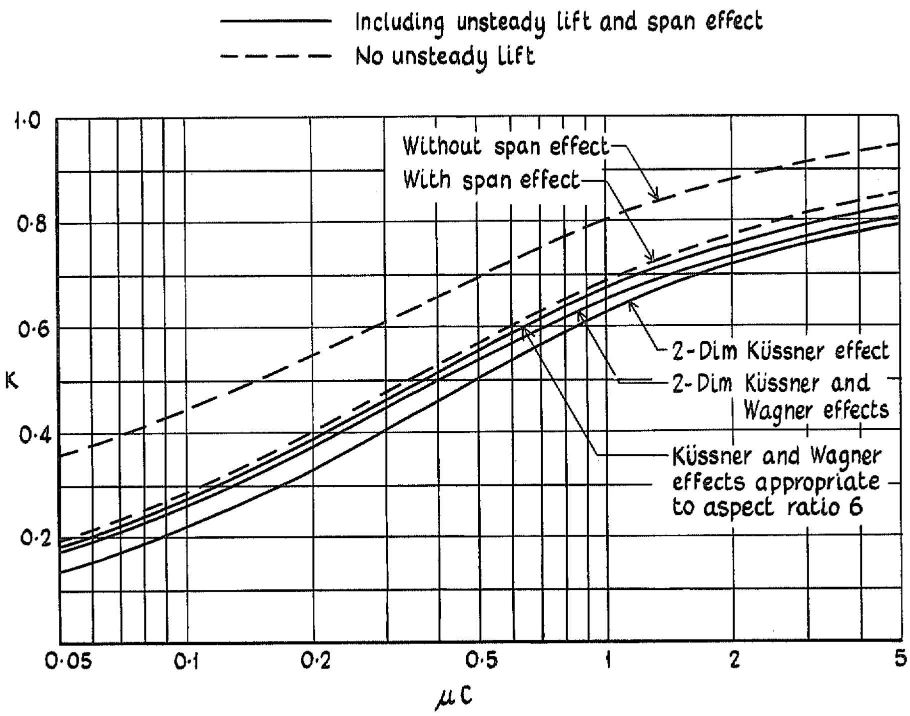

FIG. 4b. Aspect ratio 6, $C = 0{\cdot}04$.

