R. & M. No. 3607

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MINISTRY OF TECHNOLOGY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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By A. A. WOODFIELD, B.Sc., A.C.G.I., C.Eng., A.F.R.Ae.S.

Aerodynamics Dept., R.A.E., Bedford

LONDON: HER MAJESTY'S STATIONERY OFFICE
1969
PRICE £1 7*s* 0*d* [£1·35] NET

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By A. A. Woodfield, B.Sc., A.C.G.I., C.Eng., A.F.R.Ae.S.
Aerodynamics Dept., R.A.E., Bedford

Reports and Memoranda No. 3607*
February, 1968

Summary.

The equations of motion for the single-point suspension system are studied, and the method of analysis for determining the yawing moment and product of inertia given. The presence of interference from a 'rocking' mode is indicated, but the method of analysis is not affected. Design procedures are suggested which would minimise the interference from the 'rocking' mode.

Flexibility of the suspension point is shown to be unimportant, as is also the effect of using a free spreader frame to lift the aircraft.

Finally some analogue computer results are given to illustrate the motions and the method of analysis.

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*Replaces RAE Technical Report 68 044—A.R.C. 30 530.

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1. *Introduction.*

A knowledge of the moments and product of inertia of full-scale aircraft is essential for both the analysis and prediction of dynamic motion characteristics, and simple methods of measuring the rolling and pitching inertias have been in use for some time. However, measurement of both the yawing moment of inertia and the product of inertia has presented more difficulties and several different techniques have been tried.

One of the earliest and most promising of these techniques was described by Boucher, *et al.*[1]. In this method—the Single-Point Suspension Method—the aircraft is slung from a single point with springs fore and aft giving constraint in yaw, thus giving a direct measurement of yawing inertia. The product of inertia is determined from the ratio of rolling to yawing motion with the fore and aft springs in different vertical positions. This difference in spring height produces rolling moments in sympathy with the yawing motion and thus either reinforces or decreases the rolling produced by the product of inertia. Measurement of the spring positions which result in zero rolling motion determines the value of the product of inertia.

Thus, provided that the aircraft is not so large as to prohibit lifting from above, the support system can consist simply of a crane and no special structures are required. Furthermore the method of analysis appears to be straight forward.

However, despite the successful measurement of Ref. 1, two attempts to use this method at the Royal Aircraft Establishment, Bedford have disclosed major interference effects. In the first attempt in tests with the Avro 707B aircraft[2] the interference from other natural modes of the suspension system completely prevented measurements of the product of inertia, although the yawing inertia was easily determined. In the second case, tests with the Fairey Delta 2 aircraft[3], the same problems were apparent in the determination of the product of inertia. As a result the theoretical study of this Report was made to provide both a suitable method of analysis for the Fairey Delta 2 tests, and also to formulate general methods of analysis and optimization of rig design for any future tests. This has enabled the Fairey Delta 2 tests to be analysed[3] and has highlighted several undesirable features in the rig design, which were not previously appreciated.

The Report establishes the equations of motion for the complete system; indicates the required features of rig design to give reduced interference; develops a simple method of analysis which can be used even when interference is present; and then shows the method of analysis to be unaffected by flexibility of the suspension point.

No attempt is made to compare this method with the several others that are available and are discussed in Refs. 4 and 5, but it is shown to be a simple and accurate method requiring no special supporting structure, and thus has many advantages.

2. *Description of the Rig System.*

A diagram of the system is given in Fig. 1 and a photograph of a typical system used in the Fairey Delta 2 tests[3] is shown in Fig. 2. Note that for simplicity the four wires attaching the aircraft to the crane hook in Fig. 2 are shown as one link in Fig. 1. The aircraft is supported from the single point by a torsionless wire of length q ending in a hook from which the aircraft is slung with its centre of gravity a distance h below the hook. The aircraft motion in yaw is constrained by springs attached fore and aft of the centre of gravity at distances l_1 and l_2 respectively, their stiffnesses being K_1 and K_2 respectively. The vertical positions of the springs below the centre of gravity, r_1 and r_2, can be adjusted independently to provide known rolling moments in response to motions in yaw.

When the support point is fixed, the rig has three lateral degrees of freedom and the three natural modes of oscillation are:

(a) Yawing Mode: A combination of yawing and rolling motion. This is the mode which is required for the analysis.

(b) Rocking Mode: Contains nearly all rolling motion (Fig. 3a), but may contain some sideways translation depending on rig design. This mode produces most of the unwanted interference with the roll content of the yaw mode.

(c) Swaying Mode: Contains mainly sideways motion of the centre of gravity (Fig. 3b), but can have some rolling motion depending on rig design. With careful rig design this mode can be prevented from interfering with the yaw mode roll motion.

Overall motion of the rig and aircraft is defined (Fig. 1) by the two angles of rotation about the centre of gravity, yaw, ψ, and roll, ϕ, and by sideways translation of the centre of gravity, y.

3. *Theory.*

3.1. *Equations of Motion and General Rig Design Parameters.*

The general equations of motion of the system are derived in Appendix A and summarised in Table 1, which gives the characteristic determinant of the system. This determinant is rewritten in Appendix A as

$$\begin{array}{lc} \text{Parameter} & \begin{array}{ccc} \psi & \phi & y \end{array} \\ \begin{array}{l} \text{Sideforce} \\ \text{Rolling} \\ \text{Yawing} \end{array} & \begin{vmatrix} A & B & Y \\ C & \Phi & B \\ \Psi & C & A \end{vmatrix} = 0 \end{array} \qquad \text{(A.10)*}$$

where

$$\Psi = I_{zz} D^2 + \Sigma(Kl^2)$$

I_{zz} = Yawing moment of inertia

$D = d/dt$

$$\Phi = I_{xx} D^2 + \left[\Sigma(Kr^2) + Wh\left(1+\frac{h}{q}\right)\right]$$

I_{xx} = Rolling moment of inertia

$$Y = \frac{W}{g} D^2 + \left[\frac{W}{q} + \Sigma(K)\right]$$

W = Total weight

$$C = -\{I_{xz} D^2 + \Delta(Klr)\}$$

I_{xz} = Product of inertia

$$A = \Delta(Kl)$$

$$B = W\frac{h}{q} - \Sigma(Kr).$$

The prefixes Σ and Δ refer to the sum and differences respectively of the fore and aft spring terms in the brackets, e.g. $\Sigma(Kl^2) = K_1 l_1^2 + K_2 l_2^2$, or $\Delta(Klr) = K_1 l_1 r_1 - K_2 l_2 r_2$.

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*The equation numbers used in the Report with letter prefixes refer to equations of that number in the Appendix of that letter. This system has been adopted to help the reader locate the relevant portion of the Appendix more readily.

In the characteristic determinant both A and B are constants determined solely by the aircraft and rig weight, spring stiffnesses and geometry. Thus these may be controlled by the designer of the rig. They represent the yawing moment due to unit lateral translation, A, and the rolling moment due to unit lateral translation, B. Thus they are terms likely to cause interference with the yaw mode and, hence, with the inertia analysis.

By considering first the motions of a system where $A = B = 0$ (Appendix B) the basic method for measuring the yawing moment of inertia and product of inertia is obtained. Appendix B shows that when the ratio of rolling to yawing motion of the yaw mode, $(\phi/\psi)_\psi$, is zero, then the yawing moment of inertia is given by

$$I_{zz} = \frac{\Sigma(Kl^2)}{\omega_1^2} \tag{1}$$

where ω_1 = yaw mode frequency, rad/sec, and the product of inertia by

$$I_{xz} = \frac{\Delta(Klr)}{\omega_1^2}. \tag{2}$$

Appendix B also shows that the roll to yaw ratio varies linearly (to a close approximation) with $\Delta(Klr)$ and thus the value of $\Delta(Klr)$ for $(\phi/\psi)_\psi = 0$ can readily be obtained from a plot of $(\phi/\psi)_\psi$ against $\Delta(Klr)$.

This is the analysis used by Boucher, *et al.*[1], and is satisfactory for any case where $A = B = 0$. It should however be noted that equation (1) will be slightly in error when $(\phi/\psi)_\psi \neq 0$ although this will normally be less than 0·5 per cent. The error is given by the following equations from Appendix B:

$$\omega_1^2 = (\omega_1)_0^2 \left\{1+\frac{\Delta\psi}{(\omega_1)_0^2}\right\} = \frac{\Sigma(Kl^2)}{I_{zz}} \left\{1-\frac{\Delta\psi}{(\omega_1)_0^2}\right\} \tag{B.14}$$

and

$$\frac{\Delta\psi}{(\omega_1)_0^2} \simeq \left(\frac{\phi}{\psi}\right)_\psi^2 \left(\frac{I_{xx}}{I_{zz}}\right)\left(1-\frac{(\omega_2)_0^2}{(\omega_1)_0^2}\right) \tag{B.15}$$

where

$$(\omega_1)_0^2 = \Sigma(Kl^2)/I_{zz}$$

$$(\omega_2)_0^2 = \left\{\Sigma(Kr^2)+Wh\left(1+\frac{h}{q}\right)\right\}\Big/ I_{xx}.$$

Thus, the error is proportional to the square of the roll to yaw ratio for a given system.

Having shown that the normal method of analysis is correct, Appendix B then considers the characteristics of the rocking and swaying modes, and defines their frequencies and amplitude ratios. These are the characteristics which determine the degree to which these modes may be present in interference in the total motions of the rig. Appendix B shows that for $A = B = 0$ the swaying mode consists only of sideways translation motion and its frequency, ω_3, is given by:

$$(\omega_3)^2 = \left\{\frac{W}{q}+\Sigma(K)\right\}\Big/\frac{W}{g}. \tag{B.3}$$

As there is no rolling or yawing motion in this mode it cannot interfere with the rocking or yawing modes.

However the rocking mode contains both rolling and yawing motions and, as will be shown in Section 3.2, the relevant characteristics that affect the amount of interference are the frequency and the rolling to yawing ratio of the mode. These are given in Appendix B as

Frequency $\qquad \omega_2^2 \simeq \left[\Sigma(Kr^2) + Wh\left(1+\frac{h}{q}\right)\right] \Big/ I_{xx}$ (B.16)

Amplitude ratio $\qquad (\phi/\psi)_\phi \simeq I_{zz}/I_{xz}$. (B.20)

The errors in these approximations are small, with equation (B.20) being exact when the roll to yaw ratio of the yaw mode is zero, and equation (B.16) over-estimating in typical conditions by about 2 per cent at the same time. Appendix B shows that the rocking mode will not usually have any noticeable affect on the total yawing motion because of its high roll to yaw ratio. It is, however, the main cause of the interference with the rolling motion.

Having considered the basic and simplified case where both the yawing and rolling moments due to sideways translation are zero, we must now examine the effects of one or both these terms being non-zero. From the practical point of view it is generally easier to keep the yawing moment due to lateral motion, A, zero, and Appendix C considers the case where $B \neq 0$, $A = 0$, and the full case where $A \neq 0$, $B \neq 0$.

In the first case, $B \neq 0$, $A = 0$, Appendix C shows that the basic method for deriving the inertias is unaffected, but that the swaying mode now contains some rolling motion. If this swaying mode rolling motion is not to have any significant effects on the total rolling motion, it is suggested that the ratio of rolling to translational motion be kept to less than 0·05 rad/ft. From Appendix C this requires

$$|B| < |(0{\cdot}05)\, I_{xx}\, \{(\omega_2)_0^2 - (\omega_3)_0^2\}| . \qquad \text{(C.12)}$$

In the full case, with both $A \neq 0$ and $B \neq 0$, Appendix C shows that the basic method of inertia analysis of equations (1) and (2) is no longer strictly correct, although the effect on determining the yawing moment of inertia from equation (1) is negligible unless A is very large (> 1000 lb). However to keep the effect on equation (2) equivalent to less than an error of 0·05 degrees in the inclination of the principal axis, we require

$$\left|1250\, AB\right| < \left|\frac{W}{g}\,.\,\Sigma(Kl^2)\left\{(\omega_3)_0^2 - (\omega_1)_0^2\right\}\right| . \qquad \text{(C.20)}$$

3.2. *Forcing Functions.*

The relative magnitude of the motions of the three modes depend not only on their natural characteristics, but are also considerably affected by the form of the forcing function used to excite the motions. An ideal forcing function would only excite the required yaw mode, but, because of the roll motion in the yaw mode, it is not possible to achieve this, except for the one condition, when the rolling motion of the yaw mode is zero. It is desirable that the type of forcing function selected should be capable of being applied manually by one person, as this does not then add any requirement for special equipment.

The relative merits of three simple forcing functions are considered in detail in Appendix D. If the desirable condition with $A = B = 0$ applies, then it is not essential to apply a pure couple in yaw as a forcing function, because any sideways translation induced by the presence of a sideforce in the forcing function cannot induce rolling motion. Thus only a single sideforce function applied at one end of the aircraft is required. It is, however, desirable to keep the sideforce component small compared with the yawing moment by applying the force as far from the centre of gravity as possible.

Appendix D shows the best forcing function to be a pure sinusiodal input at the yaw mode frequency. This could be applied to a close approximation by one person pushing and pulling in sympathy with the yawing motion until the required amplitude is obtained. Another quite reasonable function is the half-cosine function, Fig. 4, which is closely approximated by one person pushing in sympathy with the motion, but allowing it to return freely.

Appendix D shows that a step input forcing function should always be avoided, as it will excite large rocking mode motions, because of its large input of low frequency excitation.

A further feature shown in Appendix D is the desirability of keeping the rocking and swaying mode frequencies no greater than $1/\sqrt{2}$ of the yaw mode frequency, as this attenuates their response to inputs at the yaw mode frequency.

3.3. *Determination of the Roll to Yaw Ratio, when the Rolling Motion is Distorted by the Rocking Mode.* To be able to use the simple method of Section 3.1 to determine the product of inertia, the roll/yaw ratio of the yaw mode must be measured. If the rocking mode is not excited, then this can be done directly. In general some rocking mode motion is present, particularly in the roll output, and it is necessary to account for this in determining the required roll to yaw ratio. Appendix E considers this problem and shows that, if the yaw motion is plotted against the corresponding roll motion, the envelope is a parallelogram, with the sides crossing the roll axis having a slope equal to the yaw to roll ratio of the yaw mode. This is illustrated in Figs. 5 and 6, where Fig. 5 shows the general form and Fig. 6 demonstrates the system with typical equations. In general the relation between the two frequencies will not be a simple fraction and the number of lobes will be much greater than illustrated, but the envelope is unaffected.

To obtain the required pair of envelope lines, only the points (ψ, ϕ) corresponding to peaks or troughs of the roll motion need be plotted. These are the points on the required envelope lines. The method is not affected if there is light damping present as this just tends to contract the parallelogram, and the best straight line through all the points will define the required yaw to roll ratio by its slope. This is true even if the two modes have different damping constants.

