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Asymptotic Solutions of Linear Stationary Integro-Differential Equations

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Summary.
A method is described for finding asymptotic solutions, valid for large time, of a certain class of linear, integro-differential equations which typically arise in the description of systems exhibiting heredity. The solutions are developed from a knowledge of the steady-state behaviour of the system, or more particularly of a part of the system, under harmonic excitation. The analysis is carried through for the particular case of a flexible aeroplane disturbed from steady, rectilinear flight but the main stream of the argument may be pursued independently of this physical framework.

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1. *Introduction.*

We consider a continuous, flexible aeroplane in flight through air which may itself be in motion. In the usual formulation of this problem the displacement field of the aeroplane structure relative to a set of body axes is restricted to lie in a suitably chosen linear function space of finite dimension and the 'best' solution is found by application of a generalised orthogonality relation (Ref. 1). The aerodynamic forces acting on the aeroplane exhibit heredity due to the influence of vorticity shed into the wake at earlier instants of time.

Relative to an equilibrium state of steady, trimmed, rectilinear flight the equation of first variation takes the canonical form

$$\frac{d\mathbf{x}}{dt}-\mathbf{M}\,\mathbf{x} = \int_{-\infty}^{t} \mathbf{A}\,(t-\tau)\,\mathbf{x}\,(\tau)\,d\tau+\mathbf{f}(t) \tag{1.1}$$

where $\mathbf{x}(t)$ is an n-dimensional state vector, $\mathbf{M}$ is a constant (nxn) 'mechanical' matrix and $\mathbf{A}(t)$ is an (nxn) 'aerodynamic' matrix; $\mathbf{f}(t)$ is a specified vector function of time. The equilibrium state is represented by $\mathbf{x}(t) \equiv \mathbf{0}$.

The left-hand side of equation (1.1) represents the mechanical system which is assumed to have no heredity effects; the first term on the right-hand side represents the aerodynamic forces which are dependent on the motion and takes the most general form of a stationary (invariant to time translation) linear functional. The vector $\mathbf{f}(t)$ represents the forces applied to the system by deflection of the control surfaces or by action of the air (gusts). The matrices $\mathbf{M}$ and $\mathbf{A}(t)$ and the vector $\mathbf{f}(t)$ are implicitly dependent on the equilibrium state under consideration. It is assumed that the terms of equation (1.1) are dimensionless: in particular the time, t, will be understood to be the measure of the unit c/V where c is a suitable reference length and V is the equilibrium flight speed.

Since the system is causal $\mathbf{A}(t) \equiv \mathbf{0}$ for $t < 0$ so that the upper limit on the integral in (1.1) may be replaced by $+\infty$ and we may rewrite the system equations (1.1) as

$$\frac{d\mathbf{x}}{dt}-\mathbf{M}\,\mathbf{x} = \mathbf{A} * \mathbf{x}+\mathbf{f} \tag{1.2}$$

where the convolution operator $*$ is defined for the functions $\varphi(t)$, $\psi(t)$ by,

$$\varphi * \psi = \int_{-\infty}^{t} \varphi\,(t-\tau)\,\psi\,(\tau)\,d\tau = \int_{-\infty}^{\infty} \psi\,(t-\tau)\,\varphi\,(\tau)\,d\tau = \psi * \varphi\,. \tag{1.3}$$

For equation (1.1) the initial value problem involves the specification of an initial trajectory restricted to a suitable class of functions[2]. This initial value problem will be by-passed here by assuming that the system is quiescent until time $t = 0$ and is then set in motion by external forces, that is,

$$\mathbf{x}(t) = \mathbf{f}(t) \equiv \mathbf{0} \text{ for } t < 0\,.$$

In the analysis that follows many purely technical difficulties can be simplified by interpreting $\mathbf{x}(t)$, $\mathbf{A}(t)$ and $\mathbf{f}(t)$ as distributions whose supports are bounded on the left at $t = 0$[3]. The convolution of such distributions is also in general a distribution with support bounded on the left and a convolution algebra exists having the delta functional as a unit element. It will be understood that a derivative with respect to time is interpreted (when necessary) as a distributional derivative and that the Laplace transform of the function $f(t)$ defined by

$$\bar{f}(p) = \int_0^\infty f(t)\, e^{-pt}\, dt\,, \quad p = \mu + i\nu \tag{1.4}$$

is extended to include right-sided distributions.

We define $\mathbf{X}(t)$ to be the matrix solution of

$$\frac{d\mathbf{X}}{dt} - \mathbf{M}\,\mathbf{X} = \mathbf{A} * \mathbf{X} + \mathbf{I}\,\delta(t)\,, \quad t \geqslant 0 \tag{1.5}$$

where $\delta(t)$ is the delta functional. Convoluting both sides with $\mathbf{f}(t)$ gives

$$\frac{d}{dt}(\mathbf{X} * \mathbf{f}) - \mathbf{M}\,(\mathbf{X} * \mathbf{f}) = \mathbf{A} * (\mathbf{X} * \mathbf{f}) + \mathbf{f}$$

showing that the solution of equation (1.2) is given in terms of $\mathbf{X}(t)$ by the convolution

$$\mathbf{x} = \mathbf{X} * \mathbf{f}\ . \tag{1.6}$$

It is not intended to study equation (1.2) when the form of $\mathbf{A}(t)$ is such that the elements of its Laplace transform $\overline{\mathbf{A}}(p)$ are rational functions of p: for in that case a finite number of operations consisting only of differentiation and multiplication by a real number is sufficient to bring equation (1.2) to the form

$$\frac{d\mathbf{y}}{dt} - \mathbf{C}\,\mathbf{y} = \mathbf{g}(t) \tag{1.7}$$

where the dimension of $\mathbf{y}$, $\mathbf{g}$ and $\mathbf{C}$ is not less than n. We shall, for convenience, refer to such systems as being 'instantaneous'. If the elements of $\overline{\mathbf{A}}(p)$ are rational functions of p then the elements of $\overline{\mathbf{A}}(t)$ will in general consist of a finite sum containing,

(1) the delta functional and its derivatives,

(2) exponential functions, and

(3) the product of exponential functions with polynomials.

Such forms have often been assumed as approximations to $\mathbf{A}(t)$ in applications[4,5].

For the instantaneous system (1.7) the principal matrix solution[6] has a most useful spectral representation which involves the characteristic vectors, rows and roots of the system matrix $\mathbf{C}$; these are finite in number and equal to the dimension of the system. It is the purpose of the following analysis to develop an analogue of the spectral form of the principal matrix solution for the system (1.2) so far as this is possible.

2. *Nature of the Aerodynamic Forces.*

2.1. *General.*

The aerodynamic forces acting on the deforming aeroplane are obviously extremely complex functions of the motion but when these forces are linearly dependent on the state variables as in the equation of first variation it is possible, at least to a large extent, to separate out those forces which are essentially dependent on real flow effects from those which may be estimated on the basis of an inviscid flow model[1].

Effects of unsteadiness can only be included in the latter group of forces and even in this case almost all known results apply to nearly-planar lifting surfaces. In this section we discuss only those forces which are derived from unsteady, finite wing theory as conventionally understood[7]. Two-dimensional flows are omitted from the discussion but there would be no difficulty in including such results in the general analysis.

The starting point is the (improper) integral equation,

$$v(x, y, t) = \int\int_S l(v; x', y', t) * K(x-x', y-y', t)\, dx'\, dy' \tag{2.1}$$

which relates the loading $l(v;\, x, y, t)$ on the wing to the prescribed downwash $v(x,\, y,\, t)$: the function $K(x,\, y,\, t)$ which depends implicitly on the Mach number of the trim state is termed the kernel function and the integration is (formally) over the wing planform. The prescribed downwash $v(x, y, t)$ is, in general, the sum of two contributions: one is due to the motion of the aeroplane while the other is due to motion of the air itself. In the linear theory, the first is computed as if the air were at rest and the second as if the aeroplane maintained the steady rectilinear motion of its equilibrium state.

In terms of the aeroplane motion the downwash over the wing is given by[8],

$$v(x, y, t) = w - qx + py - \left(\frac{\partial h}{\partial x} - \frac{\partial h}{\partial t}\right) \tag{2.2}$$

where $w(t)$, $q(t)$, $p(t)$ are, in the usual notation, the linear and angular velocities of the aeroplane and $h(x,\, y,\, t)$ is the elastic displacement normal to the $(x,\, y)$ plane of the body axes. Since the displacement field is approximated in a finite dimensional linear function space – $(\partial h/\partial x - \partial h/\partial t)$ will be the sum of a finite number of terms like $-\partial h_i(x,y)/\partial x\; x_i(t)$ and $h_j(x,y)x_j(t)$. Thus, so far as the motion is concerned the archetypal downwash field is of the separable form $v(x,\, y)\, \eta(t)$ and all such fields may be considered to be generated from the impulsive downwash $v(x,y)\, \delta(t)$.

Let the air itself have a vertical motion only which is described relative to space-fixed axes: when such gusts are stationary and vary only in the flight direction the downwash field is of the form $v_g(x+t)$ where the function v_g is given. This class of downwash field may be considered to be generated by the travelling impulse $\delta(x-a+t)$ where $x = a$ is the foremost point of the wing; thus the wing first encounters the spatially impulsive gust at $t = 0$.

In practice the kernel function is known only as a steady-state frequency response function $\tilde{K}(x,\, y;\, iv)$ relating the downwash $v(x,y)e^{ivt}$ to the loading $\tilde{l}(v;x,y;iv)e^{ivt}$;

$$v(x,y) = \int\int_S \tilde{l}(v; x', y'; iv)\, \tilde{K}\,(x-x', y-y'; iv)\, dx'\, dy'. \tag{2.3}$$

Computer programmes are available[10,11] which will invert equation (2.3) to give a weighted integral of the loading (a generalised force) due to a prescribed harmonic downwash: let $\tilde{Q}(v,\zeta\,;iv)e^{ivt}$ be the generalised force due to $v(x,y)e^{ivt}$ for the weighting function $\zeta(x,y)$. The corresponding convolution inverse of equation (2.1) is the generalised force $Q(v,\zeta\,;t)$ due to the downwash $v(x,y)\delta(t)$; that is, $Q(v,\zeta\,;t)$ is the generalised force generated by the impulsive downwash field $v(x,y)\delta(t)$. If $\bar{Q}(v,\zeta\,;p)$ is the Laplace transform of $Q(v,\zeta\,;t)$ it is shown in Appendix I that $\bar{Q}(v,\zeta\,;p)$ is the analytic continuation of $\tilde{Q}(v,\zeta\,;iv)$ throughout the complex plane cut where necessary to ensure that $\bar{Q}(v,\zeta\,;p)$ is single-valued.

In the case of gusts let $P(\zeta\,;t)$ be the convolution inverse of equation (2.1) for the downwash field $\delta(x-a+t)$ so that $P(\zeta\,;t)$ is the generalised force generated by the wing traversing a spatially impulsive gust. The corresponding frequency response function $\tilde{P}(\zeta\,;iv)$ is the weighted integral of the loading $\tilde{l}_g\,(x,y\,;iv)$ where $\tilde{l}_g$ satisfies

$$1 = \int\int_S \bar{l}_g(x',y';iv)e^{iv(x-a)}\,\tilde{K}(x-x', y-y'; iv)\,dx'\,dy'. \tag{2.4}$$

The Laplace transform of $P(\zeta;t)$ is similarly the analytic continuation of $\tilde{P}(\zeta;iv)$ into the complex plane.

Since $\bar{Q}(v,\zeta;p)$ and $\bar{P}(\zeta;p)$ are the Laplace transforms of real functions of t

$$\bar{Q}(v,\zeta;p^*) = \bar{Q}^*(v,\zeta;p)$$

and

$$\bar{P}(\zeta;p^*) = \bar{P}^*(\zeta;p) \tag{2.5}$$

where the superscript * denotes complex conjugate. It follows that the real parts of $\tilde{Q}$ and $\tilde{P}$ are even and the imaginary parts odd functions of v.

Throughout the paper and in the following results which are derived from the initial and final value theorems[12] σ is a positive real number;

$$\lim_{t\to\infty} Q(t)*1(t) = \lim_{\sigma\to+0} \bar{Q}(\sigma) \tag{2.6a}$$

$$\lim_{t\to+0} Q(t)*1(t) = \lim_{\sigma\to\infty} \bar{Q}(\sigma) \tag{2.6b}$$

provided the left-hand sides exist, with similar results for $P(t)$.

The elements of the aerodynamic matrix $\mathbf{A}(t)$ are the generalised forces $Q(v,\zeta;t)$ corresponding to the downwash fields associated with the elements of the state vector $\mathbf{x}(t)$ while the generalised forces $P(\zeta;t)$ will, in general, contribute to the vector $\mathbf{f}(t)$.

The nature of the aerodynamic forces is so different in the subsonic and supersonic regimes that each case will be dealt with separately.

