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An Application of Flax's Variational Principle to Lifting-Surface Theory

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Summary.

The loading on a harmonically oscillating wing is represented by a linear combination of given functions which satisfy the edge conditions. The coefficients in this linear combination are determined, by an application of the variational principle due to Flax, so that the required generalised airforces acting on the wing are obtained to the greatest accuracy possible. For the particular case of subsonic flow, it is shown that, when certain numerical integration techniques are used, the results reduce to those obtained from a normal collocation procedure for lifting-surface theory.

The procedure using the variational principle is shown to be superior to one which obtains the coefficients in the loading expression by minimising the integral of the square of the difference between the actual and calculated upwashes on the wing surface.

Illustrative examples in two-dimensional incompressible oscillatory flow are given.

CONTENTS

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1. *Introduction.*

In current lifting-surface theories it is normally the practice to assume an approximation to the loading on an oscillating wing which depends on a finite number of parameters, and to determine these parameters by equating the given upwash on the wing with that corresponding to the approximate loading at a set of points equal in number to the number of assumed parameters. The generalised airforces on the wing are then obtained by integrating the approximate loading, weighted appropriately, over the wing.

Flax[1,2] has given a variational principle which involves integrals that are closely related to generalised airforces. This variational principle is used in this report to obtain approximations to these generalised airforces.

When certain functions which occur in the final results can be closely approximated by polynomials of sufficiently low degree then the results for subsonic flow reduce to those obtained by a collocation procedure.

Examples are given for two-dimensional incompressible flow which show how the results from the variational approach compare with those from a collocation approach and with the exact results.

For three-dimensional flow Fromme[3] has advocated the use of a least squares approach. In Section 5 it is shown that the results from a least squares approach must be expected to be less accurate than those from the variational approach. For two-dimensions we illustrate the difference in accuracy by means of examples.

2. *The Basic Equations.*

We consider a flat wing and a set of right-handed rectangular Cartesian co-ordinates, fixed relative to the mean position of the wing, with the axes of x and y in the plane of the wing and the z axis perpendicular to the plane of the wing.

The wing is immersed in a stream of undisturbed speed V in the direction of the positive x-axis and oscillates harmonically in an arbitrary way with circular frequency ω about a mean position in the plane $z = 0$. Let the vertical displacement of a point x,y on the wing surface at time t be given by $Z(x,y)$ $e^{i\omega t}$ and let the corresponding loading (airforce per unit wing area in the direction of positive z) at the point x,y at time t be $l(x,y)\, e^{i\omega t}$. Then, in linearised flow, the upwash function $w(x,y)$ is given by

$$w(x,y) = \left(i\omega + V\frac{\partial}{\partial x} \right) Z(x,y) \tag{1}$$

and the loading function $l(x,y)$ satisfies the integral equation

$$w(x,y) = \frac{1}{4\pi\rho_0 V} \int_S\int l(x_0, y_0)\, K\left(x-x_0, y-y_0, \frac{\omega}{V}, M \right) dx_0\, dy_0 \tag{2}$$

where S is the wing area, ρ_0 is the density of the air in the undisturbed main stream and $K(x-x_0, y-y_0, \omega/V, M)$ is an influence function, also known as the kernel function of the integral equation. The kernel function has different forms for the three separate flow regimes $M < 1$, $M = 1$ and $M > 1$.

Consider, on the other hand, the wing to be immersed in a stream of undisturbed speed V in the direction of the negative x-axis, and again let it be oscillating harmonically with circular frequency ω about the plane $z = 0$. In this case let the vertical displacement of a point x,y on the wing surface at time t be $\bar{Z}(x,y)\, e^{i\omega t}$ and let the corresponding loading at the point x,y at time t be $\bar{l}(x,y)\, e^{i\omega t}$.

The wing is said to be in a reverse flow when the speed V of the main stream flow is in the direction of the negative x-axis, as opposed to its being in a direct flow when the speed V of the mainstream flow is in the direction of the positive x-axis.

In the reverse flow the upwash function $\bar{w}(x,y)$ of linearised flow is given by

$$\bar{w}(x,y) = \left(i\omega - V\frac{\partial}{\partial x} \right) \bar{Z}(x,y) \tag{3}$$

and the loading function $\bar{l}(x,y)$ satisfies the integral equation

$$\bar{w}(x,y) = \frac{1}{4\pi\rho_0 V} \int_S \int \bar{l}(x_0, y_0)\, K\left(x_0 - x, y - y_0, \frac{\omega}{V}, M \right) dx_0\, dy_0 \tag{4}$$

Each of the functions $Z(x,y)$ and $\bar{Z}(x,y)$ used to describe the displacement of the wing in the harmonic oscillations is usually taken to be a real function, or at most the product of a real function and a complex constant. This is not necessary for the arodynamic theory but is convenient for the dynamical theory dealing with the wing oscillation.

The loading $l(x,y)$ must satisfy appropriate edge conditions in the direct flow and the loading $\bar{l}(x,y)$ must satisfy appropriate edge conditions in the reverse flow. When these edge conditions are satisfied, the reverse flow relation

$$\int_S \int \bar{w}(x,y)\, l(x,y)\, dx\, dy = \int_S \int w(x,y)\, \bar{l}(x,y)\, dx\, dy \tag{5}$$

is satisfied (Flax[2], Heaslet and Spreiter[4]). The relation (5) follows from substituting for $\bar{w}(x,y)$ from (4) into the left-hand side of (5), inverting the order of integration, and finally using the formula (2). Inversion of the order of integration is permissible when the aforementioned edge conditions are satisfied.

The functions $w(x,y)$ and $\bar{w}(x,y)$ are taken to be given functions. Let $L(x,y)$ be a loading function which satisfies the appropriate edge conditions in the direct flow and which differs from the actual loading $l(x,y)$ corresponding to the upwash function $w(x,y)$ in the direct flow by a small variation. The upwash function corresponding to the loading function $L(x,y)$ in the direct flow will be written $W(x,y)$. The functions $L(x,y)$ and $W(x,y)$ satisfy equation (2) with $l(x_0, y_0)$ replaced by $L(x_0, y_0)$ and $w(x,y)$ replaced by $W(x,y)$.

Let $\bar{L}(x,y)$ be a loading function which satisfies the appropriate edge conditions in the reverse flow and which differs from the actual loading $\bar{l}(x,y)$ corresponding to the upwash function $\bar{w}(x,y)$ in the reverse flow by a small variation. The upwash function corresponding to the loading function $\bar{L}(x,y)$ in the reverse flow will be written $\bar{W}(x,y)$. The functions $\bar{L}(x,y)$ and $\bar{W}(x,y)$ satisfy equation (4) with $\bar{l}(x_0, y_0)$ replaced by $\bar{L}(x_0, y_0)$ and $\bar{w}(x,y)$ replaced by $\bar{W}(x,y)$.

Define the integral H by the formula

$$H = \int_S \int \Big[\bar{L}(x,y)\, w(x,y) + L(x,y)\, \bar{w}(x,y) - \tfrac{1}{2}\{\bar{L}(x,y)\, W(x,y) + L(x,y)\, \bar{W}(x,y)\} \Big] dx\, dy\,. \tag{6}$$

The variational principle of Flax[1,2] is that H is stationary for first order variations of $L(x,y)$ and $L(x,y)$ from the correct functions $l(x,y)$ and $\bar{l}(x,y)$ corresponding to the upwash functions $w(x,y)$ and $\bar{w}(x,y)$ in the direct and reverse flows respectively.

Applying the reverse flow relation to $L(x,y)$, $\bar{L}(x,y)$ $W(x,y)$ and $\bar{W}(x,y)$ we get

$$\int_S \int \bar{L}(x,y)\, W(x,y)\, dx\, dy = \int_S \int L(x,y)\, \bar{W}(x,y)\, dx\, dy \tag{7}$$

and by using (7) in relation (6), we may replace H by the formula

$$H = \int_S\int \left[\bar{L}(x,y)\, w(x,y) + L(x,y)\, \bar{w}(x,y) - \bar{L}(x,y)\, W(x,y) \right] dx\, dy\,. \tag{8}$$

The stationary value H_0 of H is given by

$$H_0 = \int_S\int \left[\bar{l}(x,y)\, w(x,y) + l(x,y)\, \bar{w}(x,y) - \bar{l}(x,y)\, w(x,y) \right] dx\, dy$$

$$= \int_S\int l(x,y)\, \bar{w}(x,y)\, dx\, dy\,. \tag{9}$$

If now $\bar{w}(x,y)$ is taken to be the function

$$\bar{w}(x,y) = Z_j(x,y) \tag{10}$$

as it can be by proper choice of $\bar{Z}(x,y)$, then

$$H_0 = \int_S\int l(x,y)\, Z_j(x,y)\, dx\, dy$$

$$= Q_{j0} \tag{11}$$

and $Q_{j0}e^{i\omega t}$ is the generalised airforce acting on the wing in mode $Z = Z_j(x,y)$ due to aerodynamic loading forces arising from the oscillation of the wing in which the vertical displacement is $Z(x,y)\, e^{i\omega t}$.

The function $\bar{Z}(x,y)$ chosen to satisfy equation (10) is, in general, a complex function of x and y.

Let us write

$$L(x,y) = l(x,y) - \varepsilon(x,y) \tag{12}$$

$$\bar{L}(x,y) = \bar{l}(x,y) - \bar{\varepsilon}(x,y) \tag{13}$$

$$W(x,y) = w(x,y) - \delta(x,y) \tag{14}$$

$$\bar{W}(x,y) = \bar{w}(x,y) - \bar{\delta}(x,y) \tag{15}$$

Then $\varepsilon(x,y)$ is a loading function which satisfies the appropriate edge conditions in the direct flow and to which corresponds the upwash function $\delta(x,y)$ in the direct flow, and $\bar{\varepsilon}(x,y)$ is a loading function which satisfies the appropriate edge conditions in the reverse flow and to which corresponds the upwash function $\bar{\delta}(x,y)$ in the reverse flow.

If we substitute (12), (13), (14) and (15) into (8) we get

$$H = \iint_S \Big[\bar{l}(x,y)\, w(x,y) - \bar{\varepsilon}(x,y)\, w(x,y) + l(x,y)\, \bar{w}(x,y) - \varepsilon(x,y)\, \bar{w}(x,y)$$

$$- \bar{l}(x,y)\, w(x,y) + \bar{\varepsilon}(x,y)\, w(x,y) + \bar{l}(x,y)\, \delta(x,y) - \bar{\varepsilon}(x,y)\, \delta(x,y) \Big]\, dx\, dy$$

$$= \iint_S \Big[l(x,y)\, \bar{w}(x,y) - \varepsilon(x,y)\, \bar{w}(x,y) + \bar{l}(x,y)\, \delta(x,y) - \bar{\varepsilon}(x,y)\, \delta(x,y) \Big]\, dx\, dy$$

$$= \iint_S \Big[l(x,y)\, \bar{w}(x,y) - \bar{\varepsilon}(x,y)\, \delta(x,y) \Big]\, dx\, dy$$

$$= H_0 - \iint_S \bar{\varepsilon}(x,y)\, \delta(x,y)\, dx\, dy \tag{16}$$

since, by the reverse flow relation for $\varepsilon(x,y)$, $\bar{l}(x,y)$, $\delta(x,y)$ and $\bar{w}(x,y)$ we have

$$\iint_S \bar{l}(x,y)\, \delta(x,y)\, dx\, dy = \iint_S \varepsilon(x,y)\, \bar{w}(x,y)\, dx\, dy. \tag{17}$$

Also, by the reverse flow relation for $\varepsilon(x,y)$, $\bar{\varepsilon}(x,y)$, $\delta(x,y)$ and $\bar{\delta}(x,y)$ we have

$$\iint_S \bar{\varepsilon}(x,y)\, \delta(x,y)\, dx\, dy = \iint_S \varepsilon(x,y)\, \bar{\delta}(x,y)\, dx\, dy \tag{18}$$

which can be used to modify the form of H given by equation (16) if desired.