4. *General Design Procedure.*

Having laid down several design criteria in Section 3.1 these may seem somewhat difficult to interpret into the design procedure required for constructing a suitable rig. This Section suggests a possible procedure which should meet the criteria, and indicates which are the more important of these requirements.

In any practical case the designer will be confronted with certain fixed values which are inherent in the equipment that is available and the aircraft whose inertias are to be measured. The most frequently fixed terms are likely to be the total height from floor to suspension point, H

$$H \simeq q+h+r, \tag{3}$$

the length of the aircraft, L

$$L \simeq l_1+l_2 \tag{4}$$

and, of course, the weight, W, and estimated values of the inertias, I_{xx}, I_{zz} and I_{xz}* of the aircraft. In addition there may well be a minimum value of the aircraft to pivot height, h, because of the endloads imparted to the airframe or spreader frame by the supporting wires, and this will be a function of the separation distance between the aircraft attachment points. As a rough estimate, it is probably reasonable to suggest that 'h' should always be greater than the distance between the two main attachment points; this implies an endload of about 20 per cent of the aircraft weight when the distances are equal.

Thus, starting with the above constraints, the other terms may now be fixed. First the mean spring stiffness, K, and the mean spring moment arm in yaw, l, are determined using the frequency criteria for the yaw and swaying modes. In the equation for the swaying mode the spring stiffness term is usually dominant and thus

$$(\omega_3)_0^2 \simeq \Sigma(K)\Big/\frac{W}{g}. \tag{5}$$

*It is important that the weights and inertias used for rig design include the expected weights and inertias of the rig as well as the aircraft. This includes the effects of the crane hook which contributes a large term to the roll inertia.

We must now define the required yaw mode frequency; this must be high enough to ease the task of keeping the other mode frequencies low, but not so high that aircraft structural modes are excited. A typical choice being about 1 Hz, or $(\omega_1)_0^2 \simeq 40$ (rad/sec)^2. The desired ratio of yaw to swaying mode frequencies is

$$(\omega_1)_0^2/(\omega_3)_0^2 \simeq 5\,.$$

Therefore from equation (5)

$$K \simeq 4\frac{W}{g}\,. \tag{6}$$

Having defined K, then l is obtained from the equation for the yaw mode frequency when $(\omega_1)_0^2 = 40$

$$l^2 \simeq 20\,I_{zz}/K = \frac{4\,I_{zz}}{W/g}\,. \tag{7}$$

If this gives $2\,l > L$, reduce l to $\frac{1}{2}\,L$ without changing K. If $(\omega_1)_0^2$ then falls below 36, increase K to give $(\omega_1)_0^2 = 36$.

Next the mean spring moment arm in roll, r, and the heights h and q are found. The two conditions that must be satisfied are

$$(\omega_2)_0^2 = \left\{\Sigma(Kr^2)+Wh\left(1+\frac{h}{q}\right)\right\}\Big/\,I_{xx} \simeq 20 \tag{8}$$

and

$$W\frac{h}{q} \simeq 2\,Kr \qquad (B \simeq 0)\,. \tag{9}$$

Combining equations (8) and (9) gives:

$$r \simeq \frac{10\,I_{xx}}{K\{h+q+r\}} \simeq \frac{10\,I_{xx}}{K\,H} \tag{10}$$

and

$$\frac{h}{q} = \frac{20\,I_{xx}}{W\,H} \tag{11}$$

or, as

$$q = H-h-r$$

$$h \simeq \frac{H\left(1-\dfrac{10\,I_{xx}}{K\,H^2}\right)}{\left(1+\dfrac{W\,H}{20\,I_{xx}}\right)} \tag{12}$$

This implies a maximum value of h when $H \to \infty$

$$(h)_{H\to\infty} = \frac{20\, I_{xx}}{W}. \tag{13}$$

If the value of h from equation (12) is less than the minimum acceptable for strength limitations, then it must be increased to the minimum acceptable, and the resulting value of $B = W\frac{h}{q} - 2\,Kr$, checked against the criterion of Section 3.1, viz.

$$B < (0{\cdot}05)\, I_{xx}\, \{(\omega_2)_0^2 - (\omega_3)_0^2\}\,. \tag{C.12}$$

These equations define the main rig measurements. We must now check the maximum allowable tolerance on $A = \Delta(Kl)$, for the maximum value of B anticipated, allowing for the maximum change in aircraft weight from the design condition. The criterion for A given in Section 3.1 is

$$1250\, AB < \frac{W}{g}\,.\,\Sigma(Kl^2)\left\{(\omega_3)_0^2 - (\omega_1)_0^2\right\}. \tag{C.20}$$

The final dimensions are the range of $\Delta(Klr)$ and hence $\Delta(r)$ required to give large enough roll to yaw ratios either side of the zero condition. Suitable values are about $(\phi/\psi)_\psi = \pm 0{\cdot}20$ to $0{\cdot}25$. From Appendix B, equation (B.6), substituting for $(\omega_2)_0$ and $(\omega_1)_0$, the roll/yaw ratio is given by

$$\left(\frac{\phi}{\psi}\right)_\psi \simeq \frac{I_{xz}\,(\omega_1)_0^2 - \Delta(Klr)}{I_{xx}\,(\omega_1)_0^2\left\{1 - \dfrac{(\omega_2)_0^2}{(\omega_1)_0^2}\right\}} \tag{14}$$

or

$$\Delta(r) \simeq \frac{1}{Kl}\left[\, I_{xz}\,(\omega_1)_0^2 \mp 0{\cdot}25\, I_{xx}\,(\omega_1)_0^2\left\{1 - \frac{(\omega_2)_0^2}{(\omega_1)_0^2}\right\}\right]$$

$$\Delta(r) \simeq 2l\frac{I_{xx}}{I_{zz}}\left[\frac{I_{xz}}{I_{xx}} \mp 0{\cdot}25\left\{1 - \frac{(\omega_2)_0^2}{(\omega_1)_0^2}\right\}\right] \tag{15}$$

5. *Effects of Suspension-Point Flexibility and using a Spreader Frame for Lifting.*

5.1. *Suspension-point flexibility.*

Usually the suspension point will have some flexibility in the lateral direction and an effective mass. Many other inertia measuring systems are affected by such flexibility and require measurement of the displacements and correction for the effects of this flexibility. This does not apply with this system, as is shown in Appendix F. Although the equations of motion are significantly modified, the basic method for determining the yaw moment and the product of inertia is not affected.

Because the support flexibility is not important it is possible to use normal equipment such as jib or gantry cranes to suspend the system—an important point in favour of this method.

Obviously the support flexibility will have a significant effect on the other modes, but in most cases the design procedure of the previous Section should still be adequate.

5.2. *Using a Spreader Frame.*

In many cases it is likely that the aircraft lifting points will not be stressed to take significant end loads, and it will be necessary to use a spreader frame, as in the tests on the Fairey Delta 2, Fig. 2, to take the sideloads. If the distance between pivoting points on the spreader and the aircraft are all equal then the spreader will always be parallel to the aircraft. This situation, shown diagrammatically in Fig. 7, is analysed in Appendix G, which shows that the use of a light spreader has no effect on the method of analysis, and the formulae for the design procedure are identical if we change q and h to q' and h', given by

$$q' = q+m \tag{G.10}$$

and

$$h' = a+b \tag{G.11}$$

where m = height between spreader and aircraft pivot points

a = height from spreader to main pivot point (hook)

b = height of (aircraft+rig) centre of gravity below aircraft pivot points.

It is important to make the height between the aircraft and spreader pivot points equal at all attachments otherwise other roll-yaw coupling terms are introduced.

Because the spreader's angular movements are identical to those of the aircraft, its inertia should be included with that of the rigs attached to the aircraft.

6. *Instrumentation and Stiffness Calibration.*

6.1. *Advantages of Angular-Rate Measurements.*

The use of angular-rate measurements, as opposed to angular displacements, has several advantages, if sensitive rate gyroscopes are available (± 5 deg/sec). These may be summarised as

(i) Easier to mount and use than displacement pick-offs.

(ii) Displacement pick-offs may need to be duplicated to correct for translational motions.

(iii) Use of angular-rate measurements gives a lower interference from the rocking mode in the roll motion. The decrease relative to displacement measurements being given by the factor, ω_2/ω_1, the ratio of the 'rocking' to 'yawing' mode frequencies (e.g. *see* Fig. 12).

Thus, in all cases it is better to use angular-rate measurements.

6.2. *Calibration of Spring Stiffnesses.*

As in all spring constrained oscillatory tests to determine inertias, the stiffnesses of the springs are the fundamental measurements that require calibration. Although it is possible to measure the stiffness of all the components, including spring attachments, independently, it is preferable to measure the overall stiffness of the system when fully assembled.

One way in which this could be done would be to prevent any lateral movement, by having a point pivot at a distance, z_0 directly beneath the centre of gravity, then prevent roll by loading the appropriate weight, G, at a known moment arm, y_0, whilst deflecting the aircraft in yaw with a known force, J, at a known moment arm, x_0. The equations for the steady state condition are considered in Appendix H, and the required calibrations are given by

$$K_1\, l_1^2 = \frac{J_1\, x_0}{\psi}$$

$$K_1\, l_1 = \frac{-\{G_1\, y_0 + J_1\, z_0\}}{(r_1 - z_0)\,\psi}$$

$$K_2\, l_2^2 = \frac{J_2\, x_0}{\psi}$$

$$K_2 l_2 = \frac{+\{\mathcal{G}_2 y_0 + J_2 z_0\}}{(r_2 - z_0)\psi}$$

where the suffices 1 and 2 apply to tests made first with only the front springs fitted, and second with only the rear springs fitted.

It is important that the restraining point pivot is under the centre of gravity of the system, and an accuracy of better than $\pm\frac{1}{4}$ inch is required.

It is also important to repeat the calibrations at all the values of $r_{1,2}$ that are to be used in the tests as the stiffnesses will usually be a function of r.

If the system is balanced to give $\Delta(Kl) = 0$, then the calibration may be performed with both sets of springs attached.

7. *Computed Example.*

To demonstrate the various points made in Section 3 a test case was designed to the requirements and method of Section 4 and programmed on an analogue computer. The aircraft and rig details are listed in Table 2*. It should be noted that it was not possible to separate the rocking mode and swaying mode frequencies by much, but, as the requirements for $A = 0$, and B nearly zero, are both met, the swaying mode effects are not significant.

The computed time histories at $\Delta(Klr) = 0$ of yawing and rolling rates following a step input are presented in Fig. 8. The form of the rolling rate time history is typical of those obtained in the tests of Refs. 2 and 3 (*see* Fig. 10, Ref. 2). The corresponding Lissajous figure obtained by plotting the yawing rate against the corresponding rolling rate is shown in Fig. 9a. From the method of Section 3.3 the yaw to roll ratio of the yaw mode is obtained from the slope of the parallelogram. From the distortion factor of the parallelogram (i.e. the inverse of the aspect ratio) and the roll to yaw ratio of the yaw mode, the interference in terms of the ratio of the rocking mode to yaw mode rolling motion is obtained. In this case the interference is about 60 per cent. The vast improvement produced by using a half-cosine forcing function is shown by the plot of the same test condition in Fig. 9b. This has an interference of only 3 per cent.

Unfortunately the interference obtained using the half-cosine input is not always so low, and the condition when $\Delta(Klr) = 42\,000$ lb-ft, shown as a time history (Fig. 10) and Lissajous figures (Fig. 11), is not so promising. The interference is 65 per cent with the step function and 60 per cent with the half-cosine function.

Fig. 12 demonstrates the advantages of using angular rate gyroscopes rather than position transducers. The interference for $\Delta(Klr) = 0$ with a step input is 60 per cent with rate gyroscopes and 105 per cent with position signals.

The actual variation of roll to yaw ratio for the yaw mode, $(\dot{\phi}/\dot{\psi})_{\psi}$, with spring rolling moment, $\Delta(Klr)$, is presented in Fig. 13. The variation is linear over the range that was calculated.

8. *Conclusions.*

The Report has studied the equations of motion of the single point suspension method for measuring the yawing moment and the product of inertia of an aircraft. It has shown that the simple method of analysis given in Ref. 1 is correct, if both the rolling moment, B, and the yawing moment, A, due to unit lateral translation of the centre of gravity are zero.

It is further shown that the errors are negligible if $A = 0$ and B is small. However, if both A and B are significant, then the simple analysis is not correct.

Interference from the rocking mode in the rolling motion is shown to be nearly always present, but a simple method of analysis is presented in Section 3.3. This only requires the plotting of yawing rate against rolling rate at the peaks and troughs of the rolling rate time history thus defining both the required

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*The aircraft and rig details are based on the Fairey Delta 2 tests of Ref. 3, but the rig has been redesigned to meet the requirements of this Report.

envelope lines (Figs. 5 and 6). The slope of the best straight line through these points defines the required roll to yaw ratio for the yaw mode and takes into account any damping of the motions.

The influence of different forcing functions on the amount of rocking mode interference (Section 3.2) shows step inputs to be very undesirable, sinusoidal inputs at the yaw mode frequency to be ideal, and half-cosine inputs at the yaw mode frequency to be acceptable.

A design procedure for future rigs is suggested in Section 4.

From the study of more general conditions it has been shown that the method of analysis is not affected by flexibility of the suspension point when $A \simeq B \simeq 0$.

A suggestion for *in-situ* calibration of the rig spring stiffnesses is presented in Section 6.2.

It is concluded that the single-point suspension method is a good simple system for the measurement of both the yawing moment and the product of inertia of aircraft that are capable of being lifted. It is not adversely affected by suspension-point flexibility and thus does not require special lifting structures.

9. *Further Developments.*

A rig is being designed on the basis of this study to measure the yawing moment of inertia and product of inertia of the Hawker Siddeley P1127 aircraft at the Royal Aircraft Establishment, Bedford. These measurements will be compared with those obtained from tests at different pitch attitudes in the rolling moment of inertia tests.

Plans are also being made to check the accuracy of these methods using a suitable large simple mass with calculable inertias.

In addition to these checks on the single-point suspension method, other studies and tests are being considered, including in-flight measurements and general purpose oscillating platforms. Both these methods are in principle not restricted by the size of the aircraft, as are any methods which require the aircraft to be supported from above.

