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The Torsional Vibrations of a Class of Thin, Tapered, Solid Wings

By

Elizabeth A. Frost

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The Torsional Vibrations of a Class of Thin, Tapered, Solid Wings

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Elizabeth A. Frost

SUMMARY

This report considers the torsional vibrations of thin solid wings of doubly-symmetrical chordwise section, with linear variation of chord, and parabolic variation of thickness.

Frequencies of symmetrical and anti-symmetrical vibrations are presented graphically for a range of values of the aspect ratio and the taper ratio.

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1 Introduction

Torsional frequencies for cantilever rectangular plates of uniform thickness have been found by Reissner and Stein[1]. In this report a similar method of analysis is used to find the frequencies for a tapered wing of doubly-symmetrical chordwise section with linear variation of chord and parabolic variation of thickness. The deflections of the wing are assumed to vary linearly across the chord. Minimisation of the potential energy then leads to an ordinary differential equation instead of the usual partial differential equation of plate theory. Taking a parabolic variation of thickness makes the solution of the differential equation tractable. The effect of constraint against axial warping is inherently included since the structure is analysed as a plate rather than as a beam.

The frequencies so obtained are compared with the frequencies obtained on the assumption that there are no constraints to axial warping, (called here the "St. Venant" method). The deflection modes calculated by these two methods are compared for a particular wing.

2 The Structure and Problems Treated

The structure considered is a thin, elastic, isotropic, solid wing of tapered thickness and chord as shown in Fig.1.

In Appendix I the frequency equations are derived for symmetrical and anti-symmetrical vibration of a wing with rectangular cross section.

In Appendix II it is shown that the frequencies for any doubly-symmetrical chordwise sections may be obtained by modifying the parameters in the results for the rectangular cross section.

In Appendix III the equations are derived for finding the frequencies by the "St. Venant" method.

The modes may be readily obtained once the frequencies are known.

3 List of Symbols

$\theta = \theta(x)$ = angular deflection

ω = circular frequency of torsional vibration

x = distance along wing from root

ℓ = semi-span

c_o = root chord

c_t = tip chord

ν = Poisson's ratio

λ = $\dfrac{\ell}{c_o}\sqrt{\dfrac{3(1-\nu)}{2}}$ = aspect ratio parameter

$\omega_{St.V_t}$ = frequency according to "St. Venant" method for tapered wings

$\omega_{St.V_o}$ = frequency according to "St. Venant" method for an untapered wing with thickness and chord equal to those at the root of tapered wing (see equation (10) of Appendix I)

k_1 k_2 = parameters depending on the chordwise section (see equation (33) of Appendix II).

Additional symbols used only in the Appendices are given before Appendix I.

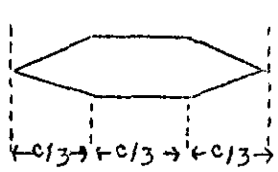
4 Presentation of Results

Figs. 2 and 3 present curves for the fundamental frequencies for symmetrical and anti-symmetrical vibration respectively. The frequencies are expressed in terms of $k_2\, \omega_{St.V_0}$ and are plotted against $k_1\lambda$ where k_1 and k_2 depend on the shape of the chordwise section. The values of k_1 and k_2 for some sections are given in Table I below. The range of $k_1\lambda$ covered is $0.5 \to 4.0$ in the symmetrical case and $0 \to 4.0$ in the anti-symmetrical case. Values taken of the taper ratio c_t/c_0 are 0.2, 0.4, 0.6, 0.8 and the untapered case $(c_t/c_0 = 1.0)$ is included from results obtained by Reissner and Stein[1] and Mansfield[2].

In Fig.5, for the purpose of comparison, the frequency is plotted as a ratio of the frequency obtained when axial warping is unconstrained. The effect of constraint against axial warping is to increase the frequency, and, as is seen from Fig.5, the smaller the value of $k_1\lambda$ the greater the increase in frequency. For the symmetrical case, as $k_1\lambda$ decreases below 2.0 the increase becomes greater than 10% for all taper ratios. For the anti-symmetrical case the increase is greater than 10% when the taper ratio is less than 0.6 and $k_1\lambda$ is small.

In Fig.6 the fundamental symmetrical and anti-symmetrical modes are given for a wing with a taper ratio of 0.6 and for which $k_1\lambda$ is unity. The modes differ considerably from each other and from that obtained when warping constraint is ignored.

Table I

Chordwise section	k_1	k_2
Rectangular	1	1
Ellipse	$\sqrt{2}$	1
Two parabolic arcs	$\sqrt{3}$	1.0690
	1.8145	1.1619
Diamond	$\sqrt{5}$	1

5 Conclusions

Fundamental frequencies have been found for the torsional vibrations of thin, solid wings of doubly-symmetrical chordwise section, linear variation of chord and parabolic variation of thickness. The frequencies are presented graphically for symmetrical and anti-symmetrical vibration over a range of values of the aspect ratio and the taper ratio. If axial warping constraints are ignored, the torsional frequencies may be considerably under-estimated, especially for the symmetrical case.

REFERENCES

No.	Author	Title, etc.
1	Reissner and Stein	Torsion and Transverse Bending of Cantilever Plates. NACA Tech Note 2369.
2	Mansfield, E.H.	The Theory of Torsional Vibrations of a Four-Boom, Thin-Walled Cylinder of Rectangular Cross-Section. R & M No. 2867.

Additional Symbols used in the Appendices

w	=	deflection normal to the plane of the wing
$D(x,y)$	=	$\dfrac{E\ t(x,y)^3}{12\ (1-\nu^2)}$, the flexural rigidity of the wing
$t(x,y)$	=	local thickness of wing
E	=	Young's modulus of elasticity
$m(x,y)$	=	mass per unit area
c	=	chord
r	=	parameter defining taper (see Fig.1)
$'$	=	differentiation w.r.t. x or x_1
$\cdot$	=	" " to time
$\delta\theta$	=	a small increase in $\theta(x)$ which is a function of x
x_1	=	$r\ell - x$
a_1, a_3, s_3	=	as defined by equation (4)
p_1, p_2, p_3, p_4	=	roots of equation (11)
A_1, A_2, A_3, A_4	=	constants occurring in equation (12)

Suffices x and y denote differentiation w.r.t. x and y

APPENDIX I

Analysis for a Plate of Varying Thickness

1 Analysis

The structure considered is a thin, elastic, isotropic, solid wing of varying thickness as shown in Fig.1 and is treated as a plate.

