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On the Formation of White-light Fringes in a Mach-Zehnder Interferometer

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Summary.

A theoretical investigation is made of the effects of dispersion in neutral gases on the formation of white-light fringes in an ideal Mach-Zehnder interferometer. It is found that the shift of the brightest fringe is determined by the change of the group refractive index at an effective mean wavelength of the white light. However, the white-light fringe technique may be used to measure changes of the phase refractive index, provided that an allowance is made for the drift of the brightest fringe caused by the effects of dispersion. Such allowance is needed whenever the fringe shift exceeds about 10.

LIST OF CONTENTS

*Replaces A.R.C. 26 635.

1. *Introduction.*

In the literature on Mach-Zehnder interferometry frequent references are made to the use of white-light fringes in situations when it is impossible to follow monochromatic fringes across discontinuities. In most cases it is tacitly assumed, and sometimes it is explicitly stated, that the 'central' white-light fringe corresponds to zero difference between the optical paths in the two beams of the interferometer (for instance, *see* Reference 5). However, this can be true only if the optical path difference is independent of the wavelength of light, that is, if the effects of dispersion in the two beams are exactly equal. There are two ways in which dispersion may have different effects in the two beams: first, because of imperfect matching of glass components and, second, due to differences in density of the gaseous media traversed by the beams.

The effects of dispersion caused by glass have been considered in some detail by Tanner[7]; they can be eliminated by proper matching of the optical components, and even when present are fairly innocuous since they normally remain unchanged in a given optical set-up and are unlikely to lead to errors in the interpretation of interferograms.

The effects of dispersion arising from differences of gas density are inevitable. They have long been recognised in refractometry[2,3] but appear to have been ignored or glossed over in the literature on interferometry in aerodynamics. This is probably due to the fact that common gases have low dispersion, and it does not seem to have been generally appreciated in the past that changes of density of such gases corresponding to fringe shifts of only 10 to 20 produce significant effects of dispersion on white-light fringes, and may cause errors in the evaluation of discontinuous density changes. Recent interferometric studies of ionized gases have led to a wider recognition of these effects, because of the high dispersion in plasmas. Alpher and White[1] used arguments based on the group velocity of light to discuss the behaviour of white-light fringes in plasmas and gave a qualitative description of the effects of dispersion in neutral gases. Glass and Kawada[4] attempted a more detailed investigation of white-light fringes in plasmas but their treatment is incorrect for reasons stated in the footnote to Section 4.1.

In the present Report we investigate the formation of fringes in an ideal Mach-Zehnder interferometer with a point source of light. We begin by considering the light of a single wavelength and obtain a relation between the refractive index in a two-dimensional disturbance in one of the beams and the location of a particular monochromatic fringe on the screen. This relation is used to deduce the displacement (or shift) of the brightest fringe when the light source contains a narrow band of wavelengths, and we find, as expected, that this fringe shift is related to the change of the *group* refractive index. In Section 3 we postulate a representative spectrum of a blue-rich source of white light (extending over the whole of the visible spectrum) and calculate the corresponding distribution of intensity in white-light fringes in the absence of dispersion; the spacing of the brightest three fringes, and the shift of the central fringe are found to be just those that would be obtained with monochromatic light of wavelength equal to an effective mean wavelength of the white light (very nearly equal to the median wavelength of the source). Finally, in Section 4 we examine the effects of dispersion in neutral gases and find that the spacing of the brightest three fringes is unaffected by dispersion and that the shift of the brightest fringe is determined by the change of the group refractive index at the effective mean wavelength of the source. This demonstrates the validity of group-velocity arguments in a situation in which one might not expect such arguments to hold, and provides a guide to the unambiguous evaluation of white-light interferograms.

2. *Monochromatic Fringes.*

The layout of a typical Mach-Zehnder interferometer is shown in Fig. 1; for simplicity, we have omitted the camera lens which focuses the working section onto the screen. Throughout this paper, the interferometer plates, windows and any other optical components are regarded as perfect, refraction is neglected and the light source is assumed to be a point.

Consider first the situation when all the mirrors are parallel, the working section of width w contains air at atmospheric conditions, the optical paths in the two beams are equal and the light is monochromatic (of wavelength λ). Then, whatever the value of λ, the light arrives at the screen in phase. If the density of air in the working section is disturbed, a fringe pattern will appear on the screen. Suppose that the disturbance is two-dimensional and let r be the density of air at any point in the working section and r_0 be the ambient air density.

Along any bright fringe, the difference between the optical paths in the two beams must be an integral multiple n of the wavelength of light used, i.e.

$$l = n\lambda .$$

The optical path is defined as the integral along a ray of the refractive index, μ, of the medium through which the light is passing. For dilute gases, μ is given by the Gladstone-Dale law

$$\mu = 1 + K\rho/\rho_0 .$$

In neutral gases the factor K is a weak function of λ, approximately of the form

$$K(\lambda) = A + B/\lambda^2 .$$

For air

$$\mu_a = 1 + K_a r/r_0$$

so that in our case

$$l = w\,[(1 + K_a r/r_0) - (1 + K_a r_0/r_0)]$$

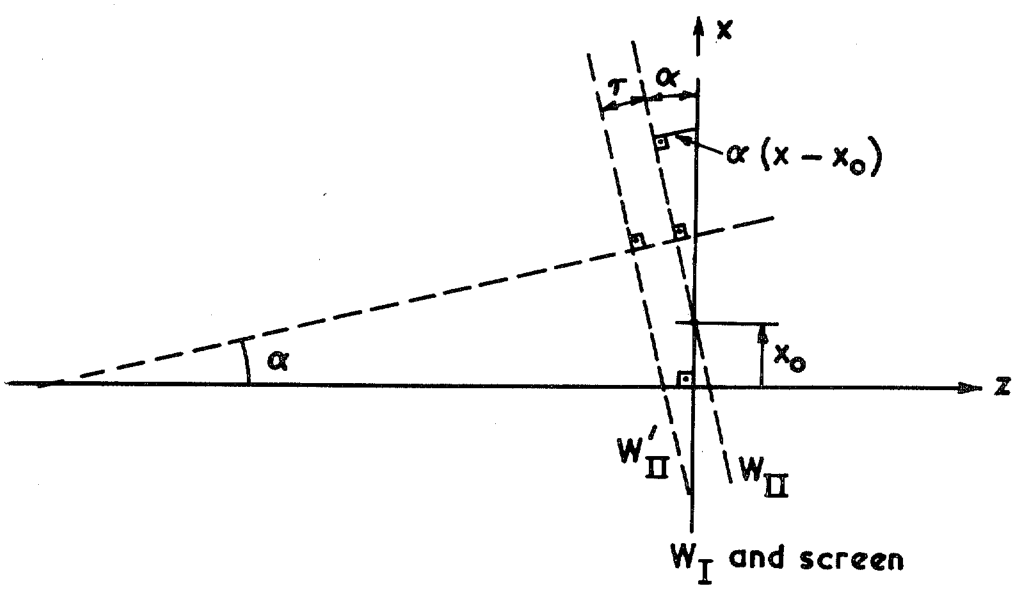
$$= w\,K_a \left(\frac{r}{r_0} - 1\right).$$

Thus,

$$\frac{r}{r_0} - 1 = \frac{n\lambda}{K_a w}$$

and if we had a means of identifying a particular fringe (i.e. ascribing to it the appropriate value of n), r/r_0 could be determined at the position of the various fringes.

