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# The Influence of Surface Waves on the Stability of a Laminar Boundary Layer with Uniform Suction

*By*

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# The influence of surface waves on the stability of a laminar boundary layer with uniform suction

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SUMMARY

In order to estimate the destabilising effect of the waves likely to be encountered on wing surfaces which will be used with boundary layer suction, calculations have been made of the effect of small sinusoidal surface waves on the stability of the asymptotic suction profile. Curves are presented of the percentage increases in local suction flow $\frac{v_s}{U}$ necessary to maintain the stability of the boundary layer at the same level as on a completely flat surface, for various values of the variables $\frac{v_s}{U}$, height:wavelength ratio $\frac{h}{L}$ and Reynolds number based on wavelength, $\frac{UL}{\nu}$. These should provide quantitative estimates for more general cases. It is found, as might have been expected, that the lower $\frac{v_s}{U}$ or the larger $\frac{h}{L}$, the larger the necessary percentage increase in $\frac{v_s}{U}$, especially for low $\frac{UL}{\nu}$. 10 per cent is a typical magnitude for the necessary increase at a high Reynolds number.

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## LIST OF SYMBOLS

$U$ Free stream velocity at infinity

$U_1$ Velocity at the edge of the boundary layer

$h$ Amplitude of the waves on the surface

$L$ Wave-length of the waves on the surface

$x$ Distance measured along surface

$y$ Distance measured normal to surface

$u$ Velocity in x-direction in the boundary layer

$v$ Velocity in y-direction in the boundary layer

$\delta^*$ Displacement thickness of the boundary layer

$\theta$ Momentum thickness of the boundary layer

$H = \frac{\delta^*}{\theta}$

$v_s$ Suction velocity

$v_{s_o}$ Suction velocity for aerodynamically flat surface

$u'$, $v'$, $x'$, $y'$, $\delta^{*'}$, $\theta'$, $U_1'$, dimensionless variables (See equation 3 Appendix I)

$a_o$ Amplitude of fluctuations produced in the velocity at the edge of the boundary layer

$a_1$, $a_2$, $a_3$, ... Coefficients in the form taken for the velocity profile

$A_n = a_n e^{ix} \quad (n = 0,1,2,....)$

$\lambda = \frac{v_s}{U}\sqrt{\frac{UL}{2\pi\nu}}$

$\sigma = \lambda^{-1/3}, \quad \tau = \frac{\sigma}{1+\sigma}$

$Y = \sigma y$

For $I$, $\eta$, $t$ and $x_1$, see (ii) in Appendix I

$R = \frac{UL}{\nu} \qquad R_{\delta^*} = \frac{U\delta^*}{\nu}$

$$\beta = \frac{R_{\delta^*}}{(R_{\delta^*})_{crit}}$$

The suffix crit to a Reynolds number means the maximum Reynolds number below which all disturbances are damped.

## 1 Introduction

The minimum suction quantities required to preserve laminar flow in the boundary layer on an aerofoil are calculated from stability theory in which it is assumed that the surface is aerodynamically clean. However it has not yet been possible to obtain a porous surface entirely satisfying this condition, and it seems probable that under production and flight conditions the presence of shallow longitudinal waves and of small excrescences will be unavoidable. Such surface imperfections are known to have a destabilising effect on the boundary layer, and it is important to have some estimate of the penalty in increased suction which is likely to be associated with them.

The effect of protuberances is discussed in a paper presented to the Boundary Layer Control Committee by P.R. Owen and Miss Klanfer[1], and the present paper contains calculations of the effect of surface waviness. To avoid mathematical complexity we consider the hypothetical case of flow over a sinusoidal surface of height $y = h \cos \frac{2\pi x}{L}$, with suction conditions asymptotic in the sense that the boundary layer thickness and shape is the same at corresponding points in successive waves. If the height : wavelength ratio $\frac{h}{L}$ were large enough such conditions would be impossible, but for practical values of $\frac{h}{L}$, of the order of $10^{-3}$, a linearised solution may be obtained by excluding squares of $\left(\frac{h}{L}\right)$. The process is thus justifiable a posteriori. The method and results are outlined in the main part of the paper, the mathematical details being given in the Appendices.

A possible objection to the applicability of the results is that asymptotic conditions are not to be expected on a wing because less suction is required for stability than is necessary to produce them, and because the boundary layer increases in thickness under the influence of pressure gradients. However the calculations have been designed to show the percentage increase in suction necessary when waves are present to preserve the stability of the worst profile which occurs at the same level as for an aerodynamically flat surface; and it may be conjectured that this percentage is relatively independent of the basic profile shape. In general different relative increases in suction will be necessary at different chordwise points, since the increase required depends on the amount already used, which follows a definite chordwise distribution (increasing considerably when the adverse pressure gradient is reached).

## 2 Stability theory for parallel flows

The mathematical theory of stability as developed by Lin[2] and others relates to parallel flows with non-dimensional velocity profiles $\frac{u}{U}(y)$, and considers the variation in the x-direction of the energy of disturbances with wave numbers $\alpha$ [$= 2\pi$/wavelength]. For each profile there is a neutral stability curve in the $(\alpha, R_{\delta_*})$ plane. Disturbances whose coordinates lie within the curve will be amplified and lead eventually to transition; all others will be damped. The curve is typically of the shape shown in Fig.1. The lower branch tends asymptotically to $\alpha = 0$ as $R_{\delta*} \to \infty$; the upper may tend to 0 or to a finite value, depending on the profile shape.

The maximum Reynolds number below which all disturbances are damped is known as $(R_{\delta*})_{critical}$. A measure of the instability of a particular profile shape at a given Reynolds number is provided by the area included between its neutral stability curve and the ordinate in question - e.g. by the shaded area in the diagram.

Parallel flow is of course never achieved with a solid boundary, but it is customary to calculate the neutral stability curve of a profile as the curve it would have in parallel flow, since the rate of growth or decay of a disturbance is considerably larger than that of the boundary layer. Similarly the influences of pressure gradients and suction on stability are calculated purely in terms of their effect on profile shape.

Suction quantities for maintaining laminar flow are calculated from the condition that $R_{\delta*}$ should be not greater than $(R_{\delta*})_{crit}$ at any point. This condition is sufficient, but by no means necessary, since even if it is not fulfilled no disturbances in the wave number range capable of amplification may be present in the boundary layer. A convenient measure of the extent to which the requirement is met is provided by the ratio $\frac{R_{\delta*}}{(R_{\delta*})_{cr}} = \beta$ (say). The present calculations have been designed to find out how much the local suction flow $\frac{v_s}{U}$ must be increased when the surface contains waves of a given height and wavelength, in order to maintain $\beta$ at the same value ($\leq 1$) as it would have if there were no waves present. These results should be quantitatively comparable to those which would be obtained if it were possible to deal with flow under more general conditions than the asymptotic.