The strain energy of bending, Π_S, is given by

$$\Pi_S = \frac{1}{2} \int_0^{\ell} \int_{-c/2}^{+c/2} D(x,y) \left[(w_{xx} + w_{yy})^2 + 2(1-\nu)(w_{xy}^2 - w_{xx}w_{yy})\right] dxdy \qquad (1)$$

The wing is vibrating torsionally with simple harmonic motion, so that the potential energy due to inertia loading, Π_ω, is given by

$$\Pi_\omega = -\frac{1}{2} \int_0^{\ell} \int_{-c/2}^{+c/2} m(x,y)\ \omega^2\ w^2\ dx\ dy \qquad (2)$$

where the function $w = w(x,y)$ is the maximum deflected shape.

Assuming that the deflection varies linearly across the chord we have

$$w = y\theta(x)$$

The total potential energy, Π, is now given by

$$\Pi = \Pi_S + \Pi_\omega$$

$$= \frac{1}{2} \int_0^{\ell} \int_{-c/2}^{+c/2} \left[D(x,y)\ \{y^2\ (\theta'')^2 + 2(1-\nu)(\theta')^2\} - m(x,y)\ \omega^2 y^2 \theta^2\right] dx\ dy$$

$$= \frac{1}{2} \int_0^{\ell} \{a_3(\theta'')^2 + 2(1-\nu)\, a_1(\theta')^2 - S_3\omega^2\ \theta^2\}\ dx \qquad (3)$$

where with the notation of Ref.1,

$$a_1 = \int_{-c/2}^{+c/2} D(x,y)dy;\quad a_3 = \int_{-c/2}^{+c/2} D(x,y)y^2dy;\quad S_3 = \int_{-c/2}^{+c/2} m(x,y)y^2dy \qquad (4)$$

The variational condition for minimum potential energy, $\delta \Pi = 0$, gives:-

$$\int_0^{\ell} \{(a_3\theta'')'' - 2(1-\nu)(a_1\theta')' - S_3\omega^2\theta\}\ \delta\theta\ dx - [\{(a_3\theta'')' - 2(1-\nu)\ a_1\theta'\}\ \delta\theta]_0^{\ell}$$

$$+ [(a_3\theta'')\ \delta\theta']_0^{\ell} = 0 \qquad (5)$$

The above equation gives a differential equation and four boundary conditions.

1.1 Solution of the Differential Equation

The differential equation to be solved is:-

$$(a_3\theta'')'' - 2(1-\nu)(a_1\theta')' - S_3\omega^2\theta = 0 \qquad (6)$$

For the tapered wing as shown in Fig.1, assuming a rectangular chordwise section,

$$\frac{c}{c_o} = \frac{x_1}{r\ell}, \quad \frac{m(x,y)}{m(0,0)} = \frac{t(x,y)}{t(0,0)} = \frac{x_1^2}{r^2\ \ell^2} \qquad (7)$$

where $x_1 = r\ell - x$. (A parabolic variation of thickness is taken so that the solution of the differential equation becomes tractable).

We now have

$$\left.\begin{aligned} a_3 &= \int_{-c/2}^{+c/2} D(x,y)y^2 dy = \frac{D(0,0)c_o^3}{12}\ \frac{x_1^9}{r^9\ \ell^9} \\ a_1 &= \int_{-c/2}^{+c/2} D(x,y)dy = D(0,0)c_o\ \frac{x_1^7}{r^7\ \ell^7} \\ S_3 &= \int_{-c/2}^{+c/2} m(x,y)y^2 dy = \frac{m(0,0)c_o^3\ x_1^5}{12\ r^5\ \ell^5} \end{aligned}\right\} \qquad (8)$$

Substituting in equation (6) gives the following differential equation in x_1:-

$$\left(\frac{x_1^9}{r^9}\,\theta''\right)'' - 16\lambda^2\left(\frac{x_1^7\,\theta'}{r^7}\right)' - 4\lambda^2\,\pi^2\left(\frac{\omega}{\omega_{St.V_o}}\right)^2 \frac{x_1^5\,\theta}{r^5} = 0 \qquad (9)$$

where

$$\omega^2_{St.V_o} = \frac{6(1-\nu)\,\pi^2\,D(0,0)}{m(0,0)\,c_o^2\,\ell^2}\;;\quad \lambda^2 = \frac{3}{2}(1-\nu)\frac{\ell^2}{c_o^2} \qquad (10)$$

A solution of equation (9) is of the form x_1^p where p satisfies the following equation:-

$$p^4 + 12p^3 + (29 - 16\lambda^2 r^2)\,p^2 - (42 + 96\lambda^2 r^2)\,p - 4\lambda^2 r^4 \pi^2\left(\frac{\omega}{\omega_{St.V_o}}\right)^2 = 0 \qquad (11)$$

The general solution of equation (9) is therefore

$$\theta = A_1 x_1^{p_1} + A_2 x_1^{p_2} + A_3 x_1^{p_3} + A_4 x_1^{p_4} \qquad (12)$$

where p_1, p_2, p_3, p_4 are the roots of equation (11).