Consider next the situation when air in the working section remains at ambient conditions but the mirror M_4 is rotated through a small angle $\alpha/2$ about an axis parallel to the y-axis. As a result the emergent beam II (or wavefront II) is rotated through the angle α about the same axis:



At the screen there is then a *geometrical path difference* between the two wavefronts given by

$$\alpha (x - x_0),$$

where $x \equiv x_0$ defines the intersection of W_I and W_{II}. A translation of the mirror M_3 along its normal would shift W_{II} to a new position, say W'_{II}. This would simply alter the value of x_0 and the geometrical path difference would become $[\alpha (x - x_0) + \tau]$, or, $\alpha (x - x'_0)$. The corresponding *optical* path difference is

$$l_i = (1 + K_a) \alpha (x - x'_0). \tag{1}$$

The suffix i denotes that the path difference is due to the adjustment of the *interferometer*. For air $K \approx 0{\cdot}0003$, therefore in (1) K_a may be neglected by comparison with 1 and we write

$$\begin{aligned} l_i &= \alpha (x - x'_0) \\ &= \alpha (x - x_0) + \tau. \end{aligned} \tag{2}$$

Instead of writing l_i in absolute terms, we can express it as a multiple N_i of a wavelength λ_m

$$\begin{aligned} l_i &\equiv N_i \lambda_m \\ &= \alpha (x - x'_0) \end{aligned} \tag{3}$$

The number N_i (which is not necessarily an integer) is thus a non-dimensional path difference.

If α and τ were known precisely, l_i or N_i would be known at *any* point on the screen. However, α and τ are too small to be accurately measurable by ordinary means, so that even with a perfect instrument the only practicable way of determining l_i or N_i would be to take a monochromatic interferogram. If the wavelength of light used for this purpose is λ_m, then *along any bright fringe* (suffix f)

$$\begin{aligned} l_i = l_{if} &= \text{integer} \,.\, \lambda_m \\ &= n_i \lambda_m. \end{aligned}$$

Hence by (3),

$$n_i \lambda_m = \alpha (x_f - x_0), \tag{4}$$

which may be regarded as an equation for the lines on the screen where the non-dimensional path difference N_i takes integral values n_i, that is, where the fringe number is n_i. Since the path difference at points which do not fall on a fringe may be determined by interpolation, we can revert to the form of equation (3) and write

$$l_i \equiv \lambda_m N_i = \alpha (x - x'_0). \tag{5}$$

In a real instrument the ideal straight line fringe pattern would be somewhat distorted by the imperfections and we would have

$$l_i \equiv \lambda_m N_i = F(x, y). \tag{5a}$$

Note that l_i is independent of λ_m, but N_i is not.

Consider next the effects of:

(a) *Uniform change of density of air in the working section.*

If the density of air in the working section has a value r_u which is uniform but different from r_0, the path difference becomes

$$\lambda_m N_i' = l_i'(\lambda) = l_i + w\, K_a(\lambda)\left(\frac{r_u}{r_0} - 1\right) \tag{6}$$

$$= \lambda_m N_i + \delta_1\, l_i(\lambda)$$

At $\lambda = \lambda_m$, $\qquad l_i'(\lambda_m) = \lambda_m N_i + \delta_1\, l_i(\lambda_m)\,.$

If an interferogram is taken with λ_m, then, *along bright fringes*

$$l_i'(\lambda_m) = l_{if}'(\lambda_m) = \lambda_m n_i'$$

$$= \lambda_m N_i + \delta_1\, l_i(\lambda_m)$$

$$= F(x, y) + \text{const.}$$

$$= F'(x, y), \text{ say,}$$

(where $\lambda_m N_i$ is still given by (5a)).

The additional term $\delta_1\, l_i$ is independent of the position on the screen and represents a bodily translation of the monochromatic fringe pattern, or, an increment to the path difference at any point. The *change* of optical path difference between one point and another (or the change of the fringe number) is therefore unaffected by a uniform change of density of air in the working section.

(b) *Substitution of another gas.*

If air in the working section is replaced by a different gas with uniform density d and Gladstone-Dale factor $K_d(\lambda)$, we have

$$\lambda_m N_i'' \equiv l_i''(\lambda) = l_i + w\left[K_d(\lambda)\frac{d}{d_0} - K_a(\lambda)\right]$$

$$= l_i + \delta_2\, l_i(\lambda)\,. \tag{7}$$

As far as the monochromatic interferogram is concerned this, again, amounts to a shift of the fringe pattern, and with $\lambda = \lambda_m$ we would have

$$\lambda_m N_i'' = l_i''(\lambda_m) = l_i(x, y) + \delta_2\, l_i(\lambda_m)$$

$$= F''(x, y)\,. \tag{8}$$

(c) *Disturbed test gas.*

Suppose now that the working section contains a gas with density $\rho(x, y)$ and Gladstone-Dale factor $K(\lambda)$. Then, the path difference at wavelength λ is

$$l(\lambda) = l_i + w\left[K(\lambda)\frac{\rho}{\rho_0} - K_a(\lambda)\right]\,. \tag{9}$$

If a *monochromatic* interferogram were taken at wavelength λ, then along bright fringes

$$l_f(\lambda) = \lambda n = l_i + w\left[K(\lambda)\frac{\rho}{\rho_0} - K_a(\lambda)\right], \tag{9a}$$

where n is an integer and λ is constant.

Therefore,
$$\frac{\rho}{\rho_0} = \frac{\lambda n - l_i}{w\,K(\lambda)} + \frac{K_a(\lambda)}{K(\lambda)}, \text{ along a fringe.} \tag{10}$$

Here $l_i = F(x, y)$ [c.f. equation (5a)] where the values of x and y are those corresponding to the fringe $l = \lambda n$. Denote the relation between x and y along this fringe by $x = f_n(y)$. Then $l_i = F[y, f_n(y)]$.

Suppose that we want to evaluate (10) by obtaining l_i from a monochromatic undisturbed interferogram taken under conditions (2). Then, dropping the double-dash superscript we have from equation (8)

$$l_i = \lambda_m N_i[y, f_n(y)] - \delta_2\, l_i(\lambda_m)$$

and from (10), along a fringe,

$$\frac{\rho}{\rho_0} = \frac{\lambda n - \lambda_m N_i - \delta_2\, l_i(\lambda_m)}{w\,K(\lambda)} + \frac{K_a(\lambda)}{K(\lambda)}, \tag{11}$$

where the term $\delta_2\, l_i$ is independent of x and y.

To obtain an absolute measure of ρ/ρ_0, we would need to know $\delta_2\, l_i(\lambda_m)$ and n. However, if we are only interested in the *difference* between the density ρ at any point on a fringe and its known value ρ_1, say, at another point on the *same* fringe, then from (11)

$$\frac{\rho - \rho_1}{\rho_0} = -\frac{\lambda_m}{w\,K(\lambda)}\left[N_i(y, f_n(y)) - (N_i)_1\right]. \tag{12}$$

The difference between the terms in the square brackets is the displacement or *shift* of a distorted fringe corresponding to the density disturbance $(\rho - \rho_1)$; this shift is expressed in terms of fringe numbers of the undisturbed monochromatic fringes of wavelength λ_m. We stress this particular definition because some confusion exists in the literature as to the precise meaning of the term 'fringe shift'.

The evaluation of a distorted monochromatic interferogram, using equation (12), is straightforward if individual fringes can be everywhere identified without ambiguity. If discontinuous or very steep density disturbances are present, it is usually impossible to follow a particular fringe across the disturbance and the evaluation of such interferograms requires the knowledge of reference densities on either side of the discontinuity. The chief aim of the white-light technique is to dispense with the need for determining such reference densities by making one fringe distinguishable from the others.