The formula (16) shows the stationary property of H at the actual flow conditions, for first order variation terms are missing in the expression for the difference $H - H_0$.

3. *Application to Lifting-Surface Theory.*

In order to have integration variables extending over standard ranges we make the changes of integration variables

$$\xi_0 = \frac{1}{c(y_0)} [x_0 - x_L(y_0)] \tag{19}$$

$$\eta_0 = \frac{1}{s} y_0 \tag{20}$$

in the integrals on the right-hand sides of equations (2) and (4). Here s is the wing semi-span (x-axis on the wing root chord), $c(y_0)$ is the local chord length and $x_L(y_0)$ is the x co-ordinate of the leading edge at the spanwise position y_0. The integral equations (2) and (4) then become

$$w(x,y) = \frac{s}{4\pi\rho_0 V}\int_{-1}^{+1} c(y_0)\, d\eta_0 \int_0^1 l(x_0, y_0)\, K\left(x-x_0, y-y_0, \frac{\omega}{V}, M\right) d\xi_0 \tag{21}$$

and

$$\bar{w}(x,y) = \frac{s}{4\pi\rho_0 V}\int_{-1}^{+1} c(y_0)\, d\eta_0 \int_0^1 \bar{l}(x_0, y_0)\, K\left(x_0-x, y-y_0, \frac{\omega}{V}, M\right) d\xi_0 . \tag{22}$$

Let us write

$$K\left(x, y, \frac{\omega}{V}, M\right) = E\left(-\frac{\omega x}{V}\right)\hat{K}\left(x, y, \frac{\omega}{V}, M\right) . \tag{23}$$

If we take

$$E(x) = \exp\,(ix) \tag{24}$$

then, for some purposes, $\hat{K}(x, y, \omega/V, M)$ is a function which is more convenient for numerical manipulation than is $K(x, y, \omega/V, M)$. For other purposes it may be better to take

$$E(x) = 1 \tag{25}$$

or

$$E(x) = \exp\left(-\frac{i\,M^2 x}{1-M^2}\right) . \tag{26}$$

For the present Report we shall leave $E(x)$ unspecified, except that we stipulate that it has the properties

$$\left.\begin{aligned} E(x_1+x_2) &= E(x_1)\,E(x_2) \\ E(o) &= 1 \end{aligned}\right\} \tag{27}$$

We assume, for subsonic flow, that $c(y_o)\, l(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right)$ and $c(y_o)\, \bar{l}(x_o, y_o)\, E\left(-\frac{\omega x_o}{V}\right)$ may be written as the convergent expansions

$$c(y_o)\, l(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{rs}\, l_r(\xi_o)\, \gamma_s(\eta_o) \sqrt{\frac{1-\xi_o}{\xi_o}}\sqrt{1-\eta_o^2} \tag{28}$$

and

$$c(y_o)\bar{l}(x_o, y_o)E\left(-\frac{\omega x_o}{V}\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \bar{A}_{rs}\, l_r(1-\xi_o)\, \gamma_s(\eta_o) \sqrt{\frac{\xi_o}{1-\xi_o}}\sqrt{1-\eta_o^2} \tag{29}$$

where the $l_r(\xi_o)$ are polynomials of degree r in ξ_o satisfying

$$\int_0^1 l_r(\xi_o)\, l_p(\xi_o) \sqrt{\frac{1-\xi_o}{\xi_o}}\, d\xi_o = \delta_{r,p} \tag{30}$$

the $\gamma_s(\eta_o)$ are polynomials of degree s in η_o satisfying

$$\int_{-1}^{+1} \gamma_s(\eta_o)\, \gamma_q(\eta_o) \sqrt{1-\eta_o^2}\, d\eta_o = \delta_{s,q} \tag{31}$$

and $\delta_{r,p}$, $\delta_{s,q}$ are Kronecker deltas. The A_{rs} and $\bar{A}_{rs}$ are constants.

For supersonic flow, the expressions (28) and (29) would need to be modified to take into account the edge conditions, which are different for supersonic flow, and also discontinuities in the pressure along Mach lines emanating from points of discontinuity of direction of the wing edges. We shall not go into any details of supersonic flow.

If we write

$$L_{rs}(\xi_o, \eta_o) = l_r(\xi_o)\, \gamma_s(\eta_o) \sqrt{\frac{1-\xi_o}{\xi_o}}\sqrt{1-\eta_o^2} \tag{32}$$

and

$$\bar{L}_{rs}(\xi_o, \eta_o) = l_r(1-\xi_o)\, \gamma_s(\eta_o) \sqrt{\frac{\xi_o}{1-\xi_o}}\sqrt{1-\eta_o^2} \tag{33}$$

then equations (28) and (29) may be written as

$$c(y_o)\, l(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{rs}\, L_{rs}(\xi_o, \eta_o) \tag{34}$$

and

$$c(y_o)\, \bar{l}(x_o, y_o)\, E\left(-\frac{\omega x_o}{V}\right) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \bar{A}_{rs}\, \bar{L}_{rs}(\xi_o, \eta_o). \tag{35}$$

Let

$$W_{rs}(x,y) = \frac{s}{4\pi\rho_o V}\int_{-1}^{+1} d\eta_o \int_0^1 L_{rs}(\xi_o, \eta_o)\, \hat{K}\left(x-x_o, y-y_o, \frac{\omega}{V}, M\right) d\xi_o \tag{36}$$

and

$$\overline{W}_{rs}(x,y) = \frac{s}{4\pi\rho_e V}\int_{-1}^{+1} d\eta_o \int_{0}^{1} \overline{L}_{rs}(\xi_o, \eta_o)\,\hat{K}\left(x_e - x, y - y_o, \frac{\omega}{V}, M\right) d\xi_o . \tag{37}$$

Then, substituting (34) and (35) into (21) and (22) and using (36) and (37) we get

$$w(x,y) = E\left(-\frac{\omega x}{V}\right)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_{rs}\, W_{rs}(x,y) \tag{38}$$

and

$$\overline{w}(x,y) = E\left(\frac{\omega x}{V}\right)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \overline{A}_{rs}\, \overline{W}_{rs}(x,y). \tag{39}$$

Let us now consider a variation $L(x_o, y_o)$ of $l(x_o, y_o)$ defined by the sum of a finite number of terms of the series (34),

$$c(y_o)\, L(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum_1 A_{rs}\, L_{rs}(\xi_o, \eta_o) \tag{40}$$

where $\sum_1$ denotes the summation of some particular finite set of terms, the set consisting of P_1 (say) terms altogether. The summation of the remaining (infinite) number of terms is denoted by $\sum_1'$. define $\varepsilon(x_o, y_o)$ by the relation

$$c(y_o)\, \varepsilon(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum_1' A_{rs}\, L_{rs}(\xi_o, \eta_o) \tag{41}$$

so that relation (12) between $L(x,y)$, $l(x,y)$ and $\varepsilon(x,y)$ is automatically satisfied.

Let us also consider a variation $\overline{L}(x_o, y_o)$ of $\overline{l}(x_o, y_o)$ defined by the sum of a finite number of terms of the series (35),

$$c(y_o)\, \overline{L}(x_o, y_o)\, E\left(-\frac{\omega x_o}{V}\right) = \sum_2 \overline{A}_{rs}\, \overline{L}_{rs}(\xi_o, \eta_o) \tag{42}$$

where $\sum_2$ denotes the summation of some particular finite set of terms, the set consisting of P_2 (say) terms altogether. The summation of the remaining (infinite) number of terms is denoted by $\sum_2'$. We define $\overline{\varepsilon}(x_o, y_o)$ by the relation

$$c(y_o)\, \overline{\varepsilon}(x_o, y_o)\, E\left(-\frac{\omega x_o}{V}\right) = \sum_2' \overline{A}_{rs}\, \overline{L}_{rs}(\xi_o, \eta_o) \tag{43}$$

so that the relation (13) between $\bar{L}(x,y)$, $\bar{l}(x,y)$ and $\bar{\varepsilon}(x,y)$ is automatically satisfied.

The upwash function $W(x,y)$ corresponding to the loading function $L(x_o, y_o)$ in the direct flow and the upwash function $\bar{W}(x,y)$ corresponding to the loading function $\bar{L}(x_o, y_o)$ in the reverse flow are obtained from (21) and (22) by replacing $l(x_o, y_o)$, $\bar{l}(x_o, y_o)$, $w(x,y)$ and $\bar{w}(x,y)$ by $L(x_o, y_o)$, $\bar{L}(x_o, y_o)$, $W(x,y)$ and $\bar{W}(x,y)$ respectively. Then on using (40) and (42) we get

$$W(x,y) = E\left(-\frac{\omega x}{V}\right) \sum\nolimits_1 A_{rs} W_{rs}(x,y) \tag{44}$$

and

$$\bar{W}(x,y) = E\left(\frac{\omega x}{V}\right) \sum\nolimits_2 \bar{A}_{rs} \bar{W}_{rs}(x,y). \tag{45}$$

Similarly, we get (*see* (14) and (15))

$$\delta(x,y) = E\left(-\frac{\omega x}{V}\right) \sum\nolimits_1' A_{rs} W_{rs}(x,y) \tag{46}$$

and

$$\bar{\delta}(x,y) = E\left(\frac{\omega x}{V}\right) \sum\nolimits_2' \bar{A}_{rs} \bar{W}_{rs}(x,y). \tag{47}$$

If we make the change of variables

$$\xi = \frac{1}{c(y)}[x - x_L(y)] \tag{48}$$

$$\eta = \frac{1}{s} y \tag{49}$$

in the integral on the right-hand side of formula (16), we get

$$H = H_o - s \int_{-1}^{+1} c(y)\, d\eta \int_0^1 \bar{\varepsilon}(x,y)\, \delta(x,y)\, d\xi$$

$$= H_o - s \int_{-1}^{+1} d\eta \int_0^1 \left\{c(y)\, \bar{\varepsilon}(x,y)\, E\left(-\frac{\omega x}{V}\right)\right\} \left\{E\left(\frac{\omega x}{V}\right) \delta(x,y)\right\} d\xi \tag{50}$$

Substituting from (43) and (46) into (50) we then get

$$H = H_o - \sum_{rs}{}_2' \sum_{pq}{}_1' \bar{A}_{rs} A_{pq} \gamma_{rs\,pq} \tag{51}$$

where

$$\gamma_{rs\,pq} = s \int_{-1}^{+1} d\eta \int_{0}^{1} \bar{L}_{rs}(\xi,\eta)\, W_{pq}(x,y)\, d\xi\,. \tag{52}$$

The formula (51) shows that the difference $H - H_o$ depends only on the coefficients which have been omitted in obtaining the finite expansions (40) and (42) from the infinite expansions (34) and (35), and, of course, in accordance with the variational principle, is a second order quantity.