1.2 Derivation of the Frequency Equation for Symmetrical Vibration

For symmetrical vibration, we have the following conditions at the root:-

$$w(0,y) = 0 \qquad (13)$$

and

$$w_x(0,y) = 0 \qquad (14)$$

so that

$$[\theta]_{x=0} = 0 \qquad (15)$$

and

$$[\theta']_{x=0} = 0 \qquad (16)$$

It follows from equations (15) and (16) that $[\delta\theta]_o = 0$ and $[\delta\theta']_o = 0$; therefore the other two boundary conditions following from the variational equation (5) are:-

$$[(a_3\theta'')' - 2(1-\nu)\,a_1\theta']_{x=\ell} = 0 \qquad (17)$$

and

$$[a_3\theta'']_{x=\ell} = 0 \qquad (18)$$

Equation (17) becomes

$$[x_1^2 \theta''' - 16 r^2 \lambda^2 \theta']_{x_1=(r-1)\ell} = 0 \qquad (19)$$

Substituting

$$\theta = \sum_{s=1-4} A_s x_1^{p_s}$$

into the boundary conditions (15), (16), (18), (19) gives

$$\sum_{s=1-4} A_s (r\ell)^{p_s} = 0 \qquad (20)$$

$$\sum_{s=1-4} A_s p_s (r\ell)^{p_s-1} = 0 \qquad (21)$$

$$\sum_{s=1-4} A_s \{p_s (p_s - 1)(p_s - 2) - 16 r^2 \lambda^2 p_s\}\{(r - 1)\ell\}^{p_s-1} = 0 \qquad (22)$$

$$\sum_{s=1-4} A_s p_s (p_s - 1) \{(r - 1)\ell\}^{p_s-2} = 0 \qquad (23)$$

The condition for a solution of equations (20)-(23) other than the vanishing of the constants A_1, A_2, A_3, A_4 is:-

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \\ (p_1^3 - p_1 - 16 r^2 \lambda^2 p_1) \left(\frac{r-1}{r}\right)^{p_1} & - & - & - \\ (p_1^2 - p_1) \left(\frac{r-1}{r}\right)^{p_1} & - & - & - \end{vmatrix} = 0 \qquad (24)$$

1.3 Derivation of the Frequency Equation for Anti-Symmetrical Vibration

For anti-symmetrical vibration, we have the following conditions at the root:-

$$w(0,y) = 0 \qquad \text{as before} \tag{13}$$

and

$$w_{xx}(0,y) = 0 \tag{25}$$

Proceeding as in Section 1.2 we again obtain equations (15), (17) and (18) as three of the boundary conditions, the fourth being

$$[\theta'']_{x=0} = 0 \tag{26}$$

Substituting

$$\theta = \sum_{s=1-4} A_s \, x_1^{\,p_s}$$

in equation (26) gives

$$\sum_{s=1-4} A_s \, p_s \, (p_s - 1) \, (r\ell)^{p_s - 2} = 0 \tag{27}$$

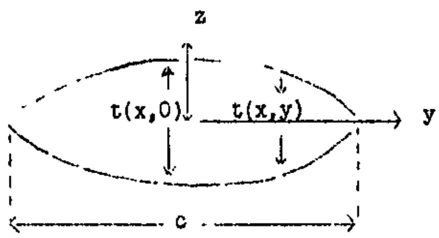
The other three boundary conditions are the same as (20), (22) and (23) of the symmetrical case.

For a solution of (20), (22), (23) and (27) other than the vanishing of the constants A_1, A_2, A_3, A_4 :-

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ p_1(p_1-1) & p_2(p_2-1) & p_3(p_3-1) & p_4(p_4-1) \\ (p_1^3 - p_1 - 16\, r^2 \lambda^2 \, p_1)\left(\frac{r-1}{r}\right)^{p_1} & - - - & - - - & - - - \\ (p_1^2 - p_1)\left(\frac{r-1}{r}\right)^{p_1} & - - - & - - - & - - - \end{vmatrix} = 0 \tag{28}$$

APPENDIX II

Chordwise Sections Other Than Rectangular



For a tapered wing with a doubly symmetrical chordwise section,

$$\left.\begin{aligned} a_3 &= \frac{D(0,0)\ c_o^3\ x_1^9}{4\ r^9\ \ell^9} \int_0^1 \{f\ (p)\}^3\ p^2\ dp \\ a_1 &= D(0,0)\ c_o \frac{x_1^7}{r^7\ \ell^7} \int_0^1 \{f\ (p)\}^3\ dp \\ \text{and} \quad S_3 &= \frac{m(0,0)\ c_o^3\ x_1^5}{4\ r^5\ \ell^5} \int_0^1 f\ (p)\ p^2\ dp \end{aligned}\right\} \qquad (29)$$

where

$$y = \frac{c}{2}p \text{ and } \frac{t(x,y)}{t(x,0)} = f\ (p) \qquad (30)$$

The differential equation (6) becomes

$$\frac{D(0,0)c_o^3}{4} \int_0^1 \{f(p)\}^3 p^2 dp \left(\frac{x_1^9}{r^9 \ell^9}\ \theta''\right)'' - 2(1-\nu)D(0,0)c_o \int_0^1 [f(p)]^3\ dp \left[\frac{x_1^7}{r^7 \ell^7}\ \theta'\right]'$$

$$- \frac{m(0,0)c_o^3\ x_1^5}{4\ r^5\ \ell^5} \int_0^1 f(p)\ p^2\ dp\ \omega^2\ \theta = 0 \quad (31),$$

which reduces to

$$\left[\frac{x_1^9}{r^9}\ \theta''\right]'' - 16\ k_1^2\ \lambda^2 \left[\frac{x_1^7\ \theta'}{r^7}\right]' - 4\ k_1^2\ \lambda^2\ \pi^2 \left(\frac{\omega}{k_2 \omega_{St.V_o}}\right)^2 \frac{x_1^5\ \theta}{r^5} = 0 \qquad (32)$$

where

$$k_1^2 = \frac{\int_0^1 [f(p)]^3 \, dp}{3\int_0^1 [f(p)]^3 \, p^2 \, dp} , \qquad k_2^2 = \frac{\int_0^1 [f(p)]^3 \, dp}{3\int_0^1 f(p) \, p^2 \, dp} . \qquad (33)$$

The parameters k_1 and k_2 are constant for a given section.

This differential equation is the same as the one obtained for the rectangular chordwise section with λ replaced by $k_1\lambda$ and $\omega_{St.V_o}$ by $k_2\omega_{St.V_o}$.

Similarly the boundary conditions can be shown to be the same after using this replacement.

The results obtained for the rectangular section therefore give results for other sections in terms of $k_1\lambda$ and $k_2\omega_{St.V_o}$ instead of λ and $\omega_{St.V_o}$.

Similarly the results obtained for the untapered wing of rectangular chordwise section can be used to obtain results for sections other than rectangular.