## 3 Outline of calculations

The effect of waves in the surface is to produce a periodic variation of the same wavelength in the velocity at the edge of the boundary layer and in the velocity profile shape. In general there is found to be a phase shift in the profile shape, different at different distances from the wall. The basic velocity profile is the asymptotic profile

$$u = U\left(1 - e^{-v_s y/\nu}\right) \tag{1}$$

or, writing $y'$ for $\frac{v_s y}{\nu}$ and $u'$ for $\frac{u}{U}$, the non-dimensional profile

$$u' = 1 - e^{-y'}$$

The effect of waves is to alter the profile shape in a way which may be represented by the equation

$$u' = 1 - e^{-y'} + \frac{h}{L} R\, F(y') e^{2\pi i x/L} \tag{2}$$

Here the (complex) function $F(y)$ represents a perturbation to the velocity profile, which vanishes at $y = 0, \infty$. Using the equations of motion and continuity, and the boundary layer momentum equation, $F(y)$ may be calculated in the form

$$F = e^{-Y}\left(1 + \frac{a_1}{a_o} Y + \frac{a_2}{a_o}\frac{Y^2}{2!} + \dots\right) \tag{3}$$

where $Y = \sigma y'$, $\sigma$ being an arbitrary constant chosen to give rapid convergence. The coefficients in the resulting series depend only on the non-dimensional variable

$$\frac{v_s}{U}\sqrt{\frac{UL}{2\pi\nu}} = \lambda \text{ (say)} \tag{4}$$

The analysis is carried out in detail in Appendix I. It is shown there that the magnitude of the perturbation function F increases as $\lambda$ decreases, so that in general the shorter the wavelength or the lower the suction velocity for a fixed $\frac{h}{L}$, the more a wave on the surface disturbs the profile shape.

The displacement and momentum thicknesses $\delta^*$ and $\theta$ and the form parameter $H = \frac{\delta^*}{\theta}$ may be calculated from the velocity profile (2), and are periodic about the mean values 1, $\frac{1}{2}$ and 2 respectively. In Fig.2 examples are shown of the velocity profiles at different points on a wavelength, together with the variation of $H, \theta$ and the (non-dimensional) velocity $U_1$ at the edge of the boundary layer, for the values $\lambda = 1$, $\lambda = 0.125$ and $\lambda = 10^{-3/2} = 0.03162$. [The value of $a_o = \frac{2\pi h}{L}$ has been chosen in each case to give an appreciable variation in profile shape; for $\lambda = 1$ this requires $a_o = 0.1$, which is much larger than the practical values anticipated.]

The profile with lowest $(R_{\delta^*})_{crit}$ in a wavelength is that for which H is greatest. $(R_{\delta^*})_{crit}$ corresponding to this profile has been calculated as a function of $\lambda$ and $\frac{h}{L}$, using the fact that $(R_{\delta^*})_{cr}$ for a range of profile shapes investigated by several authors has been found[3] to be a single function of H. The applicability of this result for a profile of the present family has been tested and found satisfactory. [Appendix II].

For the profiles considered $R_{\delta^*}$ has to the first order its value for the asymptotic profile, namely $\frac{U}{v_s}$. Thus

$$\beta = \frac{R_{\delta^*}}{(R_{\delta^*})_{cr}} = \frac{U}{v_s(R_{\delta^*})_{crit}} \tag{5}$$

For the asymptotic profile $(R_{\delta^*})_{crit} = 4 \times 10^4$. Thus the condition of keeping $\beta$ the same when there are waves in the surface of length L and height h as when the surface is devoid of waves and the suction velocity is $v_{s_o}$, may be written

$$\frac{v_s}{U}\,(R_{\delta^*})_{cr} = 4 \times 10^4\,\frac{v_{s_o}}{U} \tag{6}$$

$(R_{\delta^*})_{cr}$ is a function of $\frac{h}{L}$ and $\lambda$, and thus, for fixed $\frac{h}{L}$ and $\frac{UL}{\nu}$, of $\frac{v_s}{U}$ only. Thus for fixed $\frac{h}{L}$ and $\frac{UL}{\nu}$, the value of $\frac{v_s}{U}$ corresponding to a given $\frac{v_{s_o}}{U}$ was found graphically from equation (6). The results, in the form of curves of $\frac{v_s}{v_{s_o}}$ against $\frac{v_{s_o}}{U}$, for four values of $\frac{UL}{\nu}$ in the range $6 \times 10^3 - 10^6$, are shown in Fig.3.

In the curves shown $\frac{v_s}{v_{s_o}} \to \infty$ as $\frac{v_{s_o}}{U} \to 0$. The value of $\frac{v_s}{v_{s_o}}$ is however rather meaningless under these conditions; $v_s$ itself still has a small finite value. If $v_s$ were plotted against $v_{s_o}$, the result would be a series of lines of slope approximately 1, with intercepts on the $v_s$

axis determined by $\frac{h}{L}$, and tending to 1 as $\frac{h}{L}$ tends to 0. The curves for $\frac{UL}{\nu} = 3 \times 10^5$ drawn in this way are shown Fig.3(e).

## 4 Numerical example

Curvature gauge readings obtained by D. Johnson on the wing of a production Vampire, and by Dr. Lachmann[4] on the wind tunnel model of the sleeve for the Handley Page experiments suggest that the irreducible minimum of $\frac{h}{L}$ is approximately $10^{-3}$. The surfaces in both cases are by no means pure sinusoidal, although such a shape might well be produced by stringers at regular chordwise intervals. A harmonic analysis of the surface shape would strictly be necessary to find the wavelengths and their heights in the form assumed in the analysis.

As an example, consider an aircraft cruising at 350 knots at 40,000 feet, and a part of the wing where the local suction flow $\frac{v_{s_0}}{U} = 0.0014$. The Reynolds number per inch is $10^5$. For a wave 3 inches long and $^3/_{1000}$ inch high $\frac{UL}{\nu} = 3 \times 10^5$, $\frac{h}{L} = 10^{-3}$, and the required increase $\frac{v_s}{v_{s_0}}$ is 1.07, i.e. 7% more suction is required to maintain the same stability at the worst point in the profile as in the absence of waves. With this increase, the profile will actually be more stable than before at all other points. On the other hand, for a wave 10 inches long and $^1/_{100}$ inch high, $\frac{UL}{\nu} = 10^6$ while $\frac{h}{L}$ is still $10^{-3}$, but less than 1% increase in suction is necessary. Thus the longer the wave, the smaller the necessary increase.

Conversely if $\frac{v_{s_0}}{U}$ is smaller, a larger percentage increase is needed for waves of the same length and height. Thus with the above data, suppose $\frac{v_{s_0}}{U} = 0.0007$. For waves 3 inches long $\frac{v_s}{v_{s_0}} = 1.15$, and for waves 10 inches long $\frac{v_s}{v_{s_0}} = 1.07$.