2.2. *Wings in Subsonic Flow.*

Let $\bar{K}(x,y;p)$, $\bar{l}(v;x,y;p)$ be the analytic continuations of $\tilde{K}(x,y;iv)$, $\bar{l}(v;x,y;iv)$ into the complex plane: then equation (2.3) shows that, for separable downwash fields, $\bar{l}(v;x,y;p)$ is the operational inverse of $\bar{K}(x,y;p)$. From the known forms of $\tilde{K}(x,y;iv)$ for subsonic flow[10] it may readily be deduced that the analytic continuation $\bar{K}(x,y;p)$ has a logarithmic branch point at the origin and has no poles in the finite part of the plane; that is, $\bar{K}(x,y;p)$ is holomorphic throughout the complex plane cut along the negative real axis. In addition, by examining the change in arg $\bar{K}(p)$ over a suitable closed contour in the cut plane it may be shown that $\bar{K}(p)$ has no zeros in the finite part of the plane. Thus $\bar{l}(v;x,y;p)$ and the associated generalised force $\bar{Q}(v,\zeta;p)$ may be assumed to be holomorphic throughout the complex plane cut along the negative real axis. Since $\bar{Q}(v,\zeta;p)$ is holomorphic only in the open half-plane $\mathcal{R}e\, p > 0$, $\tilde{Q}(v,\zeta;iv)$ must be regarded as the limit of $\bar{Q}(v,\zeta;p)$ as $\mathcal{R}e\, p \to +0$; in fact $\tilde{Q}(v,\zeta;iv)$ fails to be analytic only at $v = 0$.

For the case of gusts equation (2.4) shows that $e^{p(x-a)}\bar{l}_g(x,y;p)$ is the operational inverse of $e^{px}\bar{K}(x,y;p)$. The presence of the entire function e^{px} does not alter the conclusions already arrived at so that it may be assumed that also $\bar{P}(\zeta;p)$ is holomorphic throughout the cut plane.

The generalised force $Q(v,\zeta;t)*\eta(t)$ on the wing due to the prescribed downwash $v(x,y)\,\eta(t)$ may be found by inversion of the transform $\bar{Q}(v,\zeta;p)\,\bar{\eta}(p)$.

Let $\bar{f}(p)$ be holomorphic in the half-plane $\mathcal{R}e\, p > \mu_c$ and let

$$|\bar{f}(p)| = 0(|p|^{-\alpha}),\ |p| \to \infty\ ,\ \alpha \geqslant 1\ ;$$

then $\bar{f}(p)$ is the Laplace transform of a function $f(t)$ and

$$f(t) = \frac{1}{2\pi i} \int_L \bar{f}(p) e^{pt} \, dp \tag{2.7}$$

where the path L is a vertical line in the half-plane $\mathscr{R}e\, p > \mu_c$. Under these conditions the path L may be completed at infinity (to the left for $t > 0$) due allowance being made for branch cuts. If $\alpha = -m < 1$ this result cannot be applied directly and in this case $f(t)$ will in general be a distribution bounded on the left at $t = 0$.
But let

$$\bar{f}(p) = p^{m+1} \bar{g}(p) \tag{2.8}$$

then $g(t)$, the inverse of $\bar{g}(p)$, can be found using equation (2.7) and

$$f(t) = \frac{d^{m+1} g(t)}{dt^{m+1}} . \tag{2.9}$$

For example if $\eta(t) = 1(t)$, the unit step function, the 'indicial response' $Q(v,\zeta;t) * 1(t)$ is the inverse of $\bar{Q}(v,\zeta;p)/p$. It is known[7] that, for compressible flow, the initial overpressure generated by a step change in downwash is finite hence from (2.6b)

$$\bar{Q}(|p|)/p = 0(|p|^{-1}) , |p| \to \infty$$

and equation (2.7) may be used. Since the only pole is at the origin the path L may be deformed to the path CDEFG of Figure 1 giving

$$Q(v,\zeta;t) * 1(t) = \bar{Q}(0)\, 1(t) + \frac{1(t)}{2\pi i} \int^{\infty} \frac{1}{\sigma} \{\bar{Q}(\sigma e^{i\pi}) - \bar{Q}(\sigma e^{-i\pi})\} e^{-\sigma t} d\sigma$$

and using equation (2.5)

$$Q(v,\zeta;t) * 1(t) = \bar{Q}(0)\, 1(t) + \frac{1(t)}{\pi} \int_0^{\infty} \frac{1}{\sigma} \mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})\} e^{-\sigma t} \, d\sigma . \tag{2.10}$$

The 'impulsive response' $Q(v,\zeta;t)$, which cannot be obtained directly using equation (2.7), is given from equation (2.9) as

$$Q(v,\zeta;t) = \left[\bar{Q}(0) + \frac{1}{\pi} \int_0 \frac{1}{\sigma} \mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})\} \, d\sigma \right] \delta(t)$$

$$- \frac{1(t)}{\pi} \int_0^{\infty} \mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})\} e^{-\sigma t} \, d\sigma . \tag{2.11}$$

Equations (2.10) and (2.6b) show that the amplitude of the delta functional is in fact $\lim\limits_{\sigma \to \infty} \bar{Q}(\sigma)$.

An asymptotic expansion for $Q(v,\zeta;t) * 1(t)$ valid for large t may be obtained from the series representation for $\bar{Q}(v,\zeta;p)$; this takes the form [13]

$$\bar{Q}(p) = \sum_{n=0}^{\infty} a_n p^n + p^2 \ln p \sum_{k=0}^{\infty} b_k p^k , \; a_0 = \bar{Q}(0) \tag{2.12}$$

hence, upon substituting the series (2.12) into equation (2.10) and performing a term-by-term integration, we obtain the asymptotic representation

$$Q(v,\zeta;t) * 1(t) - \bar{Q}(0) \approx \frac{1}{t^2} \sum_{k=0}^{\infty} \frac{b_k (k+1)!(-)^k}{t^k}. \tag{2.13}$$

Similarly, equation (2.11) leads to

$$Q(v,\zeta;t) \approx -\frac{1}{t^3} \sum_{k=0}^{\infty} \frac{b_k (k+2)!(-)^k}{t^k}. \tag{2.14}$$

For incompressible flow a step change in downwash generates an impulsive pressure field associated with the apparent mass of the air: in this case $\bar{Q}(p)/p^2$ satisfies the condition for using equation (2.7). A term $\mathcal{M}\delta(t)$ will be added to equation (2.10) and a term $\mathcal{M}d\delta(t)/dt$ to equation (2.11) where $\mathcal{M}$ is the appropriate apparent mass.

Another case of interest is the generalised force due to the downwash,

$$v(x,y)\, 1(t)\, e^{\lambda t}, \quad |\arg \lambda| < \pi .$$

The path L, which passes to the right of max. $(0, \mathscr{Re}\ \lambda)$, when closed at infinity now encloses a simple pole at $p = \lambda$ and we obtain

$$Q(v,\zeta;t) * 1(t)\, e^{\lambda t} = \bar{Q}(v,\zeta;\lambda)\, 1(t)\, e^{\lambda t}$$

$$+\frac{1(t)}{\pi}\int_0^{\infty} \frac{\mathscr{Im}\{\bar{Q}(\sigma e^{i\pi})\}e^{-\sigma t}}{\sigma+\lambda}\, d\sigma . \tag{2.15}$$

An asymptotic expansion for $Q(v,\zeta;t) * 1(t)e^{\lambda t}$ valid for large $|\lambda|t$ in the sector $|\arg \lambda| < \pi$ may be obtained upon using the series (2.12) combined with repeated integration by parts [14]: alternatively the term $(\sigma+\lambda)^{-1}$ may be expanded as an ascending series in σ/λ followed by term-by-term integration. We have

$$Q(v,\zeta;t) * 1(t)e^{\lambda t} - \bar{Q}(v,\zeta;\lambda)\, 1(t)e^{\lambda t}$$

$$\approx \frac{1}{t^2} \sum_{k=0}^{\infty} \left[\frac{(k+2)!(-)^k}{(\lambda t)^{k+1}} \sum_{j=0}^{k} b_j \lambda^j \right]. \tag{2.16}$$

It follows from the relations (2.5) that, if $\lambda = \gamma + i\omega$

$$Q(v,\zeta;t) * 1(t)e^{\gamma t} \begin{pmatrix} \cos \omega t \\ \sin \omega t \end{pmatrix} = \begin{matrix} \mathscr{Re} \\ \mathscr{Im} \end{matrix} (Q(v,\zeta;t) * 1(t)e^{\lambda t}) .$$

Equation (2.16) shows that for $\mathscr{Re}\ \lambda \geq 0$ the generalised force due to an exponential (time) variation of downwash is eventually exponential; clearly the case of simple harmonic motion ($\mathscr{Re}\ \lambda = 0$) is just included since the remainder term in (2.16) decreases more slowly than any exponential with negative real part no matter how small. For simple harmonic motion $|\lambda| = \omega$ and the expansion shows that the effect of finite wake length depends not only on the number of chordlengths travelled (i.e. t) but also on the number of wavelengths travelled, $\omega t/2\pi$.

When $\lambda = -\gamma, \gamma > 0$ the path of integration along both sides of the cut is indented into the upper and lower half-planes by small semicircles centred at $p = -\gamma$: in the limit as the radii of the semicircles tend to zero we obtain,

$$Q(v,\zeta;t) * 1(t)e^{-\gamma t} = \mathscr{R}e\,[\bar{Q}(v,\zeta;\gamma e^{i\pi})]\,1(t)e^{-\gamma t}$$

$$+\frac{1(t)}{\pi}\mathscr{P}\int_0^\infty \frac{\mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})\}e^{-\sigma t}}{\sigma-\gamma}d\sigma \qquad (2.17)$$

where $\mathscr{P}$ denotes the Cauchy principal value. The asymptotic expansion is that of equation (2.16) with $\lambda = -\gamma$.

A series of parallel developments may be made for the generalised forces $P(\zeta;t)$ the main differences being that,

$$\lim_{t\to 0} P(\zeta;t) * 1(t) = 0\,.$$

Obviously,

$$\lim_{t\to\infty} \bar{P}(\zeta;t) * 1(t) = \lim_{t\to\infty} Q(1,\zeta;t) * 1(t)$$

or

$$P(\zeta;+0) = \bar{Q}(1,\zeta;+0)$$

hence, by analogy with equation (2.10),

$$\bar{Q}(1,\zeta;+0) = -\frac{1}{\pi}\int_0^\infty \frac{1}{\sigma}\mathscr{I}m\{\bar{P}(\sigma e^{i\pi})\}\,d\sigma\,.$$

Since

$$\bar{Q}(1,\zeta;+0)+\frac{1}{\pi}\int_0^\infty \frac{1}{\sigma}\mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})\}\,d\sigma = \bar{Q}(1,\zeta;\infty)$$

and

$$\bar{Q}(1,\zeta;+0)+\frac{1}{\pi}\int_0^\infty \frac{1}{\sigma}\mathscr{I}m\{\bar{P}(\sigma e^{i\pi})\}\,d\sigma = 0$$

then

$$\pi\,\bar{Q}(1,\zeta;\infty) = \int_0^\infty \mathscr{I}m\{\bar{Q}(\sigma e^{i\pi})-\bar{P}(\sigma e^{i\pi})\}\,\frac{d\sigma}{\sigma}\,.$$

2.3. *Wings in Supersonic Flow.*

By an examination of the frequency response function $\tilde{K}(x,y;i\zeta)$ for supersonic flow[10] it may be seen that its analytic continuation $\bar{K}(x,y;p)$ is an entire transcendental function having no zeros in the plane: consequently $\bar{l}(v;x,y;p)$ and the associated generalised forces $\bar{Q}(v,\zeta;p)$ and $\bar{P}(\zeta,p)$ are entire transcendental functions. This is merely a reflection of the fact that in supersonic flow the transient downwash at a point can affect conditions on the wing for only a finite time after its inception. Hence $Q(v,\zeta;t)$ is a function having a bounded support and μ_c, the abscissa of convergence of its Laplace transform, is $-\infty$. Choosing the reference length c to be the maximum extent of the wing in the direction of flight then $Q(v,\zeta;t)*1(t)$ reaches its steady state at $t = M/M-1)$ where M is the flight Mach number. The initial overpressure for a step change in downwash is finite as in a subsonic compressible flow. For $t \geqslant M/(M-1)$

$$Q(v,\zeta;t)*1(t)e^{\lambda t} = \bar{Q}(v,\zeta;\lambda)e^{\lambda t} \tag{2.18}$$

for all $\lambda = \gamma + i\omega$.

Since $\bar{Q}(v,\zeta;p)$ is an entire function it has a convergent power series expansion about any point in the plane: in particular, for the origin we may write,

$$\bar{Q}(p) = \sum_{n=0}^{\infty} c_n p^n, c_0 = \bar{Q}(0). \tag{2.19}$$

Similar considerations apply to the generalised force $P(\zeta,t)$ and its transform $\bar{P}(\zeta;p)$.