APPENDIX III

St. Venant Frequency for Tapered Wings, $\omega_{St.V_t}$

When axial warping constraints are ignored, (the St. Venant method), the differential equation for torsional vibration is

$$J\ddot{\theta} = \frac{d}{dx}\left(C\,\frac{d\theta}{dx}\right) \qquad (34)$$

For a thin wing

$$J = \int_{-c/2}^{+c/2} m(x,y)\, y^2\, dy = S_3 .$$

$$C = 2(1-\nu)\int_{-c/2}^{+c/2} D(x,y)dy = 2(1-\nu)a_1$$

and equation (34) becomes

$$S_3\ddot{\theta} = 2(1-\nu)\,\frac{d}{dx}\left(a_1\,\frac{d\theta}{dx}\right) \qquad (35)$$

For the tapered wing vibrating with simple harmonic motion equation (35) becomes, on using equations (8) and (10),

$$4\,x_1^2\,\theta'' + 28\,x_1\,\theta' + \pi^2 r^2\left(\frac{\omega_{St.V_t}}{\omega_{St.V_o}}\right)^2\theta = 0 \qquad (36)$$

A solution of the above differential equation is of the form $x_1^{\,p}$ where p satisfies the following equation:-

$$4\,p^2 + 24\,p + r^2\pi^2\left(\frac{\omega_{St.V_t}}{\omega_{St.V_o}}\right)^2 = 0 \qquad (37)$$

and the general solution is

$$\theta = Ax_1^{p_1} + Bx_1^{p_2} \tag{38}$$

where p_1 and p_2 are the roots of equation (37).

The boundary conditions are

$$[\theta]_{x_1=r\ell} = 0 \tag{39}$$

$$[\theta']_{x_1=(r-1)\ell} = 0 \tag{40}$$

Substituting (38) into (39) and (40) gives

$$A(r\ell)^{p_1} + B(r\ell)^{p_2} = 0 \tag{41}$$

$$A\, p_1\, [(r-1)\ell]^{p_1-1} + B\, p_2\, [(r-1)\ell]^{p_2-1} = 0 \tag{42}$$

The frequency equation is the condition for a solution of (41) and (42) other than the vanishing of the constants A and B, i.e.

$$\begin{vmatrix} (r\ell)^{p_1} & (r\ell)^{p_2} \\ p_1\, [(r-1)\ell]^{p_1-1} & p_2\, [(r-1)\ell]^{p_2-1} \end{vmatrix} = 0 \tag{43}$$

which becomes

$$q \cos\gamma + p \sin\gamma = 0 \tag{44}$$

where

$$\left.\begin{aligned} p_1 &= p + iq \\ p_2 &= p - iq \\ \gamma &= q \log_e\left(\frac{r-1}{r}\right) \end{aligned}\right\} \tag{45}$$

and

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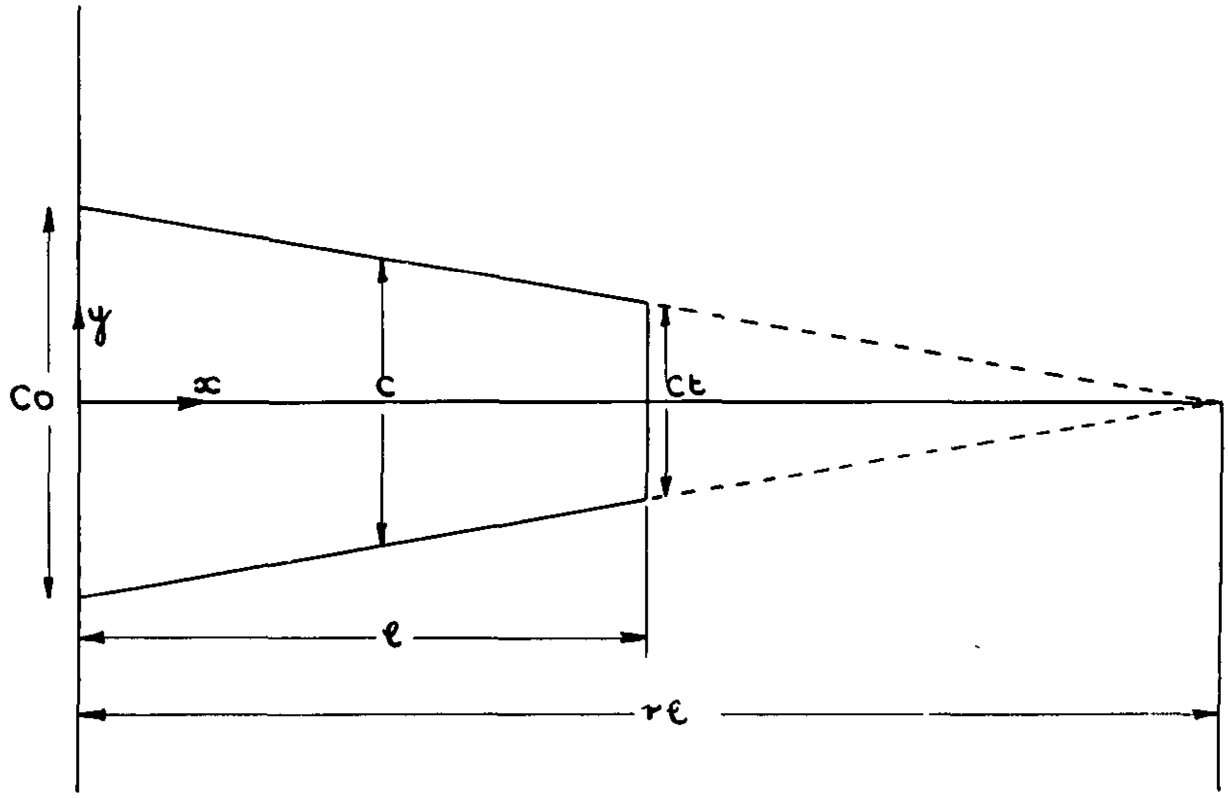
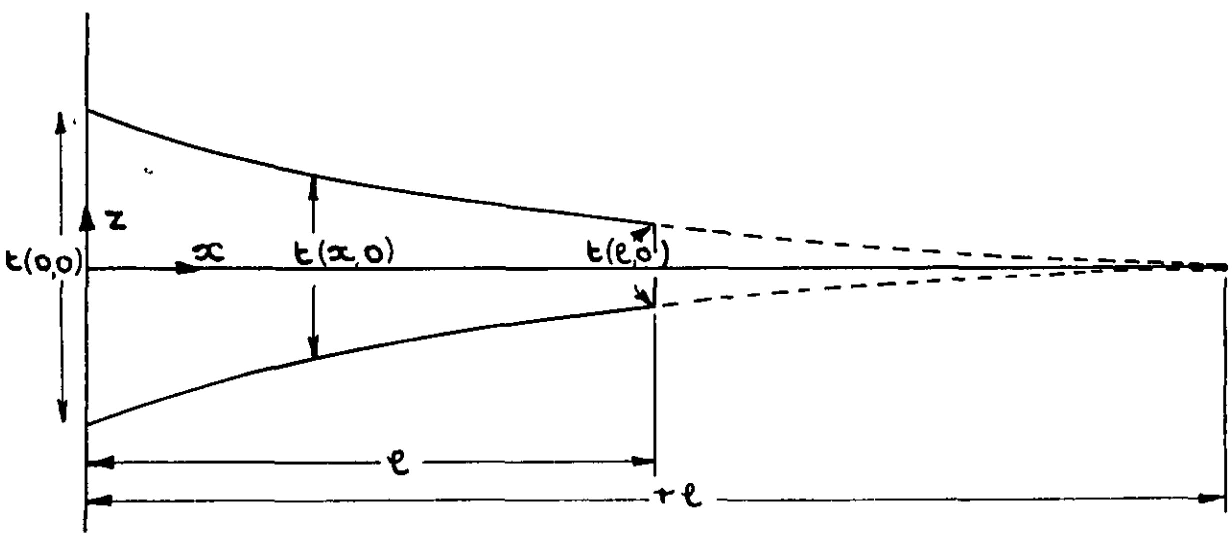
FIG.1.