## 5 Discussion

The results should provide an upper limit for the percentage increase in suction necessary to preserve stability over a wavy surface. In fact with the calculated increase in suction the profile will actually by considerably more stable than for the flat surface at all points in a wavelength except the most unfavourable. A typical value for the increase would be of the order of 10 per cent. However the limit depends on the particular level of stability chosen in calculating the suction necessary without surface waves, as well as on the variables already mentioned. It might be expected that rather smaller increases in suction than this limit would be sufficient to counteract the effect of waves, since regions of rising and falling pressure alternate along the surface. When the pressure is falling a disturbance will be damped considerably more than at the most "unfavourable" point in a wavelength, for which the calculations have been made. Attention might be drawn to Pretsch's[5] interesting explanation, for boundary layers without suction, of the fact that increase of Reynolds number can lead to transition suddenly jumping forward a whole wave length. Disturbances in the wavelength ahead of that in which transition took place at the lower Reynolds number, which were insufficiently amplified for transition in the adverse gradient, and subsequently damped in the favourable gradient, are at a higher Reynolds number sufficiently amplified in the adverse gradient. This could also apply to flow with insufficient suction to preserve a stable profile at all points in a wavelength.

The fact that waves on the surface, by disturbing the velocity profile in a periodic manner, themselves provide a potential instability, has not been considered in these calculations, since it is assumed that sufficient suction will be applied to damp out all oscillations whatever their source. Thus the results obtained are fundamentally different from those of Fage[6], which express the maximum permissible wave height in terms of the length of laminar flow before transition, and Reynolds number, on an aerofoil without suction. In Fage's case the waves themselves are responsible for the instability.

The behaviour of the boundary layer in the limit when $\frac{v_s}{U} = 0$ is not given by these calculations, since asymptotic conditions cannot then occur. In fact Quick and Schroder[7] have shown that separation will occur fairly rapidly for waves of sufficient "size" $\frac{h}{L}$.

In contrast to the effect of surface protuberances[1], which becomes more serious as suction is increased, it is seen that the destabilising effect of surface waves is alleviated by increased suction. Presumably for a surface not completely clean aerodynamically, and containing waves, a compromise must be found between the two requirements. There is also the possibility of a resonance effect between the periodicity of the profile and of the surface itself, for wave numbers near the critical. Some calculations of the suction quantities for which this might occur are given in Appendix III.

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## APPENDIX I

### Calculation of Velocity Profiles on a Wavy Surface with Asymptotic Suction

Consider the boundary layer flow over a wavy surface, with free stream velocity $U$ at infinity and $U_1$ at the edge of the boundary layer. The height of the surface is given by $y = h \cos\left(\frac{2\pi x}{L}\right)$, where $\frac{h}{L}$ is small and its square negligible.

The equations of motion and of momentum are respectively

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U_1 \frac{dU_1}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \quad , \tag{1}$$

$$\frac{d\theta}{dx} + (H+2) \frac{\theta}{U_1} \frac{dU_1}{dx} = \frac{\nu}{U_1^2} \left(\frac{\partial u}{\partial y}\right)_o - \frac{v_s}{U_1} \, , \tag{2}$$

where $H = \frac{\delta^*}{\theta}$ .

If non-dimensional variables defined by

$$u' = \frac{u}{U} \, , \; v' = \frac{v}{v_s} \, , \; x' = \frac{2\pi x}{L} \, , \; y' = \frac{v_s y}{\nu} \, , \; \theta' = \frac{v_s \theta}{\nu}$$

$$\delta^{*\prime} = \frac{v_s \delta^*}{\nu} \, , \; u_1' = \frac{U_1}{U} \tag{3}$$

are introduced, the equations may be written

$$u' \frac{\partial u'}{\partial x'} + \left(\frac{v_s}{U}\right)^2 \frac{UL}{2\pi\nu} v' \frac{\partial u'}{\partial y'} = u_1' \frac{du_1'}{dx'} + \left(\frac{v_s}{U}\right)^2 \frac{UL}{2\pi\nu} \frac{\partial^2 u'}{\partial y'^2} \tag{4}$$

$$\frac{d\theta'}{dx'} + (H+2) \frac{\theta'}{u_1'} \frac{du_1'}{dx_1'} = \left(\frac{v_s}{U}\right)^2 \frac{UL}{2\pi\nu} \left[\frac{1}{u_1'^2} \left(\frac{\partial u'}{\partial y'}\right)_{y'=0} - \frac{1}{u_1'}\right] \tag{5}$$

Dropping the primes, and introducing the non-dimensional parameter

$$\lambda = \frac{v_s}{U} \sqrt{\frac{UL}{2\pi\nu}} \tag{6}$$

the equations are

$$u \frac{\partial u}{\partial x} - u_1 \frac{du_1}{dx} = \lambda^2 \left[ \frac{\partial^2 u}{\partial y^2} - v \frac{\partial u}{\partial y} \right] \tag{7}$$

$$\frac{d\theta}{dx} + (H+2) \frac{\theta}{u_1} \frac{du_1}{dx} = \lambda^2 \left[ \frac{1}{u_1^2} \left(\frac{\partial u}{\partial y}\right)_0 - \frac{1}{u_1} \right] \tag{8}$$

With boundary conditions $u = \frac{\partial u}{\partial x} = 0$, $v = -1$, at $y = 0$, waves on the surface will produce a fluctuation in the velocity at the edge of the boundary layer so that

$$u_1 = 1 + a_o e^{ix} = 1 + A_o \text{ , say} \tag{9}$$

(We work throughout with complex quantities, of which only the real parts are relevant).

$$\frac{du_1}{dx} = i a_o e^{ix} = iA_o \text{ ,} \tag{10}$$

where $a_o$ is of the order of $\frac{h}{L}$. A reasonable form for the velocity profile is then

$$u = 1 - e^{-y} + a_o e^{ix} - e^{-Y} (a_o + a_1 Y + a_2 \frac{Y^2}{2!} + \ldots) e^{ix}$$

$$= 1 - e^{-y} + A_o - e^{-Y} (A_o + A_1 Y + A_2 \frac{Y^2}{2!} + \ldots), \tag{11}$$

where $Y = \sigma y$ and $\sigma$ is some suitably chosen constant. $A_n = a_n e^{ix}$ and $\frac{dA_n}{dx} = iA_n$. Squares and products of the $A_n$'s will be assumed negligible. This form may be made to satisfy any number of boundary conditions at $y = 0$, (i.e. at $Y = 0$), obtained by differentiating the equation of motion, and it automatically satisfies the boundary layer equations at the outer edge. Thus by finding all the coefficients of the infinite series, an exact solution would be given, since the functions $\frac{Y^n}{n!} e^{-Y}$ form a complete set from 0 to $\infty$.

Practical values of $\lambda$ may be small compared with 1, and with the obvious choice of $\sigma = 1$, it is found that the quantities $\frac{a_n}{a_o}$ increase as $\lambda^{n/3}$, so that a very large number of boundary conditions must be used to secure convergence of the series in brackets. Apart from the numerical difficulties thus raised, boundary conditions obtained by repeated differentiation with respect to $y$ become increasingly inaccurate because of the inherent approximation in the boundary layer equation. It is, therefore, desirable to use as few terms as possible, and this may best be achieved by equating $\sigma$ to a negative power of $\lambda$ . The relation

$$\sigma = \lambda^{-1/3} \qquad (12)$$

has been chosen.