LIST OF SYMBOLS

A	Difference between fore and aft spring moments, $\Delta(Kl)$ lb
a	Height from spreader to main pivot (hook), ft
$a_{1,2}$	Coefficients in equations (E.1) and (E.2), Appendix E
B	Sideforce per unit roll angle, ϕ, $B = \left(W\frac{h}{q} - \Sigma(Kr) \right)$ lb
b	Height of centre of gravity below attachment points (Appendix G)
$b_{1,2}$	Coefficients in equations (E.1) and (E.2), Appendix E
C	Portmanteau symbol, $= -(I_{xz} D^2 + \Delta(Klr))$
c	Semi-span of spreader, Appendix G
D	d/dt
E	W/q, Appendix F
F	Spring restoring force, lb, or $W\frac{h}{q}$ (Appendix F)
f	Lateral displacement of suspension point (Appendix F)
G	Rolling moment weight (Appendix H), lb
g	Gravitational acceleration, 32·2 ft/sec^2
H	Total height parameter, $\simeq q+h+r$, ft
h	Height from centre of gravity to main pivot (hook), ft
h'	Effective 'h' with spreader, $= a+b$, ft
I_{xx}	Roll moment of inertia, slug ft^2
I_{xz}	Product of inertia, slug ft^2
I_{zz}	Yaw moment of inertia, slug ft^2
J	Yaw moment force (Appendix H), lb
K	Spring stiffness, lb/ft
L	Total design length, $\simeq 2l$, ft
l	Yaw spring moment, arm, ft
M	Magnification factor
m	Height between spreader and aircraft attachment points (Appendix G)
p	Appendix B
Q	Interference factor (Appendix E)
q	Height from suspension point to main pivot (hook), ft
q'	Effective 'q' with spreader, $= q+m$, ft
r	Roll spring moment arm, ft
S	Suspension point lateral stiffness, lb/ft

$T_{1,2}$	Starboard and port tensions in spreader to aircraft attachments (Appendix G)
t	Time, sec
W	Weight of aircraft plus rig, lb
x_0	Force application arm for calibration (Appendix H)
Y	Portmanteau symbol, $= \frac{W}{g} D^2 + \left\{ \frac{W}{q} + \Sigma(K) \right\}$
y	Lateral displacement of the centre of gravity, ft
y_0	Force application arm for calibration (Appendix H)
Z	Equivalent mass of suspension point, slug
z_0	Force application arm for calibration (Appendix H)
Δ	Difference between fore and aft conditions, e.g. $\Delta(Kl) = K_1 l_1 - K_2 l_2$
$\Delta\psi, \Delta\phi, \Delta y$	Small differences in the square of the frequencies for yaw, rocking and pendulum modes
θ	Swing angle of cable at suspension point (Fig. 1)
λ	Angle between spreader to aircraft cables and the vertical
Σ	Sum of fore and aft conditions, e.g. $\Sigma(Kl^2) = K_1 l_1^2 + K_2 l_2^2$
Φ	Portmanteau symbol, $= I_{xx} D^2 + \left[\Sigma(Kr^2) + Wh\left(1 + \frac{h}{q}\right) \right]$
ϕ	Roll angle
Ψ	Portmanteau symbol, $= I_{zz} D^2 + \Sigma(Kl^2)$
ψ	Yaw angle
ω_1	'Yawing' mode frequency, rad/sec
ω_2	'Rocking' mode frequency, rad/sec
ω_3	'Swaying' mode frequency, rad/sec
ω_n	Natural frequency, rad/sec

Suffices

ϕ	'Rocking' mode
ψ	'Yawing' mode
0	Values of $\omega_{1,2,3}$ for simple modes without coupling
1	Front spring conditions
2	Rear spring conditions

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APPENDIX A

Equations of Motion for the Single-Point Suspension System.

The equations of motion are derived using the system and notation of Fig. 1. The usual assumption that all angular motions are small is made (i.e. $\sin\theta = \theta$, $\cos\theta = 1$) and the axes used are earth axes with the origin at the centre of gravity. The equations of motion are:

Sideforces:

$$\frac{W}{g} D^2 y = W\theta + F_1 + F_2 \tag{A.1}$$

where W = aircraft + rig weight, lb

g = gravitational acceleration, ft/sec^2

D = differential operator, d/dt

y = sideways displacement, ft

θ = angular displacement of suspension cable at point of suspension

F_1 = front spring restoring force, lb

F_2 = rear spring restoring force, lb.

Rolling moments:

$$I_{xx} D^2 \phi - I_{xz} D^2 \psi = Wh(\theta - \phi) - r_1 F_1 - r_2 F_2 \tag{A.2}$$

where I_{xx} = rolling moment of inertia, slug ft^2

ϕ = roll angle

I_{xz} = product of inertia, slug ft^2

ψ = yaw angle

h = height of centre of gravity below the hook, ft

r_1, r_2 = height of front (r_1) and rear (r_2) springs below centre of gravity, ft

Yawing moments:

$$I_{zz} D^2 \psi - I_{xz} D^2 \phi = l_1 F_1 - l_2 F_2 \tag{A.3}$$

where I_{zz} = yawing moment of inertia, slug ft^2

l_1, l_2 = horizontal distance of front (l_1) and rear (l_2) spring attachments from the centre of gravity, ft.

The spring restoring forces for any displacement in ψ, ϕ and y are:

$$F_1 = K_1(-l_1 \psi + r_1 \phi - y) \tag{A.4}$$

$$F_2 = K_2(l_2 \psi + r_2 \phi - y) \tag{A.5}$$

where K_1, K_2 = spring stiffnesses, lb/ft.

And from Fig. 1 the angle θ is related to ϕ and y by

$$\theta = -\frac{y}{q}-\frac{h\phi}{q}. \tag{A.6}$$

Substituting for F_1, F_2 and θ in the equations of motion (A.1), (A.2) and (A.3) we have:

Sideforce:

$$\left\{\frac{W}{g}D^2+\left[\frac{W}{q}+\Sigma(K)\right]\right\}y+\Delta(Kl)\,\psi+\left\{W\frac{h}{q}-\Sigma(Kr)\right\}\phi = 0. \tag{A.7}$$

Rolling moments:

$$\left\{I_{xx}D^2+\left[\Sigma(Kr^2)+Wh\left(1+\frac{h}{q}\right)\right]\right\}\phi-\{I_{xz}D^2+\Delta(Klr)\}\,\psi+\left\{W\frac{h}{q}-\Sigma(Kr)\right\}y = 0. \tag{A.8}$$

Yawing moments:

$$\{I_{zz}D^2+\Sigma(Kl^2)\}\,\psi-\{I_{xz}D^2+\Delta(Klr)\}\,\phi+\Delta(Kl)\,y = 0 \tag{A.9}$$

where Σ and Δ refer to the sum and differences respectively for the fore and aft spring terms in the brackets, e.g. $\Sigma(Kl^2) = K_1 l_1^2+K_2 l_2^2$ or $\Delta(Klr) = K_1 l_1 r_1-K_2 l_2 r_2$.

The characteristic determinant from equations (A.7), (A.8) and (A.9) is given in Table 1. As can be seen several coefficients are repeated and for convenience the determinant is summarised as follows:

$$\begin{array}{l c}
\text{Parameter} & \begin{array}{ccc} \psi & \phi & y \end{array} \\
\begin{array}{l} \text{Sideforce} \\ \text{Rolling} \\ \text{Yawing} \end{array} & \begin{vmatrix} A & B & Y \\ C & \Phi & B \\ \Psi & C & A \end{vmatrix}
\end{array} = 0 \tag{A.10}$$

where $\Psi = I_{zz}D^2+\Sigma(Kl^2)$

$$\Phi = I_{xx}D^2+\left[\Sigma(Kr^2)+Wh\left(1+\frac{h}{q}\right)\right]$$

$$Y = \frac{W}{g}D^2+\left[\frac{W}{q}+\Sigma(K)\right]$$

$$C = -\{I_{xz}D^2+\Delta(Klr)\}$$

$$A = \Delta(Kl)$$

$$B = W\frac{h}{q}-\Sigma(Kr).$$

The expansion of the determinant is

$$A^2\,\Phi+B^2\,\Psi+C^2\,Y-\Psi\,\Phi\,Y-2ABC = 0. \tag{A.11}$$

This is a sixth order differential equation containing only even powers. The solutions are three neutrally-damped oscillatory modes, each containing some of all three degrees of freedom (yaw, roll, and sideways translation). However, in each mode one of the degrees of freedom predominates. The modes are:

(a) Yaw mode: Predominantly yaw motion with some roll and negligible sideways motion.

This is the mode which is used for determining the inertias.

(b) Rocking mode: Predominantly roll motion with some sideways and negligible yaw motion. This is the mode which is responsible for most of the interference which distorts the rolling motion.

(c) Swaying mode: Predominantly sideways motion with some roll and negligible yaw motion.

From the solutions of the characteristic equation (A.11) the frequencies of the three modes are obtained, and on substituting any of these frequencies back in the equations of motion the amplitude ratios for that mode can be derived.

The two subsequent Appendices, B and C, consider the solutions first, (Appendix B), for the simplified case when $A\,[=\Delta(Kl)]$ and $B\left[=W\dfrac{h}{q}-\Sigma(Kr)\right]$ are both zero. This can be achieved by suitable rig design as both contain known quantities. Second, Appendix C, considers the effects of A and B when they are not zero.

APPENDIX B

Solution of the Equations of Motion when

$$A\left[= \Delta(Kl)\right] \textit{ and } B\left[= W\frac{h}{q} - \Sigma(Kr)\right]$$

are both Zero.

Having established the equations of motion for the single point suspension system in Appendix A, this Appendix considers the solution of the equations, and details the analytical equations required to obtain the yawing moment of inertia and product of inertia from measurements of the yawing and rolling motions of the rig in the yaw mode. In addition the characteristics of the rocking mode are also defined.

If

$$A = \Delta(Kl) = 0$$

and

$$B = W\frac{h}{q} - \Sigma(Kr) = 0$$

then the characteristic determinant, (A.10) in Appendix A, becomes

$$\begin{array}{l} \text{Parameter} \\ \text{Sideforce} \\ \text{Rolling} \\ \text{Yawing} \end{array} \begin{array}{c} \begin{array}{ccc} \psi & \phi & \dot{y} \end{array} \\ \begin{vmatrix} 0 & 0 & Y \\ C & \Phi & 0 \\ \Psi & C & 0 \end{vmatrix} \end{array} = 0. \tag{B.1}$$

We see that the roll and yaw coefficients in the sideforce equation, and also the lateral displacement coefficients in the rolling and yawing moment equations are all zero. Thus the swaying mode consists only of lateral displacement motion and is completely de-coupled from the remaining two modes, and thus cannot affect the inertia analysis.

The characteristic equation is thus

$$Y\,(C^2 - \Psi\,\Phi) = 0. \tag{B.2}$$

The frequency of the swaying mode, ω_3 rad/sec, is given by the solution of $Y = 0$, or

$$\omega_3^2 = \frac{g}{W}\left\{\frac{W}{q} + \Sigma(K)\right\}. \tag{B.3}$$

Having simplified the system to only two modes that affect the yawing and rolling motion, we will now consider the characteristics of the yaw mode, which is to be used for the inertia measurements. The equations of motion are:

$$\text{Rolling:} \quad C\cdot\psi + \Phi\cdot\phi = 0 \tag{B.4}$$

$$\text{Yawing:} \quad \Psi\cdot\psi + C\cdot\phi = 0\cdot \tag{B.5}$$

From the characteristic equation (B.2) the frequencies of the two remaining modes can be obtained after dividing by the common factor Y. Defining

$$(\omega_1)_0^2 = \Sigma(Kl^2)/I_{zz} \tag{B.6}$$

$$(\omega_2)_0^2 = \left\{\Sigma(Kr^2)+Wh\left(1+\frac{h}{q}\right)\right\} \Big/ I_{xx} \tag{B.7}$$

$$p^2 = \Delta(Klr)/I_{xz}. \tag{B.8}$$

Then expanding equation (B.2) and substituting from the above definition we have:

$$I_{zz}\,[D^2+(\omega_1)_0^2]\,I_{xx}\,[D^2+(\omega_2)_0^2]-I_{xz}^2\,[D^2+p^2]^2 = 0. \tag{B.9}$$

Multiplying out and rearranging, equation (B.9) becomes

$$\left(1-\frac{I_{xz}^2}{I_{zz}\,I_{xx}}\right)D^4+\left\{(\omega_1)_0^2+(\omega_2)_0^2-\frac{2\,I_{xz}^2}{I_{zz}\,I_{xx}}p^2\right\}D^2+\left\{(\omega_1)_0^2\,(\omega_2)_0^2-\frac{I_{xz}^2}{I_{zz}\,I_{xx}}p^4\right\} = 0. \tag{B.10}$$

Consider first the particular case when $p^2 = (\omega_1)_0^2$, equation (B.10) then factorises to give:

$$\{D^2+(\omega_1)_0^2\}\left[D^2+\frac{\left\{(\omega_2)_0^2-\dfrac{I_{xz}^2}{I_{zz}\,I_{xx}}(\omega_1)_0^2\right\}}{\left\{1-\dfrac{I_{xz}^2}{I_{zz}\,I_{xx}}\right\}}\right] = 0 \tag{B.11}$$

i.e. $(\omega_1)_0^2$ is a solution of the characteristic equation when $p^2 = (\omega_1)_0^2$. Similarly, from the symmetry of equation (B.10), $(\omega_2)_0^2$ is a solution when $p^2 = (\omega_2)_0^2$.

From equation (B.6) it is clear that the solution $(\omega_1)_0^2$ represents the yaw mode oscillation frequency. The equations (B.6), (B.8) and the condition $p^2 = (\omega_1)_0^2$ are then the basis of the method of measuring the yawing moment of inertia and the product of inertia from measurements of spring characteristics and the oscillation frequency when $p^2 = (\omega_1)_0^2$. The significance of this condition is shown from the equation for the roll to yaw ratio of the yaw mode, which from equation (B.4), and substituting the yaw mode solution of (B.10), $D^2 = -\omega_1^2$, is:

$$\left(\frac{\phi}{\psi}\right)_\psi = -\frac{C}{\Phi} = \frac{I_{xz}\,(\omega_1^2-p^2)}{I_{xx}\,(\omega_1^2-(\omega_2)_0^2)} \tag{B.12}$$

or as

$$\omega_1^2 = (\omega_1)_0^2 = p^2$$

$$\left(\frac{\phi}{\psi}\right)_\psi = 0.$$

Thus to determine the yawing moment and the product inertia the equations

$$I_{zz} = \Sigma(Kl^2)/(\omega_1)_0^2 \tag{B.6}$$

and

$$I_{xz} = \Delta(Klr)/(\omega_1)_0^2 \tag{B.13}$$

must be solved when

$$\left(\frac{\phi}{\psi}\right)_\psi = 0 .$$

This is the basis of the method of analysis given by Boucher, *et al.*[1].