(a) PLAN FORM



(b) SECTION ALONG THE SPAN SHOWING VARIATION OF THICKNESS.

FIG.1. THE GEOMETRY OF THE WING

FIG. 2.

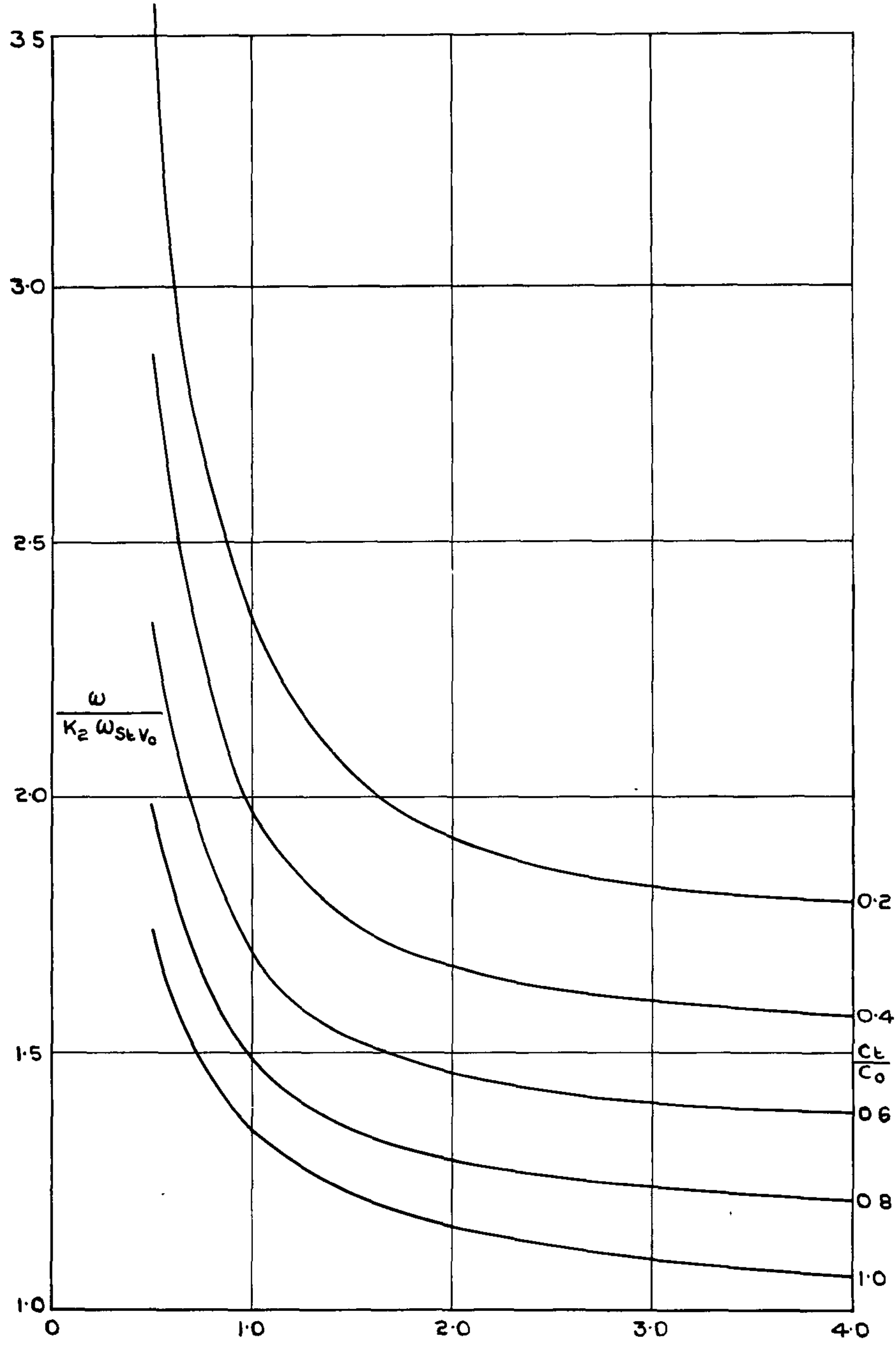


FIG. 2. FUNDAMENTAL FREQUENCIES FOR SYMMETRICAL VIBRATION.