The calculation now proceeds as follows:-

(i) The quantity $\left|\frac{H-2}{A_o}\right|$ is calculated as a function of $\lambda$ by means of the momentum equation and the equation of motion.

(ii) $A_o$ is related to $\frac{h}{L}$ by considering the potential flow past a wavy wall.

(iii) By means of the relation between H and $(R_{\delta^*})_{crit}$ found by Lin, $(R_{\delta^*})_{crit}$ is calculated for a number of values of $\frac{h}{L}$, $\frac{UL}{\nu}$ and $\frac{v_s}{U}$ as described in the main part of the report, (Section 3).

(i) <u>Coefficients in velocity profile</u>

For the profile (11),

$$u_1 - u = e^{-y} + e^{-Y}\,[A_o + A_1 Y + \ldots]$$

$$\delta^* = \frac{1}{u_1}\int_0^\infty (u_1 - u)\,dy = 1 - A_o + \frac{1}{\sigma}\,[A_o + A_1 + \ldots] \qquad (13)$$

$$\theta = \frac{1}{u_1^{\,2}}\int_0^\infty u(u_1 - u)\,dy = \int_0^\infty \Big\{e^{-y}\,(1 - e^{-y}) - 2e^{-y-Y}[A_o + A_1 Y + \ldots]$$

$$+ e^{-Y}\,[A_o + A_1 Y + \ldots]\Big\}\,dy ,$$

(excluding squares and products of the A's),

$$\text{i.e.} \qquad \theta = \frac{1}{2} + \frac{1}{\sigma}\sum_{n=0}(1 - 2\tau^{n+1})\,A_n , \qquad (14)$$

where $\tau = \dfrac{\sigma}{1+\sigma}$ ; and

$$H = \frac{\delta^*}{\theta} = 2 - 2A_o - \frac{2}{\sigma}\sum_{n=0}(1 - 4\tau^{n+1})\,A_n \qquad (15)$$

By differentiating (11) and putting $y = Y = 0$, we obtain

$$\left(\frac{\partial^p u}{\partial y^p}\right)_0 = (-1)^{p+1}\left[1 + \sigma^p \sum_{r=0}^{p} (-1)^r \binom{p}{r} A_r\right] . \qquad (16)$$

Differentiating the equation of motion (7) with respect to $y$ twice we obtain, on putting $y = 0$ and inserting the boundary conditions,

$$0 = \lambda^2 \left[\left(\frac{\partial^3 u}{\partial y^3}\right)_0 + \left(\frac{\partial^2 u}{\partial y^2}\right)_0\right] , \qquad (17)$$

and $$\left(\frac{\partial u}{\partial y}\right)_{y=0} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y=0} = \lambda^2 \left[\left(\frac{\partial^4 u}{\partial y^4}\right)_{y=0} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_{y=0} + \left(\frac{\partial^3 u}{\partial y^3}\right)_{y=0}\right] \qquad (18)$$

If the series expansion in (11) is terminated after the term in $Y^4$, equations (7), (8), (17) and (18) form four linear equations sufficient to determine $A_0 : A_1 : A_2 : A_3 : A_4$.

Making use of (16), together with the fact that

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)_0 = \frac{d}{dx}\,[1 + \sigma\,(A_0 - A_1)] = i\sigma\,(A_0 - A_1) ,$$

the momentum equation, the equations of motion and the two remaining equations become respectively

$$\frac{i}{\sigma} \sum_{n=0}^{4} (1 - 2\tau^{n+1}) A_n + 2iA_0 = \lambda^2 \, [\sigma(A_0 - A_1) - A_0] \qquad (19)$$

$$- iA_0 = \lambda^2 \, [\sigma(A_0 - A_1) - \sigma^2 \, (A_0 - 2A_1 + A_2)] \qquad (20)$$

$$0 = \lambda^2 [\sigma^2(A_0 - 2A_1 + A_2) - \sigma^3(A_0 - 3A_1 + 3A_2 - A_3)] \qquad (21)$$

$$i\sigma\,(A_0 - A_1) = \lambda^2 [\sigma^3(A_0 - 3A_1 + 3A_2 - A_3) - \sigma^4(A_0 - 4A_1 + 6A_2 - 4A_3 + A_4) + i\sigma(A_0 - A_1)] \qquad (22)$$

We now solve these equations approximately for small values of $\lambda$. The curve of $H$ against $\lambda$ for the solution so obtained will later be joined on to the value for $\lambda = \sigma = 1$, for which the equations are easily solved. Putting $\lambda^2 = \sigma^{-6}$, (12), and excluding $\sigma^{-4}$ and higher powers in comparison with 1, the equations become

$$\left[2 + \frac{1}{\sigma}(1 - 2\tau)\right] a_0 + \frac{1}{\sigma}(1 - 2\tau^2)a_1 + \frac{1}{\sigma}(1 - 2\tau^3)a_2 + \frac{1}{\sigma}(1 - 2\tau^4)a_3 + \frac{1}{\sigma}(1 - 2\tau^5)a_4 = 0$$

$$-1a_0 + \left(-\frac{2}{\sigma^4} + \frac{1}{\sigma^5}\right)a_1 + \frac{1}{\sigma^4}a_2 = 0$$

$$\left(1 - \frac{1}{\sigma}\right)a_0 + \left(-3 + \frac{2}{\sigma}\right)a_1 + \left(3 - \frac{1}{\sigma}\right)a_2 - a_3 = 0$$

$$\left(1 + \frac{1}{\sigma^3}\right)a_0 + \left(-1 - \frac{4}{\sigma^3}\right)a_1 + \frac{1}{\sigma^3}\left(6 - \frac{3}{\sigma}\right)a_2 + \frac{1}{\sigma^3}\left(-4 + \frac{1}{\sigma}\right)a_3 + \frac{1}{\sigma^3}a_4 = 0 ,$$

(23)

With the same degree of approximation, the solution of these equations is

$$\left.\begin{aligned}
a_0 &: -\frac{1}{\sigma^5}(1 - 2\tau^5) + \frac{1}{\sigma^8}(-10 + 2\tau^2 + 4\tau^3 + 6\tau^4 + 8\tau^5) . \\
= a_1 &: \frac{1}{\sigma^4}(10 - 2\tau^3 - 6\tau^4 - 12\tau^5) + \frac{1}{\sigma^5}(-6 + 2\tau^4) + \frac{1}{\sigma^6}(1 - 2\tau^5) + \frac{2}{\sigma^7} \\
= a_2 &: \frac{1}{\sigma}(1 - 2\tau^5) + \frac{1}{\sigma^4}(10 + 2\tau^2 - 6\tau^4 - 16\tau^5) \\
= a_3 &: \frac{3}{\sigma}(1 - 2\tau^5) - \frac{1}{\sigma^2}(1 - 2\tau^5) + \frac{1}{\sigma^4}(6\tau^2 + 6\tau^3 - 12\tau^5) \\
= a_4 &: \frac{1}{\sigma}(-4 + 2\tau^3 + 6\tau^4) + \frac{1}{\sigma^2}(1 - 2\tau^4) + \frac{1}{\sigma^4}(-18 + 12\tau^2 + 16\tau^3 + 12\tau^4)
\end{aligned}\right\} \quad (24)$$