3. *Asymptotic Solution of the System Equation.*

The operational form of equation (1.5) is

$$[p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)]\bar{\mathbf{X}} = \mathbf{I} \tag{3.1}$$

where $\bar{\mathbf{A}}(p)$, $\bar{\mathbf{X}}(p)$ are the Laplace transforms of $\mathbf{A}(t)$, $\mathbf{X}(t)$ respectively. Thus,

$$\bar{\mathbf{X}}(p) = [p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)]^{-1} \tag{3.2}$$

and from equation (2.7)

$$\mathbf{X}(t) = \frac{1}{2\pi i}\int_L [p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)]^{-1}e^{pt}\,dp \tag{3.3}$$

where the path L passes to the right of all the singularities of $\bar{\mathbf{X}}(p)$. The characteristic equation

$$\det[p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)] = 0$$

will in general have an infinite number of roots and hence $\bar{\mathbf{X}}(p)$ will have an infinite number of poles. The path L cannot simply be closed to the left at infinity as previously since it is not possible to find a contour of radius R for which $\|\bar{\mathbf{X}}(p)\| = 0(|p|^{-1})$ for all $|p| > R$. We shall assume that $\bar{\mathbf{X}}(p)$ is meromorphic in the whole plane (cut if necessary) including the point at infinity; that is, there are only a finite number of poles of $\bar{\mathbf{X}}(p)$ in any finite region of the plane. In this case it is possible to estimate $\mathbf{X}(t)$ by integration round an increasing sequence of contours which thread between the poles of $\bar{\mathbf{X}}(p)$. We shall deal explicitly with the subsonic case since the supersonic case is a simpler version of this treatment.

Consider the contour ABCDEFGA of Figure 2 where $\mu_M < \mu_C < \mu_L$, $\mu_C > 0$ being the abscissa of convergence of $\bar{\mathbf{X}}(t)$. Then

$$\int_{AB\ldots FGA} \bar{\mathbf{X}}(p)e^{pt}\,dp = 2\pi i \sum_n \mathbf{R}_n \tag{3.4}$$

where the $\mathbf{R}_n$ are the residues of $\overline{\mathbf{X}}(p)e^{pt}$ at its poles which lie within the contour. We now consider the various contributions to the integral from the segments of the contour as $\nu \to \infty$. Provided the segments BC, GA pass between the poles of $\overline{\mathbf{X}}(p)$ as $\nu \to \infty$ (their position need not be varied continuously) then $\|\overline{\mathbf{X}}(p)\| = 0(|p|^{-1})$ as $|p| \to \infty$ (equation (2.6b)) and the contribution from these segments vanishes (Jordan's Lemma). Since the integral along AB becomes $\mathbf{X}(t)$ we have, for $t > 0$.

$$\mathbf{X}(t) = \sum \mathbf{R}_n + \frac{1}{\pi}\int_0^{\sigma_M} \mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi})\, e^{-\sigma t}\, d\sigma \tag{3.5}$$

$$+\frac{e^{\mu_M t}}{\pi}\int_0^{\infty}[\mathscr{R}e\,\{\overline{\mathbf{X}}(\mu_M+i\nu)\}\cos \nu t - \mathscr{I}m\,\{\overline{\mathbf{X}}(\mu_M+i\nu)\}\sin \nu t]\, d\nu$$

where the $\mathbf{R}_n$ are the residues of $\overline{\mathbf{X}}(p)e^{pt}$ at its poles in the strip $\mu_M < \mu < \mu_L$, and $\sigma_M = -\mu_M > 0$. If we assume that there are a finite number of simple poles in this strip, p_n, $n = 1 \ldots\ldots M$ and that μ_M is chosen so that the path M passes between the poles of $\overline{\mathbf{X}}(p)$ we may write,

$$\mathbf{X}(t) = \sum_{n=1}^{M} \lim_{p \to p_n} [(p-p_n)\,\overline{\mathbf{X}}(p)]\, e^{p_n t} + \frac{1}{\pi}\int_0^{\sigma_M} \mathscr{I}m\,\{\overline{\mathbf{X}}(\sigma e^{i\pi})\}\, e^{-\sigma t}\, d\sigma + 0\,(e^{\mu_M t}) \tag{3.6}$$

since the integral along M is bounded. An asymptotic expansion in inverse powers of t may be developed for the second term as was done in Section 2 as the validity of that technique does not depend on the upper limit being $+\infty$[14].

Thus, assuming that the poles of $\overline{\mathbf{X}}(p)$ can be determined in order of decreasing real part equation (3.6) may be considered as an asymptotic solution of (1.5) valid for large t: in fact such a solution may be of acceptable accuracy for values of t which are not large.

The form of solution in the supersonic case will consist only of the first and third terms of equation (3.6).

The coefficient matrices of the exponentials in equation (3.6) may be expressed in terms of the associated characteristic rows and vectors of $[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]$. Let $\mathbf{u}_n{}^T$, $\mathbf{v}_n$ be the characteristic row and vector associated with the simple root p_n, that is

$$\mathbf{u}_n{}^T[p_n\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p_n)] = \mathbf{0} \tag{3.7a}$$

and

$$[p_n\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p_n)]\mathbf{v}_n = \mathbf{0}, \tag{3.7b}$$

where $\mathbf{u}_n$, $\mathbf{v}_n$ are assumed to be normalised so that

$$\mathbf{u}_n{}^T\, \mathbf{B}_n\, \mathbf{v}_n = 1$$

for a given, non-singular square matrix $\mathbf{B}_n$. It is known that[15]

$$\mathrm{Adj}\,[p_n\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p_n)] = a_n\, \mathbf{v}_n\, \mathbf{u}_n{}^T \tag{3.8}$$

where a_n is a constant. Consider the identity

$$[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]\,\mathrm{Adj}\,[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)] = \det\,[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]\,\mathbf{I};$$

differentiating with respect to p, premultiplying by $\mathrm{Adj}\,[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]$ and setting $p = p_n$ gives, upon using (3.8)

$$\mathbf{v}_n\, \mathbf{u}_n{}^T\, a_n \left[\mathbf{I} - \frac{d\overline{\mathbf{A}}(p)}{dp}\right]_{p=p_n} \mathbf{v}_n\, \mathbf{u}_n{}^T = \mathbf{v}_n\, \mathbf{u}_n{}^T \left(\frac{d}{dp}\det\,[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]_{p=p_n}\right)$$

whence

$$\left(\frac{d}{dp}\det\,[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]_{p=p_n}\right) = \mathbf{u}_n{}^T\, a_n \left[\mathbf{I} - \frac{d\overline{\mathbf{A}}(p)}{dp}\right]_{p=p_n} \mathbf{v}_n\,. \tag{3.9}$$

The coefficient of $e^{p_n t}$ in equation (3.6) is

$$\lim_{p \to p_n} [(p-p_n)\overline{\mathbf{X}}(p)] = \lim_{p \to p_n} \left[\frac{(p-p_n)\,\mathrm{Adj}\,[p\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p)]}{\det\,[p\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p)]} \right]$$

$$= \frac{\mathrm{Adj}\,[p_n\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p_n)]}{\left(\frac{d}{dp} \det\,[p\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p)]_{p=p_n} \right)}$$

and upon using equations (3.8) and (3.9) and taking

$$\mathbf{B}_n = \left[\mathbf{I} - \frac{d\overline{\mathbf{A}}(p)}{dp} \right]_{p=p_n} \tag{3.10}$$

we have,

$$\lim_{p \to p_n} [(p-p_n)\overline{\mathbf{X}}(p)] = \mathbf{v}_n \mathbf{u}_n{}^T \tag{3.11}$$

In the second term of equation (3.6) we write

$$\mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi})\} = \mathscr{I}m\{[p\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p)]^{-1}\}_{p=\sigma e^{i\pi}}$$

$$= -[\mathbf{M}-\overline{\mathbf{A}}(0)]^{-1}\,\mathscr{I}m \left[\mathbf{I} + [\mathbf{M}-\overline{\mathbf{A}}(0)]^{-1} \left[\sigma I + [\overline{\mathbf{A}}(\sigma e^{i\pi}) - \overline{\mathbf{A}}(0)] \right] \right]^{-1} \tag{3.12}$$

and expand the matrix inverse in ascending powers of

$$[\mathbf{M}-\overline{\mathbf{A}}(0)]^{-1} \left[\sigma\mathbf{I} + [\overline{\mathbf{A}}(\sigma e^{i\pi}) - \overline{\mathbf{A}}(0)] \right] :$$

together with a matrix expansion of $\overline{\mathbf{A}}(p)$ analogous to equation (2.12) this yields a suitable form for an asymptotic series for this term. If the series (3.12) is

$$\mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi})\} = \mathbf{C}_0\,\sigma^2 + \mathbf{C}_1\,\sigma^3 + O\,(\sigma^4 ln\sigma)$$

equation (3.6) can finally be written,

$$\overline{\mathbf{X}}(t) = \sum_{n=1}^{M} \mathbf{v}_n \mathbf{u}_n{}^T e^{p_n t} + \frac{1}{\pi} \left(\mathbf{C}_0 \frac{2}{t^3} + \mathbf{C}_1 \frac{6}{t^4} \right) + O\left(\frac{lnt}{t^5} \right) \tag{3.13}$$

When $\det[\mathbf{M}+\overline{\mathbf{A}}(0)] = 0$, that is the system has limiting static stability, the contour of Figure 2 is deformed around the origin and the solution (3.13) is modified to,

$$\overline{\mathbf{X}}(t) = \mathbf{v}_0 \mathbf{u}_0{}^T + \sum_{n=1}^{M} \mathbf{v}_n \mathbf{u}_n{}^T e^{p_n t} + \frac{1}{\pi} \left(\mathbf{D}_0 \frac{1}{t} + \mathbf{D}_1 \frac{1}{t^2} \right) + O\left(\frac{lnt}{t^3} \right)$$

where $\mathbf{v}_0$, $\mathbf{u}_0{}'$ are the characteristic vector and row associated with the root $p = 0$ and

$$\mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi})\} = \mathscr{I}m \left\{ \mathrm{Adj}\left[\mathbf{I} + \frac{\mathbf{M}+\overline{\mathbf{A}}(\sigma e^{i\pi})}{\sigma} \right] \times \left(1 - \frac{\Delta(\sigma e^{i\pi})/\sigma - a_0}{a_0} \right)^{-1} \right\}$$

$$= \mathbf{D}_0 + \mathbf{D}_1\sigma + O\,(\sigma^2 ln\sigma)\,.$$

Here, $\Delta(p) = \det[p\mathbf{I}-\mathbf{M}-\overline{\mathbf{A}}(p)]$ and a_0 is defined by equations (3.8) and (3.10).

The exponential part of equation (3.13) is the analogue of the spectral solution for an instantaneous system. The main difference between this part of the solution and that for an instantaneous system is that no biorthogonality relation exists for the rows and vectors $\mathbf{u}_n{}^T$, $\mathbf{v}_n$. Clearly, since the number of characteristic rows and vectors exceeds their dimension they cannot all be linearly independent and the usual reduction of the system matrix to diagonal form (for unrepeated roots) does not apply.

It should be noted that, because of the relations (2.5) the characteristic roots, rows and vectors will occur in complex conjugate pairs. The roots may also be real but whereas in supersonic flight they may be positive or negative, in subsonic flight negative real roots cannot occur because $\overline{\mathbf{X}}(p)$ is not single-valued on the negative real axis. For let,

$$\Delta(p) = \Delta_1(\mu, \nu) + i\Delta_2(\mu, \nu) = 0$$

be the characteristic equation, then if

$$\mu = -\sigma \text{ is a root,}$$

$$\Delta_1(-\sigma, 0) = 0$$

and

$$\Delta_2(-\sigma, 0) = 0\,.$$

But since $\Delta^*(p) = \Delta(p^*)$, $\Delta_2(-\sigma, 0)$ cannot be zero unless $\Delta(p)$ is single-valued in the neighbourhood of $p = -\sigma$: this is a contradiction in the subsonic case.

For the input function $\mathbf{f}(t)$ the solution is given from equation (1.6) as,

$$\mathbf{x}(t) = \sum_{n=1}^{M} \mathbf{v}_n \mathbf{u}_n^T e^{p_n t} * \mathbf{g}(t) + \frac{1}{\pi} \int_0^{\sigma_M} \mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi})\} e^{-\sigma t} d\sigma * \mathbf{g}(t)$$

$$+ 0\,(e^{\mu_M t}) \qquad (3.14)$$

where $\mathbf{f}(t) - \mathbf{g}(t) = 0\,(e^{\mu_M t})$. That is, $\mathbf{g}(t)$ is that part of the input $\mathbf{f}(t)$ which decreases in amplitude less rapidly than $e^{\mu_M t}$, $\mu_M < 0$. Alternatively, if $\mathbf{f}(p)$ is the Laplace transform of $\mathbf{f}(t)$

$$\mathbf{x}(t) = \frac{1}{2\pi i} \int_L [p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)]^{-1} \overline{\mathbf{f}}(p)\, e^{pt}\, dp$$

giving the solution

$$\mathbf{x}(t) = \sum_{n=1}^{M} \mathbf{v}_n \mathbf{u}_n^T \overline{\mathbf{f}}(p_n)\, e^{p_n t} + \frac{1}{\pi} \int_0^{\sigma_M} \mathscr{I}m\{\overline{\mathbf{X}}(\sigma e^{i\pi}) \overline{\mathbf{f}}(\sigma e^{i\pi})\} e^{-\sigma t}\, d\sigma$$

$$+ \sum_{l=1}^{N} \overline{\mathbf{X}}(r_l) \lim_{p \to r_l} [(p - r_l)\overline{\mathbf{f}}(p)] e^{r_l t} + 0(e^{\mu_M t}) \qquad (3.15)$$

where r_l, $l = 1 \ldots\ldots\ldots. N$ are the poles of $\overline{\mathbf{f}}(p)$ in the strip $\mu_M < \mu < \mu_L$, assumed simple.