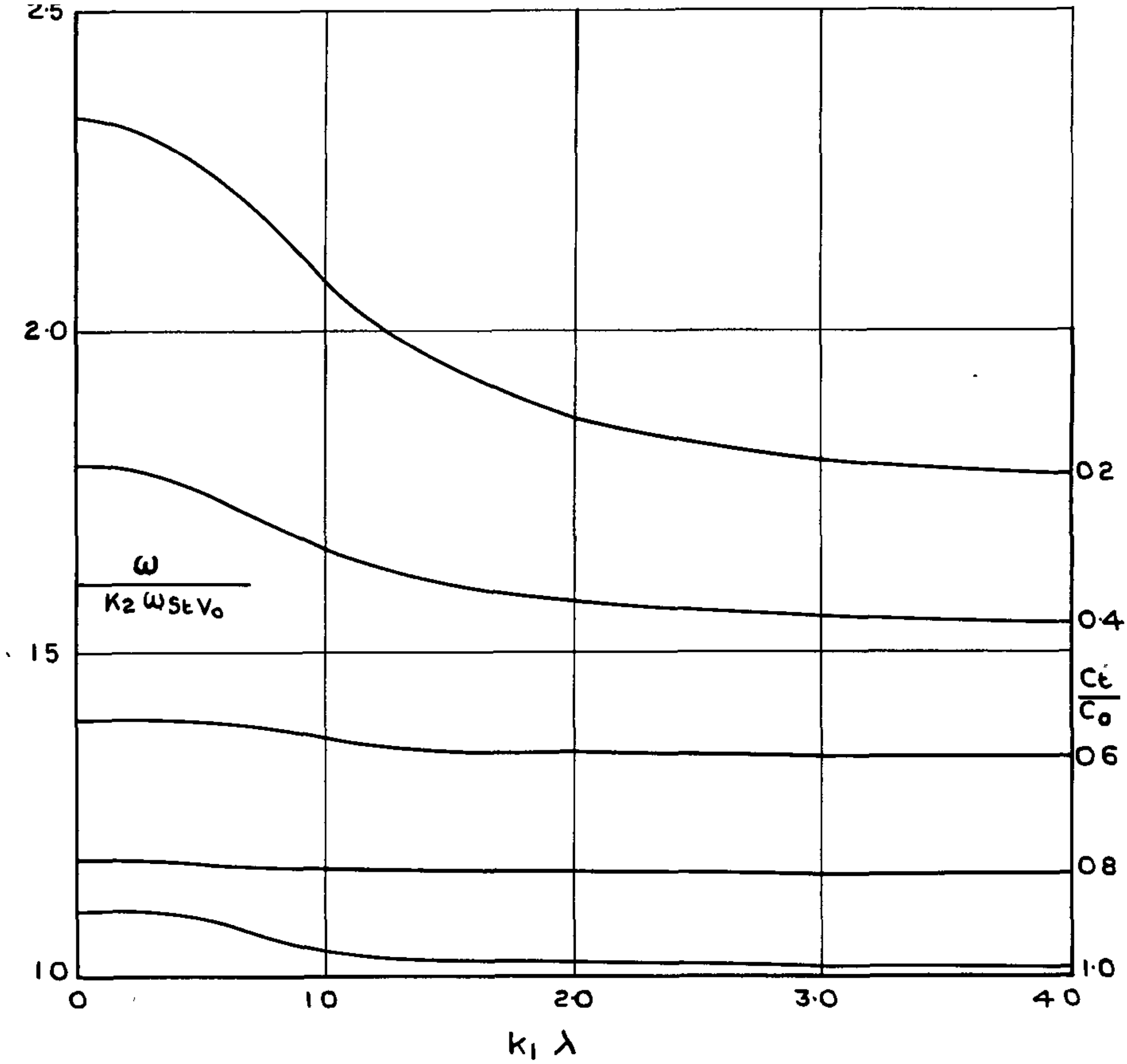


FIG. 3. FUNDAMENTAL FREQUENCIES FOR ANTI-SYMMETRICAL VIBRATION.

FIG. 4.