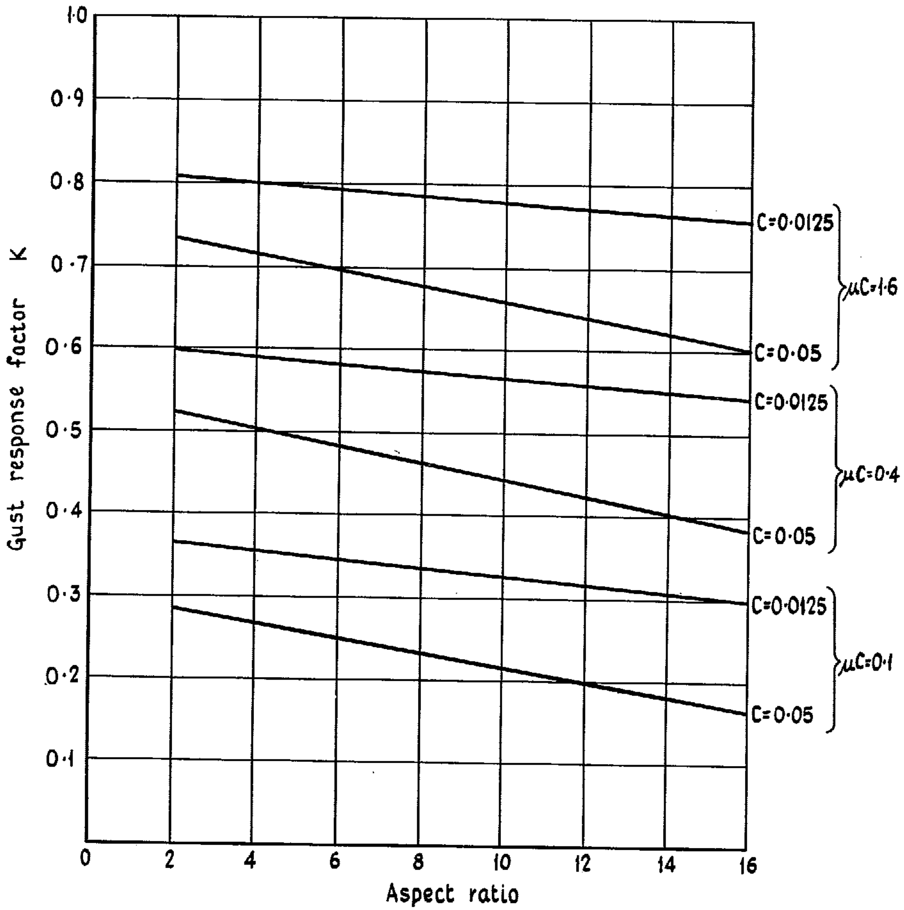

FIG. 5. Aspect ratio dependence of gust response factor.

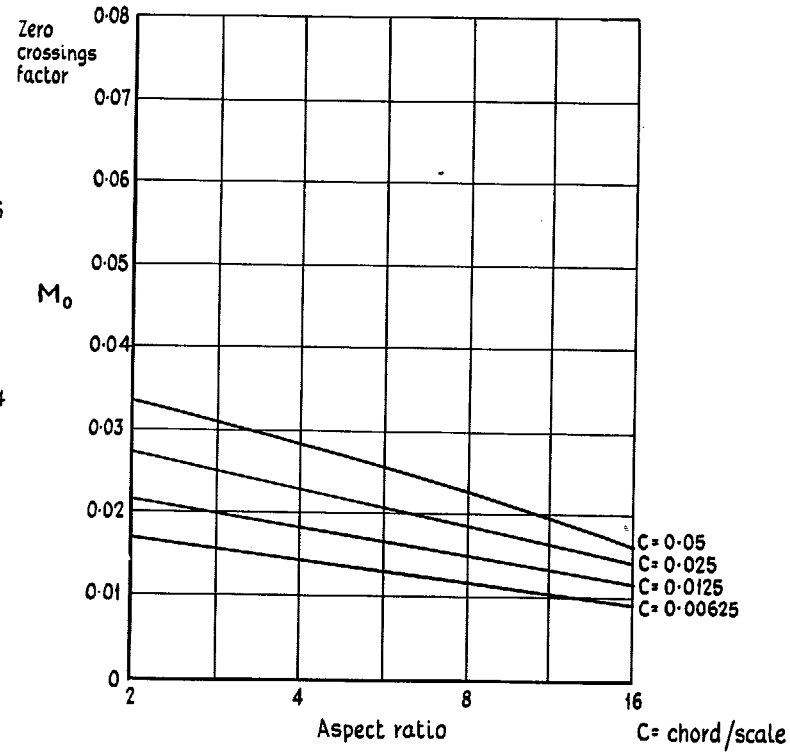

FIG. 6. Aspect ratio dependence of zero crossings factor, $\mu C = 0{\cdot}4$.