However, the coefficients A_{rs} and $\bar{A}_{rs}$ used for the evaluation of H are not known *a priori.* Therefore further variations $L_1\,(x_o,\, y_o)$ and $\bar{L}_1\,(x_o,\, y_o)$ of the loading in respectively the direct and reverse flows are considered.

These are defined by the finite summations

$$c(y_o)\, L_1\,(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum\nolimits_1 B_{rs}\, L_{rs}\,(\xi_o, \eta_o) \tag{53}$$

and

$$c(y_o)\, \bar{L}_1\,(x_o, y_o)\, E\left(-\frac{\omega x_o}{V}\right) = \sum\nolimits_2 \bar{B}_{rs}\, \bar{L}_{rs}\,(\xi_o, \eta_o) \tag{54}$$

where the coefficients B_{rs} and $\bar{B}_{rs}$ are to be determined.

The upwash function $W_1\,(x,y)$ corresponding to the loading function $L_1\,(x_o,\, y_o)$ in the direct flow and the upwash function $\bar{W}_1\,(x,y)$ corresponding to the loading function $\bar{L}_1\,(x_o,\, y_o)$ in the reverse flow are given by the formulae

$$W_1\,(x,y) = E\left(-\frac{\omega x}{V}\right) \sum\nolimits_1 B_{rs}\, W_{rs}\,(x,y) \tag{55}$$

and

$$\bar{W}_1\,(x,y) = E\left(\frac{\omega x}{V}\right) \sum\nolimits_2 \bar{B}_{rs}\, \bar{W}_{rs}\,(x,y)\,. \tag{56}$$

The value of H corresponding to the loading $L_1\,(x,y)$ in the direct flow and the loading $\bar{L}_1\,(x,y)$ in the reverse flow will be denoted by H_1.

If we make the change of variables (48) and (49) in the integrals defining H in equation (8) we get

$$H_1 = s \int_{-1}^{+1} c(y)\, d\eta \int_{0}^{1} \Big[\bar{L}_1\,(x,y)\, w(x,y) + L_1\,(x,y)\, \bar{w}(x,y) - \bar{L}_1\,(x,y)\, W_1\,(x,y) \Big]\, d\xi$$

$$= s \int_{-1}^{+1} d\eta \int_{0}^{1} \Big[\left\{ c(y)\, \bar{L}_1\,(x,y)\, E\left(-\frac{\omega x}{V}\right) \right\} \left\{ w(x,y)\, E\left(\frac{\omega x}{V}\right) \right\}$$

$$+ \left\{ c(y)\, L_1\,(x,y)\, E\left(\frac{\omega x}{V}\right) \right\} \left\{ \bar{w}(x,y)\, E\left(-\frac{\omega x}{V}\right) \right\}$$

$$-\left\{ c(y)\,\bar{L}_1(x,y)\,E\left(-\frac{\omega x}{V}\right)\right\} \quad \left\{W_1(x,y)\,E\left(\frac{\omega x}{V}\right)\right\} \quad \Big]\,d\xi\,. \tag{57}$$

If we now use formulae (53), (54), (55), (56) in (57) we get

$$H_1 = \sum_2 \bar{B}_{rs}\,\alpha_{rs} + \sum_1 B_{pq}\,\beta_{pq} - \sum_{rs}{}_2 \sum_{pq}{}_1 \bar{B}_{rs}\,B_{pq}\,\gamma_{rs\,pq} \tag{58}$$

where

$$\alpha_{rs} = s \int_{-1}^{+1} d\eta \int_0^1 \bar{L}_{rs}(\xi,\eta)\,w(x,y)\,E\left(\frac{\omega x}{V}\right) d\xi \tag{59}$$

and

$$\beta_{pq} = s \int_{-1}^{+1} d\eta \int_0^1 L_{pq}(\xi,\eta)\,\bar{w}(x,y)\,E\left(-\frac{\omega x}{V}\right) d\xi\,. \tag{60}$$

We determine the coefficients B_{pq} and $\bar{B}_{rs}$ which make H_1 stationary for first order increments in B_{pq} and $\bar{B}_{rs}$. If B_{pq} and $\bar{B}_{rs}$ respectively undergo increments δB_{pq} and $\delta\bar{B}_{rs}$ then H_1 undergoes an increment δH_1 given by

$$\begin{aligned}
\delta H_1 &= \sum_2 \delta\bar{B}_{rs}\,\alpha_{rs} + \sum_1 \delta B_{pq}\,\beta_{pq} - \sum_{rs}{}_2 \sum_{pq}{}_1 \delta\bar{B}_{rs}\,B_{pq}\,\gamma_{rs\,pq} \\
&\quad - \sum_{rs}{}_2 \sum_{pq}{}_1 \bar{B}_{rs}\,\delta B_{pq}\,\gamma_{rs\,pq} - \sum_{rs}{}_2 \sum_{pq}{}_1 \delta\bar{B}_{rs}\,\delta B_{pq}\,\gamma_{rs\,pq} \\
&= \sum_{rs}{}_2 \delta\bar{B}_{rs}\left[\alpha_{rs} - \sum_{pq}{}_1 \gamma_{rs\,pq}\,B_{pq}\right] \\
&\quad + \sum_{pq}{}_1 \delta B_{pq}\left[\beta_{pq} - \sum_{rs}{}_2 \gamma_{rs\,pq}\,\bar{B}_{rs}\right] \\
&\quad - \sum_{rs}{}_2 \sum_{pq}{}_1 \delta\bar{B}_{rs}\,\delta B_{pq}\,\gamma_{rs\,pq}\,.
\end{aligned} \tag{61}$$

If the sets of equations

$$\alpha_{rs} - \sum_{pq}{}_{1} \gamma_{rs\,pq} B_{pq} = 0 \tag{62}$$

and

$$\beta_{pq} - \sum_{rs}{}_{2} \gamma_{rs\,pq} \bar{B}_{rs} = 0 \tag{63}$$

are both satisfied, then δH_1 will contain only second order quantities so that H_1 will be stationary for first order increments in B_{pq} and $\bar{B}_{rs}$.

The set of equations (62) are a set of P_2 equations for P_1 unknowns B_{pq} and the set of equations (63) are a set of P_1 equations for P_2 unknowns $\bar{B}_{rs}$. These two sets can be solved only when $P_2 = P_1$, which henceforth we assume to be the case and let

$$P = P_1 = P_2 . \tag{64}$$

When equations (62) and (63) are satisfied, then we put

$$H_1 = H_{10} \tag{65}$$

$$\delta H_1 = -\sum_{rs}{}_{2} \sum_{pq}{}_{1} \delta \bar{B}_{rs} \, \delta B_{pq} \, \gamma_{rs\,pq} . \tag{66}$$

If we put

$$\delta B_{pq} = A_{pq} - B_{pq} \tag{67}$$

and

$$\delta \bar{B}_{rs} = \bar{A}_{rs} - \bar{B}_{rs} \tag{68}$$

then

$$\begin{aligned} H_{10} &= H - \delta H_1 \\ &= H + \sum_{rs}{}_{2} \sum_{pq}{}_{1} (\bar{A}_{rs} - \bar{B}_{rs})(A_{pq} - B_{pq}) \gamma_{rs\,pq} . \end{aligned} \tag{69}$$

Then, substituting for H from (51), we get

$$H_{10} = H_0 + \sum_{rs}{}_{2} \sum_{pq}{}_{1} (\bar{A}_{rs} - \bar{B}_{rs})(A_{pq} - B_{pq}) \gamma_{rs\,pq} - \sum_{rs}{}'_{2} \sum_{pq}{}'_{1} \bar{A}_{rs} A_{pq} \gamma_{rs\,pq} \tag{70}$$

The quantity H given by formula (51) is the same as H_1 obtained by replacing B_{pq} and $\bar{B}_{rs}$ in formula (58) by A_{pq} and $\bar{A}_{rs}$ respectively. By making small increments in B_{pq} and $\bar{B}_{rs}$ from the values A_{pq} and $\bar{A}_{rs}$

a value H'_1 for H_1 is obtained. Both quantities H and H'_1 must differ from H_0 by second order quantities for H_0 is the stationary value of the general expression H given by formula (8). Hence H'_1 must differ from H by a second order quantity and as both H'_1 and H are particular values of H_1 they must both differ by a second order quantity from the stationary value H_{10} of H_1. Thus H_{10} differs from H_0, the value of H corresponding to actual flow conditions, by a second order quantity for it may be written

$$H_{10} = (H_{10}-H)+(H-H_0)+H_0 \tag{71}$$

and both $H_{10}-H$ and $H-H_0$ are second order quantities.

The quantity H_{10} may not be the most accurate estimate of H_0 possible with only P terms, but it can be used as a good estimate for H_0. If we knew all the coefficients A_{pq} and $\bar{A}_{rs}$ we could give a better estimate.

We may arrange each set of elements α_{rs}, $\bar{B}_{rs}$, β_{pq}, B_{pq} as a column matrix of P elements, and the set of elements $\gamma_{rs\,pq}$ as a square matrix of $P \times P$ elements so that the sets of equations (62) and (63) may be written as the matrix equations

$$\alpha = \gamma\, B \tag{72}$$

and

$$\beta = \gamma'\, \bar{B} \tag{73}$$

where α, $\bar{B}$, β, B are respectively the column matrices of the α_{rs}, $\bar{B}_{rs}$, β_{pq}, B_{pq} and γ is the square matrix of the $\gamma_{rs\,pq}$. The matrix γ' is the transpose of the matrix γ.

We can now write H_{10} from equation (58) as

$$H_{10} = \bar{B}'\alpha+\beta' B-\bar{B}'\gamma B \tag{74}$$

where a dash indicates matrix transposition.

By using equation (72) in (74) we then get

$$\begin{aligned} H_{10} &= \bar{B}'\alpha+\beta' B-\bar{B}'\alpha \\ &= \beta' B \\ &= \beta'\,\gamma^{-1}\,\alpha \end{aligned} \tag{75}$$

where γ^{-1} denotes the matrix inverse to γ.

The elements of the matrix γ do not depend on the functions w and $\bar{w}$, but only on the wing geometry, the flow Mach number and the frequency ratio ω/V. The matrices α and β depend on w and $\bar{w}$ respectively, the wing geometry and the frequency ratio ω/V but not the flow Mach number.