Full analysis of equation (B.9) when $p^2 \neq (\omega_1)_0^2$ shows the variation of the general yaw mode frequency, ω_1, to be small and given by the following equations

$$\omega_1^2 = (\omega_1)_0^2 + \Delta\psi \tag{B.14}$$

where

$$\frac{\Delta\psi}{(\omega_1)_0^2} \simeq \left(\frac{\phi}{\psi}\right)_\psi^2 \left(\frac{I_{xx}}{I_{zz}}\right)\left(1-\frac{(\omega_2)_0^2}{(\omega_1)_0^2}\right). \tag{B.15}$$

From equation (B.12) the roll to yaw ratio of the yaw mode is seen to be a linear function of $\Delta(Klr)$, when the yaw mode frequency is constant. Equation (B.15) shows this to be only slightly in error, as for reasonable values of inertias and frequencies $\Delta\psi/(\omega_1)_0^2$ is less than 0·5 per cent at the reasonably high roll to yaw ratio of 0·2.

Thus the value of $\Delta(Klr)$ for zero roll to yaw ratio can readily be determined from plotting values of $(\phi/\psi)_\psi$ against $\Delta(Klr)$.

From the analysis so far it is clear that the method of analysis given by Ref. 1 is completely acceptable when $A = B = 0$. However, the main problem in analysing the tests in Refs. 2 and 3 was the presence of additional roll motion from the rocking mode, which made direct measurement of the roll to yaw ratio impossible. Despite the fact that a suitable method of analysis, which 'filters' out the unwanted rocking mode rolling motion (Appendix E), has been developed in this Report; it is highly desirable to reduce the amount of interference and thus increase the experimental accuracy.

Two main factors dictate the size of the interference; the natural frequency of the rocking mode, which from Appendix D should be less than $1/\sqrt{2}$ of the yaw mode frequency; and the roll to yaw ratio of the rocking mode, which should be as high as possible to prevent interference with the yawing motion.

From the characteristic equation (B.2) the frequency of the rocking mode, ω_2 rad/sec, is given by

$$\omega_2^2 \simeq \frac{\left[\Sigma(Kr^2) + Wh\left(1+\frac{h}{q}\right)\right]}{I_{xx}} = (\omega_2)_0^2, \qquad \text{when } C = 0 \tag{B.16}$$

and

$$\omega_2^2 = (\omega_2)_0^2 + \Delta\phi \tag{B.17}$$

where

$$\frac{\Delta\phi}{(\omega_2)_0^2} \simeq -\frac{\{\Delta(Klr) - I_{xz}(\omega_2)_0^2\}^2}{I_{xx} I_{zz} \{(\omega_1)_0^2 - (\omega_2)_0^2\}(\omega_2)_0^2}. \tag{B.18}$$

Again, the difference $\Delta\phi/(\omega_2)_0^2$ is small—being about—2 per cent for reasonable conditions.

Then from the equation of motion (B.5) the roll to yaw ratio for the rocking mode, $(\phi/\psi)_\phi$, is given by

$$\left(\frac{\phi}{\psi}\right)_\phi = \frac{I_{zz}\,\{(\omega_1)_0^2-(\omega_2)_0^2\}}{\{\Delta(Klr)-I_{xz}\,(\omega_2)_0^2\}} \tag{B.19}$$

or taking the particular and practical condition when the roll to yaw ratio of the yaw mode is zero (i.e. $\Delta(Klr) = I_{xz}\,(\omega_1)_0^2$) the equation (B.19) becomes:

$$\left(\frac{\phi}{\psi}\right)_\phi = \frac{I_{zz}}{I_{xz}}. \tag{B.20}$$

Thus, although the requirement for $\omega_2\sqrt{2} < \omega_1$ can be achieved by suitable rig design, the value of $(\phi/\psi)_\phi$ is determined by the aircraft inertia. Some variation may be obtained by tilting the aircraft, but as I_{zz}/I_{xz} is normally high this should not be necessary. Some idea of the very small amount of interference in the total yawing motion from the rocking mode yawing motion is given by taking a fairly bad condition of rolling interference of say 20 per cent rocking mode rolling amplitude to yawing mode rolling amplitude. If this occurs at a condition with the roll to yaw ratio of the yawing mode equal to 0·1, then the ratio of the rocking mode yawing motion to yawing mode yawing motion is $(0{\cdot}2)\cdot(0{\cdot}1)\cdot I_{xz}/I_{zz}$, or for a typical inertia ratio of 1/15 the interference is only 0·13 per cent. Thus the effects of the rocking mode on the total yawing motion is negligible.

APPENDIX C

Effects of: $A\,[\,=\Delta(Kl)] \neq 0$ *and* $B\left[\;= W\dfrac{h}{q}-\Sigma(Kr)\right] \neq 0$ *on the Solution of the Equations of Motion.*

Although both A and B contain only known rig and aircraft parameters and can thus be made zero by suitable design, it is important to study the effects of them being non-zero on the method of analysis for inertia measurements. In practice it will generally be fairly straightforward to keep $A\,[\,=\Delta(Kl)]$ equal to zero. Thus we shall consider only two cases; first that $B \neq 0$, $A = 0$, the most probable condition; second that $B \neq 0$, $A \neq 0$, the general condition which has occurred in the tests of Refs. 2 and 3.

For the case with $B \neq 0$, $A = 0$ the characteristic determinant, characteristic equation and equations of motion are:

$$\begin{array}{l|ccc|}
\text{Parameter} & \psi & \phi & y \\
\text{Sideforce} & 0 & B & Y \\
\text{Rolling} & C & \Phi & B \\
\text{Yawing} & \Psi & C & 0
\end{array} = 0 \tag{C.1}$$

$$B^2\,\Psi + C^2\,Y - \Psi\,\Phi\,Y = 0 \tag{C.2}$$

$$\text{Sideforce} \qquad B\cdot\phi + Y\cdot y = 0 \tag{C.3}$$

$$\text{Rolling} \qquad C\cdot\psi + \Phi\cdot\phi + B\cdot y = 0 \tag{C.4}$$

$$\text{Yawing} \qquad \Psi\cdot\psi + C\cdot\phi \qquad = 0. \tag{C.5}$$

Taking the analysis of the yaw mode, as in Appendix B, we can show the effects of B on equations for the yaw mode frequency, ω_1, and the yaw mode roll to yaw ratio, $(\phi/\psi)_\psi$. If we define the yaw mode frequency, as in Appendix B, as

$$\omega_1^2 = (\omega_1)_0^2 + \Delta\psi\,. \tag{B.14}$$

Then, by substituting $D^2 = -\omega_1^2$, in the characteristic equation (C.2) the equation for the frequency ω_1, is obtained, making the assumption that $\Delta\psi$ is small

$$\omega_1^2 = (\omega_1)_0^2 + (\Delta\psi)_{B=0}\cdot\frac{1}{\left\{1-\dfrac{B^2}{\left(\dfrac{W}{g}\right)I_{xx}\,[(\omega_1)_0^2-(\omega_2)_0^2]\,[(\omega_1)_0^2-(\omega_3)_0^2]}\right\}} \tag{C.6}$$

where (as in Appendix B)

$$(\omega_1)_0^2 = \Sigma(Kl^2)/I_{zz}$$

$$(\omega_2)_0^2 = \left[\Sigma(Kr^2) + Wh\left(1+\frac{h}{q}\right)\right]\Big/ I_{xx}$$

$$(\omega_3)_0^2 = \left[\frac{W}{q} + \Sigma(K)\right]\Big/\frac{W}{g}$$

and $(\Delta\psi)_{B=0}$ is defined in equation (B.15).

Also the roll to yaw ratio is found from the equation of motion (C.3) and (C.4) using the same substitution of $D^2 = -\omega_1^2$ and assuming $\Delta\psi$ is small. This gives

$$\left(\frac{\phi}{\psi}\right)_\psi = \left(\frac{\phi}{\psi}\right)_{\substack{\psi \\ B=0}} \cdot \frac{1}{\left\{1 - \dfrac{B^2}{\left(\dfrac{W}{g}\right) I_{xx} \left[(\omega_1)_0^2 - (\omega_2)_0^2\right]\left[(\omega_1)_0^2 - (\omega_3)_0^2\right]}\right\}} \tag{C.7}$$

Thus the basic method of analysis of Appendix B is unaffected by $B \neq 0$ when $A = 0$.

However it can be seen from the equations of motion (C.3), (C.4) and (C.5) that the swaying mode is no longer only sideways motion but contains some rolling and yawing. The method of filtering the rolling motion interference can only be used when the motion contains two frequencies, and not three. Thus, although the method of analysing the yaw mode is unaffected by $B \neq 0$, the extraction of the roll to yaw ratio for the yaw mode from the total motion will be affected. As for the rocking mode, in Appendix B, the two main criteria for low interference are that the frequency, ω_3, should be less than $1/\sqrt{2}\,\omega_2$ (ω_2 = rocking mode frequency) and the ratio of rolling to lateral translation motion, $(\phi/y)_y$, should be small. Defining the frequency of the swaying mode in the usual way

$$\omega_3^2 = (\omega_3)_0^2 + \Delta y \tag{C.8}$$

where $(\omega_3)_0^2 = \left\{\dfrac{W}{q} + \Sigma(K)\right\} \Big/ \dfrac{W}{g}$ the solution of $Y = 0$.

Then

$$\frac{\Delta y}{(\omega_3)_0^2} \simeq \frac{-\dfrac{W}{g} B^2}{I_{xx}\left\{\dfrac{W}{q} + \Sigma(K)\right\}^2 \left\{\dfrac{(\omega_2)_0^2}{(\omega_3)_0^2} - 1\right\}}. \tag{C.9}$$

Now from equation (C.3) the roll to translation ratio $(\phi/y)_y$ is given by

$$\left(\frac{\phi}{y}\right)_y = -\frac{Y}{B} \tag{C.10}$$

or substituting $D^2 = -\omega_3^2$ in Y

$$\left(\frac{\phi}{y}\right)_y \doteq \frac{\dfrac{W}{g}\Delta y}{B} = \frac{W}{g}(\omega_3)_0^2 \left(\frac{\Delta y}{(\omega_3)_0^2}\right) \Big/ B.$$

Substituting for $\Delta y/(\omega_3)_0^2$ from equation (C.9) we have

$$\left(\frac{\phi}{y}\right)_y = -\frac{\dfrac{W}{g} B}{I_{xx}\left\{\dfrac{W}{q} + \Sigma(K)\right\} \left\{\dfrac{(\omega_2)_0^2}{(\omega_3)_0^2} - 1\right\}}. \tag{C.11}$$

Thus, if we aim to keep $(\phi/y)_y < |0{\cdot}05|$ rad/ft, then we must have

$$|B| < |(0{\cdot}05)\, I_{xx}\, \{(\omega_2)_0^2 - (\omega_3)_0^2\}|\,. \tag{C.12}$$

The yaw to roll ratio for the swaying mode can also be derived from equation (C.5)

$$\left(\frac{\psi}{\phi}\right)_y = -\frac{C}{\Psi} \tag{C.13}$$

or taking the particular but relevant condition when the roll to yaw ratio of the yaw mode is zero (i.e. $\Delta(Klr) = I_{xz}\,(\omega_1)_0^2$) equation (C.13) becomes:

$$\left(\frac{\psi}{\phi}\right)_y = \frac{I_{xz}}{I_{zz}}\,. \tag{C.14}$$

Thus the pendulum mode yaw motion is negligible.

The addition of $B \neq 0$ will also change the characteristics of the rocking mode from those given in Appendix B, but if B meets the criterion of equation (C.12) the effects will be small and need not be considered.

Finally we consider the general case where both A and B are non-zero. The characteristic determinant and equations, and the equations of motion are then those given in Appendix A, as follows:

$$\begin{array}{l|ccc|l} \text{Parameter} & \psi & \phi & y & \\ \text{Sideforce} & A & B & Y & \\ \text{Rolling} & C & \Phi & B & = 0 \\ \text{Yawing} & \Psi & C & A & \end{array} \tag{A.10}$$

$$A^2\,\Phi + B^2\,\Psi + C^2\,Y - \Psi\,\Phi\,Y - 2\,A\,B\,C = 0 \tag{A.11}$$

$$\text{Sideforce} \qquad A\cdot\psi + B\cdot\phi + Y\cdot y = 0 \tag{C.15}$$

$$\text{Rolling} \qquad C\cdot\psi + \Phi\cdot\phi + B\cdot y = 0 \tag{C.16}$$

$$\text{Yawing} \qquad \Psi\cdot\psi + C\cdot\phi + A\cdot y = 0\,. \tag{C.17}$$

The effects on the yaw mode characteristics used for inertia analysis are considered at the analysis point when $(\phi/\psi)_\psi$ is zero. Then

$$\Delta\psi = -A^2/I_{zz}\left(\frac{W}{g}\right)\{(\omega_3)_0^2 - (\omega_1)_0^2\}\,. \tag{C.18}$$

This means that there will be an error in the determination of the yawing inertia of $\Delta\psi/(\omega_1)_0^2$ times the true inertia. However, except for very large values of A the value of $\Delta\psi/(\omega_1)_0^2$ is negligible.

Also at $(\phi/\psi)_\psi = 0$ then

$$I_{xz} = \frac{\Delta(Klr) + AB\Big/\dfrac{W}{g}((\omega_3)_0^2 - \omega_1^2)}{\omega_1^2}\,. \tag{C.19}$$

This gives an error in I_{xz} of $\Delta I_{xz} = AB\Big/\frac{W}{g}\cdot\omega_1^2\left((\omega_3)_0^2-\omega_1^2\right)$. Thus if we are to keep the consequent error in the inclination of the principle axis to less than 0·05 degrees then $\Delta I_{xz}/I_{zz}$ must be less than about 0·08 per cent. This means that we must have

$$\left|1250\right| AB < \left|\frac{W}{g}\Sigma(Kl^2)\cdot\left\{(\omega_3)_0^2-(\omega_1)_0^2\right\}\right|. \tag{C.20}$$

Thus the permissible maximum value of AB is proportional to the mass, the yaw stiffness and the difference between the yaw and pendulum mode frequencies. Thus to reduce the errors with given values of A and B both the yaw stiffness and the difference between the frequencies should be the maximum possible.

The analysis of the general equations for the characteristics of the other two modes is not attempted here as the aim should always be to meet the criterion of equation (C.20) and in these circumstances the analysis of the earlier part of this Appendix is sufficiently accurate. In practice, as A and B are always likely to have some value even in a well designed system, then the two criteria are equations (C.12) for B and (C.20) for A.

APPENDIX D

Influence of Various Forcing Functions on the Interference between the Rocking and Yaw Modes.

Throughout this Appendix we will consider only the design case when the yawing moment for unit sideways displacement, A, and the rolling moment for unit sideways displacement, B, are both zero. Thus any sideways displacement produced by a forcing function which is not a pure couple in yaw cannot affect the rocking or yaw modes, because these contain no translational motion. The swaying mode, which will be excited by any sideforce component in the forcing function, contains no rolling or yawing motion.