4. *An Approximation to the Roots of the Characteristic Equation.*

As has already been mentioned in Section 2 there exist programmed numerical techniques for generating the frequency response functions $\tilde{Q}(\nu, \zeta; i\nu)$ for both subsonic and supersonic flow, at least for frequency parameters which are not too high. In practice the computed results are invariably given in terms of classical flutter derivatives which are not directly related to downwash but to displacement of the aerofoil surface; their relation to the generalised forces $Q(v,\zeta;t)$ is given in Appendix II. The generalised frequency response functions will be considered to be the basic data for the computation of the solution (3.6).

Since the characteristic roots in (3.6) are supposed to be determined in order of decreasing real part the approximation is based on a power series expansion of $\bar{Q}(v,\zeta;p)$ about points on the imaginary axis. The solution (3.6) will be supposed to include explicitly only those roots which can be determined with acceptable accuracy in this way. In practice the part of the solution of interest contains only those roots of det $[p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)]=0$ which are closely associated with the mechanical system and the rigid aeroplane; these are equal in number to the dimension of the state vector. Fortunately, except at very low speeds the (reduced) decay rates for aero-mechanical systems are usually small: a limit to the application of the technique described here would probably be a decay rate corresponding to about three chord-lengths to half-amplitude but any such estimate depends on the equilibrium flight Mach number and the aspect ratio of the wing.

In the case of supersonic flow $\bar{Q}(v,\zeta;p)$ has a convergent power series expansion about any point in the plane: for subsonic flow a power series expansion about the point p_0 will be valid only within the circle of radius $\mathscr{I}m\{p_0\}$ although since $\bar{Q}(p)$ is holomorphic in the cut plane an analytic continuation may be used to approach the negative real axis. An expansion about the origin takes the form (2.12).

The power series expansion of $\bar{Q}(p)$ about p_0 is

$$\bar{Q}(p)=\bar{Q}(p_0)+\left.\frac{d\bar{Q}}{dp}\right|_{p=p_0}(p-p_0)+\left.\frac{d^2\bar{Q}}{dp^2}\right|_{p=p_0}\frac{(p-p_0)^2}{2!}+\ldots..$$

and if, in particular, $p_0=i\omega$, a point on the imaginary axis,

$$\bar{Q}(p)=\bar{Q}(i\omega)+\left.\frac{d\bar{Q}}{dp}\right|_{p=i\omega}(p-i\omega)+\left.\frac{d^2\bar{Q}}{dp^2}\right|_{p=i\omega}\frac{(p-i\omega)^2}{2!}+\ldots.. \quad (4.1)$$

Let

$$\bar{Q}(p)=\bar{Q}'(\mu,\nu)+i\bar{Q}''(\mu,\nu)$$

then

$$\frac{d\bar{Q}(p)}{dp}=-i\frac{\partial\bar{Q}'}{\partial\nu}+\frac{\partial\bar{Q}''}{\partial\nu}. \quad (4.2)$$

We may write the frequency response function $\tilde{Q}(i\nu)$ as

$$\tilde{Q}(i\nu)=\tilde{Q}'(\nu)+i\nu\,\tilde{Q}''(\nu) \quad (4.3)$$

and using the identity between $\tilde{Q}(i\nu)$ and $\left.\bar{Q}(p)\right|_{p=i\nu}$,

$$\left.\frac{d\bar{Q}(p)}{dp}\right|_{p=i\omega}=\left[-i\frac{d\tilde{Q}'}{d\nu}+\tilde{Q}''+\nu\frac{d\tilde{Q}''}{d\nu}\right]_{\nu=\omega} \quad (4.4)$$

By using equation (4.4) repeatedly in (4.1) and rearranging terms we obtain

$$\bar{Q}(p)=\tilde{Q}'(\omega)+p\tilde{Q}''(\omega)-i\left(\frac{d\tilde{Q}'}{d\nu}+p\frac{d\tilde{Q}''}{d\nu}\right)_{\nu=\omega}(p-i\omega)$$

$$-\left(\frac{d^2\tilde{Q}'}{dv^2}+p\frac{d^2\tilde{Q}''}{dv^2}\right)_{v=\omega}\frac{(p-i\omega)^2}{2!}-\ldots\ldots\ldots . \tag{4.5}$$

Now let p be the point $\gamma+i\omega$ so that the line pp_0 is parallel to the real axis, then we write (4.5) as

$$\begin{aligned}\bar{Q}(p) &= \tilde{Q}'(\omega)-i\gamma\frac{dQ'}{dv}\bigg|_{v=\omega}-\frac{\gamma^2}{2!}\frac{d^2\tilde{Q}'}{dv^2}\bigg|_{v=\omega}-\ldots\ldots\ldots\ldots \\ &+p\left(\tilde{Q}''(\omega)-i\gamma\frac{d\tilde{Q}''}{dv}\bigg|_{v=\omega}-\frac{\gamma^2}{2!}\frac{d^2\tilde{Q}''}{dv^2}\bigg|_{v=\omega}-\ldots\ldots\right).\end{aligned} \tag{4.6}$$

This is the most convenient form for the elements of $\bar{\mathbf{A}}(p)$ in computing, by an iterative technique, those characteristic roots of $\det[p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)]=0$ which have sufficiently small real part: also, this form for $\bar{\mathbf{A}}(p)$ does not (locally) increase the order of the chracteristic equation. In effect the characteristic roots are determined as selected roots from a sequence of (nxn) matrices rather than from one single (nxn) matrix as for an instantaneous system.

For a chosen airspeed assume that an approximation to a characteristic root has been found. The approximation may be improved by successive recalculation of this root when $\bar{\mathbf{A}}(p)$ is, at each step, updated by the use of equation (4.6) when $\gamma+i\omega$ is the current approximation: at each step the other characteristic roots are discarded (*see* Appendix III). If a matrix iterative technique is used a good estimate of the required root is available at each step. With an obvious extension of the notation of equation (4.6) we define the first approximation to $\bar{\mathbf{A}}(p)$ as,

$$\bar{\mathbf{A}}(p)=\tilde{\mathbf{A}}'(\omega)+p\tilde{\mathbf{A}}''(\omega) \tag{4.7}$$

for the (approximate) root $\gamma+i\omega$. When, by a number of iterations the change in $\bar{\mathbf{A}}(p)$ becomes sufficiently small the resulting root and characteristic row and vector may be termed the first approximation: for this approximation the characteristic matrix is real and the roots appear as complex conjugates. For a root with zero or very small real part this approximation is sufficient.

The second approximation to $\bar{\mathbf{A}}(p)$ is similarly

$$\bar{\mathbf{A}}(p)=\tilde{\mathbf{A}}'(\omega)-i\gamma\frac{d\tilde{\mathbf{A}}'}{dv}\bigg|_{v=\omega}+p\left(\tilde{\mathbf{A}}''(\omega)-i\gamma\frac{d\tilde{\mathbf{A}}''}{dv}\bigg|_{v=\omega}\right) \tag{4.8}$$

for the root $\gamma+i\omega$. In this case the elements of the characteristic matrix are complex and hence the roots do not appear in conjugate pairs. However associated with the approximate root $\gamma+i\omega$ will be its conjugate $\gamma-i\omega$ and relative to the latter

$$\begin{aligned}\bar{\mathbf{A}}(p^*) &= \tilde{\mathbf{A}}'(-\omega)-i\gamma\frac{d\tilde{\mathbf{A}}'}{dv}\bigg|_{v=-\omega}+p\left(\tilde{\mathbf{A}}''(-\omega)-i\gamma\frac{d\tilde{\mathbf{A}}''}{dv}\bigg|_{v=-\omega}\right) \\ &= \tilde{\mathbf{A}}'(\omega)+i\gamma\frac{d\tilde{\mathbf{A}}'}{dv}\bigg|_{v=\omega}+p\left(\tilde{\mathbf{A}}''(\omega)+i\gamma\frac{d\tilde{\mathbf{A}}''}{dv}\bigg|_{v=\omega}\right) \\ &= \bar{\mathbf{A}}^*(p).\end{aligned}$$

Hence a successive approximation to the root $\gamma+i\omega$ will be mirrored by an approximation to its conjugate $\gamma-i\omega$: in practice only the roots for $\omega\geq 0$ need be computed since the conjugate root, row and vector may be synthesised. A suggested computational procedure is outlined in Appendix III. A third approximation may be constructed but the necessary derivatives in equation (4.6) become difficult to compute accurately: fortunately, for wings of low and moderate aspect ratio the variation of $\tilde{Q}'$ and $\tilde{Q}''$ with v is not large.

In subsonic flow when $|\omega| \leq |\gamma|$ the expansion (2.12) must be used rather than (4.6). Corresponding to the first approximation (cf equation (4.7)) is

$$\bar{\mathbf{A}}(p) = \tilde{\mathbf{A}}'(0) + p\tilde{\mathbf{A}}''(0) \tag{4.9}$$

while corresponding to the second approximation (cf. equation (4.8)) is

$$\bar{\mathbf{A}}(p) = \tilde{\mathbf{A}}'(0) - \frac{2}{\pi}\left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v \to +0} (\gamma + i\omega)^2 ln\, r$$

$$+ p\left(\tilde{\mathbf{A}}''(0) - i(\gamma + i\omega)\left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v \to +0} \frac{\theta}{\pi/2}\right) \tag{4.10}$$

where $re^{i\theta} \equiv \gamma + i\omega$.
In particular, for $\omega = +0$ and $\gamma = -\sigma < 0$,

$$\bar{\mathbf{A}}(p) = \tilde{\mathbf{A}}'(0) - \frac{2}{\pi}\left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v \to +0} \sigma^2 ln\sigma + p\left(\tilde{\mathbf{A}}''(0) + 2i\sigma\left(\frac{d\mathbf{A}''}{dv}\right)_{v \to +0}\right)$$

The approximation (4.9) corresponds to the well-known use of constant aerodynamic derivatives in stability and control of rigid aeroplanes[9]. Indeed it should be remembered that we are dealing here only with those aerodynamic forces which are derived from potential flow and for all other forces a 'steady' approximation is the only one available[1].

For two-dimensional flows a simple first approximation like equation (4.9) is not available since in this case

$$\tilde{\mathbf{A}}''(0) = \frac{1}{v} \operatorname*{Lim}_{v \to +0} \mathcal{I} m\{\tilde{\mathbf{A}}(iv)\}$$

does not exist.

When, in subsonic flow, the first approximation leads to a real, negative root it is shown in Appendix III that two possibilities arise. Either the root is spurious or it corresponds to a complex pair having a real part differing little from the root in question and a very small imaginary part (cf. the discussion at the end of Section 3).

Having found the characteristic roots, vectors and rows the exponential part of the solution (3.13) is known. Loss of asymptotic stability is indicated when, for some airspeed, a pair of roots approaches the imaginary axis. In the remainder of the solution for the subsonic case, equation (3.12), the dominant term $\frac{2}{\pi}\frac{\mathbf{C}_0}{t^3}$ arises from the term

$$-\frac{2}{\pi}\left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v \to +0} p^2\, ln\, p$$

in the expansion for $\bar{\mathbf{A}}(p)$ (cf. equations (2.12), (4.10)). This leading term can always be found since the right-hand derivative $(d\tilde{\mathbf{A}}''/dv)_{v \to +0}$ is given by quasi-steady wing theory[13]: the higher order terms cannot readily be estimated. It would appear to be inconsistent to retain the exponential part of the solution at all when only the leading term of the asymptotic series is available. Although it is true that this part of the solution will, in theory, eventually dominate any decaying exponentials, in practice it rapidly becomes insignificantly small (*see* Section 5.2).

Assuming the strip $\mu_M < \mu < \mu_C$ to contain all the roots associated with the mechanical system (and derivable from the first approximation) the magnitude of μ_M will depend on the location of other roots of $\det[p\mathbf{I} - \mathbf{M} - \overline{\mathbf{A}}(p)] = 0$. Such roots cannot be found using the technique of this section and without further specification of $\overline{\mathbf{A}}(p)$ for sufficiently large $\mathscr{Re}\ p$ no estimation of μ_M is available.

The solution for the input function $\mathbf{f}(t)$, equation (3.15), requires the evaluation of $\overline{\mathbf{X}}(p)$ at the poles of $\bar{\mathbf{f}}(p)$, hence it is implied that, using the technique of this section, only those poles of $\bar{\mathbf{f}}(p)$ can be included which do not lie too far from the imaginary axis. In the case when $\mathbf{f}(t)$ is due to flight through a gust structure $v_g(x+t)$ the elements of $\bar{\mathbf{f}}(p)$ take the form $\bar{P}(\zeta,p)\bar{v}_g(p)$ and in subsonic flow $\bar{\mathbf{f}}(p)$ has a logarithmic branch point at the origin: the poles of $\bar{\mathbf{f}}(p)$ are the poles of $\bar{v}_g(p)$. By the nature of its definition (equation (2.4)) the frequency response function $\tilde{P}(\zeta\,;i\nu)$ shows much larger variations with ν than the functions $\tilde{Q}(\nu,\zeta\,;i\nu)$ and the use of the second approximation is probably essential.