Substituting these values into (15), we obtain for small $\lambda$,

$$\left|\frac{H-2}{a_0}\right| = 4\,\frac{\sigma^3(-\tau^3 - 3\tau^4 + 4\tau^5) + \sigma^2(\tau^4 - \tau^5)}{1 - 2\tau^5} \qquad (25)$$

$$= 4\,\frac{(-\tau^3 - 3\tau^4 + 4\tau^5) + \lambda^{1/3}(\tau^4 - \tau^5)}{\lambda(1 - 2\tau^5)}$$

where $\tau = \dfrac{1}{1 + \lambda^{1/3}}$

For $\lambda = 1$ , the equations (19) to (22) become particularly simple, with the solution

$$a_0 : (-49+i) = a_1 : (32+70i) = a_2 : (31+21i) = a_3 : (30-28i) = a_4 : (-40+4i) \qquad (26)$$

The value of $\left|\frac{H-2}{a_0}\right|$ for this case and that for $\lambda = 0.4$ are plotted in Fig.5, together with the curve obtained from (25) for smaller values of $\lambda$.

The variation of $\bar{H}$ along the surface is given by $H = 2 + |H-2| e^{i(x+\varepsilon)}$, say, where $\varepsilon$ is some phase angle. The stability is least for the profile with greatest $H$ , so that in calculating the maximum Reynolds number for stability at all points in a wave-length, it is sufficient to use the curve of Fig.6 with

$$H = 2 + |H-2| .$$

## (ii) Velocity fluctuation at infinity

In this section it is more convenient to work with dimensional quantities.

The flow over the wavy surface with boundary layer present may be represented by the potential flow, with uniform velocity at infinity over a surface of height

$$y = \eta + \delta^* ,$$

where $\eta = he^{2\pi ix/L}$ and

$$\delta^* = \frac{\nu}{v_s}\{1 - A_0 + \frac{1}{\sigma}(A_0 + A_1 + A_2 + A_3 + A_4)\} \qquad \text{from (13)}$$

$$= \frac{\nu}{v_s}\{1 + A_0\Delta\}, \text{ say, where } \Delta = O(1),$$

$$= \frac{\nu}{v_s}\{1 + a_0\Delta e^{2\pi ix/L}\}. \qquad (27)$$

The flow is calculated by representing the surface by a source distribution of magnitude

$$\frac{U}{\pi}\frac{dy}{dx} = \frac{U}{\pi}\frac{2\pi i}{L}e^{2\pi ix/L}\left[h + \frac{\nu}{v_s}a_0\Delta\right]. \qquad (28)$$

The velocity in the x direction at a point $(x_1, y_1)$ is then

$$\left[h + \frac{\nu}{v_s}a_0\Delta\right]\frac{U}{\pi}\frac{2\pi i}{L}\int_{-\infty}^{\infty}\frac{(x_1 - x)e^{2\pi ix/L}dx}{y_1^2 + (x_1 - x)^2} \qquad (29)$$

Writing $t = x - x_1$, this may be written

$$- \frac{2\pi i}{L} \frac{U}{\pi} \cdot \left[ h + \frac{\nu}{v_s} a_o \Delta \right] \int_{-\infty}^{\infty} \frac{e^{2\pi i t/L} \, t \, dt}{y_1^2 + t^2} \, . \tag{30}$$

To evaluate the integral consider

$$I = \int_{-\infty}^{\infty} \frac{e^{i\mu t} \, dt}{y_1^2 + t^2} \, . \tag{31}$$

This may be integrated round a semi-circle of large radius $R$ in the upper half plane. The contribution from the curved part is $O(\frac{1}{R})$. The only pole inside the semi-circle is at $t = iy_1$, where the residue is $e^{-\mu y_1}/2iy_1$.

$$\therefore \quad I = \frac{\pi}{y_1} e^{-\mu y_1} \tag{32}$$

$$\frac{dI}{d\mu} = - \pi e^{-\mu y_1}$$

$$= \int_{-\infty}^{\infty} \frac{i t e^{i\mu t} dt}{t^2 + y_1^2} \, . \tag{33}$$

Thus,

$$[illegible]\, i \left(-\frac{1}{\pi}\right) \int_{-\infty}^{\infty} \frac{t e^{2\pi i t/L} dt}{t^2 + y_1^2} = e^{-2\pi y_1/L} \tag{34}$$

and the velocity due to the source distribution is

$$U_1 - U = \frac{2\pi U}{L} \left\{ h + \frac{\nu}{v_s} a_o \Delta \right\} e^{2\pi i x/L} \, e^{-2\pi y_1/L} \tag{35}$$

Now let us choose $y_1$ so that

$$\left. \begin{aligned} \frac{2\pi y_1}{L} &\ll 1 \\ \frac{v_s y_1}{\nu} &> 5 \end{aligned} \right\} \tag{36}$$

From Fig.6, for this value of H $(R_{\delta^*})_{crit} = 1.1 \times 10^4$. Lin's equation has two roots,

$$y_{c_1} = 0.064 \quad \text{and} \quad y_{c_2} = 0.293$$

For these

$$u_{c_1} = 0.0296\ , \quad u_{c_2} = 0.2071$$

The former of these corresponds to $(R_{\delta^*})_{crit} = 3.26 \times 10^7$, the latter to $(R_{\delta^*})_{crit} = 1.36 \times 10^4$. It seems reasonable to treat the lower of these as the relevant value On this basis $(R_{\delta^*})_{crit}$ is estimated to within 20% of its correct value and $\log_{10} (R_{\delta^*})_{crit}$ , a quantity which means more, to within $2\frac{1}{2}$%. Higher accuracy in stability theory would be quite fortuitous.

The slope and curvature parameters

$$\ell = \frac{\theta}{U} \left(\frac{\partial u}{\partial y}\right)_{y=0}$$

$$m = \frac{\theta^2}{U} \left(\frac{\partial^2 u}{\partial y^2}\right)_{y=0}$$

for the profile are respectively 0.512, -0.674. The value 0.674 for $-m$ is much larger than those of profiles investigated by other authors, and it might at first seem surprising that such good agreement with the $[(R_{\delta^*})_{cr}\ , H]$ curve is achieved in this case. However the profile curvature could be arbitrarily changed in the neighbourhood of the origin quite sufficiently to bring $m$ into the range of values usually considered, with negligible changes in the value of H , and without affecting the outer root of Lin's equation at all.