In practice, however, it will be better to keep the amount of any sideforce component low relative to the yawing moment, although it is not necessary to make it zero. This implies the application of any single sideforce at the farthest possible distance from the centre of gravity. It is clear that, as only the yaw mode is required, any forcing function which contains a rolling moment component should be avoided.

Considering now the influence of three types of forcing function:

(i) Step function—obtained by releasing the system from an initial steady displacement.

(ii) Sinusiodal function—force varying sinusoidally with maximum at zero deplacement.

(iii) Half-cosine function—this is similar to input (ii) but only the push forces are applied (Fig. 4).

We can study the response of the two modes (rocking and yaw) by considering the harmonic analysis of the three inputs, and the frequency response characteristics of the modes.

The variation of the magnification factor, M—the ratio of output motion to input motion—with frequency for a neutrally damped oscillatory system is given by

$$M = \frac{1}{\left|\left(\frac{\omega}{\omega_n}\right)^2 - 1\right|} \tag{D.1}$$

where ω = impressed frequency

ω_n = natural frequency.

The variation of M with frequency is shown in Fig. 14. It is seen that for M to be less than unity the impressed frequency must be greater than the natural frequency by a factor of $\sqrt{2}$. Thus, as the main impressed frequency for exciting the yaw mode will be the yaw mode frequency, the rocking mode response will be attenuated if its natural frequency is less than $1/\sqrt{2}$ of the yaw mode frequency. The rocking mode motion will be mainly excited by the yaw mode roll motion, when $(\phi/\psi)_\psi \neq 0$, and also by any small amounts of direct rolling moment inputs that may be present in the basic forcing function, due to inaccuracies in the input positioning. Thus, providing positioning errors are negligible, the rocking mode motion will be proportional to $(\phi/\psi)_\psi$ for a given form of forcing function.

The relative amounts of rocking mode motion produced by a given forcing function may be determined from an harmonic analysis of the forcing function. This harmonic analysis together with the frequency response characteristics for the rocking mode determines the magnitude of the rocking mode oscillations. However, the relevant criterion is the ratio of rocking mode to yaw mode motion, which should be zero or at least very small. To obtain the relative responses the harmonic analysis should be used together with the ratio of the rocking-mode to yaw-mode magnification factors, M_ϕ/M_ψ, which is illustrated in Fig. 15 for a case where the rocking mode natural frequency, ω_2 is one-half of the yaw mode natural frequency, ω_1. From this it is clear that the ideal forcing function would be a pure sinusoidal function at the yaw mode frequency, as this will contain no harmonics and M_ϕ/M_ψ is zero at this frequency.

The other two suggested forcing functions do contain harmonics. Analysing the step function as a Fourier series gives the variation of relative amplitudes at different frequencies as the fundamental frequency tends to zero and is shown in Fig. 16. Thus the step input is a powerful way of producing interference because of its large harmonic content at the lower frequencies. This is well demonstrated by the product of the relative amplitudes and M_ϕ/M_ψ which is shown in Fig. 17.

Harmonic analysis of the half-cosine function gives discrete amplitudes for the various harmonics which are shown in Fig. 18. This assumes a continuous forcing function, but is valid for a sufficiently long train of inputs. This form of forcing function has a large content at the fundamental frequency, and none below this frequency. If the fundamental frequency is that of the yaw mode and the rocking mode frequency is one-half of this frequency, then the relative response will be as indicated by Fig. 19. Thus, this half-cosine input is much more suitable than the step input, although not quite as good as the pure sinusiodal input.

In practice, if the sinusoidal or half-cosine functions are produced in the simplest way by a person pushing and pulling, or just pushing, in sympathy with the yaw mode motion, then the function will not be purely sinusoidal. However the main component will still be at the yaw mode frequency and most distortions of the motion will tend to increase only the higher frequency components of the input.

APPENDIX E

Determination of Amplitude Ratios from time Histories of Two Parameters Containing only Two Frequency Components with Zero Damping.

One of the problems of analysis of the inertia results is the determination of the amplitude ratio of roll to yaw motion in the yawing mode in the presence of interference from the rocking mode. If we denote the two frequencies by ω_1 (yaw mode) and ω_2 (rocking mode) the yaw and roll motions are given by

$$\text{Yaw motion,} \quad \psi = a_1 \sin \omega_1 t + a_2 \sin \omega_2 t \tag{E.1}$$

$$\text{Roll motion,} \quad \phi = b_1 \sin \omega_1 t + b_2 \sin \omega_2 t \tag{E.2}$$

where the required roll to yaw ratio is b_1/a_1.

It should be noted that as the modes are neutrally damped the motion equations do not contain any phase angle terms. These can only occur when the systems are damped.

If time is eliminated from equations (E.1) and (E.2) and the yaw is plotted against roll. the resulting curve is a form of Lissajous figure contained in a parallelogram as shown in Fig. 5. If $a_2 = b_2 = 0$ the parallelogram reduces to a line given by

$$\phi = \frac{b_1}{a_1} \psi . \tag{E.3}$$

The difference caused by the interference $\delta\phi$ is given from equations (E.1) and (E.2) by:

$$\delta\phi = \left(b_2 - \frac{b_1}{a_1} a_2 \right) \sin \omega_2 t . \tag{E.4}$$

This is seen to be independent of the yaw magnitude; thus the envelope for $\sin \omega_2 t = \pm 1$ (i.e. the two sides of the parallelogram) is at a distance $\pm \left(b_2 - \frac{b_1}{a_1} a_2 \right)$ from the line $\phi = \frac{b_1}{a_1} \psi$ and is parallel to that line.

Similarly, it can be shown that the top and bottom of the parallelogram are at a distance of $\left(a_1 - \frac{a_2}{b_2} b_1 \right)$ from the line $\psi = \frac{a_2}{b_2} \phi$ are parallel to it.

Thus, the required roll to yaw ratio is given by the slope of the sides of the parallelogram:

$$\text{slope} = \text{yaw/roll ratio} .$$

Also the level of interference is shown by the slenderness of the parallelogram and may be defined by the interference factor, Q, given by

$$Q = \frac{b_2 - \frac{b_1}{a_1} a_2}{a_1 - \frac{a_2}{b_2} b_1}$$

or

$$Q = \frac{b_2}{a_1} . \tag{E.5}$$

Fig. 6 shows the Lissajous figure for the equations

$$\psi = 5 \sin 5t + 0{\cdot}004 \sin 3t \tag{E.6}$$

$$\phi = \sin 5t + 0{\cdot}2 \sin 3t \tag{E.7}$$

which illustrate a typical roll/yaw ratio in the yaw mode of 0·2 ($= b_1/a_1$), a distortion factor, Q, of 0·04 ($Q = b_2/a_1$), and a roll/yaw ratio in the rocking mode of 50 ($= b_2/a_2$). In this case the two frequencies are related by a simple fraction and the figure soon repeats. In practice the two frequencies are unlikely to be simply related and the figures will be less simple as shown in the examples of Fig. 9.

Thus, to derive the required roll/yaw ratio from measurements of roll and yaw motion as time histories, a plot of yaw against roll is required. However as only the enveloping lines are required the only points that need be plotted are those corresponding to the peaks (+ve and −ve) of the roll motion. These are all points which lie on the enveloping lines. The effects of light damping may be accounted for by plotting the best straight line between the two envelope lines.

APPENDIX F

Effects of Suspension Point Flexibility.

For many aircraft the suspension point can be provided by a mobile crane and this Appendix considers the effects of suspension point flexibility on the equations of motion. We will consider the general case of a suspension point which is capable of deflecting sideways. Defining the distance as f in the y direction with a stiffness, S lb/ft, and an equivalent mass, Z slugs.

Thus taking the general condition with the suspension point deflected and the attachment wire at angle θ the equation of motion for lateral movement of the suspension point is

$$(ZD^2+S)f+W\theta = 0. \tag{F.1}$$

The equations of Appendix A for the general system remain unchanged, except that

$$\theta = \{-y-h\phi+f\}/q. \tag{F.2}$$

We now have a fourth equation and a fourth variable and the characteristic determinant becomes

$$\begin{array}{l} \text{Parameter} \\ \\ \text{Sideforce} \\ \\ \text{Rolling} \\ \\ \text{Yawing} \\ \\ \text{Suspension} \end{array} \begin{array}{c} \begin{array}{cccc} \underline{\psi} & \underline{\phi} & \quad & \underline{f} \end{array} \\ \begin{vmatrix} A & B & y & -\dfrac{W}{q} \\ C & \Phi & B & -W\dfrac{h}{q} \\ \Psi & C & A & 0 \\ 0 & -W\dfrac{h}{q} & -\dfrac{W}{q} & ZD^2+\left(S+\dfrac{W}{q}\right) \end{vmatrix} \end{array} = 0 \tag{F.3}$$

or putting

$$E = -\frac{W}{q}$$

$$F = -W\frac{h}{q}$$

$$\Delta = ZD^2+\left(S+\frac{W}{q}\right)$$

and taking the design case with $A = B = 0$. The determinant becomes:

$$\begin{vmatrix} 0 & 0 & Y & E \\ C & \Phi & 0 & F \\ \Psi & C & 0 & 0 \\ 0 & F & E & \Delta \end{vmatrix} = 0. \tag{F.4}$$

This gives the characteristic equation as

$$\{C^2-\Psi\,\Phi\}\left\{Y\Delta+E^2-\frac{F^2\,Y}{\Phi}\right\}+C^2\frac{F^2\,Y}{\Phi}=0\,. \tag{F.5}$$

The roll to yaw ratio for the yaw mode is given by

$$\left(\frac{\phi}{\psi}\right)_\psi=\frac{-C}{\left\{\Phi+\dfrac{F^2\,Y}{E^2-\Delta Y}\right\}}\,. \tag{F.6}$$

Thus the basic method of analysis of Appendix B still applies, because when $C=0$, then $(\phi/\psi)_\psi=0$, and from equation (F.5), $\Psi=0$.

There will of course be interference with and changes to the characteristics of the rocking and swaying modes. This can then give some difficulties in the analysis of the rolling motion if the changes are large. For this reason it is desirable to keep the suspension point reasonably stiff. The main criterion being to keep the factor

$$\left(\frac{\phi}{f}\right)_f=\frac{-F}{\left\{I_{xx}\,[(\omega_2)_0^2-(\omega_4)_0^2]-\dfrac{\{\Delta(Klr)-I_{xz}\,(\omega_4)_0^2\}^2}{I_{zz}\,[(\omega_1)_0^2-(\omega_4)_0^2]}\right\}} \tag{F.7}$$

small, and thus reduce the rolling motion contribution from the motion of the suspension. The frequency of the suspension point mode is taken as

$$(\omega_4)_0^2=\left(S+\frac{W}{q}\right)\Big/Z\,. \tag{F.8}$$

As equation (F.7) reduces to

$$\left(\frac{\phi}{f}\right)_f=\frac{-F}{\left\{I_{xx}\,[(\omega_2)_0^2-(\omega_4)_0^2]-\dfrac{I_{xz}^2}{I_{zz}}\,[(\omega_1)_0^2-(\omega_4)_0^2]\right\}} \tag{F.9}$$

when $(\phi/\psi)_\psi=0$, it can be seen that the best way of keeping $(\phi/f)_f$ small is to have $(\omega_4)_0$ as high as possible (i.e. the stiffest possible suspension point). If $(\omega_4)_0>>(\omega_2)_0$ and $(\omega_1)_0$, then

$$\left(\frac{\phi}{f}\right)_f\simeq\frac{F}{I_{xx}\,(\omega_4)_0^2}=\frac{Wh}{I_{xx}\,q\,(\omega_4)_0^2}\,. \tag{F.10}$$

This implies that the desirable suspension point frequency should be of the order of 2 Hz or greater, as this should make $(\phi/f)_f$ less than about 0·005.

Thus, the stiffness of the suspension point is not a critical factor in the use of the single point suspension method, provided the rig is designed to have $A=B=0$.

APPENDIX G

Effects of a 'Light' Spreader Frame.

Considering the geometry shown in Fig. 7 and the equations of Appendix A, it is clear that only the sideforce and rolling moment equation are affected by the use of the spreader frame. The weight of the spreader is assumed zero. The equations then become

Sideforce:

$$\frac{W}{g} D^2 y = F_1 + F_2 - (T_1 + T_2)\lambda . \tag{G.1}$$

Rolling moment:

$$I_{xx} D^2 \phi - I_{xz} D^2 \psi = (T_2 - T_1) c - (T_1 + T_2) b (\phi + \lambda) - r_1 F_1 - r_2 F_2 \tag{G.2}$$

where T_1, T_2 are the tensions in the wires between the aircraft and spreader frame

λ is the angle between the normal to the spreader frame and the wires

c is the semi-span of the spreader

b is the height of the aircraft + rig centre of gravity below the aircraft attachment pivots.

Other force and moment equations may then be derived to eliminate T_1, T_2, c and λ from equations (G.1) and (G.2). These are:

Vertical force: $$T_1 + T_2 = W \tag{G.3}$$

Lateral force on spreader: $$W\theta = -(T_1 + T_2)\lambda \tag{G.4}$$

or from (G.3)

$$\theta = -\lambda . \tag{G.5}$$

Moments about the mid point of the spreader:

$$(T_2 - T_1) c = W a (\theta - \phi) \tag{G.6}$$

where a is the height from the spreader to the main pivot (hook).

Using these results equations (G.1) and (G.2) become:

$$\frac{W}{g} D^2 y = W\theta + F_1 + F_2 \tag{G.7}$$

$$I_{xx} D^2 \phi - I_{xz} D^2 \psi = W (a + B)(\theta - \phi) - r_1 F_1 - r_2 F_2 \tag{G.8}$$

and the geometric relation between θ, y, ϕ is

$$\theta = -\frac{y}{(q+m)} - \frac{(a+b)}{(q+m)} \phi . \tag{G.9}$$

Comparing equations (G.7), (G.8) and (G.9) with equations (A.1), (A.2) and (A.6) of Appendix A it can be seen that if we substitute

$$q' = q+m \tag{G.10}$$

and

$$h' = a+h \tag{G.11}$$

for q and h in Appendix A then the equations are identical.