In lifting surface theory it is usual for the finite summations in (53) and (54) to consist of double summations over a rectangular array. The sets for the summations $\sum_1$ and $\sum_2$ will be taken to be the same rectangular array and we shall write instead of (53) and (54) the formulae

$$c(y_o)\, L_1\,(x_o, y_o)\, E\left(\frac{\omega x_o}{V}\right) = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} B_{rs}\, L_{rs}\,(\xi_o, \eta_o) \tag{76}$$

and

$$c(y_o)\,\bar{L}_1\,(x_o, y_o)\,E\left(-\frac{\omega x_o}{V}\right) = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \bar{B}_{rs}\,\bar{L}_{rs}\,(\xi_o, \eta_o)\,. \tag{77}$$

The functions L_{rs} (ξ_o, η_o) and $\bar{L}_{rs}$ (ξ_o, η_o) were chosen for their orthogonality properties so that the infinite series on the right of (34) and (35) should be convergent in general. The right-hand sides of (76) and (77), however, are finite series and in these we may replace the $L_{rs}\,(\xi_o, \eta_o)$ by any nm linear combinations of the L_{rs} (ξ_o, η_o) and the $\bar{L}_{rs}$ (ξ_o, η_o) by any nm independent linear combinations of the $\bar{L}_{rs}$ (ξ_o, η_o). For example we could write, instead of (76) and (77),

$$c(y_o)\,L_1\,(x_o, y_o)\,E\left(\frac{\omega x_o}{V}\right) = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} C_{rs}\,j_r(\xi_o)\,k_s(\eta_o) \tag{78}$$

and

$$c(y_o)\,\bar{L}_1\,(x_o, y_o)\,E\left(-\frac{\omega x_o}{V}\right) = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \bar{C}_{rs}\,j_r(1-\xi_o)\,k_s(\eta_o) \tag{79}$$

where

$$j_r(\xi_o) = \sum_{p=0}^{n-1} a_{rp}\,\xi_o^p \sqrt{\frac{1-\xi_o}{\xi_o}} \qquad r = 0, 1, 2, \ldots, n-1 \tag{80}$$

$$k_s(\eta_o) = \sum_{q=0}^{m-1} b_{sq}\,\eta_o^q \sqrt{1-\eta_o^2} \qquad s = 0, 1, 2, \ldots, m-1 \tag{81}$$

and the a_{rp} and b_{sq} are any sets of coefficients such that the determinants $|a_{rp}|$ and $|b_{sq}|$ are non-zero

The pairs of formulae (76), (77) and (78), (79) are completely equivalent to each other. One could obtain the $\bar{C}_{rs}$ as a linear combination of the $\bar{B}_{rs}$ and the C_{rs} as a linear combination of the B_{rs}. On the other hand we could obtain the values $\bar{C}_{rs}$ and C_{rs} by using (78) and (79) to evaluate H_1 from (57) and finding the stationary value of the resulting expression. We do this below.

Let

$$U_{rs}\,(x,y) = \frac{s}{4\pi\rho_o V}\int_{-1}^{+1} d\eta_o \int_0^1 j_r(\xi_o)\,k_s(\eta_o)\,\hat{K}\left(x-x_o, y-y_o, \frac{\omega}{V}, M\right) d\xi_o \tag{82}$$

so that

$$W_1(x,y) = E\left(-\frac{\omega x}{V}\right)\sum_{r=0}^{n-1}\sum_{s=0}^{m-1} C_{rs}\,U_{rs}\,(x,y)\,. \tag{83}$$

If we now substitute from (78), (79) and (83) into (57) we get

$$H_1 = \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \bar{C}_{rs}\,\theta_{rs} + \sum_{p=0}^{n-1}\sum_{q=0}^{m-1} C_{pq}\,\phi_{pq} - \sum_{r=0}^{n-1}\sum_{s=0}^{m-1}\sum_{p=0}^{n-1}\sum_{q=0}^{m-1} \bar{C}_{rs}\,C_{pq}\,\psi_{rs\,pq} \tag{84}$$

where

$$\theta_{rs} = s\int_{-1}^{+1} d\eta \int_{0}^{1} j_r(1-\xi)\,k_s(\eta)\,w(x,y)\,E\left(\frac{\omega x}{V}\right)d\xi \tag{85}$$

$$\phi_{pq} = s\int_{-1}^{+1} d\eta \int_{0}^{1} j_p(\xi)\,k_q(\eta)\,\bar{w}(x,y)\,E\left(-\frac{\omega x}{V}\right)d\xi \tag{86}$$

and

$$\psi_{rs\,pq} = s\int_{-1}^{+1} d\eta \int_{0}^{1} j_r(1-\xi)\,k_s(\eta)\,U_{pq}(x,y)\,d\xi\,. \tag{87}$$

The function H_1 is stationary when

$$\theta_{rs} - \sum_{p=0}^{n-1}\sum_{q=0}^{m-1} \psi_{rs\,pq}\,C_{pq} = 0 \qquad \begin{array}{l} r = 0, 1, 2, \ldots, n-1, \\ s = 0, 1, 2, \ldots, m-1. \end{array} \tag{88}$$

and

$$\phi_{pq} - \sum_{r=0}^{n-1}\sum_{s=0}^{m-1} \psi_{rs\,pq}\,\bar{C}_{rs} = 0 \qquad \begin{array}{l} r = 0, 1, 2, \ldots, n-1, \\ s = 0, 1, 2, \ldots, m-1. \end{array} \tag{89}$$

The sets of equations (88) and (89) may be written as the matrix equations

$$\theta = \psi C \tag{90}$$

and

$$\phi = \psi' \bar{C} \tag{91}$$

where θ, ϕ, $\bar{C}$ and C are respectively column matrices of the θ_{rs}, ϕ_{pq}, $\bar{C}_{rs}$ and C_{pq}, and ψ is a square matrix of the $\gamma_{rs\,pq}$. The matrix ψ' is the transpose of the matrix ψ.

We can now write H_{10} from equation (84) as

$$
\begin{aligned}
H_{10} &= \bar{C}'\theta + \phi' C - \bar{C}'\psi C \\
&= \bar{C}'\theta + \phi' C - \bar{C}'\theta \\
&= \phi' C \\
&= \phi' \psi^{-1} \theta .
\end{aligned} \tag{92}
$$

elements of ψ, given by formula (87) are more complicated to evaluate because the function $U_{pq}(x,y)$ must first be determined from formula (82) and it is a tedious process to evaluate $U_{pq}(x,y)$ for any given values of x and y.

4. *Reduction to a Current Lifting-Surface Theory.*

In order to evaluate H_{10} from formula (92) the elements of the matrices ϕ, θ and ψ must be determined. The elements of θ and ϕ given in formulae (85) and (86) are quite straightforward to evaluate, but the elements of ψ, given by formula (87) are more complicated to evaluate because the function $U_{pq}(x,y)$ must first be determined from formula (82) and it is a tedious process to evaluate $U_{pq}(x,y)$ for any given values of x and y.

It may be possible to evaluate the elements of θ and ϕ analytically from formulae (85) and (86) if $w(x,y)$ and $\bar{w}(x,y)$ are functions of sufficiently simple form. Otherwise the elements may be obtained from these formulae by means of numerical integration. If the values of the integrand at a finite number of points is to be used for the numerical integration and if the integrand is to be approximated by the product of a polynomial function in ξ and η with the function $\sqrt{\frac{1-\xi}{\xi}}\sqrt{1-\eta^2}$, then there is an optimum position for these points to give best accuracy. Let ξ_u, $u = 1,2,\ldots n$ be the n zeroes of $l_n(\xi)$ and let η_v, $v = 1,2,\ldots,m$ be the m zeroes of $\gamma_m(\eta)$. If mn points only are to be used for the numerical integration then for the numerical integration of the integral on the right-hand side of (85) it is best to take these mn points to be the points

$$
1-\xi_u, \eta_v \qquad \begin{matrix} u = 1,2,\ldots,n \\ v = 1,2,\ldots,m \end{matrix} \tag{93}
$$

and for the numerical integration of the integral on the right-hand side of (86) it is best to take these mn points to be the points

$$
\xi_u, \eta_v \qquad \begin{matrix} u = 1,2,\ldots n \\ v = 1,2,\ldots m \end{matrix} \tag{94}
$$

The elements of ψ will have to be obtained by numerical integration of the integral on the right-hand side of equation (87) since $U_{pq}(x,y)$ can only be obtained numerically. In this case the integration points (93) are the optimum ones to choose in general.

Let us define

$$
\left.\begin{aligned}
x_{uv} &= c(y_v)\,\xi_u + x_L(y_v) \\
y_v &= s\,\eta_v
\end{aligned}\right\} \qquad \begin{matrix} u = 1,2,\ldots,n \\ v = 1,2,\ldots,m \end{matrix} \tag{95}
$$

and

$$\left.\begin{aligned} \check{x}_{uv} &= c(y_v)(1-\xi_u)+x_L(y_v) \\ y_v &= s\,\eta_v \end{aligned}\right\} \quad \begin{aligned} u &= 1,2,\ldots,n \\ v &= 1,2,\ldots,m \end{aligned} \tag{96}$$

Then the results of performing numerically the integrations on the right-hand sides of (85), (86) and (87) using mn points at the optimum location are

$$\theta_{rs} = s \sum_{u=1}^{n} \sum_{v=1}^{m} H_u^{(n)}\, G_v^{(m)}\, j_r(\xi_u)\, k_s(\eta_v)\; w(\check{x}_{uv}, y_v)\, E\left(\frac{\omega \check{x}_{uv}}{V}\right) \tag{97}$$

$$\phi_{pq} = s \sum_{u=1}^{n} \sum_{v=1}^{m} H_u^{(n)}\, G_v^{(m)}\, j_p(\xi_u)\, k_q(\eta_v)\, \overline{w}(x_{uv}, y_v)\, E\left(-\frac{\omega x_{uv}}{V}\right) \tag{98}$$

$$\psi_{rs\,pq} = s \sum_{u=1}^{n} \sum_{v=1}^{m} H_u^{(n)}\, G_v^{(m)}\, j_r(\xi_u)\, k_s(\eta_v)\, U_{pq}(\check{x}_{uv}, y_v) \tag{99}$$

and the constants $H_u^{(n)}$, $u = 1,2,\ldots,n$; $G_v^{(m)}$, $v = 1,2,\ldots,m$; are the integration weighting constants.

We can define functions $h_r^{(n)}(\xi)$, $r = 1,2,\ldots,n$, by the formula

$$h_r^{(n)}(\xi) = \sum_{p=0}^{n-1} h_{rp}^{(n)}\, \xi^p \sqrt{\frac{1-\xi}{\xi}} \tag{100}$$

which are such that

$$h_r^{(n)}(\xi_u) = \delta_{ru} \tag{101}$$

and we can define functions $g_s^{(m)}(\eta)$, $s = 1, 2, \ldots, m$, by the formula

$$g_s^{(m)}(\eta) = \sum_{q=0}^{m-1} g_{sp}^{(m)}\, \eta^q \sqrt{1-\eta^2} \tag{102}$$

which are such that

$$g_s^{(m)}(\eta_v) = \delta_{sv}\,. \tag{103}$$

Then

$$H_u^{(n)} = \int_0^1 h_u^{(n)}(\xi)\, d\xi \tag{104}$$

and

$$G_v^{(m)} = \int_{-1}^{+1} g_v^{(m)}(\eta)\, d\eta\,. \tag{105}$$

It will be noticed that the definitions of the functions $h_r^{(n)}(\xi)$ and $g_s^{(m)}(\eta)$ given in formulae (100) and (102) are analogous respectively to those of $j_r(\xi_0)$ and $k_s(\eta_0)$ given in formulae (80) and (81). Indeed, as particular cases, we may take

$$j_r(\xi_o) = h_r^{(n)}(\xi_o) \tag{106}$$

and

$$k_s(\eta_o) = g_s^{(m)}(\eta_o)\,. \tag{107}$$

If we do this, then formulae (97), (98) and (99) reduce to

$$\theta_{rs} = s\, H_r\, G_s\, w(\check{x}_{rs}, y_s)\, E\left(\frac{\omega \check{x}_{rs}}{V}\right) \tag{108}$$

$$\phi_{pq} = s\, H_p\, G_q\, \bar{w}(x_{pq}, y_q)\, E\left(-\frac{\omega x_{pq}}{V}\right) \tag{109}$$

and

$$\psi_{rs\,pq} = s\, H_r\, G_s\, U_{pq}(\check{x}_{rs}, y_s)\,. \tag{110}$$

If we use expressions (108), (109) and (110) for the elements of the matrices θ, ϕ and ψ respectively then the expression (92) for H_{10} is exactly the expression which would be obtained for the corresponding generalised force in, for example, Ref. 5. Thus the results obtained using the variational technique are closely related to those of a collocation procedure.