## APPENDIX III

### Surface Waves as a Source of Disturbance

As is pointed out in section 5, regular waves on the surface, or indeed any discontinuities with which a wavelength can be associated, provide disturbances which could be amplified under critical conditions. The suction quantities calculated in the report are large enough to prevent these conditions ever arising, i.e. $R_{\delta*} < (R_{\delta*})_{crit}$ always, but it is interesting to compare the length of the surface waves discussed with that of the critical disturbance.

For the asymptotic profile, exact computation is stated in Reference 3, Figure 21, to give the critical wave number

$$\alpha_{cr} = 0.17$$

Here

$$\alpha_{cr} = \frac{2\pi\delta*}{\ell_{cr}}$$

and

$$\delta* = \frac{\nu}{V_s}$$

Thus

$$\ell_{cr} = \frac{2\pi\nu}{V_s} \cdot \frac{1}{0.17}$$

$$\frac{\ell_{cr}}{L} = 37 \left(\frac{U}{V_s}\right)\left(\frac{\nu}{UL}\right)$$

And if the surface wave is of critical wavelength, $\ell_{cr} = L$,

$$\frac{V_s}{U} = 37 \Bigg/ \left(\frac{UL}{\nu}\right)$$

Thus for the values of $\frac{UL}{\nu}$ plotted in Figure 3, the surface will provide a critical disturbance for the following values of $\frac{V_{so}}{U}$:

| | | | | |
|---|---|---|---|---|
| $\frac{UL}{\nu}$ | $10^6$ | $3 \times 10^5$ | $3 \times 10^4$ | $6 \times 10^3$ |
| $\frac{V_{so}}{U} \times 10^4$ | 0.37 | 1.23 | 12.3 | 61.7 |

The calculations have not taken into account the possibility of a resonance between profile- and surface-disturbance at this wavelength, but it now seems possible that some such effect may play a part in the transition caused by surface excrescences.

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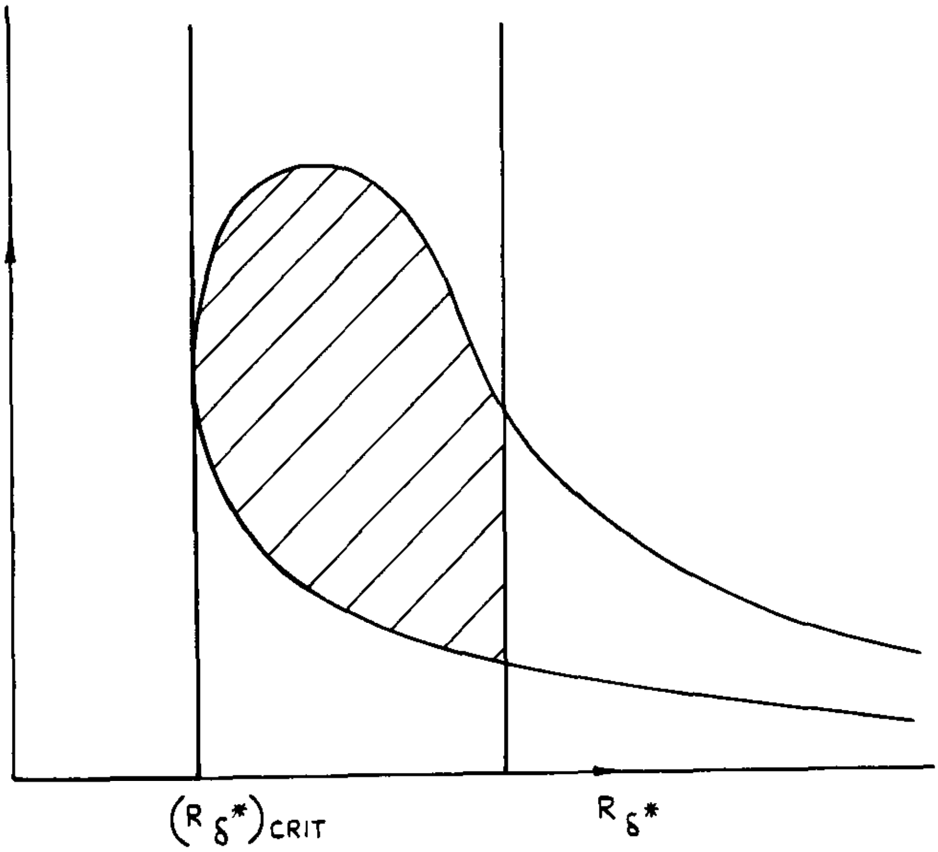



FIG. 1. TYPICAL NEUTRAL STABILITY CURVE.