APPENDIX H

Calibration of Spring Stiffnesses in Situ

Consider the steady state condition where $\phi = 0$, $y = 0$ and the aircraft is displaced in yaw. Let a restraint with lateral force R be placed at a distance z_0 directly below the centre of gravity; a weight G be placed at an arm of y_0 to one side of the centre of gravity with zero arm in the x directions; and a lateral force J be applied at a horizontal distance x_0 from the centre of gravity with no z displacement. Then the equations of motion from Appendix A give

$$-\Delta(Kl) \cdot \psi + J + R = 0 \tag{H.1}$$

$$-\Sigma(Kl^2) \cdot \psi + J\, X_0 = 0 \tag{H.2}$$

$$+\Delta(Klr) \cdot \psi + G\, y_0 - R\, z_0 = 0\,. \tag{H.3}$$

Considering the condition with only the front springs attached and eliminating R we get

$$K_1\, l_1^2 = \frac{J}{\psi} X_0 \tag{H.4}$$

and

$$K_1\, l_1 = -\frac{G\, y_0 + J\, z_0}{(r_1 - z_0)\, \psi} \tag{H.5}$$

and similarly if only the rear springs are attached

$$K_2\, l_2^2 = \frac{J}{\psi} X_0 \tag{H.4a}$$

$$K_2\, l_2 = +\frac{G\, y_0 + J\, z_0}{(r_2 - z_0)\, \psi}\,. \tag{H.5a}$$

Alternatively, if the system is initially balanced to give $\Delta(Kl) = 0$, then the tests may be done with both sets of spring attached, and

$$\Sigma(Kl^2) = \frac{J}{\psi} X_0 \tag{H.6}$$

$$\Delta(Klr) = \frac{G\, y_0 + J\, z_0}{\psi}\,. \tag{H.7}$$

The balancing may be done by applying a sideforce at the centre of gravity and adjusting one of the spring moment arms until only sideways displacement is present with zero yaw.

TABLE 1

Characteristic Determinant of the Equations of Motion.

Parameter	ψ	ϕ	y	
Sideforce	$\Delta(Kl)$	$W\dfrac{h}{q}-\Sigma(Kr)$	$\dfrac{W}{g}D^2+\left(\dfrac{W}{q}+\Sigma(K)\right)$	
Rolling moment	$-(I_{xz}D^2+\Delta(Klr))$	$I_{xx}D^2+\left[\Sigma(Kr^2)+Wh\left(1+\dfrac{h}{q}\right)\right]$	$W\dfrac{h}{q}-\Sigma(Kr)$	$=0$
Yawing moment	$I_{zz}D^2+\Sigma(Kl^2)$	$-(I_{xz}D^2+\Delta(Klr))$	$\Delta(Kl)$	

TABLE 2

Case used in Analogue Computer Study.
(Section 7)

W lb	11645	$\frac{\Sigma(r)}{2}$ ft	0·353
I_{zz} slug ft^2	29900	h ft	6
I_{xx} slug ft^2	5500	q ft	34
I_{xz} slug ft^2	800	B lb	−65
K_1 lb/ft	3000	A lb	0
K_2 lb/ft	3000	$(\omega_1)_0^2$ sec^{-2}	39·3
l_1 ft	14	$(\omega_2)_0^2$ sec^{-2}	15·1
l_2 ft	14	$(\omega_3)_0^2$ sec^{-2}	17·8

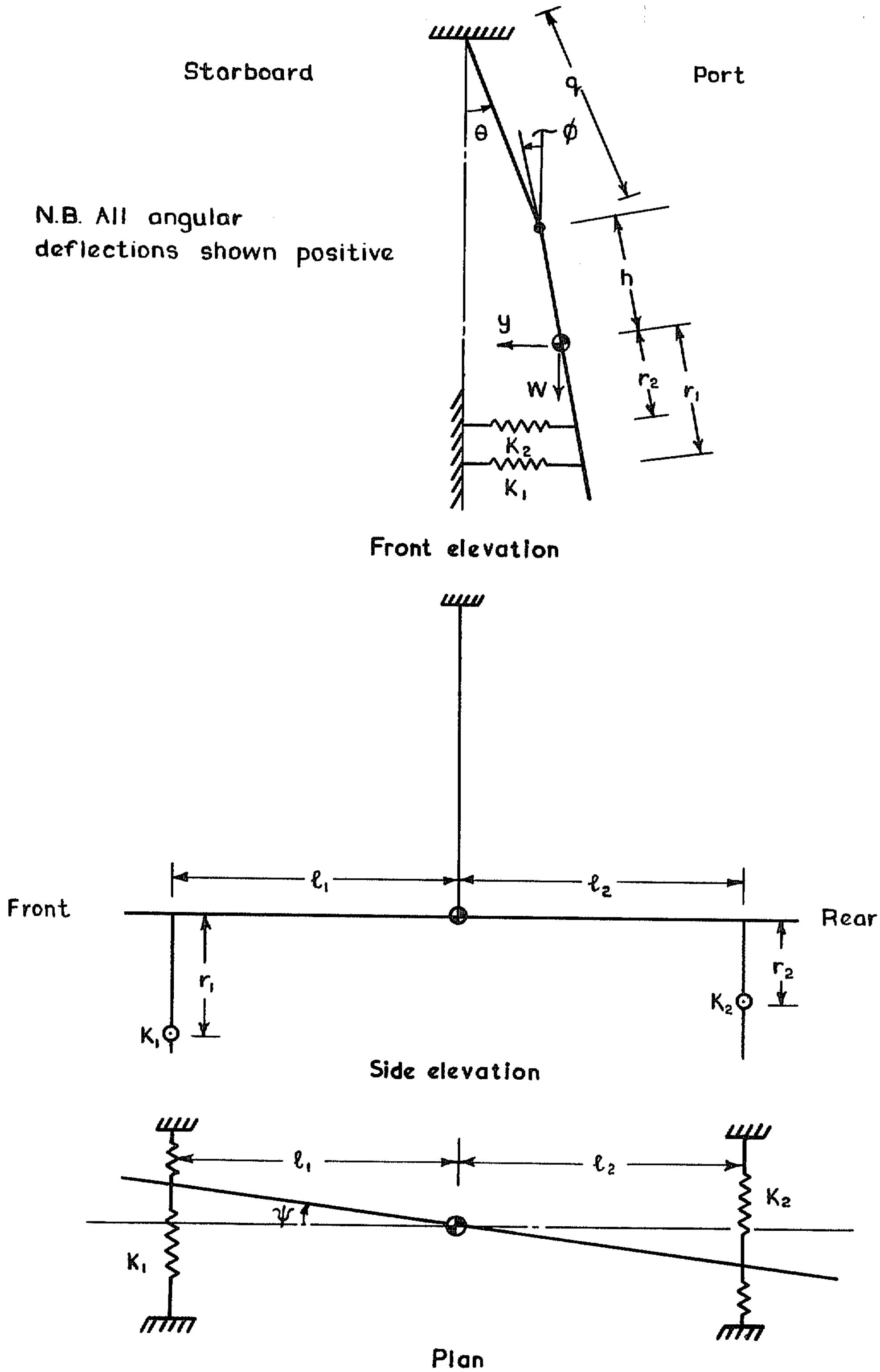

FIG. 1. Diagram of the test rig.

FIG. 2. Rig with spreader frame used for the Fairey Delta 2 inertia measurements.

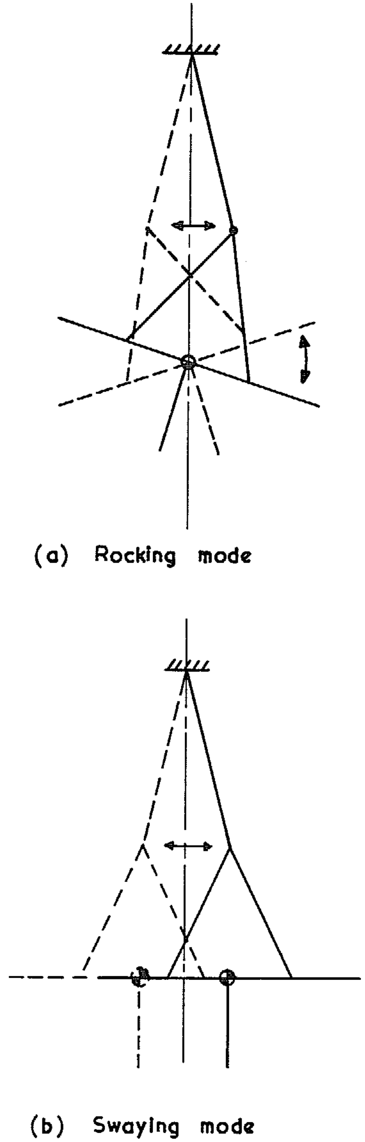

FIG. 3. Diagrams of the subsidiary modes.

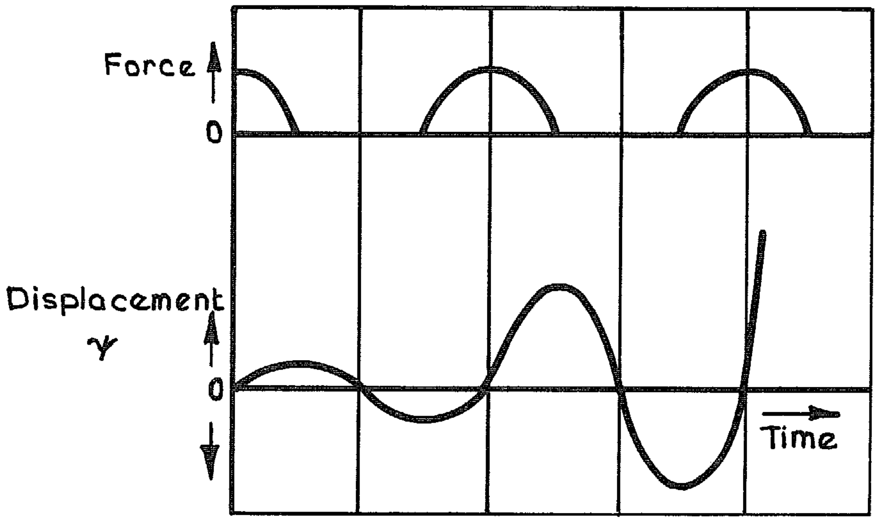

FIG. 4. Half cosine forcing function.

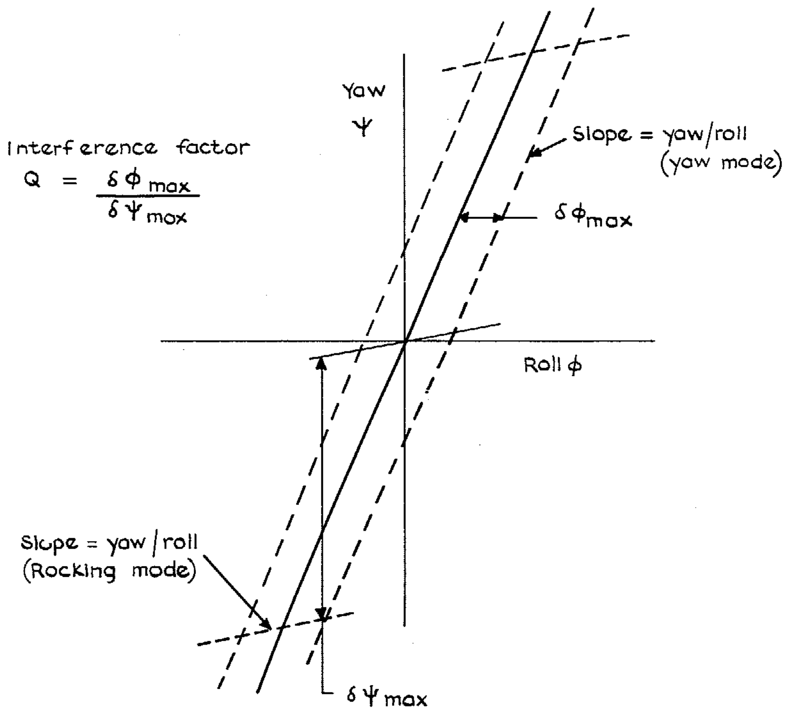

FIG. 5. Yaw *vs.* roll envelope (Appendix E).

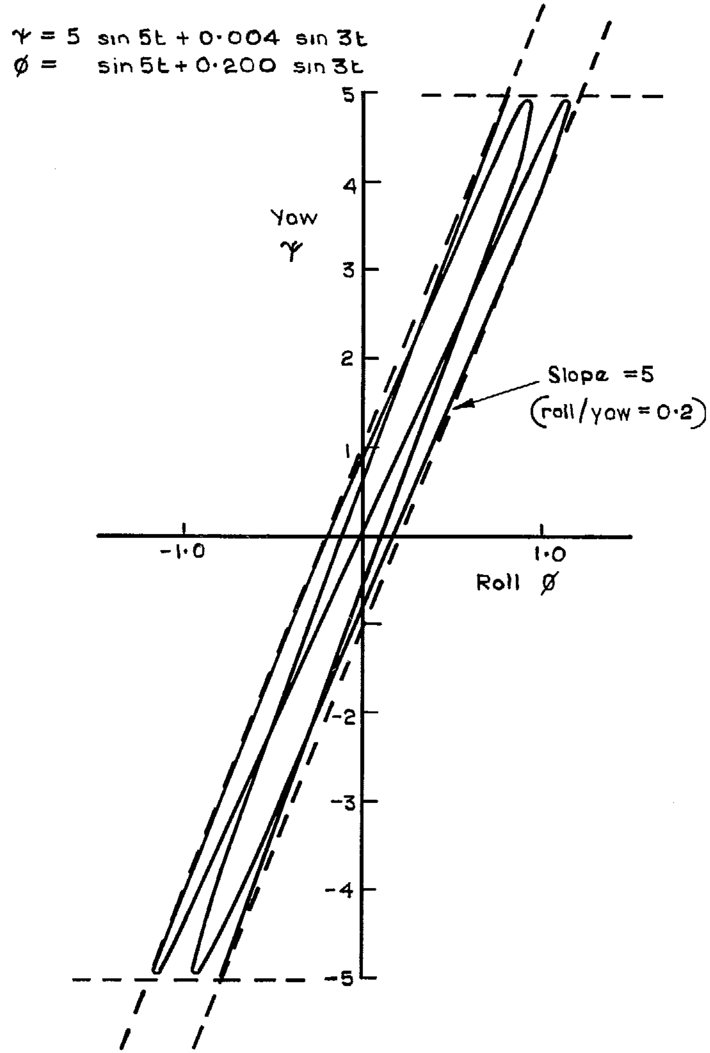

FIG. 6. Yaw *vs.* roll plot for typical equations.

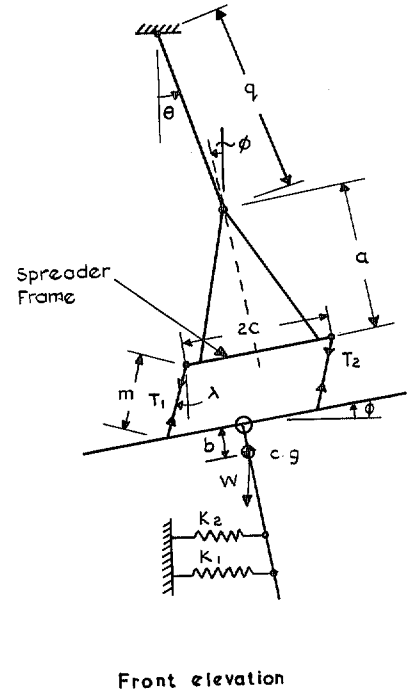

FIG. 7. Diagram of rig with spreader frame.