The quantities $U_{pq}(\check{x}_{rs}, y_s)$ appearing in equation (110) may be obtained from formula (82) by means of an approximation technique such as the one described in Ref. 5. The accuracy with which these quantities are obtained using this technique may not be as good as is desirable, in particular for points $\check{x}_{rs}$, y_s near to the wing edges. Garner and Fox[6] have suggested a more accurate procedure for obtaining these quantities, but even this procedure cannot give these quantities in the ultimate when the edges are approached. Possibly a completely different procedure for obtaining these quantities $U_{pq}(\check{x}_{rs}, y_s)$ is necessary.

A procedure such as that of Watkins *et al*[7], or a modification of it, may also produce results that are more accurate than the ones obtained by the technique of Ref. 5.

The accuracy with which the elements of θ and ψ are obtained can be increased by using more than mn points for the evaluation of the integrals on the right-hand sides of equations (85) and (86), or performing analytically these evaluations if this is possible. Also the elements of ψ may be obtained with greater accuracy by using more than mn points for the evaluation of the integral on the right-hand side of (87). In this case some of the integration points might tend to get very close to the edges and a good procedure for evaluating $U_{pq}(x,y)$ at the integration points should be used.

When there are control surfaces present then one or both of $w(x,y)$ and $\bar{w}(x,y)$ may be discontinuous and as a result the formulae (97) and (98) or (108) and (109) may be rather inaccurate. In this case it is advisable to perform analytically the integrations on the right of (85) and (86), or at least to use a numerical integration procedure which takes the discontinuities into account. The basis of equivalent upwash and displacement calculations rests on the accurate evaluation of the θ_{rs} and ϕ_{pq}.

Of course, if there are control surfaces present, the number of terms required in the summations in (53) and (54) to give a small difference $H_{10}-H_0$ may be relatively high compared with the number required if only smooth modes of displacement occur.

If for the distributions of loading $L_1(x_0, y_0)$ and $\bar{L}_1(x_0, y_0)$ corresponding to the stationary value H_{10} we write

$$L_1(x, y_o) = l(x_o, y_o) - \varepsilon_1(x_o, y_o) \tag{111}$$

and

$$\bar{L}_1(x_o, y_o) = \bar{l}(x_o, y_o) - \bar{\varepsilon}_1(x_o, y_o) \tag{112}$$

and if the upwashes $W_1(x,y)$ and $\bar{W}_1(x,y)$ corresponding respectively to the loading $L_1(x_0, y_0)$ in the direct flow and $\bar{L}_1(x_0, y_0)$ in the reverse flow, are written

$$W_1(x,y) = w(x,y) - \delta_1(x,y) \tag{113}$$

and

$$\bar{W}_1(x,y) = \bar{w}(x,y) - \bar{\delta}_1(x,y) \tag{114}$$

then, following the process leading to equation (16), we have

$$H_{10} = H_0 - \int\int_S \bar{\varepsilon}_1(x,y)\,\delta_1(x,y)\,dx\,dy\,. \tag{115}$$

The difference $H_{10}-H_0$ therefore depends on the quantity

$$\int\int_S \bar{\varepsilon}_1(x,y)\,\delta_1(x,y)\,dx\,dy = \int\int_S \varepsilon_1(x,y)\,\bar{\delta}_1(x,y)\,dx\,dy\,. \tag{116}$$

If the functions $w(x,y)$ and $\bar{w}(x,y)$ are smooth functions, then it may be expected that for a relatively small number of terms in the summations (53) and (54) the difference $H_{10}-H_0$ is very small. If one of the functions $w(x,y)$ or $\bar{w}(x,y)$ is not smooth, then, with the same summations, the difference $H_{10}-H_0$ may be expected to be larger, while if both $w(x,y)$ and $\bar{w}(x,y)$ are not smooth the difference $H_{10}-H_0$ may be expected to be larger still.

If, for example, the loading $l(x_0, y_0)$ is such that it is given exactly by a finite series of the form (53), then

$$\delta_1(x,y) = 0\,, \qquad \varepsilon_1(x,y) = 0 \tag{117}$$

and

$$H_{10} = H_0 \tag{118}$$

whatever the form of $\bar{w}(x,y)$. In three-dimensional wing theory this is not in general the case, but it does occur in two-dimensional incompressible flow.

If $l(x_0, y_0)$ is not given exactly by a finite series of the form (53) nor is $\bar{l}(x_0, y_0)$ given by a finite series of the form (54) then the results obtained using equivalent upwash and displacement functions may not give the generalised airforces to the desired accuracy for a relatively small number of terms in the summations, when one of $w(x,y)$ and $\bar{w}(x,y)$ is discontinuous. If both $w(x,y)$ and $\bar{w}(x,y)$ are discontinuous the error may be worse.

5. *A Least Squares Approach.*

A process which has been suggested (*see*, e.g. Fromme[3]) for obtaining the generalised force is to take an approximation to the loading given by formula (53) and to obtain the coefficients B_{rs} by minimising the expression

$$\int\int_S |w(x,y) - W_1(x,y)|^2 f(x,y)\, dx\, dy \tag{119}$$

where $W_1(x,y)$ is given by formula (55) and $f(x,y)$ is some function which one might consider desirable to introduce and might for convenience be taken to be unity. The estimate Q_j for the generalised force Q_{jo} is then taken to be

$$Q_j = \int\int_S L_1(x,y)\, Z_j(x,y)\, dx\, dy\,. \tag{120}$$

If we write

$$L_1(x_o, y_o) = l(x_o, y_o) - \varepsilon_1(x_o, y_o) \tag{121}$$

then

$$Q_j = \int\int_S l(x,y)\, Z_j(x,y)\, dx\, dy - \int\int_S \varepsilon_1(x,y)\, Z_j(x,y)\, dx\, dy$$

$$= Q_{jo} - \int\int_S \varepsilon_1(x,y)\, Z_j(x,y)\,. \tag{122}$$

Thus, in general, the quantity Q_j approaches Q_{jo} only to within terms of first order, so this procedure procedure is not as good an estimate of a generalised force as is obtained by using Flax's variational principle. This will be illustrated by an example in the next section.

6. *Example for Two-Dimensional Incompressible Flow.*

Consider a two-dimensional wing of chord length c. Introduce co-ordinate axes x and z such that the x-axis is along the direction of the wing chord and the z-axis is perpendicular to the wing chord.

Corresponding to the x co-ordinate of a point P on the wing we can introduce a transformed co-ordinate ξ by means of the formula

$$\xi = \frac{x - x_{L.}}{c} \tag{123}$$

where $x_{L.}$ is the x-co-ordinate of the leading edge.

The two-dimensional wing is assumed to be oscillating harmonically with circular frequency ω in a flow of subsonic main stream speed V.

Corresponding to the upwash distributions $w(x)$ and $\bar{w}(x)$ there are respectively the loading function $l(x)$ in the direct flow and the loading function $\bar{l}(x)$ in the reverse flow.

We wish to estimate the value of H_0 where

$$H_0 = \int_{\substack{\text{wing}\\ \text{chord}}} l(x)\,\bar{w}(x)\,dx\,. \tag{124}$$

We shall take $E(x)$ to be unity and take the approximate loading distributions $L_1(x)$ and $\bar{L}_1(x)$ in the direct and reverse flows to be given by

$$L_1(x) = \sum_{r=1}^{n-1} C_r\,j_r(\xi) \tag{125}$$

and

$$\bar{L}_1(x) = \sum_{r=0}^{n-1} \bar{C}_r\,j_r(1-\xi) \tag{126}$$

where

$$j_r(\xi) = \rho V^2\,\xi^r \sqrt{\frac{1-\xi}{\xi}}\,. \tag{127}$$

The upwash corresponding to the loading $j_r(\xi)$ in the direct flow is denoted by $U_r(x)$. The estimate H_{10} of H_0, using the variational principle is then given by (*see* equation (92))

$$H_{10} = c\phi'\,\psi^{-1}\,\theta \tag{128}$$

where θ and ϕ are column matrices with elements θ_r and ϕ_p respectively and ψ is a square matrix with elements ψ_{rp} and

$$\theta_r = \int_0^1 j_r(1-\xi)\,w(x)\,d\xi \tag{129}$$

$$\phi_p = \int_0^1 j_p(\xi)\,\bar{w}(x)\,d\xi \tag{130}$$

$$\psi_{rp} = \int_0^1 j_r(1-\xi)\,U_p(x)\,d\xi\,. \tag{131}$$

From two-dimensional incompressible flow theory we know the functions $U_p(x)$ corresponding to the loading distributions $j_p(\xi)$. They are

$$U_0(x) = V\left\{-\frac{\pi}{4}(iv)\left[H_1^{(2)}(v)+i\,H_0^{(2)}(v)\right]e^{-iv(2\xi-1)}\right\} \tag{132}$$

$$U_1(x) = V \left\{ \frac{1}{4iv} + \frac{\pi}{8} H_1^{(2)}(v)\, e^{-iv(2\xi-1)} \right\} \tag{133}$$

$$U_2(x) = V \left\{ -\frac{1}{4(iv)^2}(1+iv) + \frac{1}{2iv}\xi - \frac{\pi}{2}\left[\left(\frac{1}{4iv} - 1 \right) H_1^{(2)}(v) + \frac{i}{8} H_0^{(2)}(v) \right] e^{-iv(2\xi-1)} \right\} \tag{134}$$

etc., where

$$v = \left(\frac{\omega c}{2V} \right). \tag{135}$$

We can work out ψ_{rp} from equation (131) to get

$$\psi_{00} = -\frac{\pi}{4} \rho V^3 (iv) \left[H_1^{(2)}(v) + i H_0^{(2)}(v) \right] \int_0^1 e^{-iv(2\xi-1)} \sqrt{\frac{\xi}{1-\xi}}\, d\xi \tag{136}$$

$$\psi_{01} = \rho V^3 \left[\frac{1}{4iv} \int_0^1 \sqrt{\frac{\xi}{1-\xi}}\, d\xi + \frac{\pi}{8} H_1^{(2)}(v) \int_0^1 e^{-iv(2\xi-1)} \sqrt{\frac{\xi}{1-\xi}}\, d\xi \right] \tag{137}$$

$$\psi_{10} = -\frac{\pi}{4} \rho V^3 (iv) \left[H_1^{(2)}(v) + i H_0^{(2)}(v) \right] \int_0^1 e^{-iv(2\xi-1)} (1-\xi) \sqrt{\frac{\xi}{1-\xi}}\, d\xi \tag{138}$$

$$\psi_{11} = \rho V^3 \left[\frac{1}{4iv} \int_0^1 (1-\xi) \sqrt{\frac{\xi}{1-\xi}}\, d\xi + \frac{\pi}{8} H_1^{(2)}(v) \int_0^1 e^{-iv(2\xi-1)} (1-\xi) \sqrt{\frac{\xi}{1-\xi}}\, d\xi \right] \tag{139}$$