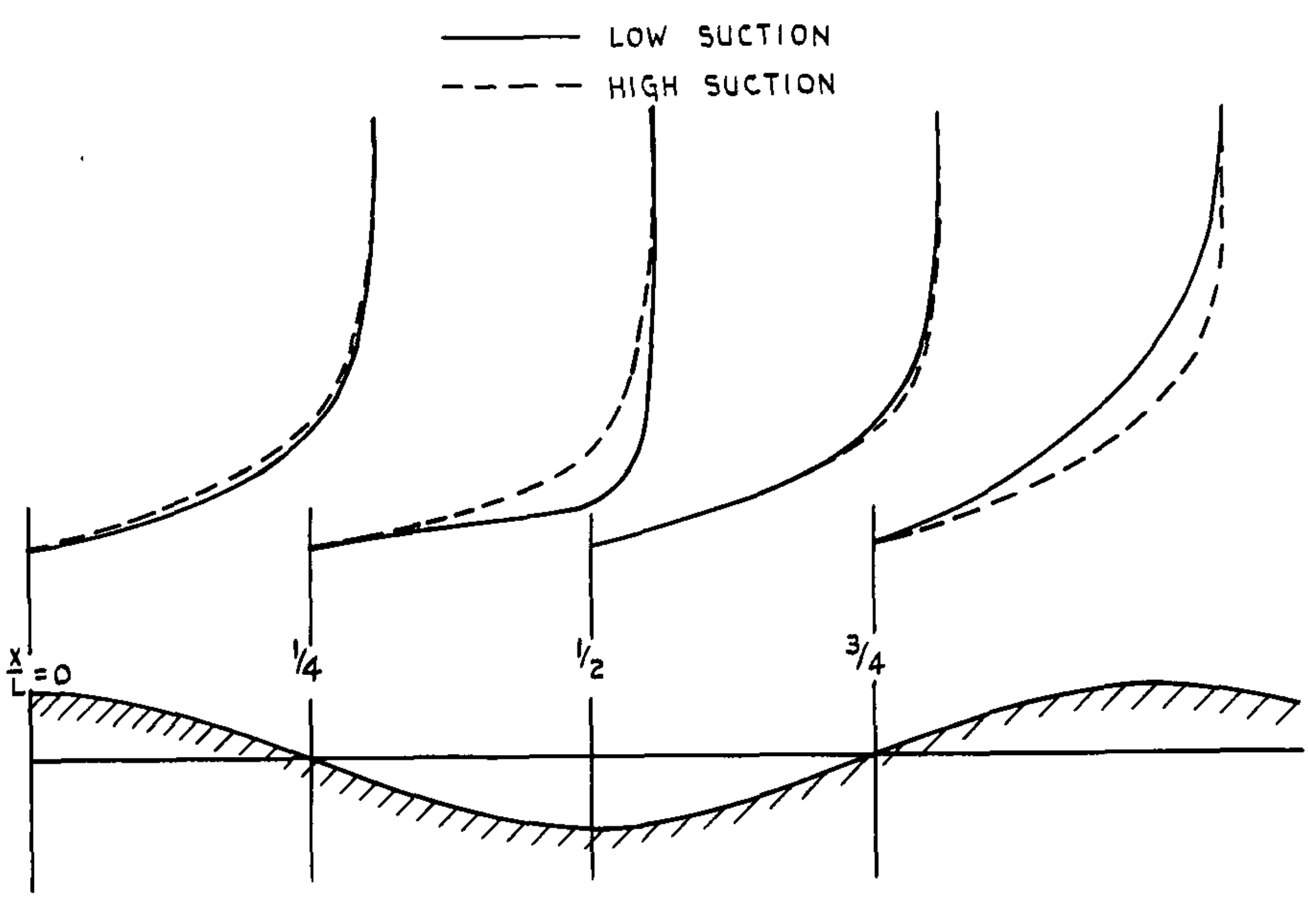



FIG. 2(a) SCHEMATIC VELOCITY PROFILES ALONG WAVY WALL.

FIG. 2 (b).

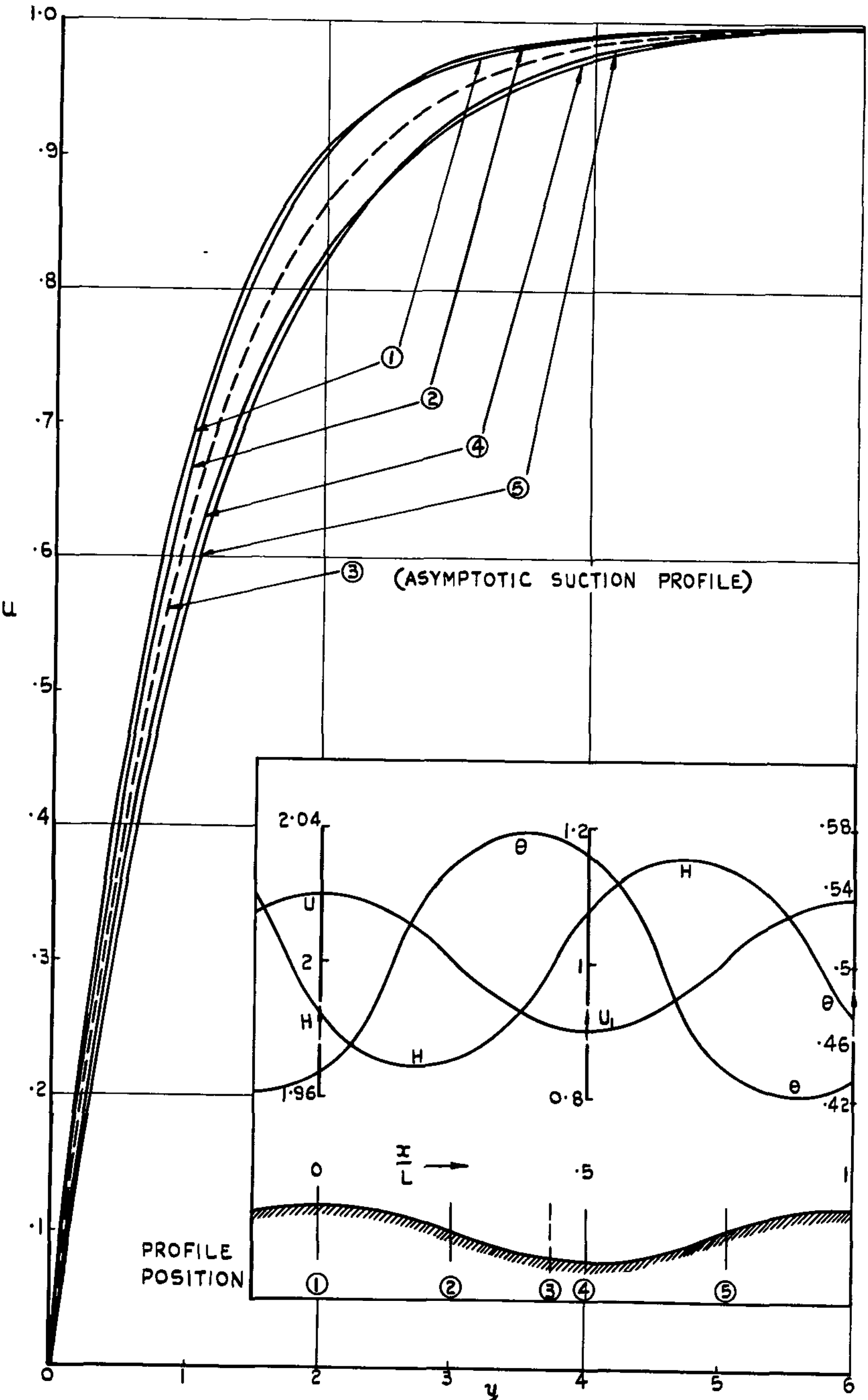



FIG. 2 (b). PROFILES FOR $\frac{V_s}{U}\sqrt{\frac{UL}{2\pi\nu}} = 1$, $\frac{h}{L} = 0{\cdot}016$.

FIG. 2 (c).