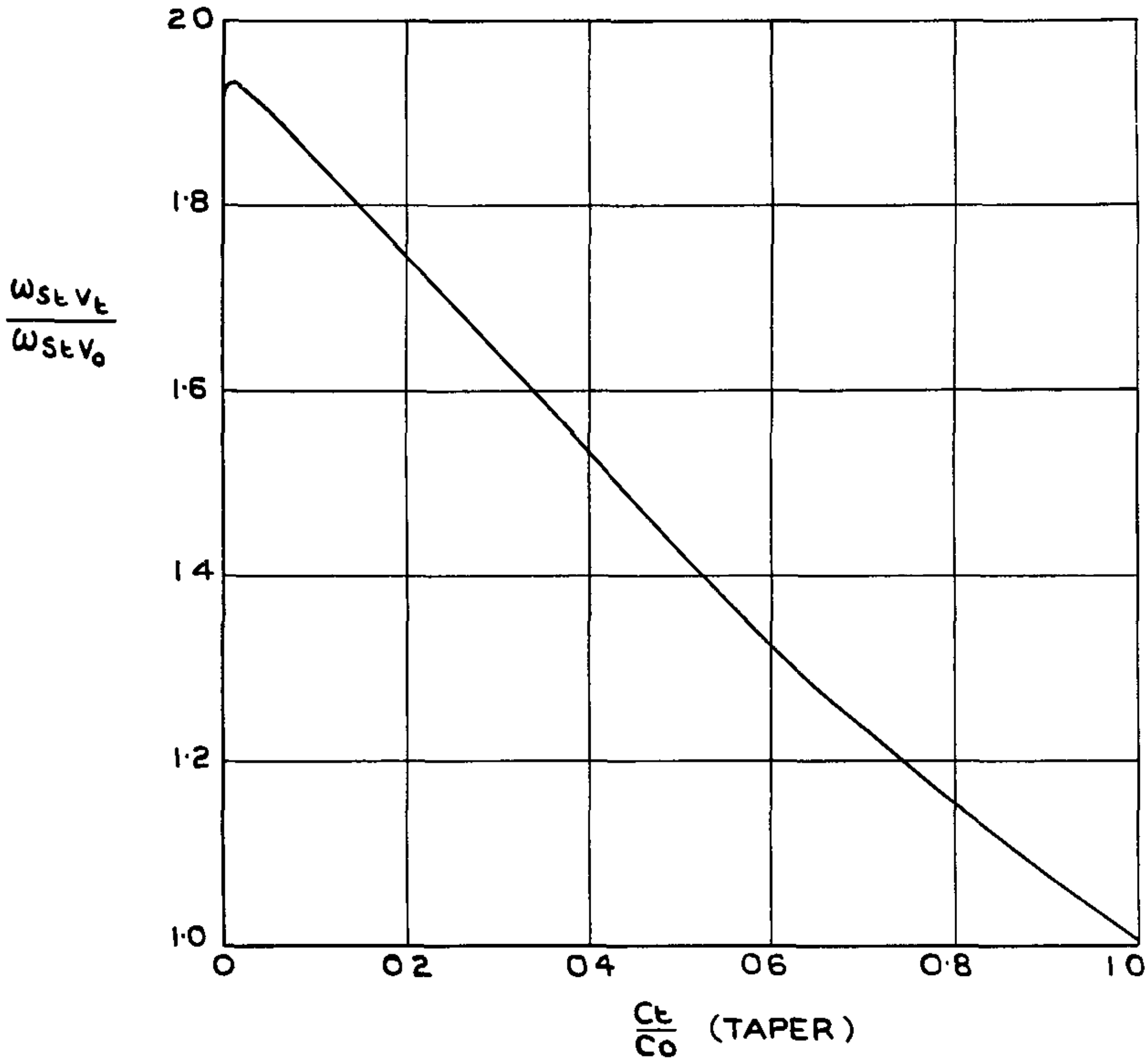


FIG.4. VARIATION OF THE "ST VENANT" FREQUENCY WITH TAPER.

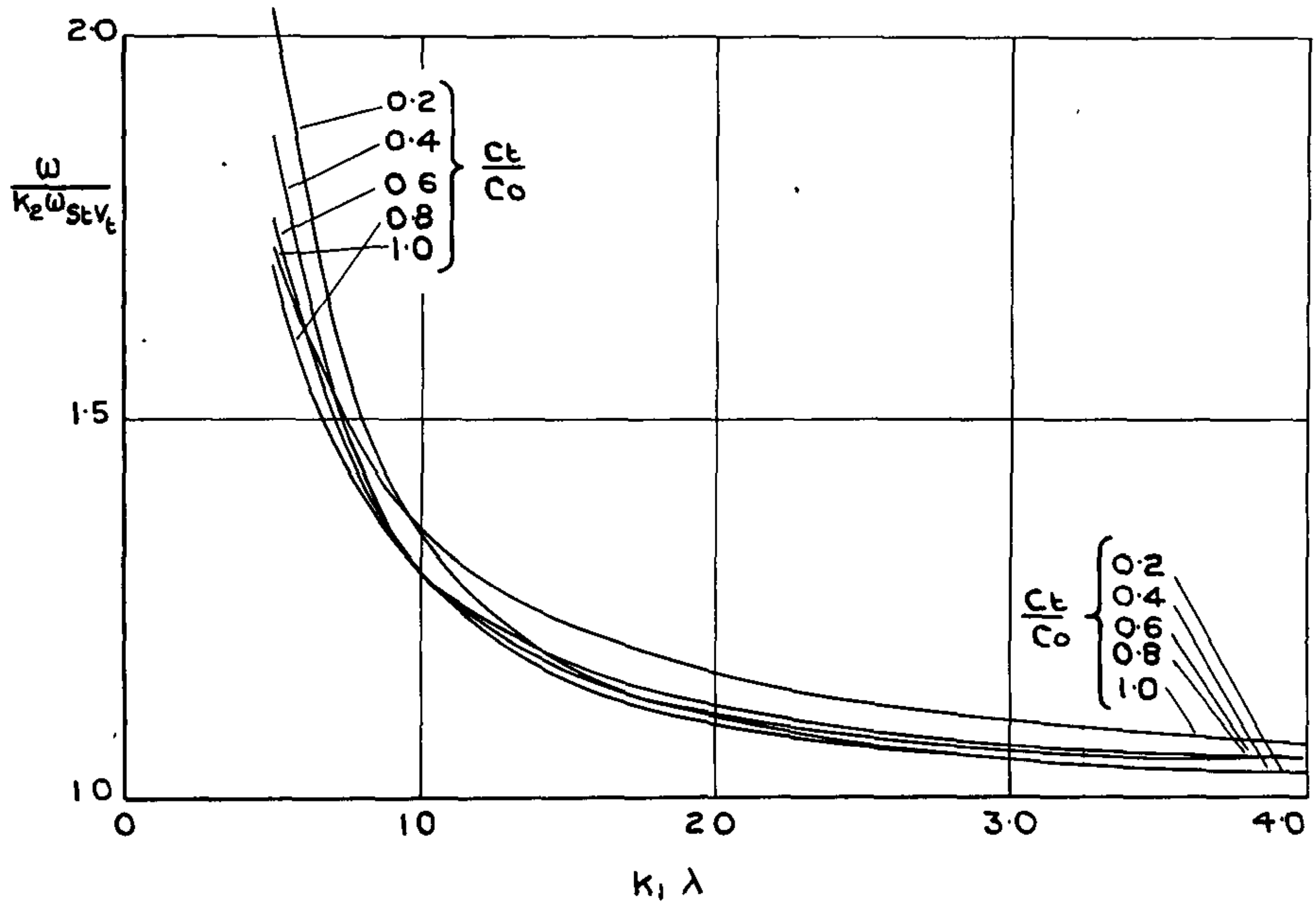


FIG.5.(a) FUNDAMENTAL FREQUENCIES FOR SYMMETRICAL VIBRATION PLOTTED AS RATIOS OF THE "ST. VENANT" FREQUENCIES.