If we take $n = 1$, ψ is a 1×1 matrix consisting of the element ψ_{00}.
Hence

$$\psi^{-1} = -\frac{4}{\pi \rho V^3 (iv) \{ H_1^{(2)}(v) + i H_0^{(2)}(v) \} \int_0^1 e^{-iv(2\xi-1)} \sqrt{\frac{\xi}{1-\xi}}\, d\xi}. \tag{140}$$

If we take $n = 2$, ψ is the 2×2 matrix

$$\begin{bmatrix} \psi_{00} & \psi_{01} \\ \psi_{10} & \psi_{11} \end{bmatrix} \tag{141}$$

Then ψ^{-1} is the 2×2 matrix

$$\begin{bmatrix} \gamma_{00} & \gamma_{01} \\ \gamma_{10} & \gamma_{11} \end{bmatrix} \tag{142}$$

where

$$\gamma_{00} = -\frac{\dfrac{4}{(iv)} \displaystyle\int_0^1 (1-\xi)\sqrt{\frac{\xi}{1-\xi}}\, d\xi + 2\pi\, H_1^{(2)}(v) \int_0^1 e^{-iv(2\xi-1)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\, d\xi}{\pi\rho V^3 \{H_1^{(2)}(v) + i\, H_0^{(2)}(v)\}\, P(v)} \tag{143}$$

$$\gamma_{01} = -\frac{\dfrac{4}{(iv)} \displaystyle\int_0^1 \sqrt{\frac{\xi}{1-\xi}}\, d\xi + 2\pi\, H_1^{(2)}(v) \int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\, d\xi}{\pi\rho V^3 \{H_1^{(2)}(v) + i\, H_0^{(2)}(v)\}\, P(v)} \tag{144}$$

$$\gamma_{10} = \frac{4(iv) \displaystyle\int_0^1 e^{-iv(2\xi-1)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\, d\xi}{\rho V^3\, P(v)} \tag{145}$$

$$\gamma_{11} = -\frac{4(iv) \displaystyle\int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\, d\xi}{\rho V^3\, P(v)} \tag{146}$$

and

$$P(v) = \int_0^1 e^{-iv(2\xi-1)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\, d\xi \int_0^1 \sqrt{\frac{\xi}{1-\xi}}\, d\xi$$

$$- \int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\, d\xi \int_0^1 (1-\xi)\sqrt{\frac{\xi}{1-\xi}}\, d\xi\,. \tag{147}$$

If $n = 1$ we get from (128), (129), (130) and (131)

$$H_{10} = -\frac{4}{\pi}\rho V c \frac{\int_0^1 \bar{w}(x)\sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 w(x)\sqrt{\frac{\xi}{1-\xi}}\,d\xi}{(iv)\,\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\,d\xi}. \tag{148}$$

If $n = 2$ we get from (128)

$$H_{10} = c\,[\gamma_{00}\,\theta_0\,\phi_0+\gamma_{01}\,\theta_1\,\phi_0+\gamma_{10}\,\theta_0\,\phi_1+\gamma_{11}\,\theta_1\,\phi_1] \tag{149}$$

where θ_0, θ_1 are obtained from (129); ϕ_0, ϕ_1 from (130); and γ_{00}, γ_{01}, γ_{10}, γ_{11} from (143) to (146) respectively.

The integrals occurring in (140) and (143) to (147) can all be evaluated analytically, but we do not choose to do this for the present because the values obtained by numerical evaluation are also of interest.

In the least squares procedure, an approximation function $L_1(x)$ of the form (125) is taken for $l(x)$ and the coefficients C_r are chosen such that the expression

$$\int_0^1 |w(x)-W_1(x)|^2\,dx \tag{150}$$

is a minimum, where

$$W_1(x) = \sum_{r=0}^{n-1} C_r\,U_r(x). \tag{151}$$

If $n = 1$, then there is only one coefficient in the formula (125), namely C_0, and if the expression (150) is to be a minimum we must have

$$C_0 = \frac{\int_0^1 w(x)\,U_0^*(x)\,d\xi}{\int_0^1 |U_0(\xi)|^2\,d\xi}$$

$$= -\frac{4}{\pi V}\,\frac{1}{(iv)}\,\frac{\int_0^1 w(x)\,e^{iv(2\xi-1)}\,d\xi}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \tag{152}$$

where the asterisk * attached to a quantity denotes that its complex conjugate is to be taken.

The estimate H_s for H_0 is then

$$H_s = C_0\, c \int_0^1 j_o(\xi)\, \overline{w}(x)\, d\xi$$

$$= -\frac{4}{\pi}\rho V c \frac{1}{(iv)} \frac{\int_0^1 \overline{w}(x)\sqrt{\frac{1-\xi}{\xi}}\, d\xi \int_0^1 w(x)\, e^{iv(2\xi-1)}\, d\xi}{\{H_1^{(2)}(v)+i\, H_0^{(2)}(v)\}}. \tag{153}$$

If $n = 2$, there are two coefficients in formula (125), namely C_0 and C_1. If expression (150) is to be a minimum we must have

$$C_0 = \frac{\alpha_{11}}{V^2} \int_0^1 w(x)\, U_0^*(x)\, d\xi + \frac{\alpha_{10}}{V^2} \int_0^1 w(x)\, U_1^*(x)\, d\xi \tag{154}$$

$$C_1 = \frac{\alpha_{01}}{V^2} \int_0^1 w(x)\, U_0^*(x)\, d\xi + \frac{\alpha_{00}}{V^2} \int_0^1 w(x)\, U_1^*(x)\, d\xi \tag{155}$$

where

$$a_{00} = -\frac{(iv)^2}{\left\{1-\left(\frac{\sin v}{v}\right)^2\right\}} \tag{156}$$

$$\alpha_{01} = \frac{2iv\, H_1^{(1)}(v) - \frac{8}{\pi}\left(\frac{\sin v}{v}\right)}{\left\{H_1^{(1)}(v) - i\, H_0^{(1)}(v)\right\}\left\{1-\left(\frac{\sin v}{v}\right)^2\right\}} \tag{157}$$

$$\alpha_{10} = -\frac{2iv\, H_1^{(2)}(v) + \frac{8}{\pi}\left(\frac{\sin v}{v}\right)}{\left\{H_1^{(2)}(v) + i\, H_0^{(2)}(v)\right\}\left\{1-\left(\frac{\sin v}{v}\right)^2\right\}} \tag{158}$$

$$\alpha_{11} = \frac{4H_1^{(2)}(v)\, H_1^{(1)}(v) + \frac{8}{\pi}\frac{1}{(iv)}\left(\frac{\sin v}{v}\right)\left\{H_1^{(1)}(v) - H_1^{(2)}(v)\right\} - \frac{16}{\pi^2}\frac{1}{(iv)^2}}{\left\{H_1^{(2)}(v) + i\, H_0^{(2)}(v)\right\}\cdot\left\{H_1^{(1)}(v) - i\, H_0^{(1)}(v)\right\}\left\{1-\left(\frac{\sin v}{v}\right)^2\right\}}. \tag{159}$$

The estimate H_s for H_0 is then

$$H_s = C_0\, c \int_0^1 j_0(\xi)\,\overline{w}(x)\,d\xi + C_1\, c \int_0^1 j_1(\xi)\,\overline{w}(x)\,d\xi$$

$$= \rho c \left[\alpha_{11} \int_0^1 \overline{w}(x) \sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 w(x)\,U_0^*(x)\,d\xi \right.$$

$$+\alpha_{10} \int_0^1 \overline{w}(x) \sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 w(x)\,U_1^*(x)\,d\xi$$

$$+\alpha_{01} \int_0^1 \overline{w}(x)\,\xi \sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 w(x)\,U_0^*(x)\,d\xi$$

$$\left. +\alpha_{00} \int_0^1 \overline{w}(x)\,\xi \sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 w(x)\,U_1^*(x)\,d\xi \right]. \qquad (160)$$

Example 1.

Let us consider the wing to be in heaving oscillatory motion so that

$$Z(x,t) = ce^{i\omega t}. \qquad (161)$$

Then

$$w(x) = 2ivV. \qquad (162)$$

The actual two-dimensional incompressible flow solution gives for the loading function $l(x)$ the expression

$$l(x) = 4\rho V^2 (iv) \left[\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v) + i\,H_0^{(2)}(v)\}} + 2iv\,\xi \right] \sqrt{\frac{1-\xi}{\xi}}. \qquad (163)$$

The total lift per unit span is $Le^{i\omega t}$ where

$$L = c \int_0^1 l(x)\,d\xi$$

$$= 2\pi\rho V^2 c(iv) \left[\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v) + i\,H_0^{(2)}(v)\}} + \tfrac{1}{2}(iv) \right]. \qquad (164)$$

The total pitching moment per unit span, about the mid-chord line, is $Me^{i\omega t}$ where

$$M = c^2 \int_0^1 (\tfrac{1}{2}-\xi)\, l(x)\, d\xi$$

$$= \tfrac{1}{2}\pi\rho V^2 c^2 (iv) \left[\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \right]. \tag{165}$$

We may also write L and M in the forms

$$L = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \left[-4-2(iv)+(iv)^2+0(v^3)+0\left(v^3 \log\frac{v}{2}\right) \right] \tag{166}$$

$$M = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \left[-1+0(v^2)+0\left(v^2 \log\frac{v}{2}\right) \right]. \tag{167}$$

To estimate L we must take $\bar{w}(x) = 1$ and to estimate M we must take $\bar{w}(x) = c(\frac{1}{2}-\xi)$ in both the procedure using the variational principle and that using least squares.

We shall consider results for $n = 1$ and $n = 2$ only in the following cases.