3. *Nearly-monochromatic Fringes.*

Since for a fixed y there is a unique relation between x and N_i along a fringe, the values of y and N_i may be used as co-ordinates of a point on the screen, rather than the values of y and x at that point (*see* Fig. 2). Therefore, if we substitute for l_i in (9a) from (8),

$$\lambda n = \lambda_m N_i - \delta_2\, l_i(\lambda_m) + w\left[K(\lambda)\frac{\rho(y)}{\rho_0} - K_a(\lambda)\right],$$

along a fringe.

This may be regarded as an equation for N_i as a function of y, i.e. an equation describing *the location of the n^{th} fringe in the (N_i, y) co-ordinates*:

$$N_i = \frac{\lambda n}{\lambda_m} + \frac{\delta_2 l_i(\lambda_m)}{\lambda_m} + \frac{w}{\lambda_m}\left[K(\lambda)\frac{\rho(y)}{\rho_0} - K_a(\lambda)\right] . \tag{13}$$

When white light is used, each wavelength of the source produces a fringe system described by (13). In the absence of dispersion (constant K and K_a) *all* such systems would reinforce at the $n = 0$ fringe (since (13) would then be independent of λ). Away from the fringe corresponding to $n = 0$, the fringe systems produced by the various wavelengths of the source would fail to overlap, so that the intensity of light would be highest at the central fringe ($n = 0$) corresponding to zero path difference for all wavelengths. With dispersion, when K and K_a depend on λ, there is, in general, no position on the screen where the optical path difference is the same for all wavelengths and the position of the brightest fringe cannot be determined without a more detailed investigation.

Before attempting such an investigation in the next part of the Report, let us examine briefly the formation of non-monochromatic fringes on the assumption that the light source covers a narrow band centred on a mean wavelength $\bar{\lambda}$. One can then argue that fringes produced by the different wavelengths of the source will reinforce when $dN_i/d\lambda = 0$. Differentiating (13) with respect to λ and setting $dN_i/d\lambda = 0$ leads to

$$n_{\max} = w\left(\frac{dK_a}{d\lambda} - \frac{\rho}{\rho_0}\frac{dK}{d\lambda}\right)_{\lambda = \bar{\lambda}} .$$

Again, in the absence of dispersion this gives $n_{\max} = 0$, so that all the fringe systems overlap at the fringe corresponding to zero path difference.

In terms of the group Gladstone-Dale factor defined by

$$K_g = K - \lambda\frac{dK}{d\lambda} \tag{14}$$

the above equation becomes

$$n_{\max} = \frac{w}{\bar{\lambda}}\left[\frac{\rho}{\rho_0}(\bar{K}_g - \bar{K}) - (\bar{K}_{ag} - \bar{K}_a)\right] . \tag{15}$$

This equation asserts that, with dispersion, the fringe systems produced by wavelengths close to $\bar{\lambda}$ reinforce at the position on the screen where the path difference is $l = n_{\max}\bar{\lambda}$, or, where $l - n_{\max}\bar{\lambda} = 0$. Using the form (9a) for l and substituting for $n_{\max}$ from (15) we find

$$0 = l - n_{\max}\bar{\lambda} = l_i + w\left(\frac{\rho}{\rho_0}\bar{K}_g - \bar{K}_{ga}\right) . \tag{16}$$

The right-hand side of this equation is the optical path difference based on the *group refractive index*. Thus, in the presence of dispersion, the fringes produced by a light source with a narrow band of wavelengths reinforce where the *group* rather than the *phase* path difference is zero. It will be shown in the next Section that this is also so in the case of fringes produced by white light with a spectrum extending over the whole visible region, provided that one ascribes to the white light an effective mean wavelength corresponding to the wavelength $\bar{\lambda}$ of the nearly-monochromatic source considered in this Section.

4. *White-light Fringes.*

4.1. *The Expression for the Intensity at the Screen.*

Let $\frac{1}{2}\, i(\lambda)$ be the intensity of light, per unit wavelength, arriving at a point on the screen *via* each beam separately. Then, it can be readily shown that the total intensity I is given by

$$I = 2\int_{\lambda_1}^{\lambda_2} i(\lambda)\cos^2\left[\frac{\pi\, l(\lambda)}{\lambda}\right] d\lambda,$$

$$= I_0\left[1 + \int_{\lambda_1}^{\lambda_2} \frac{i}{I_0}\cos\frac{2\pi l}{\lambda}\, d\lambda\right] \tag{17}$$

where $l(\lambda)$ is, as before, the optical path difference at the point in question,

$$I_0 = \int_{\lambda_1}^{\lambda_2} i(\lambda)\, d\lambda \tag{18}$$

is the total intensity in the absence of interference, and the spectrum extends between the limits λ_1 and λ_2.

This result can be obtained by combining, according to Young's principle of superposition, two simple-harmonic wave trains of equal amplitudes but different phases, squaring the resultant amplitude to obtain the instantaneous intensity per unit wavelength*, averaging the result over a time interval large by comparison with a representative periodic time and, finally, integrating over the spectrum from λ_1 to λ_2.

4.2. *The Intensity Distribution Function* $i(\lambda)$.

The variation of i with λ depends not only on the emission spectrum of the source but also on the transmission characteristics of the optical system. The fringe pattern is usually recorded photographically and the final effect produced by the light arriving at the screen with a particular distribution of $i(\lambda)$ is also dependent on the sensitivity of the photographic emulsion. It is obviously convenient to think of the transmission characteristics of the optical system and the spectral response of the photographic emulsion as modifications of the source spectrum and we shall regard $i(\lambda)$ as the effective emission spectrum of the source.

In this Report we are concerned mainly with short-duration sources of light provided by spark and flash-tube discharge. It is difficult to make general statements about the spectra of such sources, because their detailed spectral characteristics depend on the gas in which the discharge takes place and on the material of the electrodes. In most cases, however, the emission from such sources (especially from spark discharges in argon or air) is relatively much richer in violet and blue light than is sunlight or the light from tungsten filaments. A typical spectrum of light from spark sources was given by North and North[6] and is reproduced here in Fig. 3. This is, clearly, a simplified distribution in which all the emission is represented by continuum radiation. More recently, some spectra of an argon spark discharge were obtained by Mr. Townsend of the Aerodynamics Division, NPL (private communication from Dr. K. C. Lapworth, 1963). These show a large amount of continuum radiation interspersed with many strong argon emission lines; most of these lines lie in the violet, blue and green parts of the spectrum. No quantitative measurements have yet been made, but the general appearance of the spectra confirms the conclusion that spark sources are rich in the violet and blue light and that the distribution suggested by North and North is a reasonable representation of their spectra.

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*Glass and Kawada[4] consider the formation of white-light fringes with dispersion, but their analysis is incorrect, for they fail to distinguish between amplitude and intensity and between instantaneous and time-averaged quantities. They operate with a quantity to which they refer variously as 'amplitude', 'intensity' and 'brightness' and which is, in fact, quite meaningless, because it represents the instantaneous resultant amplitude integrated over the spectrum. The light of different wavelengths is not coherent and it is the intensity and not the amplitude that should be integrated over the spectrum.

As regards the transmission characteristics of the optical system and the sensitivity of the photographic emulsion, the important effects are, on the one hand, the absorption of the ulta-violet radiation (of wavelength below, say, 0·4 μ) by the large total thickness of glass usually present in the optical system and, on the other hand, the loss of sensitivity of panchromatic emulsions in the deep-red and infra-red (beyond about 0·65 to 0·7 μ).