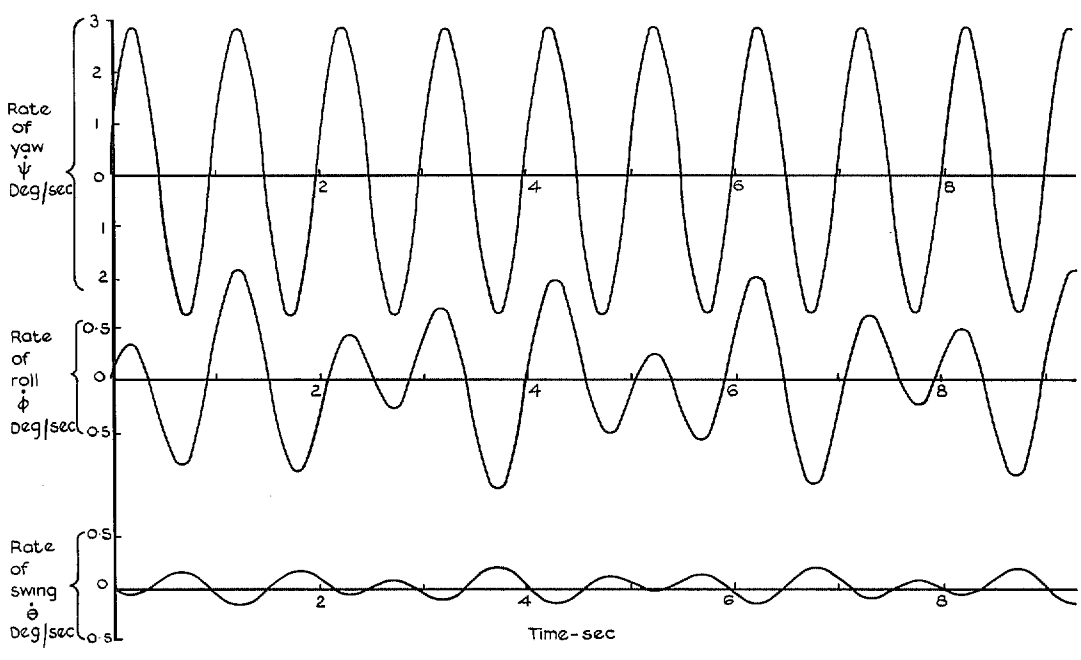

FIG. 8. Variation of rates of yaw, roll and swing with time following a step input: $\Delta(Klr) = 0$ with $r_1 = r_2 = 0{\cdot}353$ ft.

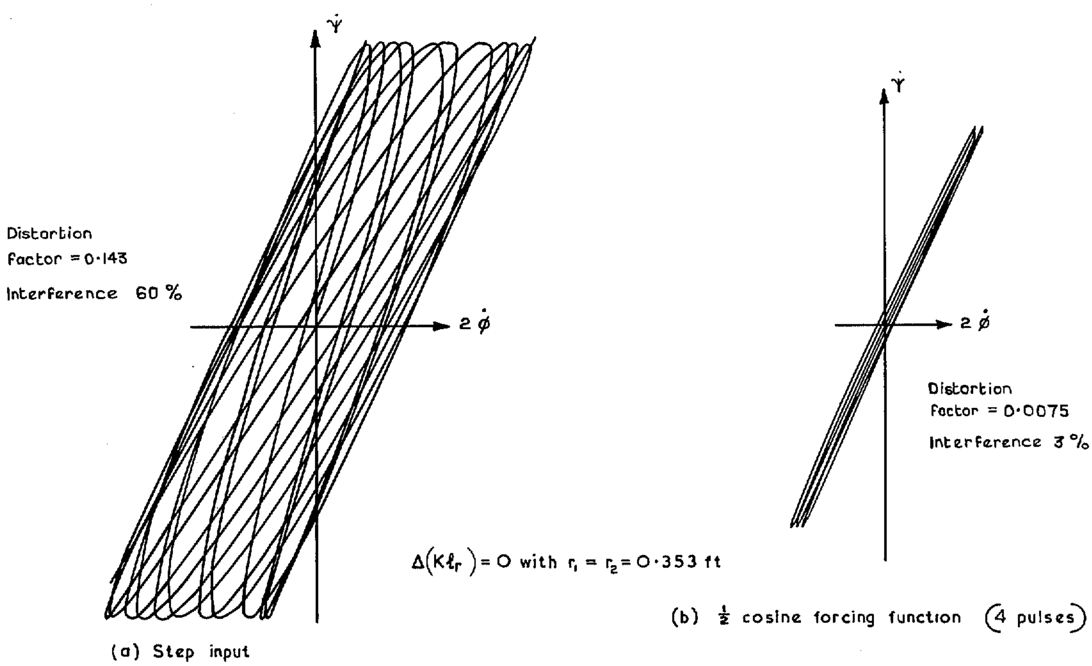

FIG. 9. Variation of rate of roll ($\dot{\phi}$) with rate of yaw ($\dot{\psi}$) following (a) Step input; (b) $\frac{1}{2}$ cosine forcing function (4 pulses).

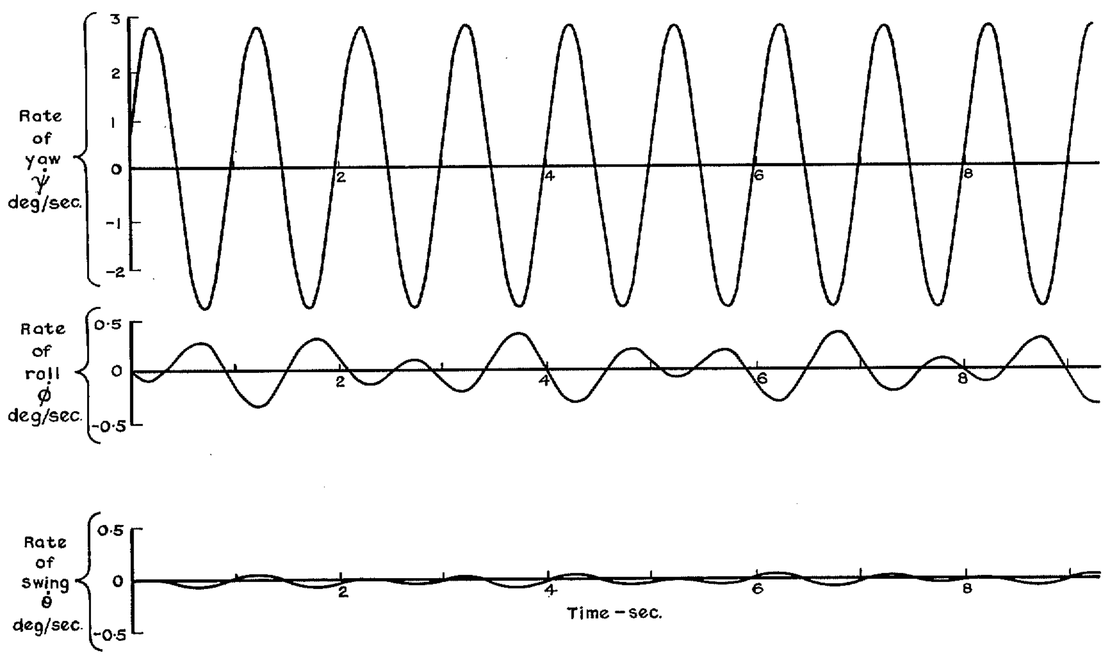

FIG. 10. Variation of rates of Yaw, Roll and Swing with time following a step input $\Delta(Klr) = 42\,000$ lb-ft with $r_1 = 0{\cdot}853$ ft and $r_2 = -0{\cdot}147$ ft.

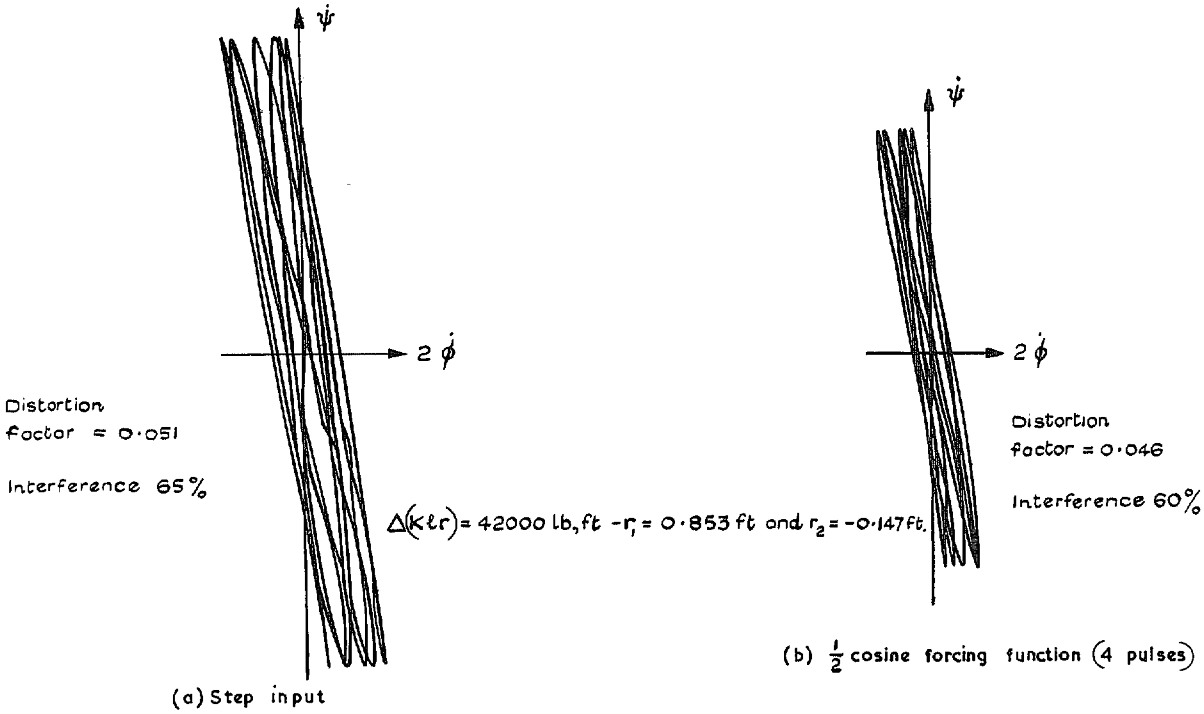

FIG. 11. Variation of rate of roll ($\dot\phi$) with rate of yaw ($\dot\psi$) following
(a) Step input; (b) $\frac{1}{2}$ cosine forcing function (4 pulses).

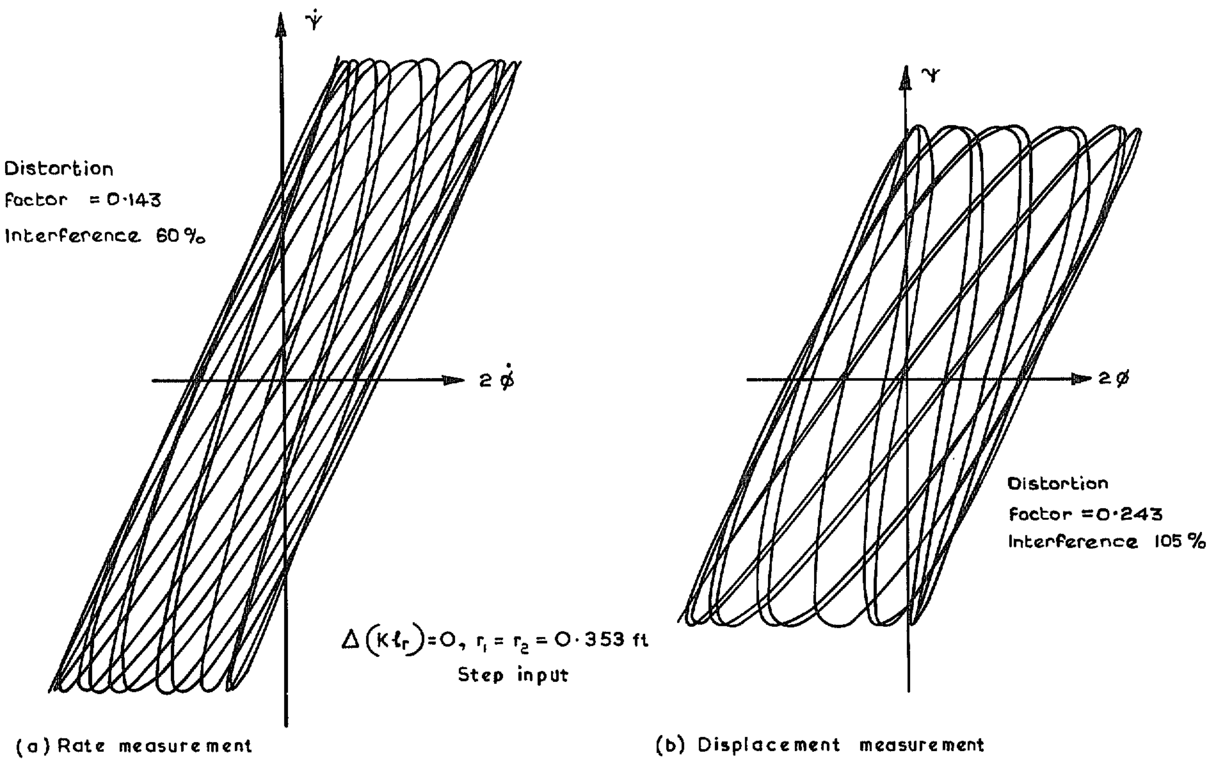

FIG. 12. Effect of measuring angular rate of angular displacement on the distortion factor
(a) Rate measurement; (b) Displacement measurement.