Case (1.*i*).

$$n = 1 \qquad w(x) = 2ivV \qquad \bar{w}(x) = 1$$

The estimate H_{10} for L obtained from the variational principle is given from the formula (148) as

$$H_{10} = -\frac{8}{\pi}\rho V^2 c \frac{\displaystyle\int_0^1 \sqrt{\frac{1-\xi}{\xi}}\, d\xi \int_0^1 \sqrt{\frac{\xi}{1-\xi}}\, d\xi}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\} \displaystyle\int_0^1 e^{-iv(2\xi-1)} \sqrt{\frac{\xi}{1-\xi}}\, d\xi}. \tag{168}$$

The integrals in formula (168) can be evaluated analytically, or numerically. The numerical integrations are carried out using a one-point integration formulae. The result obtained using the numerical values of these integrals in (168) is exactly that which is obtained by the collocation method and will be denoted by $H_{10}^{(c)}$. Thus

$$H_{10} = -4\rho V^2 c \frac{1}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\,\{J_0(v)-i\,J_1(v)\}} \tag{169}$$

and

$$H_{10}^{(c)} = -4\rho V^2 c \frac{1}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\, e^{-iv/2}}. \tag{170}$$

The estimate H_s for H_0 obtained from the least squares procedure is given from formula (153) as

$$H_s = -\frac{8}{\pi}\rho V^2 c \frac{\int_0^1 \sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 e^{iv(2\xi-1)}\,d\xi}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}$$

$$= -4\rho V^2 c \frac{\left(\frac{\sin v}{v}\right)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}. \tag{171}$$

The expressions for H_{10}, $H_{10}^{(c)}$ and H_s given in (169), (170) and (171) may be written

$$H_{10} = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}[-4-2iv+0(v^3)] \tag{172}$$

$$H_{10}^{(c)} = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}[-4-2iv-\tfrac{1}{2}(iv)^2+0(v^3)] \tag{173}$$

$$H_s = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}[-4-\frac{2}{3}(iv)^2+0(v^4)] \tag{174}$$

The expressions (172), (173), (174) are estimates for L which is given accurately by formula (166). The expression H_{10} is closest to L, differing from it by a term of $0(v^2)$ in the square brackets. The expression $H_{10}^{(c)}$ also differs from L by a term of $0(v^2)$ in the square brackets, whereas the expression H_s differs from L by a term of $0(v)$ in the square brackets.

Case (1.*ii*).

$$n = 1 \qquad w(x) = 2ivV \qquad \bar{w}(x) = c(\tfrac{1}{2}-\xi)$$

The estimate H_{10} for M obtained from the variational principle is given from formula (148) as

$$H_{10} = -\frac{8}{\pi}\rho V^2 c^2 \frac{\int_0^1 (\tfrac{1}{2}-\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 \sqrt{\frac{\xi}{1-\xi}}\,d\xi}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\,d\xi}. \tag{175}$$

If the integrals are evaluated analytically we get

$$H_{10} = -\rho V^2 c^2 \frac{1}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\,\{J_0(v)-i\,J_1(v)\}} \tag{176}$$

and if they are evaluated numerically using one-point integration formulae we get

$$H_{10}^{(c)} = -\rho V^2 c^2 \frac{1}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\,e^{-iv/2}}. \tag{177}$$

The estimate H_s for H_0 obtained from the least squares procedure is given by formula (154) as

$$H_s = -\frac{8}{\pi}\rho V^2 c^2 \frac{\int_0^1 (\tfrac{1}{2}-\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 e^{iv(2\xi-1)}\,d\xi}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}$$

$$= -\rho V^2 c^2 \frac{\left(\frac{\sin v}{v}\right)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}. \tag{178}$$

The expressions for H_{10}, $H_{10}^{(c)}$ and H_s given by (176), (177) and (178) may be written

$$H_{10} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[-1-\tfrac{1}{2}(iv)+0(v^2)\right] \tag{179}$$

$$H_{10}^{(c)} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[-1-\tfrac{1}{2}(iv)+0(v^2)\right] \tag{180}$$

$$H_s = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[-1+0(v^2)\right]. \tag{181}$$

The expressions (179), (180), (181) are estimates for M which is given accurately by formula (167). The expression H_s is closest to M, differing from it by a term of $0\left(v^2 \log\frac{v}{2}\right)$ in the square brackets. The expressions H_{10} and $H_{10}^{(c)}$ differ from M by a term of $0(v)$ in the square brackets.

Case (1.*iii*).

$$n = 2 \qquad w(x) = 2ivV \qquad \overline{w}(x) \text{ unspecified}$$

The estimate H_{10} for H_0 obtained from the variational principle is given by formula (149). On substituting for the quantities involved and simplifying we get

$$H_{10} = 4\rho V^2 c\,(iv) \int_0^1 \left[\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}+2iv\xi\right]\overline{w}(x)\sqrt{\frac{1-\xi}{\xi}}\,d\xi$$

$$= \rho V^2 c \int_0^1 l(x)\,\overline{w}(x)\,d\xi. \tag{182}$$

Thus the generalised force is obtained exactly from the variational principle, irrespective of the function $\bar{w}(x)$, provided the integral (182) is evaluated exactly. If numerical evaluation of (182) using two integration points is used, then the generalised force is obtained exactly provided $\bar{w}(x)$ is a polynomial of degree not higher than the second in ξ, but not otherwise.

The estimate H_s of the generalised force obtained from the least squares procedure is also exact in this case.

Example 2.

Let us consider the wing to be in pitching oscillatory motion about its mid-chord line so that

$$Z(x,t) = c(\tfrac{1}{2}-\xi)\, e^{i\omega t} . \tag{183}$$

Then

$$w(x) = V[-1+iv(1-2\xi)] . \tag{184}$$

The actual two-dimensional incompressible flow solution gives for the loading $l(x)$ the expression

$$l(x) = 2\rho V^2 \left[-\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} + \tfrac{1}{2}(iv)\frac{i\,H_0^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} - 4(iv)\xi + (iv)^2\,(\xi-2\xi^2) \right] \sqrt{\frac{1-\xi}{\xi}} . \tag{185}$$

The total lift per unit span is $Le^{i\omega t}$ where

$$L = \pi\rho V^2\, c \left[-(1+\tfrac{1}{2}iv)\frac{H_1^{(2)}(v)}{\{H^{(2)}(v)+i\,H^{(2)}(v)\}} - \tfrac{1}{2}iv \right] \tag{186}$$

and the total pitching moment per unit span, about the mid-chord line, is $Me^{i\omega t}$ where

$$M = \tfrac{1}{4}\pi\rho V^2\, c^2 \left[-\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} + \tfrac{1}{2}(iv)\frac{i\,H_0^{(2)}(v)}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} + \tfrac{1}{8}(iv)^2 \right] \tag{187}$$

We may write L and M in the forms

$$L = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \left[\frac{2}{(iv)} + 2 - \tfrac{1}{2}(iv) - \tfrac{1}{2}(iv)^2 + (iv)^2 \left\{ \gamma + \frac{i\pi}{2} + \log\left(\frac{v}{2}\right) \right\} + 0(v^3) + 0\left(v^3 \log\frac{v}{2} \right) \right] \tag{188}$$

$$M = \frac{\rho V^2\, c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}} \left[\frac{1}{2iv} - \frac{3}{16}(iv) + \left\{ \frac{1}{2}(iv) + \frac{1}{16}(iv)^2 \right\} \times \left\{ \gamma + \frac{i\pi}{2} + \log\left(\frac{v}{2}\right) \right\} + 0(v^3) + 0\left(v^3 \log\frac{v}{2} \right) \right] . \tag{189}$$

To estimate L we must take $\bar{w}(x) = 1$ and to estimate M we must take $\bar{w}(x) = c(\tfrac{1}{2}-\xi)$ in both the procedure using the variational principle and that using least squares.

We shall consider results for $n = 1$ and $n = 2$ only in the following cases.

Case (2.*i*).

$$n = 1 \qquad w(x) = V[-1+iv(1-2\xi)] \qquad \bar{w}(x) = 1$$

The estimate H_{10} for L obtained from the variational principle is given from formula (148) as

$$H_{10} = -\frac{4}{\pi}\rho V^2 c \frac{\int_0^1 \sqrt{\frac{1-\xi}{\xi}}\, d\xi \int_0^1 [-1+iv(1-2\xi)] \sqrt{\frac{\xi}{1-\xi}}\, d\xi}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\} \int_0^1 e^{-iv(2\xi-1)} \sqrt{\frac{\xi}{1-\xi}}\, d\xi}. \tag{190}$$

If the integrals are evaluated analytically we get

$$H_{10} = 2\rho V^2 c \frac{[1+\frac{1}{2}iv]}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\{J_0(v)-i\,J_1(v)\}} \tag{191}$$

and if they are evaluated numerically using one-point integration formulae we get

$$H_{10}^{(c)} = 2\rho V^2 c \frac{[1+\frac{1}{2}iv]}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\, e^{-iv/2}}. \tag{192}$$

The estimate H_s for L obtained from the least squares procedure is given from formula (153) as

$$H_s = -\frac{4}{\pi}\rho V^2 c \frac{\int_0^1 \sqrt{\frac{1-\xi}{\xi}}\, d\xi \int_0^1 [-1+iv(1-2\xi)]\, e^{iv(2\xi-1)}\, d\xi}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}$$

$$= 2\rho V^2 c \frac{\left[\left(\frac{\sin v}{v}\right)+\frac{1}{2iv}\left\{\left(\frac{\sin v}{v}\right)-\cos v\right\}\right]}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}. \tag{193}$$

The expressions for H_{10}, $H_{10}^{(c)}$ and H_s given by (191), (192) and (193) may be written

$$H_{10} = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{2}{iv}+2+\tfrac{1}{2}(iv)+0(v^2)\right] \tag{194}$$

$$H_{10}^{(c)} = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{2}{iv}+2+\tfrac{3}{4}(iv)+0(v^2)\right] \tag{195}$$

$$H_s = \frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{2}{iv}-\frac{1}{3}+0(v^2)\right]. \tag{196}$$

The expressions (194), (195), (196) are estimates for L which is given accurately by formula (188). The expression H_{10} is closest to L differing from it by a term of $0(v)$ in the square brackets. The expression $H_{10}^{(c)}$ also differs from L by a term of $0(v)$ in the square brackets, whereas the expression H_s differs from L by a term of $0(1)$ in the square brackets.

Case (2.*ii*).

$$n = 1 \qquad w(x) = V[-1+iv(1-2\xi)] \qquad \overline{W}(x) = c(\tfrac{1}{2}-\xi)$$

The estimate H_{10} for M obtained from the variational principle is given from formula (148) as

$$H_{10} = -\frac{4}{\pi}\rho V^2 c^2 \frac{\int_0^1 (\tfrac{1}{2}-\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 [-1+iv(1-2\xi)]\sqrt{\frac{\xi}{1-\xi}}\,d\xi}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\,d\xi}. \tag{197}$$

If the integrals are evaluated analytically we get

$$H_{10} = \tfrac{1}{2}\rho V^2 c^2 \frac{[1+\tfrac{1}{2}iv]}{(iv)\{H_1^{(2)}(v)+i\,H_1^{(2)}(v)\}\{J_0(v)-i\,J_1(v)\}} \tag{198}$$

and if they are evaluated numerically using one-point integration formulae we get

$$H_{10}^{(c)} = \tfrac{1}{2}\rho V^2 c^2 \frac{[1+\tfrac{1}{2}iv]}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\,e^{-iv/2}}. \tag{199}$$

The estimate H_s for L obtained from the least squares procedure is given by formula (153) as

$$H_s = -\frac{4}{\pi}\rho V^2 c^2 \frac{\int_0^1 (\tfrac{1}{2}-\xi)\sqrt{\frac{1-\xi}{\xi}}\,d\xi \int_0^1 [-1+iv(1-2\xi)]\,e^{iv(2\xi-1)}\,d\xi}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}$$

$$= \tfrac{1}{2}\rho V^2 c^2 \frac{\left[\left(\frac{\sin v}{v}\right)+\frac{1}{2iv}\left\{\left(\frac{\sin v}{v}\right)-\cos v\right\}\right]}{(iv)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}. \tag{200}$$

The expressions for H_{10}, $H_{10}^{(c)}$ and H_s given by (198), (199) and (200) may be written

$$H_{10} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{1}{2iv}+\tfrac{1}{2}+0(v)\right] \tag{201}$$

$$H_{10}^{(c)} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{1}{2iv}+\tfrac{1}{2}+0(v)\right] \tag{202}$$

$$H_s = \frac{\rho V^2 c^2}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{1}{2iv}-\frac{1}{12}+0(v)\right]. \tag{203}$$

The expressions (201), (202) and (203) are estimates for M which is given accurately by formula (189). All the expressions differ from M by a term of 0(1) in the square brackets.