In view of these considerations, we have chosen to represent the intensity distribution $i(\lambda)$ by the function (also plotted in Fig. 3):

$$i(\lambda) = k/\lambda^2, - \lambda_1 \leqslant \lambda \leqslant \lambda_2$$

$$= 0, - \lambda < \lambda_1, - \lambda > \lambda_2 \qquad (19)$$

where $\lambda_1 = 2/5\ \mu$, $\lambda_2 = 2/3\ \mu$ and k is a constant. Substituting in (18) we can relate k to I_0, the intensity at the screen in the absence of interference:

$$k = \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} I_0$$

so that $k = I_0$, when $\lambda_1 = 2/5$ and $\lambda_2 = 2/3$.

The main virtue of this dependence of i on λ is that it provides a reasonable quantitative description of a typical wavelength distribution of intensity whilst making the evaluation of (17) particularly easy. In fact, in terms of the wave number* $\nu = 1/\lambda$ equation (17) becomes

$$\frac{I}{I_0} = 1 + \frac{1}{I_0} \int_{\nu_2}^{\nu_1} \frac{i(1/\nu)}{\nu^2} \cos(2\pi l \nu)\, d\nu,$$

and substituting $i(1/\nu) = I_0 \nu^2/(\nu_1 - \nu_2)$, we have

$$\frac{I}{I_0} = 1 + \frac{1}{\nu_1 - \nu_2} \int_{\nu_2}^{\nu_1} \cos(2\pi l \nu)\, d\nu. \qquad (20)$$

Of course, our choice of $i(\lambda)$, equation (19), amounts to assuming a rectangular distribution of the intensity per unit wave-number, $j(\nu)$:

$$j(\nu) = I_0/(\nu_1 - \nu_2), - \nu_2 \leqslant \nu \leqslant \nu_1,$$

$$= 0, - \nu < \nu_2, - \nu > \nu_1,$$

where $\nu_1 = 2{\cdot}5\ \mu^{-1}$, $\nu_2 = 1{\cdot}5\ \mu^{-1}$.

4.3. *White-light Fringes in the Absence of Dispersion.*

When there is no dispersion or, more correctly, when the dispersion in the two beams is exactly equal, the optical path difference l is independent of ν and equation (20) yields

$$\frac{I}{I_0} = 1 + \frac{1}{\pi l \Delta \nu} \sin(\pi l \Delta \nu) \cos(2\pi l \bar{\nu}), \qquad (21)$$

where $$\Delta\nu = \nu_1 - \nu_2,$$

and $$\bar{\nu} = \tfrac{1}{2}(\nu_1 + \nu_2). \qquad (22)$$

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*For numerical convenience we define the wave number as the reciprocal of the wavelength measured in microns.

The value of λ corresponding to $\bar{\nu}$ is

$$\bar{\lambda} = 2\lambda_1 \lambda_2/(\lambda_1+\lambda_2) \tag{23}$$

and it is easy to verify that

$$\int_{\lambda_1}^{\bar{\lambda}} i(\lambda)\, d\lambda = \int_{\bar{\lambda}}^{\lambda_2} i(\lambda)\, d\lambda\,,$$

that is, $\bar{\lambda}$ is the median wavelength of the intensity distribution equation (19). As we shall see later, $\bar{\lambda}$ may be regarded as an effective mean wavelength of the white light.

It will be more convenient to write (21) in the form

$$\frac{I}{I_0} = 1+\frac{1}{\pi Lv}\sin(\pi Lv)\cos(2\pi L) \tag{24}$$

where

$$L = l/\bar{\lambda} \tag{25}$$

is the path difference in terms of the median wavelength $\bar{\lambda}$, and

$$v = \Delta\nu/\bar{\nu} \tag{26}$$

is the intensity band-width expressed as a fraction of the median wave-number $\bar{\nu}$.

It is clear that the variation of intensity with L can be regarded as an oscillation of unit period about the average intensity* I_0, modulated with the period $2/v$ and decaying like $(Lv)^{-1}$; the amplitude of the oscillation is maximum at $L = 0$ and is equal to I_0.

The intensity distribution represented by equation (24) is symmetric about the maximum corresponding to zero path difference ($L = 0$). The actual fringe pattern on the screen would, of course, depend on the manner of variation of l with the position on the screen. For the purpose of the present discussion we shall assume that l is proportional to the distance from a straight reference line (where $l = 0$) on the screen, so that equation (24) with L proportional to the distance from the central fringe will also represent the spatial distribution of intensity, Fig. 4 shows the variation of I/I_0 with L for $v = 0{\cdot}5$. Two important features of the white-light fringes are immediately apparent:

(1) The fringe 'visibility' (defined as the ratio of the adjacent maxima and minima of I) decreases so rapidly with L that one would expect to see clearly only 3 bright fringes ($L = 0$ and $L \approx \pm 1$),

(2) The spacing of the central fringe and its two immediate neighbours is very nearly the same as would be obtained with monochromatic light of wavelength $\bar{\lambda}$; this suggests a method for the experimental determination of the median wavelength.

To obtain accurate positions of the maxima and minima of I we differentiate (24) with respect to L and set $dI/dL = 0$. This leads to

$$2\pi L\tan(2\pi L) = \pi Lv\cot(\pi Lv)-1 \tag{27}$$

which is of the form

$$y\tan y = x\cot x-1\,.$$

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*It can be verified that $\int_{-\infty}^{\infty} IdL = I_0$, which satisfies the conservation of energy.

For a given v, equation (27) has an infinite number of roots

$$L = L_{n/2}, n = 0, \pm 1, \ldots\ldots \pm\infty,$$

with $L_0 = 0$ and $L_{(-n/2)} = L_{(n/2)}$. We have determined the variation of L_1 and L_2 with v for $0 \leqslant v \leqslant 0{\cdot}9$ (the values of v greater than 0·5 are of no practical interest, except for optical systems sensitive to the infra-red or ultra-violet as well as to the visible light); these are shown as dotted lines in Fig. 4. As v is increased from 0 to 0·5, L_1 varies only from 1 to 0·98. Thus the positions of the 3 white-light fringes of highest contrast (corresponding to L_0, L_1 and L_{-1}) are practically independent of the band-width of the source for $v \leqslant 0{\cdot}5$ and differ by not more than 2 per cent of the fringe width from the locations that would be obtained using monochromatic light of wavelength $\bar{\lambda}$; this wavelength may be regarded as an effective mean wavelength of the white light.

It is interesting to note that a similar conclusion is reached if we assume that the relative band-width v is small. To the first order in Lv, equation (27) reduces to

$$2\pi L \sin(2\pi L) = 0,$$

whose roots are

$$L = \pm n/2, \quad n = 0, 1, 2 \ldots,$$

provided $nv \ll 1$.

Carrying the approximation a stage further, it can be shown that to the second order in Lv, equation (27) reads

$$2\pi L \tan(2\pi L) = -\frac{1}{3}(\pi L v)^2$$

whence it follows that

$$L_{\pm 1} = \pm 1 \mp \frac{1}{12} v^2, \tag{28}$$

i.e. the effective mean wavelength of the light is

$$\lambda_e = \left(1 - \frac{1}{12} v^2\right) \bar{\lambda}. \tag{29}$$

Thus, although λ_e is, formally, independent of the band-width only to the first order in v, the coefficient of the second-order term in (29) is so small that the first approximation is very accurate even for v as large as 0·5 and the second approximation to $L_{\pm 1}$ (equation (28)) is so accurate that for $v \leqslant 0{\cdot}5$ it is practically indistinguishable from the exact solution plotted in Fig. 4.