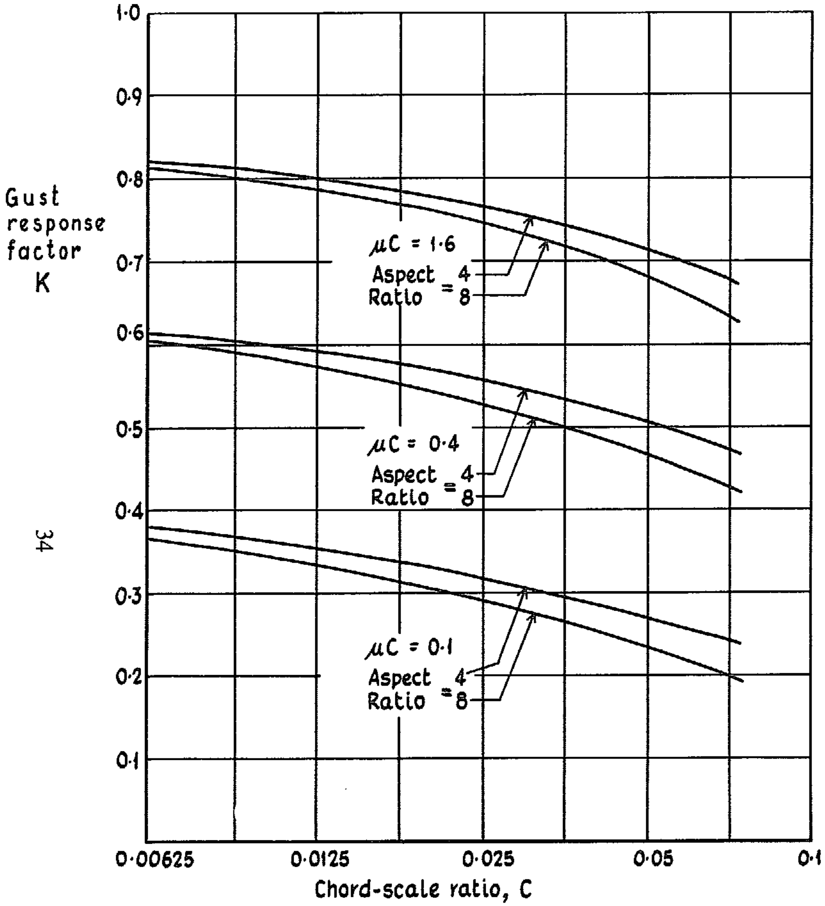

FIG. 7. Variation of gust response factor with chord-scale ratio.

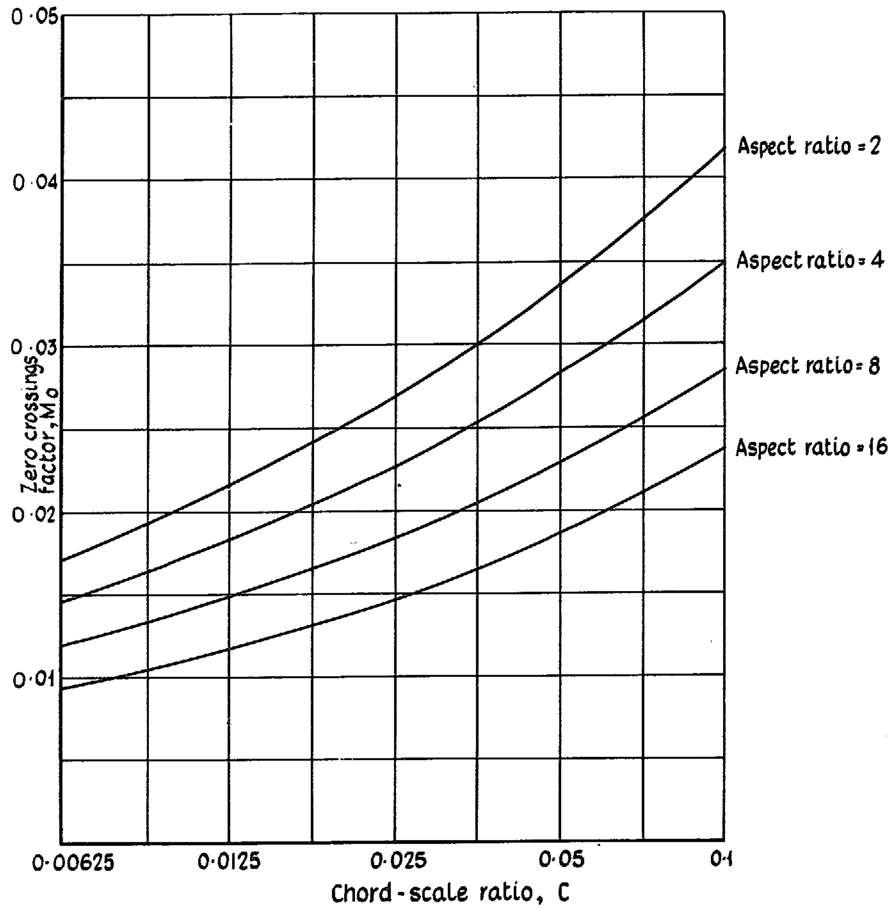

FIG. 8. Variation of zero crossings factor with chord-scale ratio.