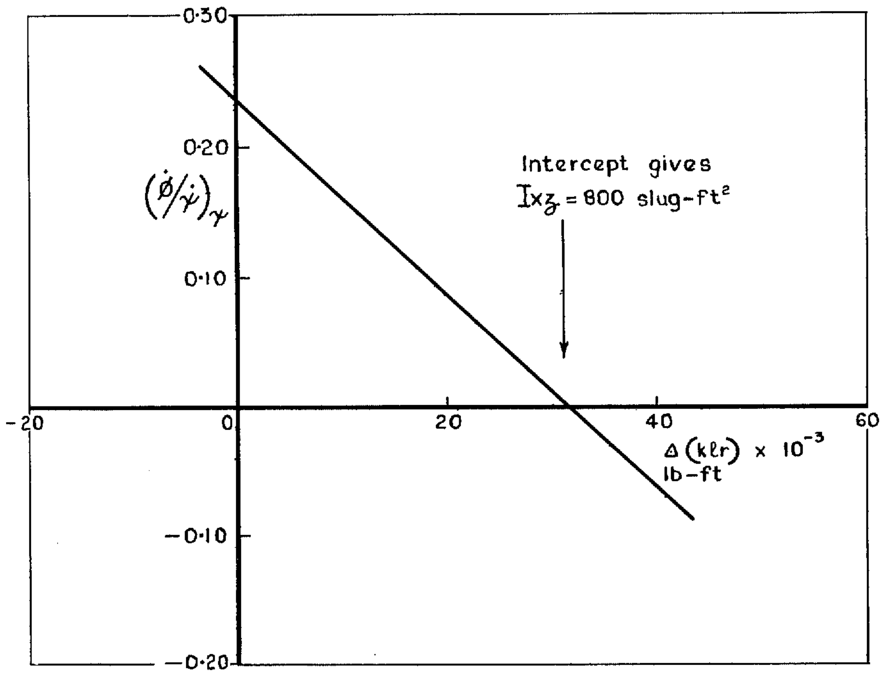

FIG. 13. Variation of roll/yaw ratio with spring position.

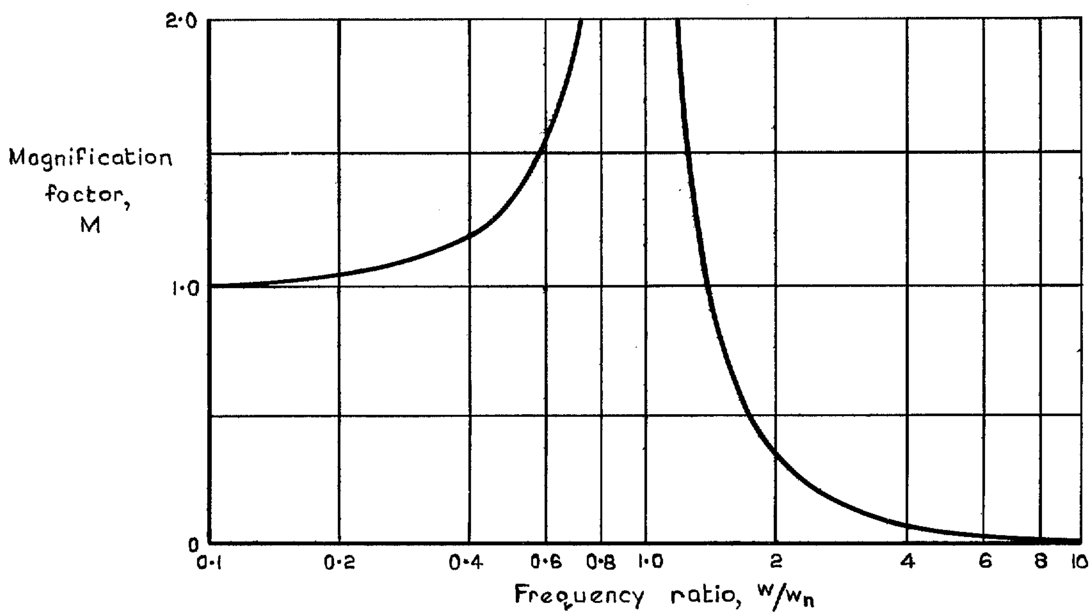

FIG. 14. Response of an undamped second order (oscillatory) system.

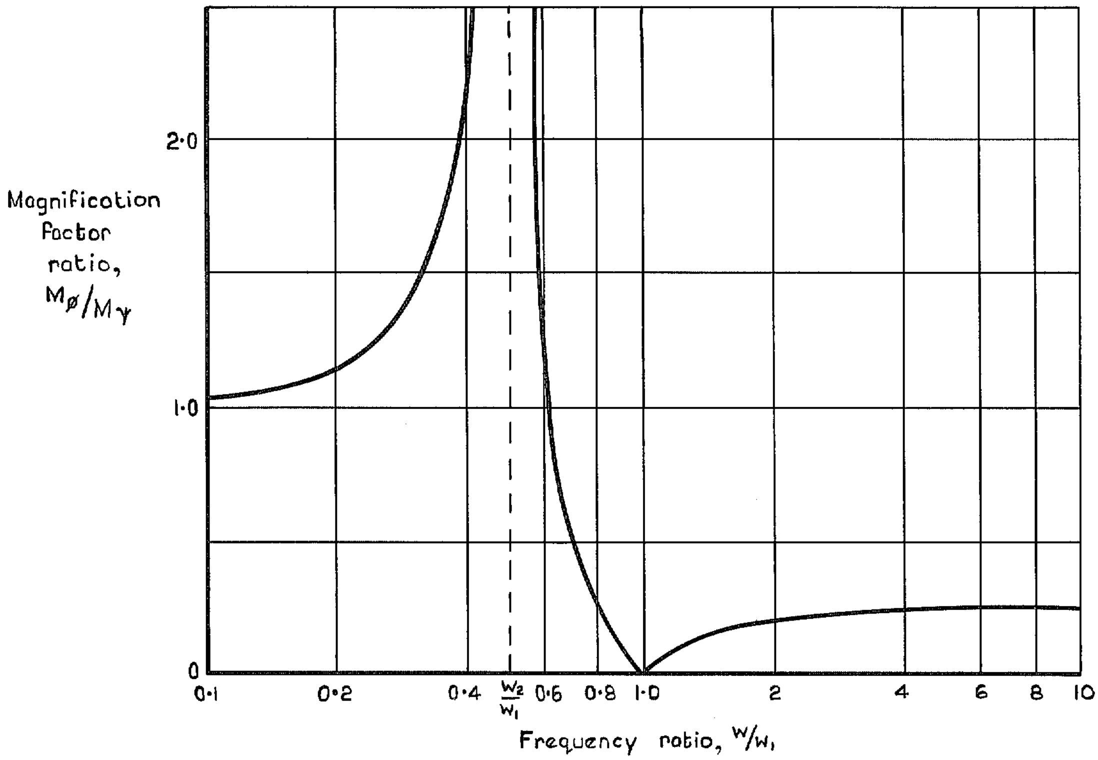

FIG. 15. Relative response of 'rocking' to 'yawing' modes as a function of frequency: $w_1/w_2 = 2$.

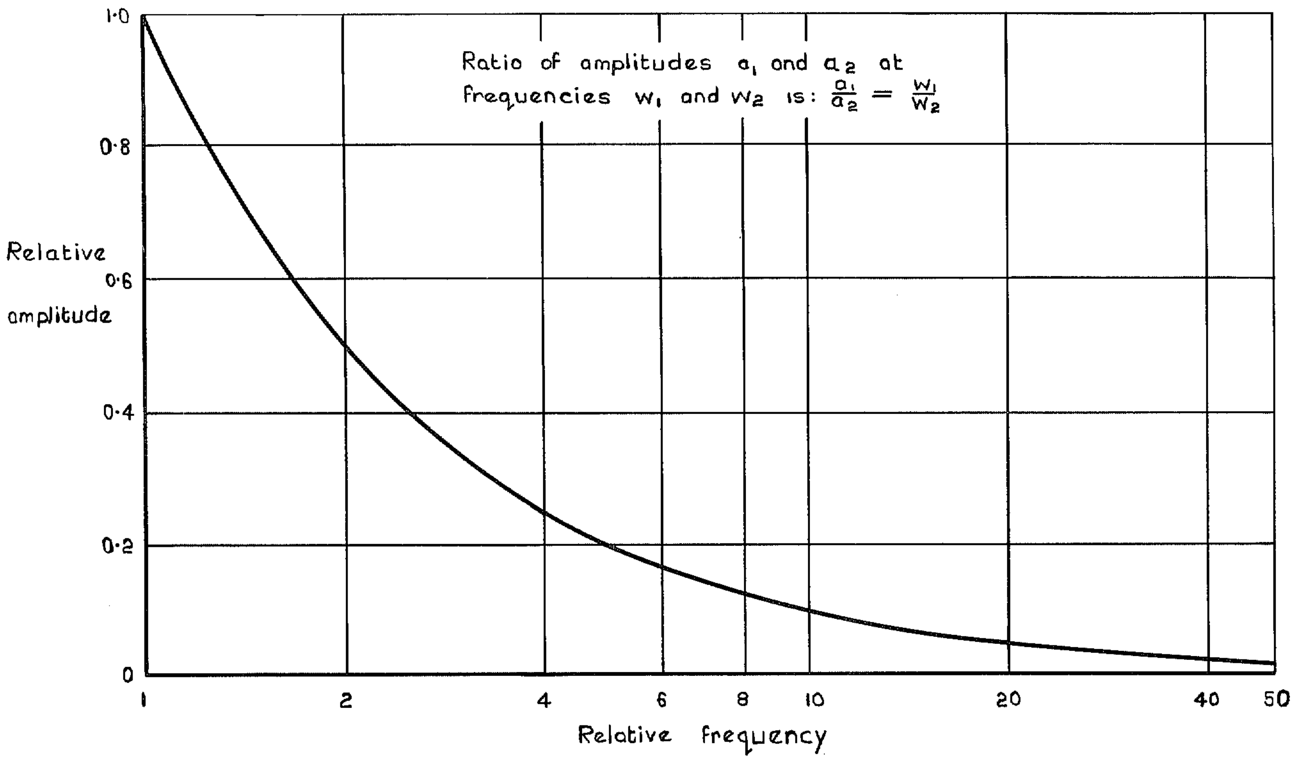

FIG. 16. Harmonic analysis of a step input—relative amplitudes when the fundamental frequency tends to zero.

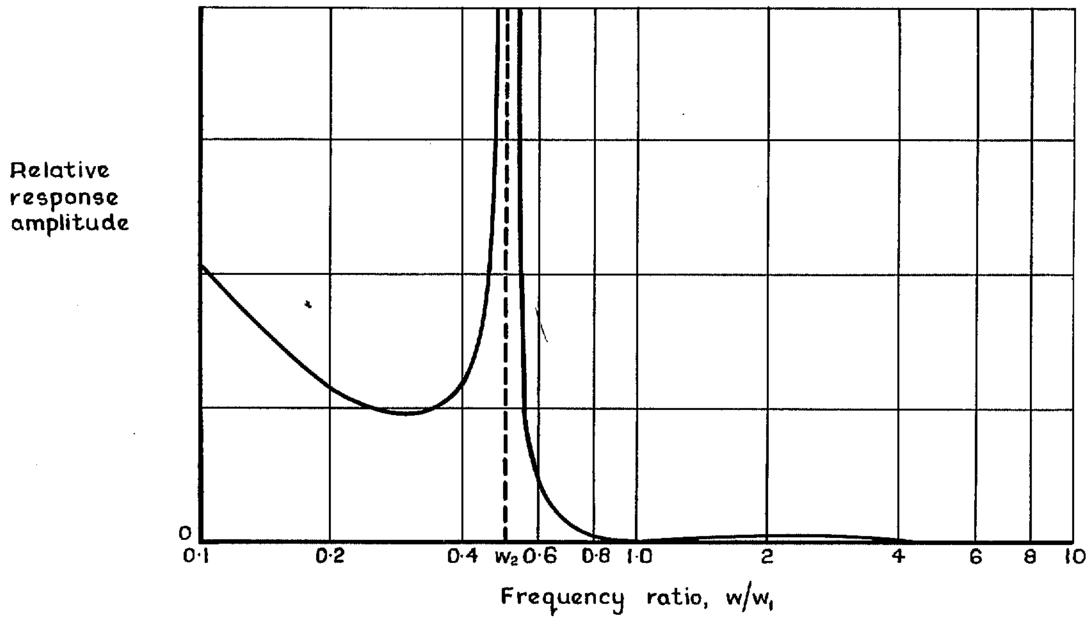

FIG. 17. Relative response of 'rocking' and 'yawing' modes to a step input, when $w_1/w_2 = 2$.

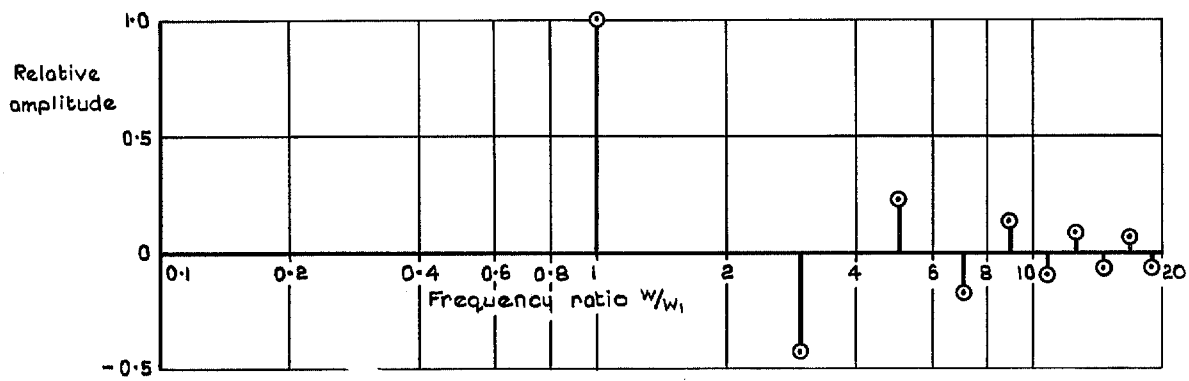

FIG. 18. Harmonic analysis of a continuous half cosine function with a fundamental frequency w_1.

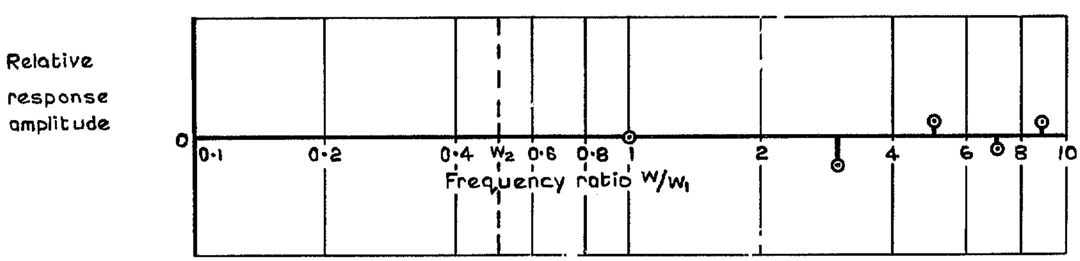

FIG. 19. Relative response of 'rocking' and 'yawing' modes to a half-cosine input of frequency w_1, when $w_1/w_2 = 2$.

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Dd. 146916 K.5.

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