4.4. *White-light Fringes with Dispersion.*

In general, dispersion in the two beams of the interferometer is not equal and l depends on λ. Then, as we saw in Section 1 (equation (9)), the optical path difference is

$$l(\lambda) = l_i + w\left[K(\lambda)\frac{\rho}{\rho_0} - K_a(\lambda)\right] \tag{30}$$

where l_i is the path difference due to the adjustment of the interferometer. We recall that l_i is, to a very high accuracy, independent of the refractive index and, therefore, independent of λ.

In the visible part of the spectrum, dispersion in most neutral dilute gases is low and the Gladstone-Dale factors of such gases may be represented very accurately by the approximate Cauchy formula

$$K = A + B/\lambda^2$$

which may be written in the form

$$K = K(\bar{\lambda}) + B\left(\frac{1}{\lambda^2} - \frac{1}{\bar{\lambda}^2}\right),$$

$$= K(\bar{\lambda}) + Bg(\lambda). \tag{31}$$

Throughout the visible spectrum, $Bg(\lambda)$ amounts to no more than a few per cent of $K(\bar{\lambda})$, and we shall find it convenient later to approximate $g(\lambda)$ by the straight line

$$\frac{2}{\bar{\lambda}^2}\left(1 - \frac{\lambda}{\bar{\lambda}}\right) + c.$$

This line has the same slope at $\lambda = \bar{\lambda}$ as $g(\lambda)$ in equation (31). Determining the constant c by the least-squares method we find that $c = 0{\cdot}283$ for $\frac{2}{5} \leqslant \lambda \leqslant \frac{2}{3}$; the two forms of $g(\lambda)$ are compared in Fig. 5. The constant c may, of course, be absorbed in $K(\bar{\lambda})$, so that (31) can be rewritten

$$K = K'(\bar{\lambda}) + Bg(\lambda) \tag{32}$$

where

$$K'(\bar{\lambda}) = K(\bar{\lambda}) + Bc$$

and

$$g(\lambda) = 2(1 - \lambda/\bar{\lambda})/\bar{\lambda}^2. \tag{33}$$

Inserting these in equation (30), we have

$$l(\lambda) = l_i + w\left(K'\frac{\rho}{\rho_0} - K'_a\right) + w\left(B\frac{\rho}{\rho_0} - B_a\right)g(\lambda)$$

$$= \bar{l} + Cg(\lambda) \tag{34}$$

$$= \bar{l} + Cf(v)$$

where

$$\bar{l} = l(\bar{\lambda}),$$

$$C = w\left(B\frac{\rho}{\rho_0} - B_a\right), \tag{35}$$

$$f(v) = g(\lambda) = 2\bar{v}^2(1 - \bar{v}/v).$$

Substituting equation (34) into (20)

$$\frac{I}{I_0} = 1 + \frac{1}{\Delta v}\int_{v_2}^{v_1} \cos\left[2\pi(\bar{l} + Cf(v))v\right] dv$$

and integrating*, we find that

$$\left.\begin{aligned}\frac{I}{I_0} &= 1+\frac{1}{\pi L'v}\sin(\pi L'v)\cos(2\pi\bar{L})\\ &= 1+\frac{1}{\pi L'v}\sin(\pi L'v)\cos 2\pi(L'-D),\end{aligned}\right\} \tag{36}$$

where
$$\left.\begin{aligned}\bar{L} &= \bar{l}/\bar{\lambda},\\ L' &= \bar{L}+2C\bar{v}^3\\ &= \bar{L}+D.\end{aligned}\right\} \tag{37}$$

The maxima and minima of (36) occur when

$$2\pi L'\tan 2\pi(L'-D) = \pi L'v\cot(\pi L'v)-1. \tag{38}$$

To appreciate the significance of L', we substitute for C in (37) from (35). Then, substituting for $\bar{l}$ from (34), we have

$$\begin{aligned}\bar{\lambda}L' &= l(\bar{\lambda})+2C/\bar{\lambda}^2\\ &= l_i+w\left[\left(K+\frac{2}{\bar{\lambda}^2}B\right)\frac{\rho}{\rho_0}-\left(K_a+\frac{2}{\bar{\lambda}^2}B_a\right)\right].\end{aligned}$$

But $(K+\frac{2}{\bar{\lambda}^2}B)$ is the group Gladstone-Dale factor (*see* equation (14), Section 2). Therefore,

$$\bar{\lambda}L' = l_i+w\left(K_g\frac{\rho}{\rho_0}-K_{ga}\right).$$

Thus, $\bar{\lambda}L'$ is the optical path difference based on the group refractive-index, at the wavelength $\bar{\lambda}$.

Comparing (36) and (24), we see that in the presence of dispersion the variation of intensity with $\bar{L}$ can still be regarded as an oscillation of unit period, but now the modulation depends on $(\bar{L}+D)v$ rather than $\bar{L}v$.

Detailed examination of (38) shows that when D is equal to an integer, say m, I has an overall maximum at $L' = \bar{L}+m = 0$, that is at $\bar{L} = -m$, or, $\bar{l} = -m\bar{\lambda}$. In other words, the brightest fringe coincides with the position on the screen where $\bar{l}$, the *phase* path-difference at the median wavelength $\bar{\lambda}$, is $(-m)$ times this wavelength, or where the *group* path-difference is zero. This is just the conclusion reached in Section 2, in the case of fringes produced by a source of narrow band-width.

When $D = m+\frac{1}{2}$, $L' = 0$ corresponds to the *overall minimum* of illumination, that is the *darkest* fringe; there are then two equally bright fringes adjacent to the darkest fringe.

---

* The Cauchy formula, equation (31) would lead to an integral of the form $\int x^{-2/3}\cos x\,dx$ which would have to be evaluated numerically.

When D is neither an integer nor a half integer, there is *one* overall *maximum* at $\bar{L}+\{D\} = 0$, where $\{D\}$ denotes the integer nearest to D. As in the case of $D = 0$, the locations of the two light fringes adjacent to the brightest fringe can be obtained very accurately by expanding equation (38) to the second order in $\bar{L}v$:

$$\tan 2\pi \bar{L} = -\frac{\pi}{6} \bar{L}v^2 \tag{39}$$

whence it follows that the brightest fringe is at

$$\bar{L}_0 = -m\left(1-\frac{1}{12}v^2\right)-\frac{1}{12}Dv^2$$

and its two neighbours at

$$\bar{L}_{+1} = (-m+1)\left(1-\frac{1}{12}v^2\right)-\frac{1}{12}Dv^2,$$

and

$$\bar{L}_{-1} = (-m-1)\left(1-\frac{1}{12}v^2\right)-\frac{1}{12}Dv^2,$$

where

$$m = \{D\}.$$

Thus, the spacing of the three brightest fringes is unaffected by dispersion and is determined by the median wavelength of the light, $\bar{\lambda}$.