Case (2.*iii*).

$$n = 2 \qquad w(x) = V[-1+iv(1-2\xi)] \qquad \bar{w}(x) = 1$$

The estimate H_{10} for L obtained from the variational principle is given by formula (149). If the integrals occurring are evaluated analytically, the resulting expression is exactly that given for L in formula (186).

If the integrals are evaluated numerically using two-point integration formulae, then θ_0, θ_1, ϕ_0, ϕ_1 are obtained exactly but the numerical estimates

$$\int_0^1 e^{-iv(2\xi-1)}\sqrt{\frac{\xi}{1-\xi}}\,d\xi \simeq \frac{\pi}{2}\{J_0(v)-i\,J_1(v)+E_0(v)\} \tag{204}$$

$$\int_0^1 e^{-iv(2\xi-1)}(1-\xi)\sqrt{\frac{\xi}{1-\xi}}\,d\xi \simeq \frac{\pi}{4}\left\{\frac{J_1(v)}{v}+E_1(v)\right\} \tag{205}$$

where

$$E_0(v) = -\frac{1}{384}(iv)^4+0(v^5) \tag{206}$$

$$E_1(v) = -\frac{1}{96}(iv)^3+0(v^4) \tag{207}$$

which are obtained from two-point integration formula, are only approximate and lead to approximate values of γ_{00}, γ_{01}, γ_{10} and γ_{11}.

Using the approximations (204) and (205) to evaluate γ_{00}, γ_{01}, γ_{10} and γ_{11} we get

$$H_{10}^{(c)} = L-\frac{\pi\rho V^2\,c(iv)}{4\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\,\frac{[H_1^{(2)}(v)\,E_0(v)+iv\{H_1^{(2)}(v)+iH_0^{(2)}(v)\}\,E_1(v)]}{\left[2\dfrac{J_1(v)}{v}-J_0(v)+iJ_1(v)+2E_1(v)-E_0(v)\right]}$$

$$= L-\frac{\rho V^2 c}{\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}}\left[\frac{5}{768}(iv)^4+0(v^5)+0\left(v^5\log\frac{v}{2}\right)\right]. \tag{208}$$

The estimate H_s for L obtained from the least squares procedure is given from formula (160), which in this case reduces to

$$H_s = \pi\rho V^2 c\left[-\frac{H_1^{(2)}(v)}{\{H_1^{(2)}(v)+iH_0^{(2)}(v)\}}-\tfrac{1}{2}iv+\frac{\cos v-\left(\dfrac{\sin v}{v}\right)}{1-\left(\dfrac{\sin v}{v}\right)^2\quad\{H_1^{(2)}(v)+iH_0^{(2)}(v)\}}\times\right.$$

$$\left.\times\left(\frac{2}{\pi iv}+\left(\frac{\sin v}{v}\right)H_1^{(2)}(v)+\tfrac{1}{2}(iv)\left(\frac{\sin v}{v}\right)\{H_1^{(2)}(v)+i\,H_0^{(2)}(v)\}\right)\right]$$

$$= \frac{\rho V^2 c}{\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\left[\frac{2}{i\nu}+2-\frac{2}{3}(i\nu)+(i\nu)^2\left\{\gamma+\frac{i\pi}{2}+\log\frac{\nu}{2}\right\}\right.$$

$$\left.+0(\nu^3)+0\left(\nu^3\log\frac{\nu}{2}\right)\right]. \qquad (209)$$

The estimate $H_{10}^{(c)}$ is much closer to L than is H_s.

Case (2.*iv*).

$$n = w(x) = V[-1+i\nu(1-2\xi)] \qquad \overline{w}(x) = c(\tfrac{1}{2}-\xi)$$

The estimate H_{10} for M obtained from the variational principle is given by formula (149). If the integrals are evaluated analytically we get

$$H_{10} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\left[\frac{1}{2i\nu}-\frac{3}{16}(i\nu)+\frac{1}{64}(i\nu)^2+\left\{\frac{1}{2}(i\nu)+\frac{1}{16}(i\nu)^2\right\}\times\right.$$

$$\left.\times\left\{\gamma+\frac{i\pi}{2}+\log\frac{\nu}{2}\right\}+0(\nu^3)+0\left(\nu^3\log\frac{\nu}{2}\right)\right] \qquad (210)$$

and if they are evaluated numerically using two-point integration formulae we get

$$H_{10}^{(c)} = \frac{\rho V^2 c^2}{\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\left[\frac{1}{2i\nu}-\frac{3}{16}(i\nu)+\frac{1}{64}(i\nu)^2+\left\{\frac{1}{2}(i\nu)+\frac{1}{16}(i\nu)^2\right\}\times\right.$$

$$\left.\times\left\{\gamma+\frac{i\pi}{2}+\log\frac{\nu}{2}\right\}+0(\nu^3)+0\left(\nu^3\log\frac{\nu}{2}\right)\right] \qquad (211)$$

which is exactly the same as H_{10} to the accuracy given.

The estimate H_s for M obtained from the least squares procedure is given by formula (160), which in this case reduces to

$$H_s = \pi\rho V^2 c^2\left[-\frac{H_1^{(2)}(\nu)}{4\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\right.$$

$$\left.+\frac{\left\{\cos\nu-\dfrac{\sin\nu}{\nu}\right\}\left\{\dfrac{1}{2\pi i\nu}+\dfrac{1}{4}\left(\dfrac{\sin\nu}{\nu}\right)H_1^{(2)}(\nu)\right\}}{\left\{1-\left(\dfrac{\sin\nu}{\nu}\right)^2\right\}\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\right]$$

$$= \frac{\rho V^2 c^2}{\{H_1^{(2)}(\nu)+i\,H_0^{(2)}(\nu)\}}\left[\frac{1}{2i\nu}-\frac{1}{6}i\nu+\frac{1}{2}(i\nu)\left\{\gamma+\frac{i\pi}{2}+\log\frac{\nu}{2}\right\}+0(\nu^2)\right.$$

$$\left.+0\left(\nu^2\log\frac{\nu}{2}\right)\right]. \qquad (212)$$

The expressions (210), (211) and (212) are estimates for M which is given accurately by formula (189). The expressions H_{10} and $H_{10}^{(c)}$ differ from M by a term of $0(\nu^2)$ in the square brackets, whereas the expression H_s differs from M by a term of $0(\nu)$ in the square brackets.

7. *Concluding Remarks.*

A method, based on the variational principle of Flax, for obtaining approximately the generalised airforces on an oscillating wing in a stream flow has been described. Attention has been confined mainly to subsonic flow, where results are shown to reduce to those of normal lifting-surface theory using a collocation procedure when certain further approximations are made.

A method based on least squares has been suggested[3] as a means of obtaining greater accuracy than that which is obtained with the collocation procedure. It is asserted here that the least squares approach is not, in general, as accurate as the procedure based on the variational principle.

Examples in two-dimensional incompressible flow have been given, which show clearly that the results obtained from the variational principle are in general superior to those obtained from the least squares procedure. Case (1.ii) is an exception.

The results from the variational principle are only marginally more accurate than those from the collocation procedure, the order of error being the same in each case considered with the exception of Case (2.iii) where the result from the variational approach is exact.

In two-dimensional incompressible subsonic flow the order of error for both the results from the variational procedure and from the collocation procedure is the same in all cases if the same approximations (125) and (126) to the loading are used.

In three-dimensional flow, possibly the errors from the results using the variational procedure and from the results using the collocation procedure are of different orders of magnitude. This can only be determined by means of extensive calculations.

Since the present Report was written Stark has published a paper (Ref. 8) which covers similar ground.

LIST OF SYMBOLS

c	Chord length of a two-dimensional wing
$c(y)$	Wing chord at spanwise position y
$E(x)$	Function having the properties (27)
$G_v^{(m)}$	Spanwise weighting function
$H_u^{(n)}$	Chordwise weighting function
H	Integral defined in equation (8)
H_0	Stationary value of H
$j_r(\xi_o)$	Defined in equation (80) or (127)
$k_s(\eta_o)$	Defined in equation (81)
$K\left(x, y, \frac{\omega}{v}, M\right)$	Kernel function
$l(x,y)$	Loading function in the direct flow
$\bar{l}(x,y)$	Loading function in the reverse flow
$L(x,y)$	Loading function in the direct flow
$L(x,y)$	Loading function in the reverse flow
$L_1(x,y)$	Loading function in the direct flow
$\bar{L}_1(x,y)$	Loading function in the reverse flow
$l_r(\xi)$	Polynomial of degree r in ξ satisfying equation (30)
M	Mach number
Q_j	Generalised airforce (*see* equation (11))
s	Semi-span of wing
S	Wing area
t	Time
V	Speed of undisturbed flow
$w(x,y)$	Upwash function in the direct flow
$\bar{w}(x,y)$	Upwash function in the reverse flow
$W(x,y)$	Upwash function in the direct flow
$\bar{W}(x,y)$	Upwash function in the reverse flow
$W_1(x,y)$	Upwash function in the direct flow
$\bar{W}_1(x,y)$	Upwash function in the reverse flow
x,y,z	Cartesian co-ordinates
x_o, y_o	Cartesian co-ordinates
$X_L(y)$	Abscissa of leading edge at spanwise position y
$Z(x,y)$	Displacement function

$\bar{Z}(x,y)$	Displacement function
$Z_j(x,y)$	Displacement function
$\gamma = 0{\cdot}57721\ldots$	Euler's constant
$\gamma_s(\eta)$	Polynomial of degree s in η satisfying equation (31)
$\delta(x,y)$	Defined in equation (14)
$\bar{\delta}(x,y)$	Defined in equation (15)
$\varepsilon(x,y)$	Defined in equation (12)
$\bar{\varepsilon}(x,y)$	Defined in equation (13)
η	Defined in equation (49)
η_o	Defined in equation (20)
η_v	Zeroes of $\gamma_m(\eta)$
$\nu = \dfrac{\omega c}{2V}$	Frequency parameter
ξ	Defined in equation (48)
ξ_o	Defined in equation (19)
ξ_u	Zeroes of $l_n(\xi)$
ρ_o	Density of fluid in the undisturbed flow
ω	Circular frequency
*	Denotes conjugate of a complex quantity

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