5. *Two Simple Examples.*

5.1. *Short-period Motion of a Rigid Aeroplane.*

For some speed ranges it may be important to include frequency effects when calculating the short-period motion of an aeroplane; this is particularly so in the transonic range. The unsteady aerodynamic forces discussed in Section 2 are based on linearised theories which do not hold in the transonic speed range. A linearised theory does exist for simple harmonic motion in the transonic speed range provided the frequency parameter ν is bounded away from zero. Thus the method described in Section 4 can be applied in those cases for which $|\omega| > |\gamma|$ by using equation (4.8) but an expansion analogous to that of equation (4.10) is not available when $|\omega| < |\gamma|$.

With this important reservation we may discuss the application of the method to the classical short-period approximation wherein the motion is assumed to be described completely in terms of incidence change w and pitch rate q. In this section only, the notation of Ref. 22 is used, extended where necessary to include the transcendental nature of the aerodynamic 'derivatives'. The operational form of the dynamic-normalised equation of motion corresponding to equation (3.1) is Ref. 22, p. 21,

$$\left(p\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \bar{z}_w(p) & -1+\bar{z}_q(p) \\ m_w(p) & m_q(p) \end{bmatrix} \right) \begin{bmatrix} \bar{w} \\ \bar{p} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.1}$$

The first approximation to, for example, $\bar{z}_w(p)$ is given by equation (4.7),

$$\bar{z}_w(p) = \tilde{z}'_w(\nu) + p\tilde{z}''_w(\nu) \tag{5.2}$$

where (equation (4.3))

$$\tilde{z}_w(i\nu) = \tilde{z}'_w(\nu) + i\nu\tilde{z}''_w(\nu)\,. \tag{5.3}$$

In the spirit of the derivative notation we shall write

$$\tilde{z}'_w(\nu) \equiv \tilde{z}_w(\nu)$$

and

$$\tilde{z}''_w(\nu) \equiv \tilde{z}_{\dot{w}}(\nu)$$

etc., retaining the tilde superscript to remind us that these are frequency dependent 'derivatives'. The first approximation for equation (5.1) is then

$$\left(P\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} \tilde{z}_w(\nu)+p\tilde{z}_{\dot{w}}(\nu), -1+\tilde{z}_q(\nu)+p\tilde{z}_{\dot{q}}(\nu) \\ \tilde{m}_w(\nu)+p\tilde{m}_{\dot{w}}(\nu), \tilde{m}_q(\nu)+p\tilde{m}_{\dot{q}}(\nu) \end{bmatrix} \right) \begin{bmatrix} \bar{w} \\ \bar{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5.4}$$

with characteristic equation,

$$\left((1+\tilde{z}_{\dot{w}})(1+\tilde{m}_{\dot{q}}) \right) p^2$$
$$+ \left((1+\tilde{z}_{\dot{w}})\tilde{m}_q+\tilde{z}_w(1+\tilde{m}_{\dot{q}})+\tilde{m}_{\dot{w}}(1-\tilde{z}_q)-\tilde{m}_w\tilde{z}_{\dot{q}} \right) p$$
$$+ \left(\tilde{z}_w\tilde{m}_q+\tilde{m}_w(1-\tilde{z}_q) \right) = 0 .$$

If, as is usual, we neglect $\tilde{z}_{\dot{w}}$, $\tilde{m}_{\dot{q}}$ and $\tilde{z}_q$ compared to unity and consider the derivative $\tilde{z}_{\dot{q}}$ to be negligible, we have finally the characteristic equation,

$$p^2+B(v)p+C(v) = 0 \tag{5.5}$$

where

$$B(v) = \tilde{z}_w+\tilde{m}_q+\tilde{m}_{\dot{w}},$$
$$C(v) = \tilde{m}_w+\tilde{m}_q\tilde{z}_w .$$

The first approximation roots $p_1^{(1)}$, $p_1^{(2)}$ are solutions of the equation

$$p_1^2+B(\omega_1)p_1+C(\omega_1) = 0$$

where $|\omega_1| = |\mathscr{I}m(p_1^{(1)}, p_1^{(2)})|$ if the roots are a complex pair,
$= 0$ if the roots are real.

Assuming the first approximation roots to be found, we now consider the second approximation, firstly for the case of a complex pair of roots with $|\mathscr{I}m(p_1^{(1)}, p_1^{(2)}| > |\mathscr{R}e\,(p_1^{(1)}, p_1^{(2)})|$ (equation (4.8)) and secondly with this inequality reversed (equation (4.10)) which includes the case of a real pair of roots. This distinction is only necessary for subsonic speeds.

(a) complex pair of roots, $p_1^{(1)}, p_1^{(2)} = \gamma_1 \pm i\omega_1$; $|\omega_1| > |\gamma_1|$.

Ignoring terms of $0(\gamma_1^2)$ the characteristic equation becomes simply,

$$p^2+(B(\omega_1)-i\gamma_1\delta B(\omega_1))p+(C(\omega_1)-i\gamma_1\delta C(\omega_1)) = 0$$

where

$$\delta B(\omega_1) = \left.\frac{dB}{dv}\right|_{v=\omega_1}, \delta C(\omega_1) = \left.\frac{dC}{dv}\right|_{v=\omega_1} .$$

In order to obtain an approximate analytical form for the second approximation roots, assume that $B^2(\omega_1)/4 << C(\omega_1)$ so that

$$\gamma_1 = -\frac{B(\omega_1)}{2}, \omega_1^2-C(\omega_1) \approx 0,$$

giving the second approximation characteristic equation

$$p^2-2\gamma_1\left(1+i\frac{\delta B(\omega_1)}{2}\right)p+\omega_1\left(1-i\gamma_1\frac{\delta C(\omega_1)}{\omega_1^2}\right) = 0 .$$

This equation has two roots which are not complex conjugates: we retain only that root which has a positive imaginary part since we are improving the root $\gamma_1+i\omega_1$. The improved root is (cf. equation (AIII.1))

$$p_2^{(1)} = \gamma_1\left(1+\frac{1}{\omega_1}\frac{\delta C(\omega_1)}{2}\right)+i\omega_1\left(1+\frac{\gamma_1}{\omega_1}\frac{\delta B(\omega_1)}{2}\right)$$

and

$$p_2^{(2)} = p_2^{(1)*}.$$

It is interesting to note that the change in the 'frequency' term of the characteristic equation affects the damping and *vice-versa.*

(b) Pair of roots, $p_1^{(1)}, p_1^{(2)}$; $|\mathscr{R}e(p_1^{(1)}, p_1^{(2)})| > |\mathscr{I}m(p_1^{(1)}, p_1^{(2)})|$.

For the root $p_1^{(1)} = \gamma_1+i\omega_1 = r_1e^{i\theta_1}$, for example, the second approximation characteristic equation is

$$p^2+(B(0)-i\frac{2}{\pi}(\gamma_1+i\omega_1)\,\theta_1\,\delta B(+0))p$$

$$+(C(0)-\frac{2}{\pi}(\gamma_1+i\omega_1)^2\,ln\,r_1\,\delta B(+0)) = 0\,.$$

In this case, to obtain an analytical form for the improved roots, assume that

$$\left(\frac{B^2(0)}{4}\right) >> C(0)\ ;\ B(0) > 0\,,\ C(0) > 0$$

so that the two (real) roots of the first approximation are

$$p_1^{(1)} \approx -B(0)+\frac{C(0)}{B(0)},\ p_1^{(2)} \approx -\frac{C(0)}{B(0)}.$$

For the root $p_1^{(1)}$ we take $r_1 = B(0)$, $\theta_1 = \pi$ in the expansion (4.10) giving the characteristic equation,

$$p^2+B(1+2i\delta B)p+(C-\frac{2}{\pi}\delta B\,B^2\,ln\,B) = 0$$

with the appropriate root (cf. equation (AIII.2)),

$$p_2^{(1)}= p_1^{(1)}\left(1-\frac{2}{\pi}\left(\frac{\delta B}{B}\right)B\,ln\,B\right)\left(1+2i\left(\frac{\delta B}{B}\right)B\right)$$

and its complex conjugate $p_2^{(1)*}$. For the root $p_1^{(2)}$, take $r_1 = C(0)/B(0)$ giving the characteristic equation,

$$p^2+\left(B+2i\frac{C}{B}\delta B\right)+\left(C-\frac{2}{\pi}\delta B\frac{C^2}{B^2}ln\frac{C}{B}\right)= 0$$

with the appropriate root

$$p_2^{(2)} = p_1^{(2)}\left(1-\frac{2}{\pi}\left(\frac{\delta B}{B}\right)\frac{C}{B}ln\frac{C}{B}\right)\left(1-2i\left(\frac{\delta B}{B}\right)\frac{C}{B}\right)$$

and its complex conjugate $p_2^{(2)*}$.

As discussed in Appendix III, only one pair of these roots is genuine: since the expansion used was for $\omega_1 = +0$ we can accept only that root, $p_2^{(1)}$ or $p_2^{(2)}$, which has positive imaginary part. Thus if $\delta B < 0$ the root pair $p_2^{(2)}$, $p_2^{(2)*}$ are genuine while if $\delta B > 0$ the pair $p_2^{(1)}$, $p_2^{(1)*}$.

In this analysis we have retained only one quadrature derivative component, namely $\tilde{m}_w'' \equiv \tilde{m}_{\dot{w}}$ and

$$\delta B(+0) = \frac{dB}{dv}\bigg|_{v\to+0} = \frac{dm_{\dot{w}}}{dv}\bigg|_{v\to+0}$$

For a conventional aeroplane the main contribution to $\tilde{m}_{\dot{w}}$ is from the 'lag in downwash' effect at the tailplane so that the right hand side of the above expression cannot simply be estimated from $\tilde{z}_w(0)$ and $\tilde{m}_w(0)$ by using the result quoted at the end of Appendix II.

5.2. *Classical, Binary Aeroelastic System.*

The example chosen here is that of a simple, classical binary system for a rigid wing having plunging and pitching freedoms. Because of its familiarity a Lagrangean formulation is used. The flutter derivatives are those for a rectangular wing of aspect ratio 4 in incompressible flow with the axis of pitch at half-chord: they are derived from the results of Lawrence and Gerber[21].

The parameters of the system are deliberately chosen to give wide frequency separation of the roots and no mechanical inertial or stiffness coupling is included.

The operational form of the homogeneous system equations is, with an obvious extension of the usual British notation for flutter derivatives,

$$[p^2 a_z + p\bar{l}_{\dot{z}}(p) + a_z v_z^2 + \bar{l}_z(p)]\frac{\bar{z}}{c} + [p\bar{l}_{\dot{\theta}}(p) + \bar{l}_\theta(p)]\,\bar{\theta} = 0$$

$$[-p\bar{m}_{\dot{z}}(p) - \bar{m}_z(p)]\frac{\bar{z}}{c} + [p^2 a_\theta - p\bar{m}_{\dot{\theta}}(p) + a_\theta v_\theta^2 - \bar{m}_\theta(p)]\,\bar{\theta} = 0 \tag{5.6}$$

with $a_z = 15{\cdot}4$ and $a_\theta = 3{\cdot}85$. The in-vacuo natural frequencies in plunging and pitching respectively are $\omega_z = 5$ rad/sec and $\omega_\theta = 30$ rad/sec.

Figure 3 shows the variations of true frequency with airspeed for the two roots of the system which originate from the uncoupled motions at zero airspeed: the suffices z and θ are used in this sense. No other roots of the system were sought. Figure 4 shows the inverse of time to half or double amplitude for the two modes; the flutter speed is 500 ft/sec. Both first and second approximations are shown when the difference is discernible.

The non-dimensional root loci are shown in Figure 5. As the pair of roots of smaller frequency approach the real axis the second approximation shows that the more negative root of the first approximation is genuine but its imaginary part is extremely small.

For an airspeed of 440 ft/sec the response in θ due to a unit impulse applied in the pitching freedom is,

$$a_\theta \theta_\theta(t) = 0{\cdot}156 e^{-{\cdot}018t} \sin(0{\cdot}086t - 0{\cdot}810)$$

$$+ 1{\cdot}674^{-0{\cdot}0053t} \sin(0{\cdot}595t + 0{\cdot}056) + \frac{0{\cdot}057}{t^3} + O(1/t^4)\,. \tag{5.7}$$

After two chordlengths the term in $1/t^3$ is extremely small.

The frequency response function $\tilde{\theta}_\theta(iv)$ due to unit amplitude excitation in pitch is given directly from eqn. (5.6) by

$$\tilde{\theta}_\theta(iv) = \frac{-v^2a_z+ivl_{\dot z}(v)+a_zv_z{}^2+l_z(v)}{\begin{vmatrix} (-v^2a_z+ivl_{\dot z}(v)+a_zv_z{}^2+l_z(v)), (ivl_{\dot\theta}(v)+l_\theta(v)) \\ (-ivm_{\dot z}(v)-m_z(v)), (-v^2a_\theta-ivm_{\dot\theta}(v)+a_\theta v_\theta{}^2-m_\theta(v)) \end{vmatrix}} \quad . \tag{5.8}$$

For an airspeed of 440 ft/sec the function $a_\theta\mathscr{Re}\{\tilde{\theta}_\theta(iv)\}$ is plotted in Figure 6. This function gives the data required for a calculation of $\theta_\theta(t)$ by a numerical evaluation of the integral [16,23]

$$a_\theta\theta_\theta(t) = \frac{2}{\pi}\int_0^\infty a_\theta\,\mathscr{Re}\,\{\tilde{\theta}_\theta\,(iv)\}\cos vt\,dv.$$

This evaluation would serve as a basis of comparison with the analytical solution, equation (5.7). However numerical inversion brings its own difficulties and in order that these do not cloud the issue a comparison is made by calculating the real part of the analytical Fourier transform of the solution, equation (5.7). The difference between the curves is everywhere very small and cannot be shown on Figure 6. Instead a series of selected values are compared in Table 1. The term $0{\cdot}057/t^3$ yields, for small v, the contribution $0{\cdot}0091\ v^2 lnv$ and this is always less than 0·0017 in magnitude: it would be undetectable in a numerical inversion.