As an example, consider the situation as D is increased from 0 to a positive value. At $D = 0$ the brightest fringe is located at $\bar{L} = 0$ and its two neighbours at $\bar{L} = 1-\frac{1}{12}v^2$ and $\bar{L} = -1+\frac{1}{12}v^2$. At $D = \frac{1}{2}$ there are two equally bright fringes at $\bar{L} = -\frac{1}{24}v^2$ and $\bar{L} = -1+\frac{1}{24}v^2$, adjacent to the darkest fringe at $\bar{L} = -\frac{1}{2}$. When $D = 1$, the brightest fringe is at $\bar{L} = -1$ and its neighbours at $\bar{L} = -\frac{1}{12}v^2$ and $\bar{L} = -2+\frac{1}{12}v^2$. This situation recurs as D passes through integral and half-integral values and the brightest fringe 'drifts' from $\bar{L} = 0$ to $\bar{L} = -1$, -2, etc., its successive locations coinciding with the positions of monochromatic fringes that would be obtained with light of wavelength $\bar{\lambda}$; this drift of the brightest fringe does not depend on the absolute value of D, but only on the *change* of D. When D is not an integer, the position of the brightest fringe is not quite coincident with that of the corresponding monochromatic fringe because

$$\bar{L}_0 = -m+\frac{1}{12}v^2(m-D).$$

However, even for $v = 0{\cdot}5$, the term $\frac{1}{12}v^2(m-D)$ amounts to only about $\pm 0{\cdot}01$ (because $|m-D| < \frac{1}{2}$), and this is well within the accuracy with which fringe locations can be determined in practice.

As an illustration, we have calculated the intensity distribution for $D = 0{\cdot}4$; this is shown in Fig. 6. To see whether the straight-line approximation to $K(\lambda)$, equations (16) and (17), has any significant effect on the fringe location, we have also calculated the distribution of intensity with $K(\lambda)$ approximated by two straight lines secant to the curve of $K(\lambda)$ at $(\lambda_1, \bar{\lambda})$ and $(\bar{\lambda}, \lambda_2)$ (*see* Fig. 5). The two intensity distributions for $D = 0{\cdot}4$ are compared in Fig. 6, from which it is seen that improving the approximation to $K(\lambda)$ has very little effect on the predicted fringe positions.

It now remains for us to interpret the significance of the dispersion parameter D in terms of the density changes in the gas. This is most conveniently done by relating the density changes to the corresponding fringe shifts that would be obtained with monochromatic light of wave-length $\bar{\lambda}$ (cf. equation (12)). Then, referring to the second equation (35), we can write

$$\frac{C}{w} = B\left(\frac{\rho-\rho_0}{\rho_0}\right)+B-B_a)$$

$$= \frac{B\bar{\lambda}}{w\bar{K}}(N_0-N)+(B-B_a)$$

$$= B\left[\frac{\bar{\lambda}(N_0-N)}{w\bar{K}}+\left(1-\frac{B_a}{B}\right)\right].$$

Therefore, from (37),

$$D = 2\frac{B}{\bar{\lambda}^2}\frac{w}{\bar{\lambda}}\left[\frac{\bar{\lambda}(N_0-N)}{w\bar{K}}+\left(1-\frac{B_a}{B}\right)\right]. \tag{40}$$

As we saw above, the maximum of illumination will drift from one fringe to the adjacent bright fringe when D changes by unity. Therefore, the change of N that will produce such a drift is given by

$$\Delta N = \tfrac{1}{2}\frac{\bar{\lambda}^2}{B}\cdot\frac{\bar{\lambda}}{w}\cdot\frac{w\bar{K}}{\bar{\lambda}},$$

or

$$\Delta N = \tfrac{1}{2}\frac{\bar{\lambda}^2}{B}\bar{K}. \tag{41}$$

Taking $\bar{\lambda} = 0{\cdot}5\,\mu$, we have

$$\Delta N = \frac{1}{8}\frac{\bar{K}}{B}.$$

For *carbon dioxide*, $\bar{K} = 4{\cdot}52\times10^{-4}$ and $B = 2{\cdot}85\times10^{-6}\,(\mu^{-2})$, hence $(\Delta N)_{CO_2} = 19{\cdot}8$. Thus, a density change corresponding to a fringe shift of 20 produces, in CO_2, a drift of the maximum of illumination from the 'central' white-light fringe to its neighbour which, in turn, becomes the apparent 'central' fringe; the drift is such that the apparent fringe shift is one *more* than it would be without dispersion. Of course, with a density change corresponding to a fringe shift of 10 ambiguity will arise, because two fringes will then appear to be equally bright.

For *air*, $\bar{K} = 2{\cdot}94\times10^{-4}$ and $B = 1{\cdot}76\times10^{-6}$, hence $(\Delta N)_{air} = 20{\cdot}9$. Similar values of ΔN obtain for other common gases.

5. *Conclusion.*

The results of our investigation lead us to conclude that if the drift of the location of the brightest fringe is recognised and allowed for, the white-light fringe technique can be used to measure changes of the *phase* refractive index at the effective mean wavelength of white light. This mean wavelength can be determined experimentally by comparing the spacing of the three brightest white-light fringes with the spacing of the corresponding monochromatic fringes produced by a source of known wavelength. Such measurements will yield λ_e with an accuracy of about 5 to 10 per cent, which is quite satisfactory because the variation of the Gladstone-Dale factor with the wavelength usually amounts to less than 5 per cent throughout the visible spectrum.

For most neutral gases the allowance for the drift of the brightest fringe is required whenever the fringe shift exceeds 10.

LIST OF SYMBOLS

A, B	Constants in the Cauchy formula for the Gladstone-Dale factor.
C	Quantity defined in equation (35)
D	Dispersion parameter defined in equation (37)
$f(\nu), g(\lambda)$	Functions defined in equations (31) to (35)
I	Total intensity at a point on the screen
I_0	Value of I in absence of interference
$i(\lambda)$	Intensity distribution function
K	Phase Gladstone-Dale factor
K_g	Group Gladstone-Dale factor
$\bar{K} = K(\bar{\lambda})$	
$L = l/\bar{\lambda}$	
L_n	Value of L corresponding to a maximum of I
l	Optical path difference
l_i	Optical path difference due to adjustment of interferometer
$\bar{l} = l(\bar{\lambda})$	
m	An integer
$N_i = l_i/\lambda_m$	Non-dimensional path difference
n	Fringe number
r	Density of air
r_o	Density of ambient air
$v = \Delta\nu/\bar{\nu}$	
w	Width of working section along light beam
x, y, z	Cartesian co-ordinates (*see* Fig. 1)
$\delta_1 l_i, \delta_2 l_i$	Increments of l_i
λ	Wavelength of light
$\bar{\lambda}$	Median wavelength of white light
λ_e	Effective mean wavelength of white light
λ_m	Wavelength of monochromatic light
λ_1, λ_2	Lower and upper limits of white-light spectrum
$\nu = 1/\lambda$	Wave number
$\Delta\nu = \nu_1 - \nu_2$	
ρ	Density of disturbed gas
ρ_0	Reference density corresponding to ambient temperature and pressure