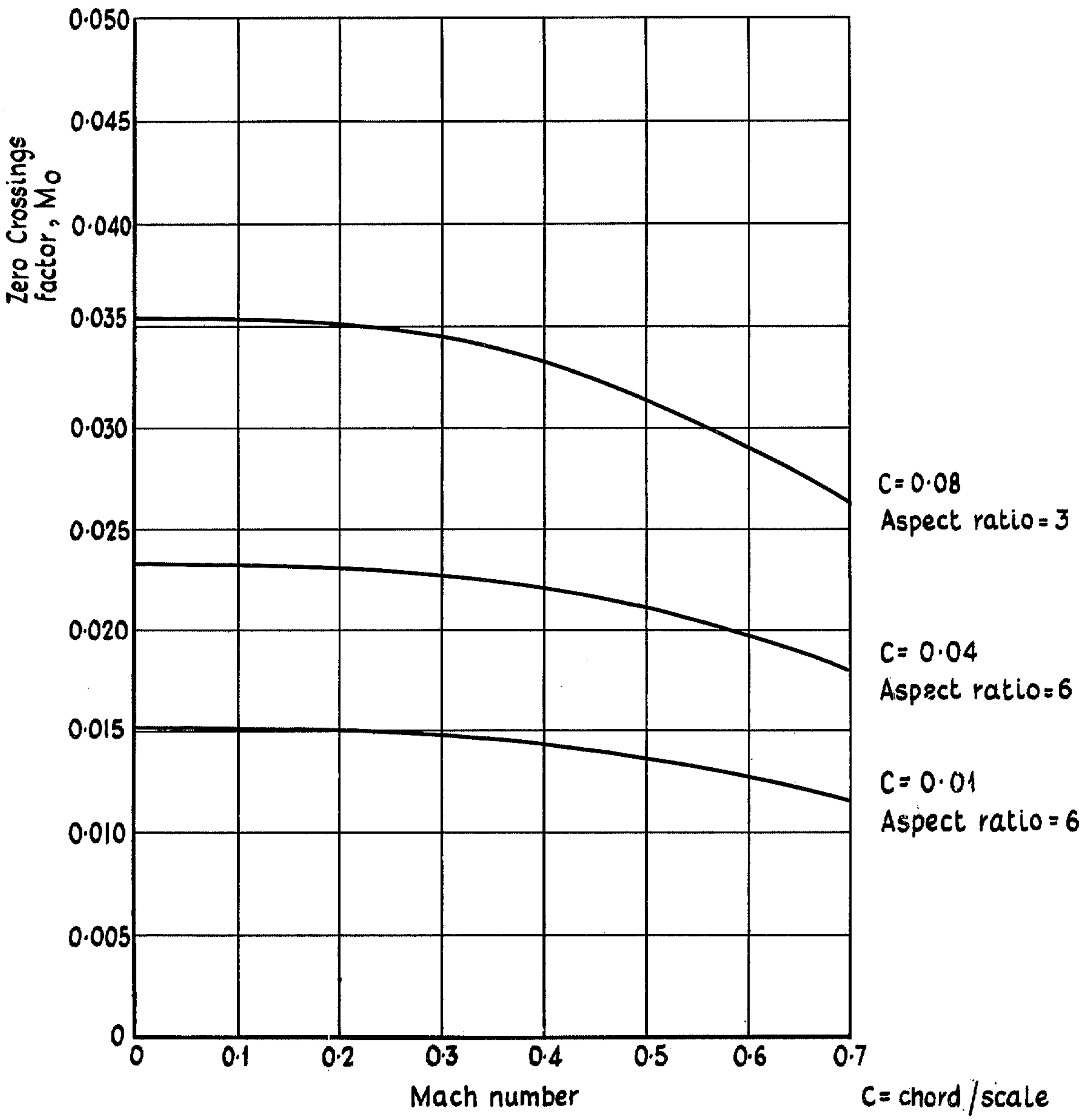

FIG. 9. Compressibility effect on zero crossings factor.

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