6. *Concluding Remarks.*

An alternative way of dealing with the system (1.1) when $\tilde{\mathbf{A}}(iv)$ is known is by a numerical evaluation of equation (3.3) when the path L is the imaginary axis. Clearly, a necessary condition is that the system is asymptotically stable.

If in equation (3.3) we write

$$\bar{\mathbf{X}}(p)\,\Big|_{p=iv} = \tilde{\mathbf{X}}(iv) = \tilde{\mathbf{X}}'(v)+i\tilde{\mathbf{X}}''(v)$$

where $\tilde{\mathbf{X}}'(v)$ is an even and $\tilde{\mathbf{X}}''(v)$ an odd function of v then using the fact that $\mathbf{X}(t) = 0$ for $t < 0$ we have, for $t \geq 0$,

$$\mathbf{X}(t) = \frac{2}{\pi}\int_0^\infty \tilde{\mathbf{X}}'(v)\cos vt\,dv = -\frac{2}{\pi}\int_0^\infty \tilde{\mathbf{X}}''(v)\sin vt\,dv\ .$$

Numerous investigations have been made which exploit these relations [16,17,23], particularly to derive the generalised forces $Q(v,\zeta;t)*1(t)$ and $P(\zeta;t)*1(t)$ from oscillatory aerodynamic derivatives[5,18,19]. This technique is probably best suited to intermediate ranges of time since for large time accuracy will suffer due to the rapidly oscillating character of the integrals while for small time the necessary curtailment of the range of integration and unreliable aerodynamic data at high frequency parameters makes the estimation of error difficult. For very small times it is probably best to use indicial aerodynamic data[7] which is not obtained by Fourier inversion procedures and to integrate the equation of motion numerically. In this respect the situation for supersonic flight is easier to deal with than that for subsonic flight.

The numerical inversion of the frequency response function $\tilde{\mathbf{X}}(iv)$ will not give that part of the solution which, at subsonic speeds, is dependent on the singular behaviour of $\tilde{\mathbf{X}}(iv)$ at $v = 0$. The asymptotic behaviour of this part of the solution can however be found directly from $\tilde{\mathbf{X}}'(v)$, $\tilde{\mathbf{X}}''(v)$ by using the expansion (2.12) with $p = iv$ and the known results for the Fourier transforms of the pseudo-functions $(1/|t|^n)$ and $(1/|t|^n)$ sgn t, where n is an integer greater than unity[20].

Perhaps the main attraction of the type of solution presented here is that it is not purely numerical but retains the familiar features associated with the analysis of linear, stationary, instantaneous systems; that is, the exponential solution with the attendant root locus and mode of motion so familiar in rigid aeroplane stability and control.

LIST OF SYMBOLS

$h(x,y,t)$	Transverse displacement of wing camber surface
$l(v;x,y,t)$	Wing loading due to the downwash field $v(x,y,t)$
$l_g(x,y,t)$	Wing loading due to impulsive, travelling gust
p	Laplace transform parameter
$p(t)$	Rate of roll
$q(t)$	Rate of pitch
r	Modulus of complex variable
t	Time measure
$v(x,y,t)$	Downwash velocity over wing planform
$v_g(x+t)$	Vertical gust velocity distribution
w	Incidence
$x_i(t)$	Element of state vector
x,y,z	Rectangular Cartesian body axes with x-axis along equilibrium flight vector and z-axis downward
$\mathbf{f}(t)$	Forcing vector – equation (1.1)
$\mathbf{x}(t)$	State vector – equation (1.1)
$\mathbf{u}_n^T$	Characteristic row } for the simple root, p_n
$\mathbf{v}_n$	Characteristic vector } for the simple root, p_n
$K(x,y,t)$	Aerodynamic kernel function
$Q(v,\zeta;t)$	Weighted integral over the wing of $l(v;x,y,t)$, $v = v(x,y)\delta(t)$
$P(\zeta;t)$	Weighted integral over the wing of $l_g(x,y,t)$
M	Mach number
$\mathscr{M}$	Apparent mass (incompressible flow)
$\mathbf{A}(t)$	Aerodynamic matrix – equation (1.1)
$\mathbf{B}_n$	Normalising matrix for the root p_n – equation (3.10)
$\mathbf{M}$	Mechanical matrix – equation (1.1)
$\mathbf{X}(t)$	Matrix solution of equation (1.5)
γ	Real part of typical characteristic root
$\delta(t)$	Delta functional
$\zeta(x,y)$	Weighting function for $Q(v,\zeta;t)$ and $P(\zeta;t)$
$\eta(t)$	Generalised co-ordinate – amplitude of displacement
θ	Argument of complex variable
μ	Real part of complex variable, p

LIST OF SYMBOLS-continued

ν	Imaginary part of complex variable, p
σ	Positive real variable
ω	Imaginary part of typical characteristic root
$\Delta(p) = 0$	Characteristic equation

Superscripts.

$-$	Laplace transform
$\sim$	Complex amplitude of simple harmonic quantity
$'$	Real part
$''$	Imaginary part
$*$	Complex conjugate
T	Matrix transpose
$\| \ \|$	Matrix norm

REFERENCES

No.	*Author(s)*	*Title, etc.*
1	R. D. Milne	Dynamics of the deformable aeroplane. A.R.C. R. & M. 3345, September 1962.
2	N. N. Krasovskii	*Stability of motion.* Stanford University Press, 1963.
3	A. H. Zemanian	*Distribution theory and transform analysis.* McGraw Hill, 1965.
4	J. R. Richardson	*A more realistic method for routine flutter calculations.* *Proc. A.I.A.A.* Symposium on Structural Dynamics and Aero-elasticity, 1965.
5	W. P. Jones	Aerodynamic forces on wings in non-uniform motion. A.R.C. R. & M. 2117, August 1945.
6	R. A. Struble	*Nonlinear differential equations.* McGraw Hill, 1962.
7	J. W. Miles	*The potential theory of unsteady supersonic flow.* Cambridge University Press, 1959.
8	R. D. Milne	On the use of linearised unsteady aerofoil theory in arbitrarily moving axis systems. *Aeronaut. Quart.*, Vol. 13, Feb. 1962.
9	W. J. Duncan	*The principles of control and stability of aircraft.* Cambridge University Press, 1952.
10	D. E. Davies	Calculation of unsteady generalised airforces on a thin wing oscillating harmonically in subsonic flow. A.R.C. R. & M. 3409, August 1963.

11	D. J. Allen D. S. Sadler	Oscillatory aerodynamic forces in linearised supersonic flow for arbitrary frequencies, planforms and Mach numbers. A.R.C. R. & M. 3415, January 1963.
12	B. A. Fuchs and V. I. Levin	*Functions of a complex variable*, Vol. 2. Pergamon Press, 1961.
13	H. C. Garner and R. D. Milne	Asymptotic expansion for transient forces from quasi-steady subsonic wing theory. *Aeronaut. Quart.*, Vol. 17, Nov. 1966.
14	A. Erdelyi	*Asymptotic expansions.* Dover, 1956.
15	R. A. Frazer, W. J. Duncan and A. R. Collar	*Elementary matrices.* Cambridge University Press, 1952.
16	C. G. B. Mitchell	Calculation of the response of a flexible aircraft to harmonic and discrete gusts by a transform method. A.R.C. R. & M. 3498, Nov. 1965.
17	A. E. Brown and A. A. Schy	Analysis of the dynamic lateral stability of a delta-wing airplane with frequency-dependent stability derivatives. N.A.S.A. Tech. Note D–113, Nov. 1959.
18	B. Mazelsky	Numerical determination of indicial lift of a two-dimensional sinking aerofoil at subsonic Mach numbers from oscillatory lift coefficients with calculations for Mach number 0·7. N.A.C.A. Tech. Note 2562, 1951.
19.	B. Mazelsky	Determination of indicial lift and moment of a two-dimensional pitching aerofoil at subsonic Mach numbers from oscillatory coefficients with numerical calculation for a Mach number of 0·7. N.A.C.A. Tech. Note 2613, 1952.
20	M. J. Lighthill	*An introduction to Fourier analysis and generalised functions.* Cambridge University Press, 1958.
21	H. R. Lawrence and E. H. Gerber	The aerodynamic forces on low aspect ratio wings oscillating in an incompressible flow. *Aeronaut. Sci.*, Vol. 19, 1952.
22	H. R. Hopkin	A scheme of notation and nomenclature for aircraft dynamics and associated aerodynamics. R.A.E. Tech. Report 66200, June 1966.
23	J. J. Donegan and C. R. Huss	Comparison of several methods for obtaining the time response of linear systems to either a unit impulse or arbitrary input from frequency response data. N.A.C.A. Tech. Note 3701, 1957.

APPENDIX I

Continuation of the Frequency Response Function.

Lemma:

$$\text{Let } y(t) = h(t) * x(t) \; ; \quad h(t) = 0 \, , t < 0$$

and let the Laplace transform $\bar{h}(p)$ of $h(t)$ have ascissa of convergence μ_c: if

(1) $x(t) = 1(t)\, e^{pt}$

(2) $\mathscr{Re}\, p > \mu_c$

then

$$\lim_{t \to \infty} y(t) = \bar{h}(p) e^{pt}$$

Proof:

$$y(t) = \int_0^t h(\tau) e^{p(t-\tau)}\, d\tau$$

$$= e^{pt} \int_0^t h(\tau)\, e^{-p\tau}\, d\tau$$

and

$$\lim_{t \to \infty} y(t)\, e^{-pt} = \lim_{t \to \infty} \int_0^t h(\tau)\, e^{-p\tau}\, d\tau \, .$$

The limit will exist if $\mathscr{Re}\, p > \mu_c$ and is then equal to $\bar{h}(p)$ hence,

$$\lim_{t \to \infty} y(t)\, e^{-pt} = \bar{h}(p) \, .$$

In particular, if the half-plane of convergence of $\bar{h}(p)$ contains the imaginary axis then $h(iv)$, the frequency response function, exists and is equal to $\bar{h}(p)$ evaluated on the imaginary axis, that is,

$$h(iv) = [\bar{h}(p)]_{p=iv} \, , \mu_c < 0 \, .$$

Conversely, $h(iv)$ is an analytic function and its continuation into the half-plane $\mu > \mu_c$ is the holomorphic function $\bar{h}(p)$. The function $\bar{h}(p)$ may also be continued into the half-plane $\mu \leqslant \mu_c$ but will not be holomorphic there. When $\mu_c > 0$ then $[\bar{h}(p)]_{p=iv}$ may be defined by continuation as above but this function has no meaning as a frequency response function.

APPENDIX II

The Definition of Flutter Derivatives.

The axis system referred to here is that normally used in aeroplane stability and control having the x-axis along the flight-direction and the z-axis downwards.