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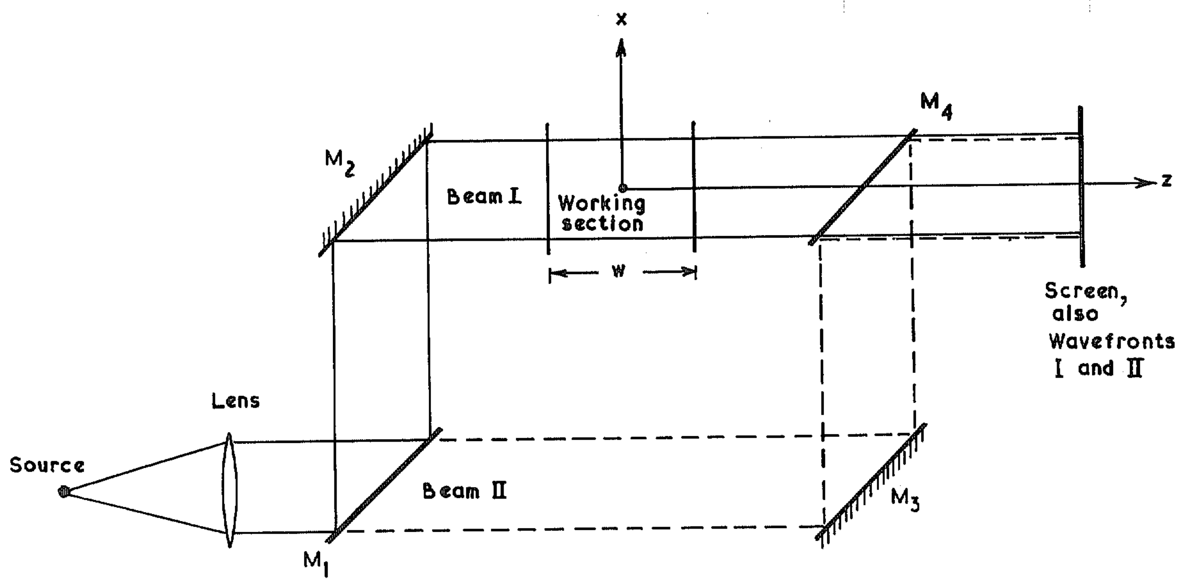

FIG. 1. Layout of Mach-Zehnder interferometer.

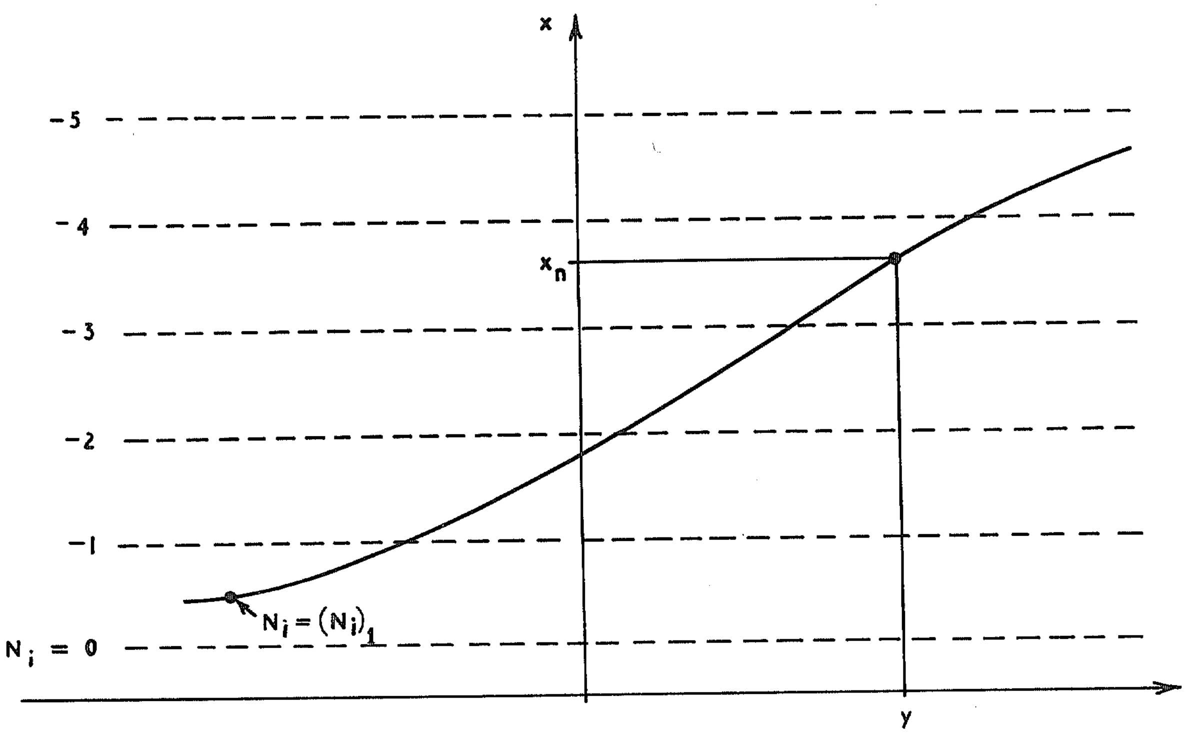

FIG. 2. Distorted n^{th} fringe superimposed on undisturbed monochromatic fringes. Along the n^{th} fringe $l = n\lambda$, $x_n = f_n(y), -N_i = N_i(y, f_n(y))$.

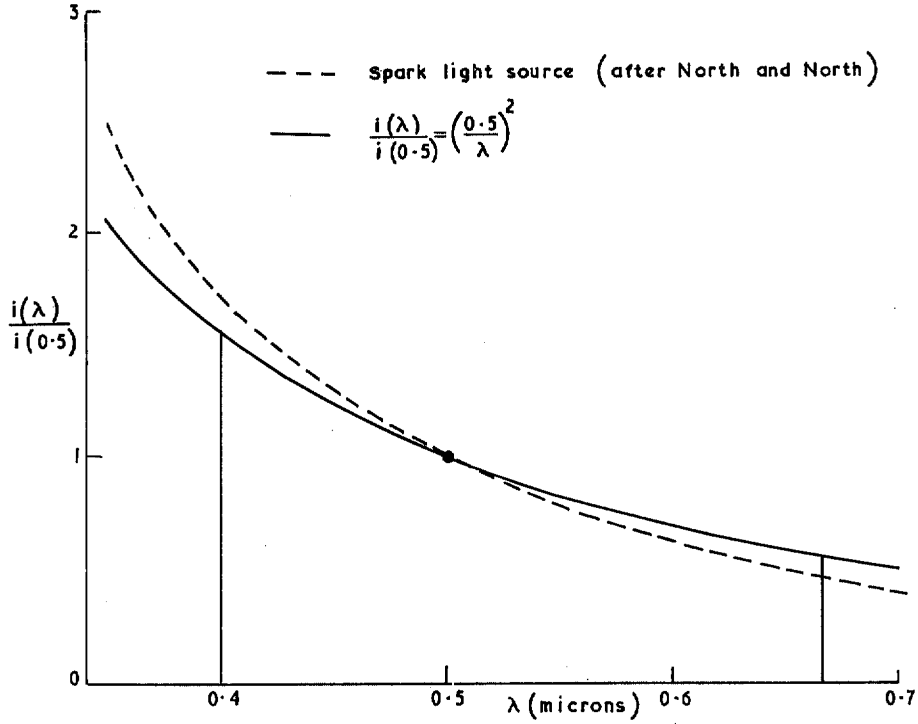

FIG. 3. The intensity distribution function.