FIG. 5.(b)

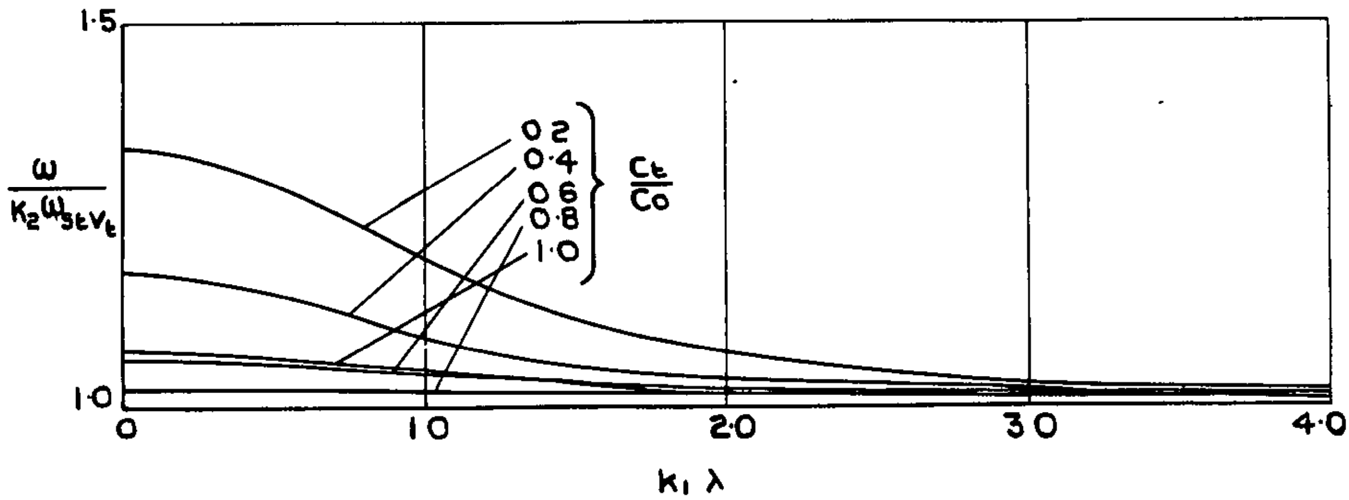


FIG 5 (b) FUNDAMENTAL FREQUENCIES FOR ANTI-SYMMETRICAL VIBRATION PLOTTED AS RATIOS OF THE "ST VENANT" FREQUENCIES.

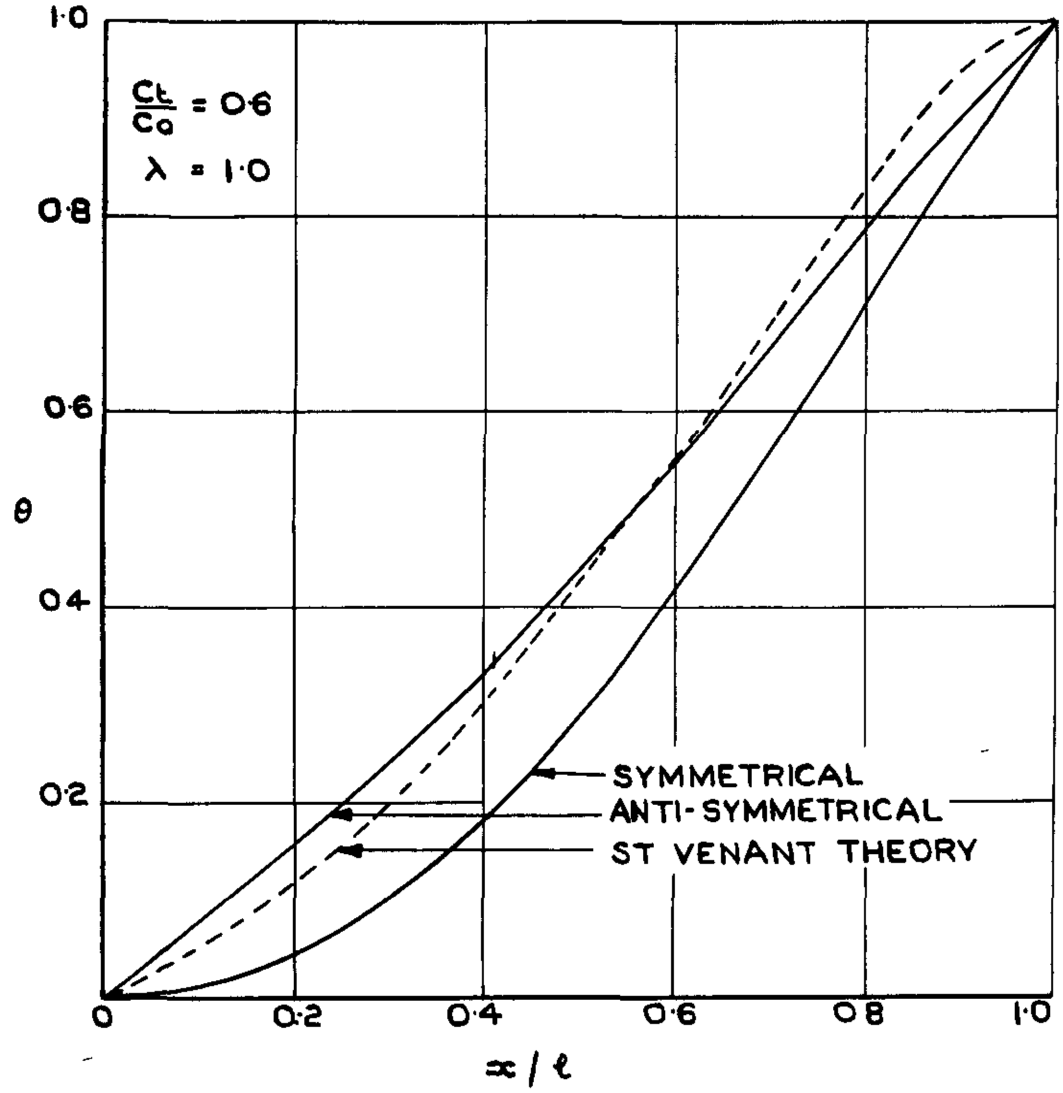


FIG.6. DEFLECTION MODES FOR A PARTICULAR WING.

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