In classical aeroelasticity a Lagrangean formulation of the equation of motion is invariably adopted. The result is that the flutter derivatives in the displacement mode $h_i(x,y)$ are defined as a linear combination of the generalised forces $\tilde{Q}(iv)$ for the downwash fields $\partial h_j/\partial x$ and h_j. Let $F_{ij}(t)$ be the generalised force in mode h_i due to the displacement $h_j(x,y)\delta(t)$. Then by definition (equation (2.2))

$$F_{ij}(t) = -Q\,(\partial h_j/\partial x, h_i; t) + \frac{d}{dt}Q(h_j, h_i; t)$$

and for the motion $h_j(x,y)\,\eta_j(t)$,

$$F_{ij}(t)_*\,q_j(t) = \left[-Q\,(\frac{\partial h_j}{\partial x}, h_i; t) + \frac{d}{dt}Q(h_j, h_i; t)\right]_*\eta_j(t)$$

$$= \left[-Q\,(\frac{\partial h_j}{\partial x}, h_i; t)_*\,1(t) + Q(h_j, h_i; t)\right]*\frac{d\eta_j}{dt}$$

Where, in compressible flow, $Q(t)_*1(t)$ shows the behaviour $k_1 1(t)$ at $t = 0$, $F_{ij}(t)_*1(t)$ will show the behaviour $k_1\delta(t) + k_2 1(t)$ while in incompressible flow it will be $k_1 d\delta(t)/dt + k_2\delta(t) + k_3 1(t)$.
When

$$\eta_j = \eta_j e^{ivt},$$

$$\tilde{F}_{ij}(iv) = -\tilde{Q}\,(\frac{\partial h_j}{\partial x}, h_i; iv) + iv\tilde{Q}\,(h_j, h_i; iv)$$

and writing

$$\tilde{Q}(iv) = \tilde{Q}'(v) + iv\tilde{Q}''(v),$$

$$\tilde{F}_{ij}(iv) = \tilde{F}_{ij}'(v) + iv\tilde{F}_{ij}''(v)$$

(cf. equation (4.3)) the flutter derivatives are given by

$$\tilde{F}_{ij}'(v) = -\tilde{Q}'\,(\frac{\partial h_j}{\partial x}, h_i; v) - v^2\tilde{Q}''\,(h_j, h_i; v)$$

$$\tilde{F}_{ij}''(v) = -\tilde{Q}''(\frac{\partial h_j}{\partial x}, h_i; v) + \tilde{Q}'(h_j, h_i; v).$$

If, in the expansion (2.12), the coefficients $a_n^{(1)}$, $b_k^{(1)}$ refer to $\bar{Q}(\partial h_j/\partial x, h_i; p)$ and the coefficients $a_n^{(2)}$, $b_k^{(2)}$ to $\bar{Q}(h_j, h_i; p)$ we have,

$$\bar{F}_{ij}(p) = -a_0^{(1)} - \sum_{n=1}^{\infty} (a_n^{(1)} - a_{n-1}^{(2)})p^n$$

$$+ p^2 \ln p(-b_0^{(1)} - \sum_{k=1}^{\infty} (b_k^{(1)} - b_{k-1}^{(2)})p^k.$$

Quasi-steady wing theory gives the coefficients $a_0^{(1)}$, $(a_1^{(1)} - a_0^{(2)})$ and $b_0^{(1)}$. In the notation of this paper the central result of Ref. 13 is,

$$\left.\frac{d\tilde{Q}''(v,\zeta;iv)}{dv}\right|_{v\to+0} = \frac{A}{16}\tilde{Q}'(v,1;0)\,\tilde{Q}'(1,\zeta;0)$$

where A is wing aspect ratio. Since

$$\left.\frac{d\tilde{F}_{ij}''}{dv}\right|_{v\to+0} = -\left.\frac{d\tilde{Q}''(\frac{\partial h_j}{\partial x}, h_i;iv)}{dv}\right|_{v\to+0},$$

then

$$\left.\frac{d\tilde{F}_{ij}''}{dv}\right|_{v\to+0} = -\frac{A}{16}\tilde{Q}'(\frac{\partial h_j}{\partial x}, 1;0)\,\tilde{Q}'(1,h_i;0)$$

$$= -\frac{A}{16}\tilde{F}_{1j}'(0)\quad \tilde{F}_{i2}'(0)$$

if we take

$$h_1 = 1\,, h_2 = x\,.$$

APPENDIX III

Computation of the Characteristic Roots.

In this appendix is outlined a computational procedure for estimating those roots of $\det[p\mathbf{I}-\mathbf{M}-\bar{\mathbf{A}}(p)] = 0$ having sufficiently small real part.

(a) First approximation (equations (4.7) and (4.9))

For initial estimates of those roots which correspond to modes involving mainly overall aeroplane motion use equation (4.9). For an initial estimate of the pair of roots which correspond to a normal vibration mode use equation (4.7) with ω equal to the reduced natural frequency at the speed considered.

By matrix iteration or otherwise compute the corresponding roots and vectors one at a time by translation of the estimated root to the origin: adjust $\tilde{\mathbf{A}}'(\omega)$, $\tilde{\mathbf{A}}''(\omega)$ by look-up or interpolation and recompute the root until the difference in the imaginary part is small. If, in the subsonic case, $|\gamma| \geq |\omega|$ the computation should be stopped. If particular roots are followed through a speed range the value of the root at each previous speed provides an estimate for the current speed. Finally compute the characteristic rows as vectors of the transposed matrix and normalise with (equation (3.10)), $\mathbf{B}_n = \mathbf{I}-\tilde{\mathbf{A}}''(\omega)$

(b) Second approximation (equations (4.8) and (4.10))

This approximation makes $\bar{\mathbf{A}}(p)$ complex. The roots and vectors of complex matrices are usually computed from a real matrix of twice the dimension whose roots and vectors are the required roots and vectors together with their conjugates. This method is unsuitable here since there is no simple way of selecting those roots which belong to the appropriate expansion (4.8) in the upper or lower half-plane. However since γ is by the very character of the method of solution considered to be small, a perturbation technique is very suitable.

Let $p_n = \gamma_n+i\omega_n$, $\mathbf{v}_n$, $\mathbf{u}_n{}^T$ be the first approximation root, vector and row. For supersonic flow and in subsonic flow when $|\gamma_n|<|\omega_n$ we obtain, by the usual perturbation argument, the correction to p_n,

$$\delta p_n = -i\gamma_n\mathbf{u}_n{}^T \left(\frac{d\tilde{\mathbf{A}}'}{d\nu}\bigg|_{\nu=\omega_n} - p_n \frac{d\tilde{\mathbf{A}}''}{d\nu}\bigg|_{\nu=\omega_n} \right) \mathbf{v}_n . \qquad \text{(AIII.1)}$$

It follows that, for supersonic flow, when $\omega_n = 0$, $\delta p_n = 0$ to this order of approximation. In subsonic flow when $|\gamma_n| \geqslant |\omega_n|$ (and hence ω_n also is small) we have,

$$\delta p_n = \mathbf{u}_n{}^T \left(-\frac{d\tilde{\mathbf{A}}''}{d\nu}\bigg|_{\nu\to+0} p_n^2 \left(\frac{\ln r_n}{\pi/2} - i\frac{\theta_n}{\pi/2} \right) \right) \mathbf{v}_n \qquad \text{(AIII.2)}$$

where $r_n e^{i\theta_n} \equiv \gamma_n + i\omega_n$

Equations (AIII.1) and (AIII.2) need only be used for $\omega_n > 0$ since $\delta p^*_n = (\delta p_n)^*$.

As usual, difficulties will arise with both first and second approximations when equal or nearly equal roots occur: however, since calculations will normally be carried through for a range of airspeed such situations can be avoided by making a small change in the chosen airspeed. The perturbation technique usually also yields a simple expression for the change in the characteristic vectors or rows but since this result depends on the biorthogonality relation and the existence of a finite dimensional vector basis, it does not apply here. If the vectors are required there seems no alternative to a direct calculation using the root already determined.

An interesting situation arises when, in the subsonic case, the first approximation yields real negative roots. Let $p_n = -\sigma_n$, $\sigma_n > 0$ be such a root; then by equation (AIII.2), taking $p_n = -\sigma_n + i0 = \sigma_n e^{i\pi}$,

$$\mathscr{I}m\, \delta p_n = 2\sigma_n{}^2\mathbf{u}_n{}^T \left(\frac{d\tilde{\mathbf{A}}''}{d\nu} \right)_{\nu\to+0} \mathbf{v}_n$$

since $\mathbf{u}_n$, $\mathbf{v}_n$ are real. If $\mathscr{I}m\, \delta p_n > 0$ the second approximation shows that to the first approximation root $-\sigma_n$ corresponds the complex pair

$$\left(-\sigma_n - \frac{2\sigma_n^2}{\pi} ln|\sigma_n| \mathbf{u}_n^T \left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v\to+0} \mathbf{v}_n\right) \pm i2\sigma_n^2 \mathbf{u}_n^T \left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v\to+0} \mathbf{v}_n \tag{AIII.3}$$

whereas if $\mathscr{I}m\ \delta p_n < 0$ this root must be regarded as spurious. In fact such roots certainly exist but they lie on other Riemann surfaces to the one being considered and thus have no physical significance. As an example suppose a pair of complex roots to degenerate to a real negative pair in the first approximation either through a continuous change in speed or other parameter. Let the roots be $-\sigma_n$, $-\sigma_m$ with $-\sigma_n < -\sigma_e < -\sigma_m$ where $-\sigma_e$ is that point at which the root locus first reached the real axis. Let $\Delta_1(p)$ be the first approximation characteristic equation and let

$$\Delta_2(p) = \Delta_1(p) + \delta\Delta_1(p)$$

be the second approximation. Then

$$\delta p_n = \frac{-\delta\Delta_1''(-\sigma_n,+0) - i\,\delta\Delta_1''(-\sigma_n,+0)}{\left.\dfrac{d\Delta_1}{dp}\right|_{p=-\sigma_n}}$$

with a similar expression for δp_m. Now $\delta\Delta_1''(-\sigma,+0)$ is not zero for any $\sigma > 0$ and hence

$$\text{sgn}\delta\Delta_1''(-\sigma_n,+0) = \text{sgn}\ \delta\Delta_1''(-\sigma_m,+0)$$

while

$$\text{sgn}\left.\frac{d\Delta_1}{dp}\right|_{p=-\sigma_n} = -\text{sgn}\left.\frac{d\Delta_1}{dp}\right|_{p=-\sigma_m}$$

since there are, by assumption, no real roots of $\Delta_1(p)$ in the interval $(-\sigma_n, -\sigma_m)$. Hence

$$\text{sgn}\ \mathscr{I}m\ \delta p_n = -\text{sgn}\ \mathscr{I}m\ \delta p_m$$

and one of the roots is spurious. Equation (AIII.3) gives, for small σ, an upper bound to the imaginary part of such a pair of roots. We have

$$|\mathscr{I}m\ p_n| = |\mathscr{I}m\ \delta p_n| = \left|2\sigma_n^2 \mathbf{u}_n^T \left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v\to+0} \mathbf{v}_n\right|$$

$$\leq 2\sigma_n^2 \|\mathbf{u}_n\| \left\|\left(\frac{d\tilde{\mathbf{A}}''}{dv}\right)_{v\to+0}\right\| \|\mathbf{v}_n\|$$

and using the normalising condition (3.10),

$$\leq 2\sigma_n^2 \frac{\left\|\left(\dfrac{d\tilde{\mathbf{A}}''}{dv}\right)_{v\to+0}\right\|}{\|\mathbf{I}-\tilde{\mathbf{A}}''(0)\|}$$

where $\|\ \ \|$ denotes the linear norm of a matrix.

TABLE 1

v	2	3	4	5
0	0·022	2·822	2·844	2·921
0·1	−0·742	2·894	2·152	2·223
0·2	−0·441	3·170	2·729	2·797
0·4	−0·106	5·132	5·026	5·101
0·56	−0·058	24·274	24·216	24·216
0·58	−0·055	51·272	51·217	51·151
0·59	−0·053	84·310	84·257	85·489
0·60	−0·051	−73·485	−73·536	−74·578
0·62	−0·049	−30·945	−30·994	−31·103
0·64	−0·047	−17·532	−17·579	−17·583

Col. 2 is the real part of the Fourier transform of $0{\cdot}156e^{-{\cdot}081t}\sin(0{\cdot}086t-0{\cdot}810)$

Col. 3 is the real part of the Fourier transform of $1{\cdot}674e^{-0{\cdot}0053t}\sin(0{\cdot}595t+0{\cdot}056)$

Col. 4 is the sum of cols. 2 and 3 (Eq. (5.7))

Col. 5 is the real part of the frequency response, Eq. (5.8)

$$V = 440 \text{ ft/sec}$$

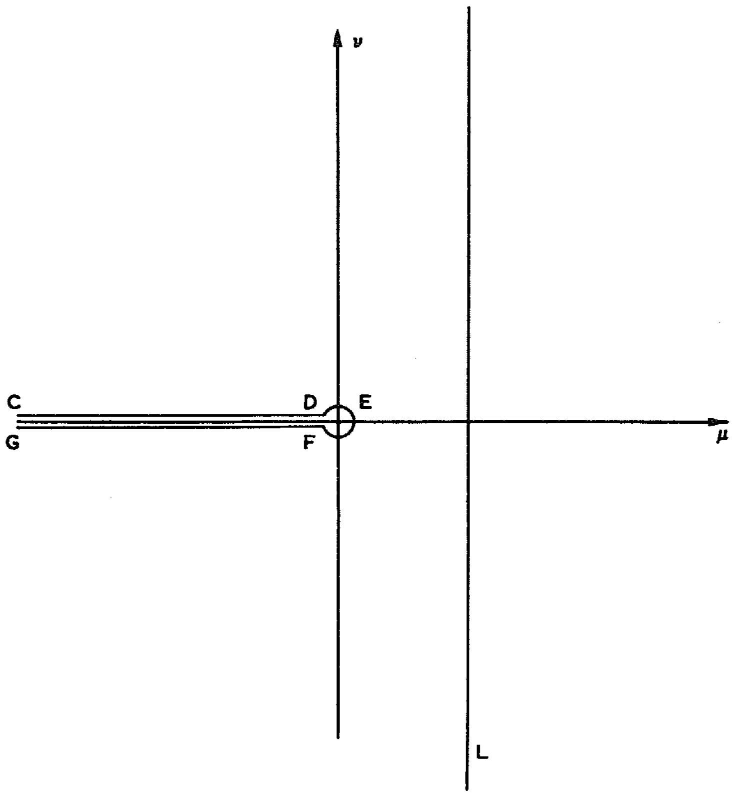

FIG. 1.