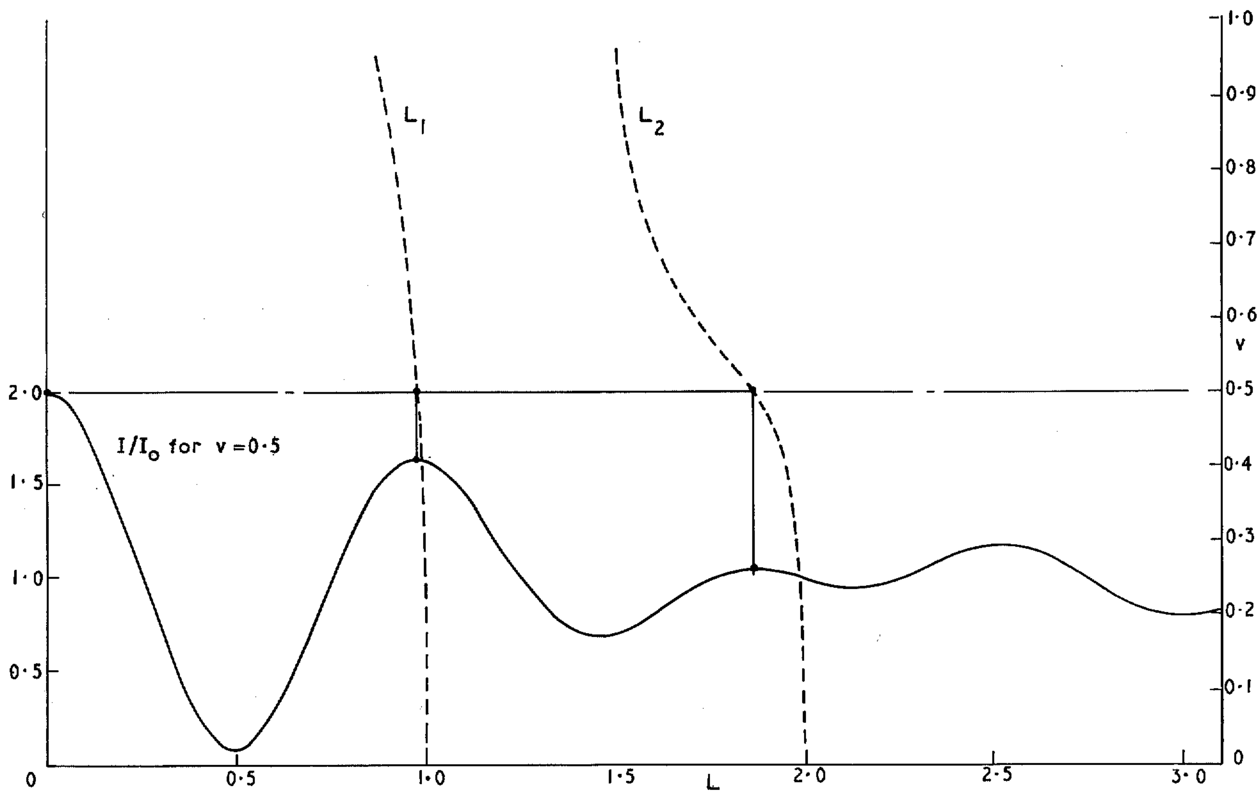

FIG. 4. White-light fringes without dispersion.

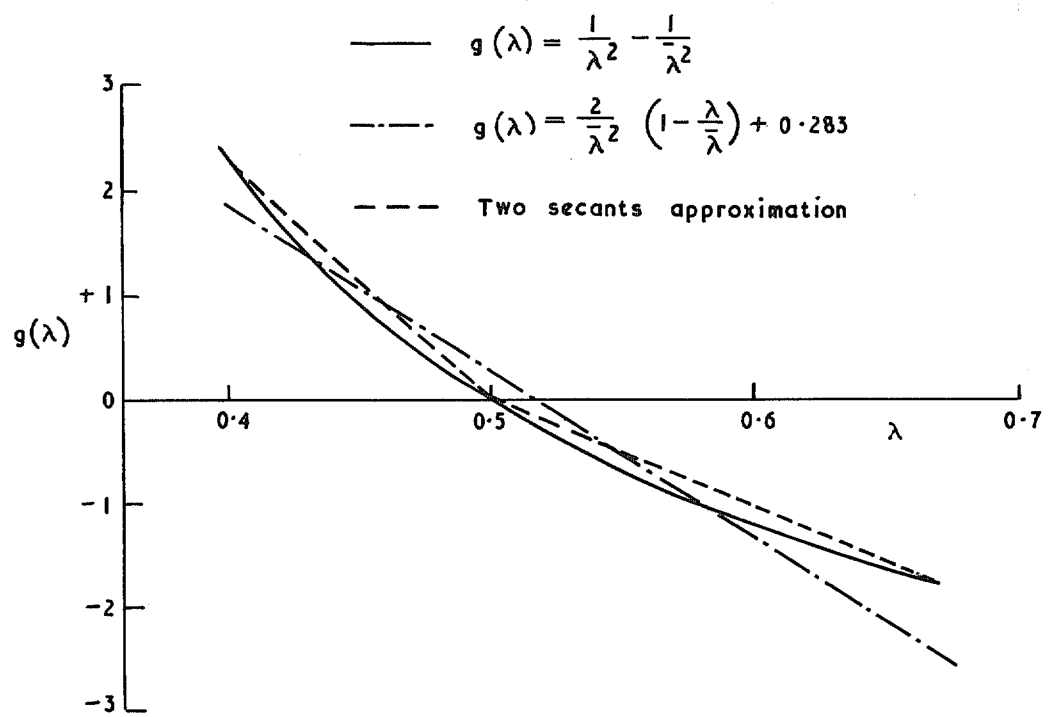

FIG. 5. The function $g(\lambda)$.

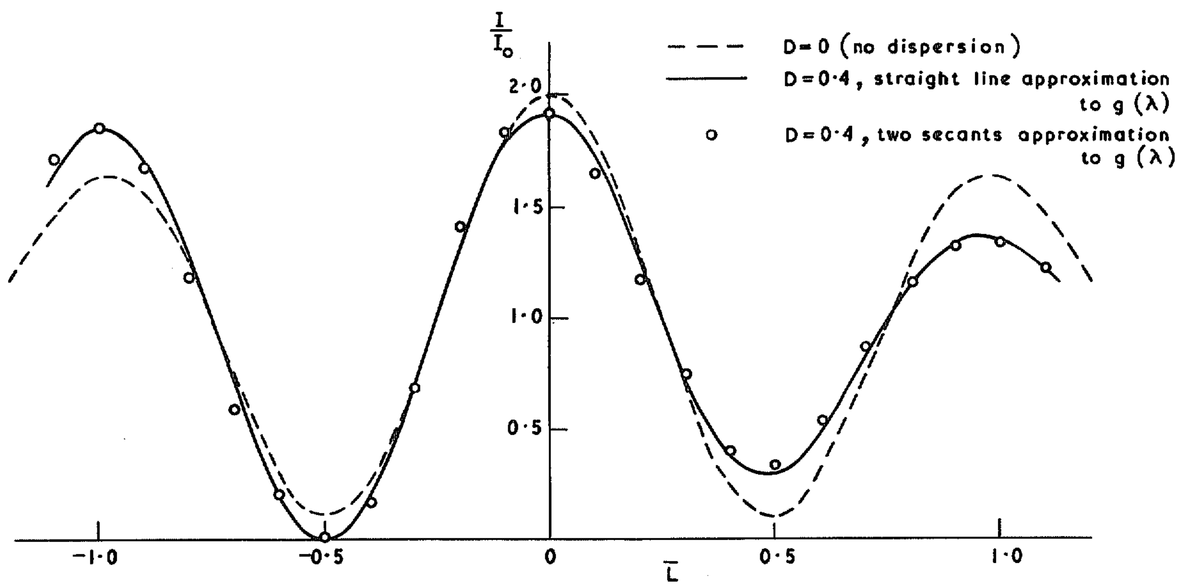

FIG. 6. White-light fringes with dispersion.

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