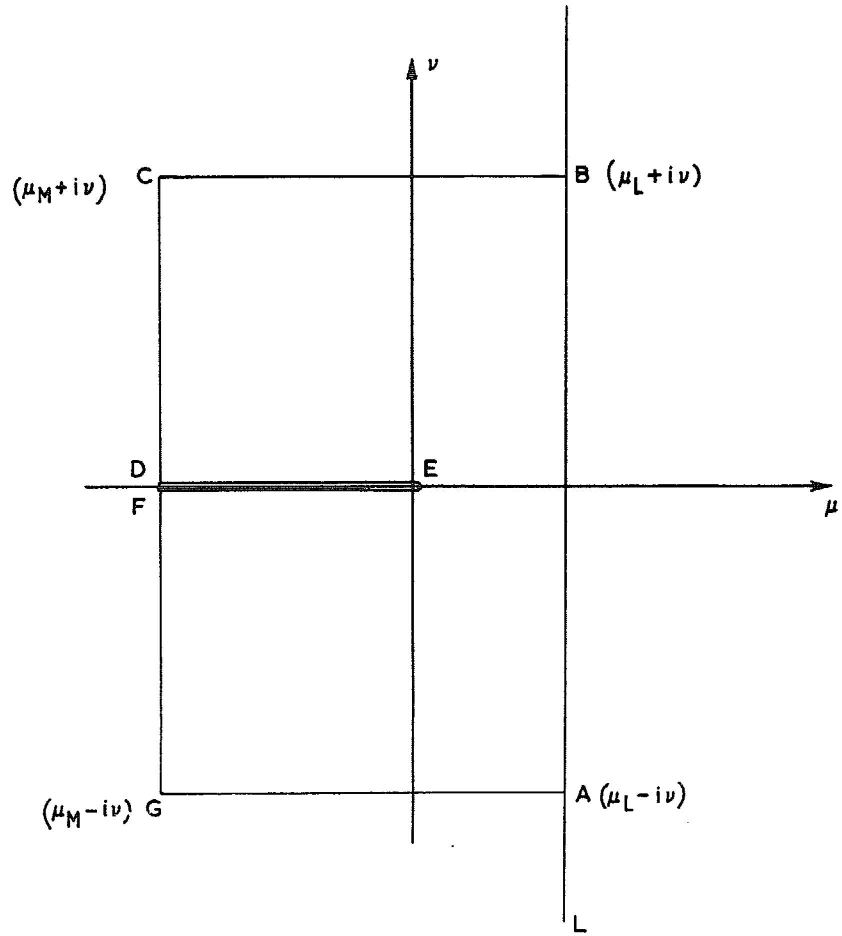

FIG. 2.

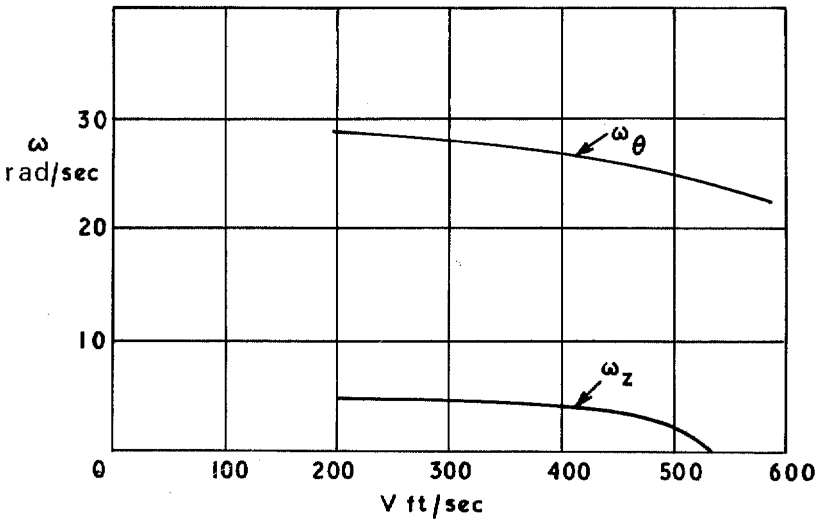

FIG. 3. Variation of frequency with airspeed.

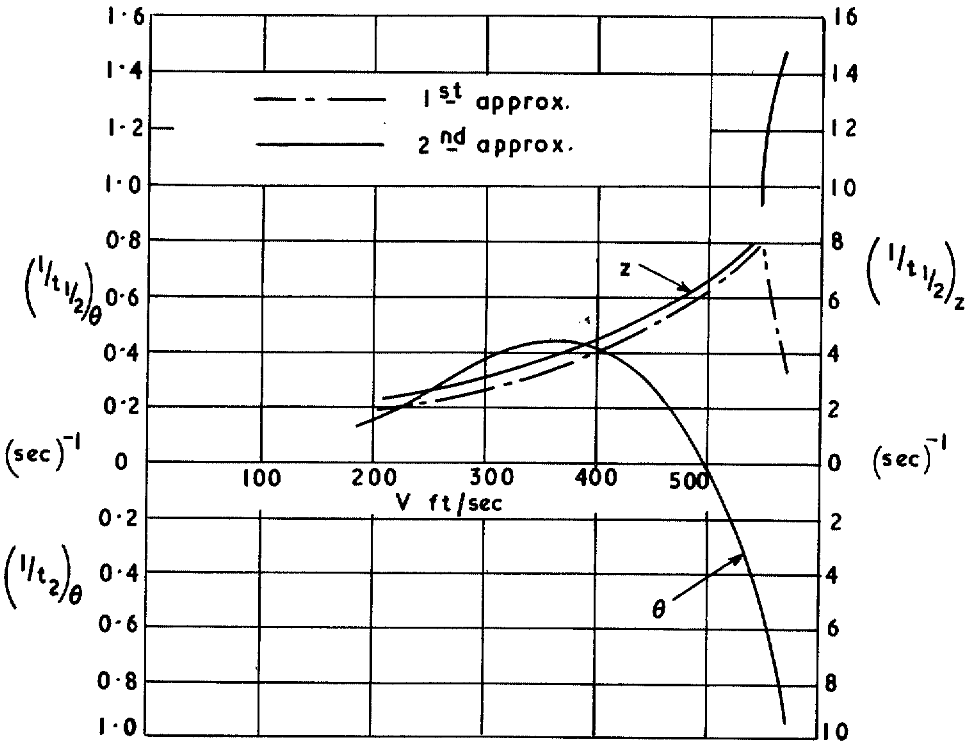

FIG. 4. Variation of decay/growth rate with airspeed.

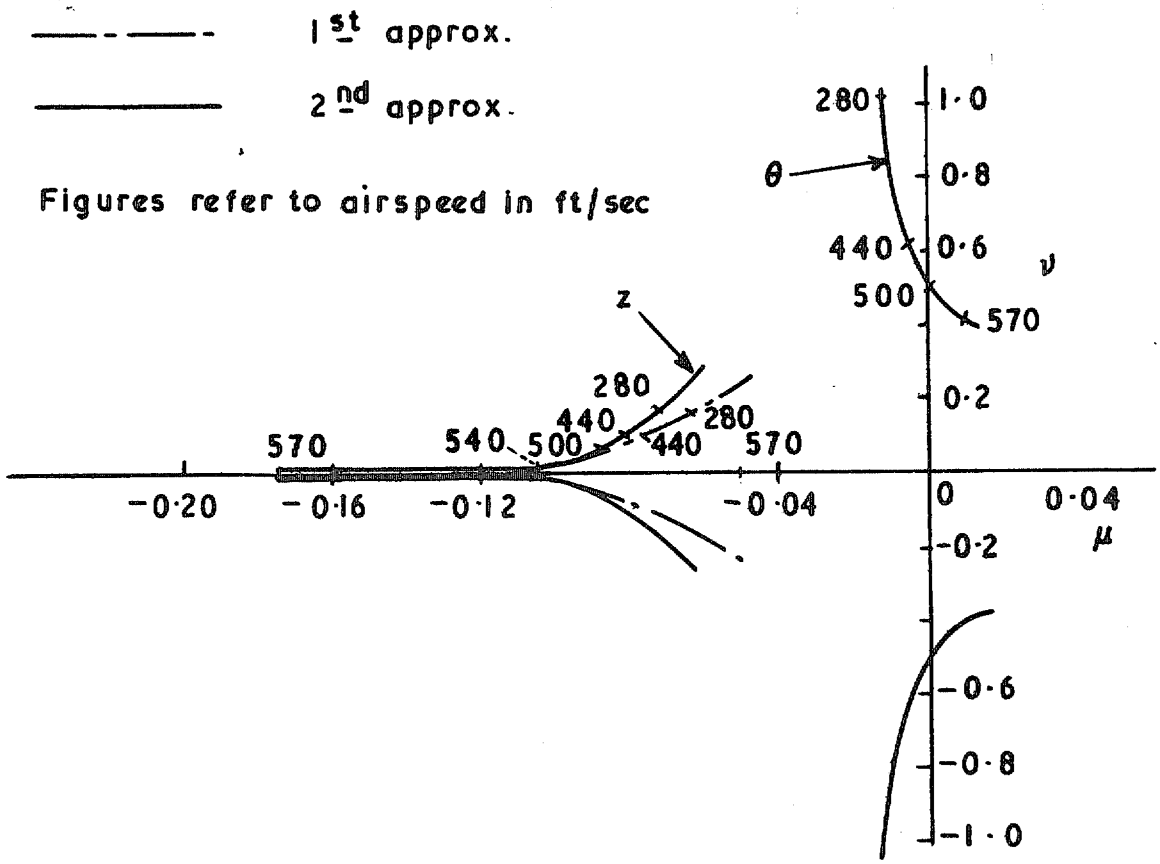

FIG. 5. Root loci.

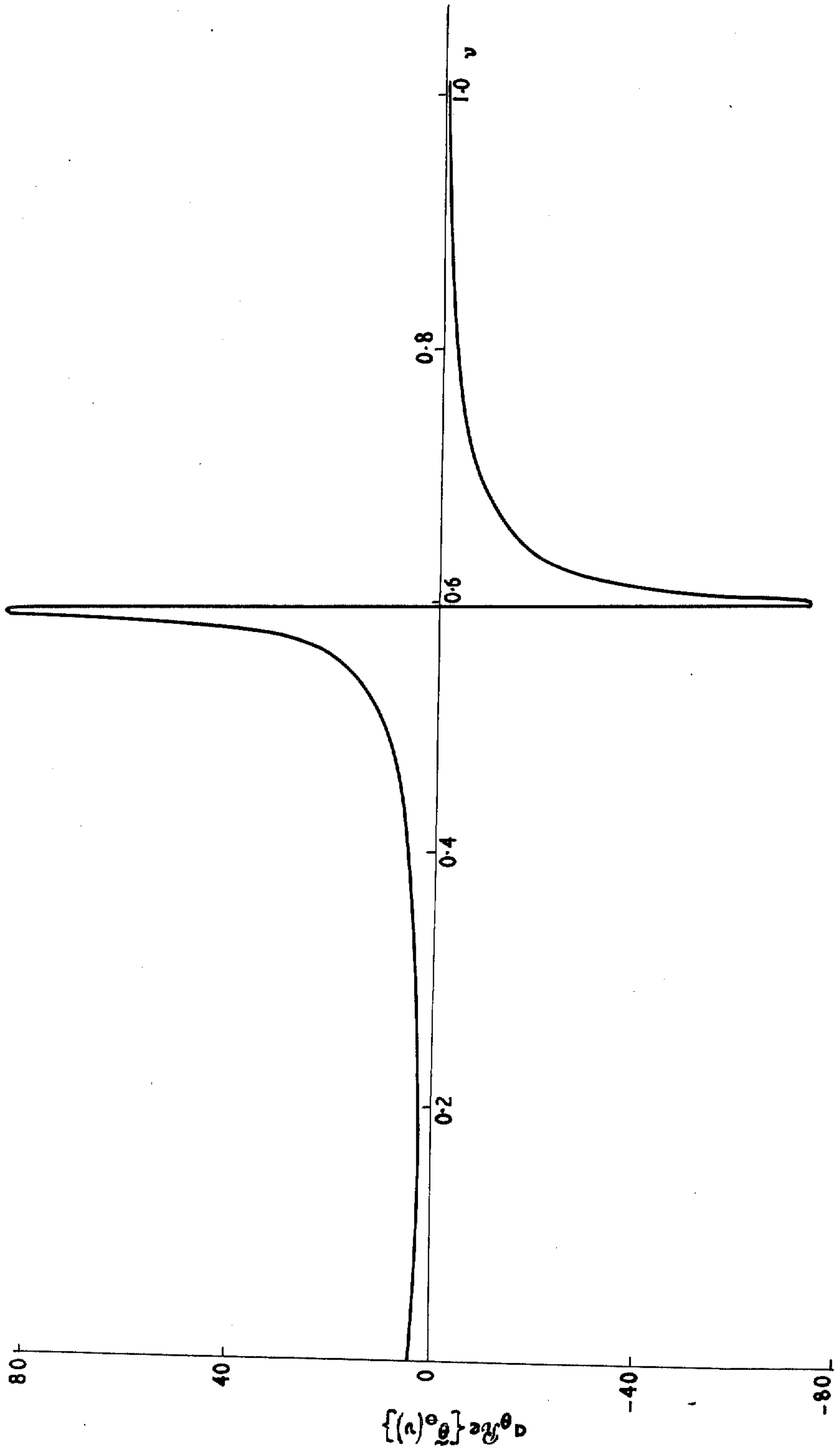

FIG. 6. Real part of frequency response in θ due to excitation in θ.

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