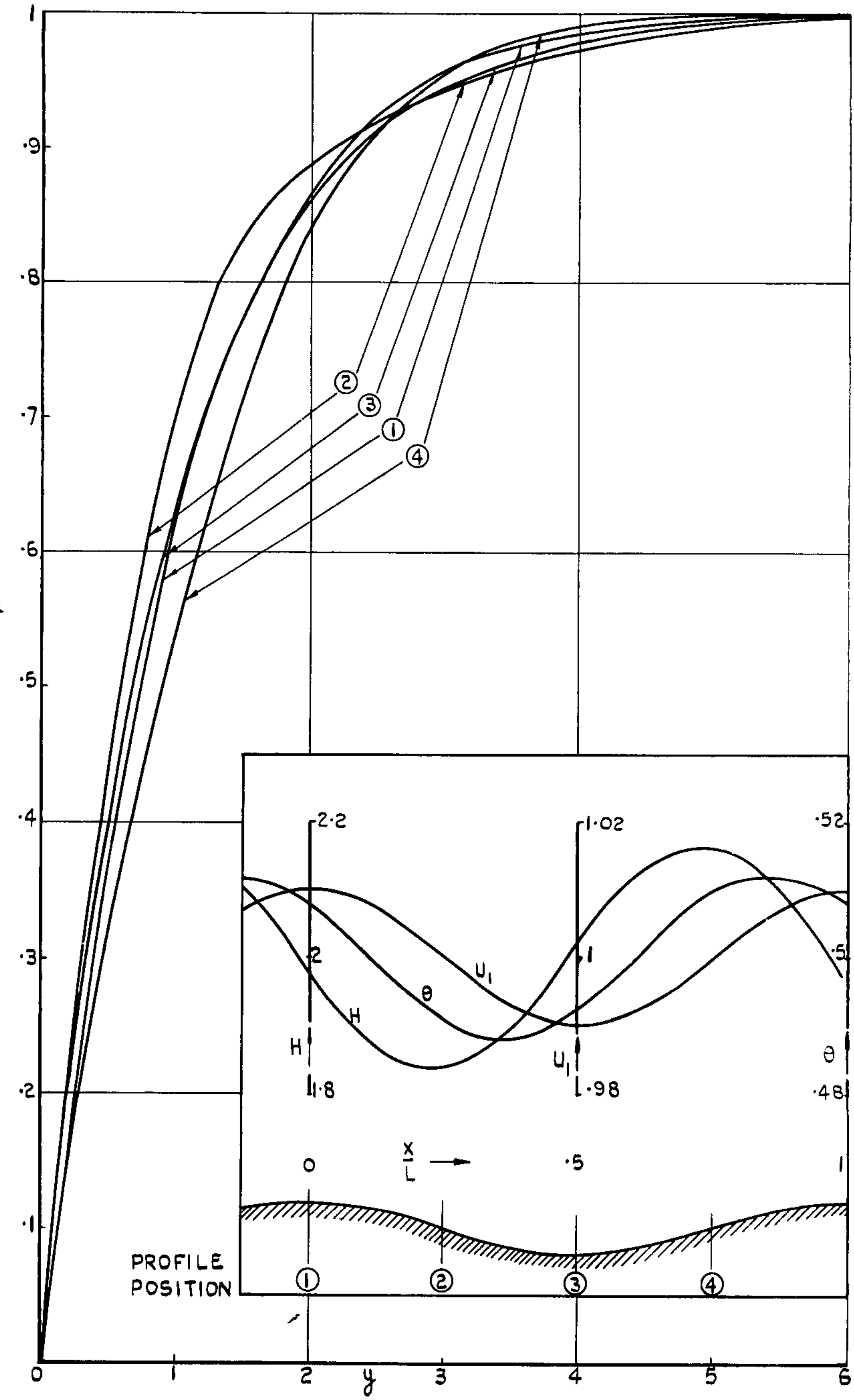



FIG. 2(c). PROFILES FOR $\frac{v_s}{U}\sqrt{\frac{UL}{2\pi\nu}} = \cdot 125$, $\frac{h}{L} = \cdot 0016$.

FIG. 2 (d).

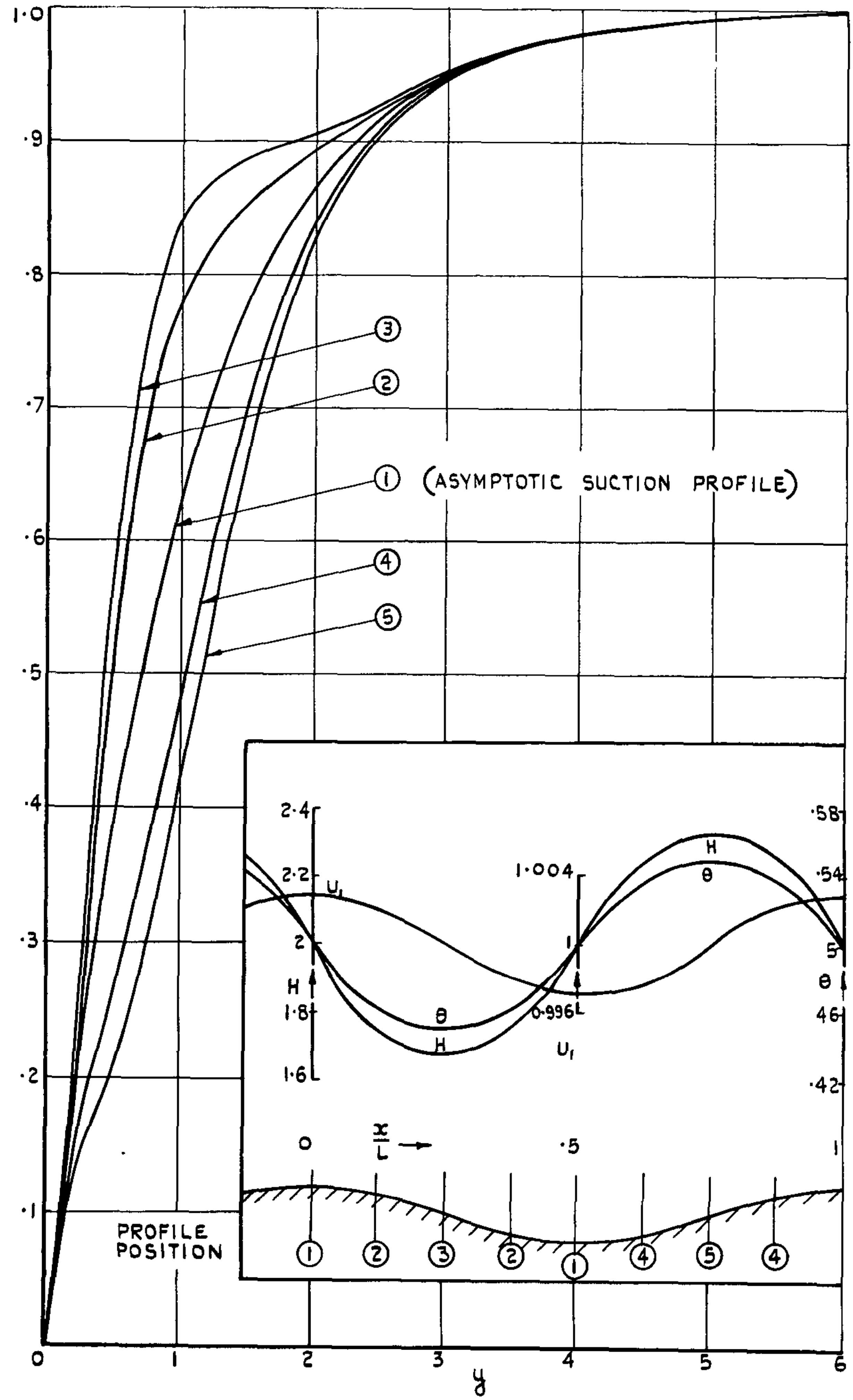



FIG. 2 (d) PROFILES FOR $\frac{V_S}{U} \frac{UL}{2\pi\nu} = 10^{-3/2}$

$\frac{h}{L} = \cdot 0005$

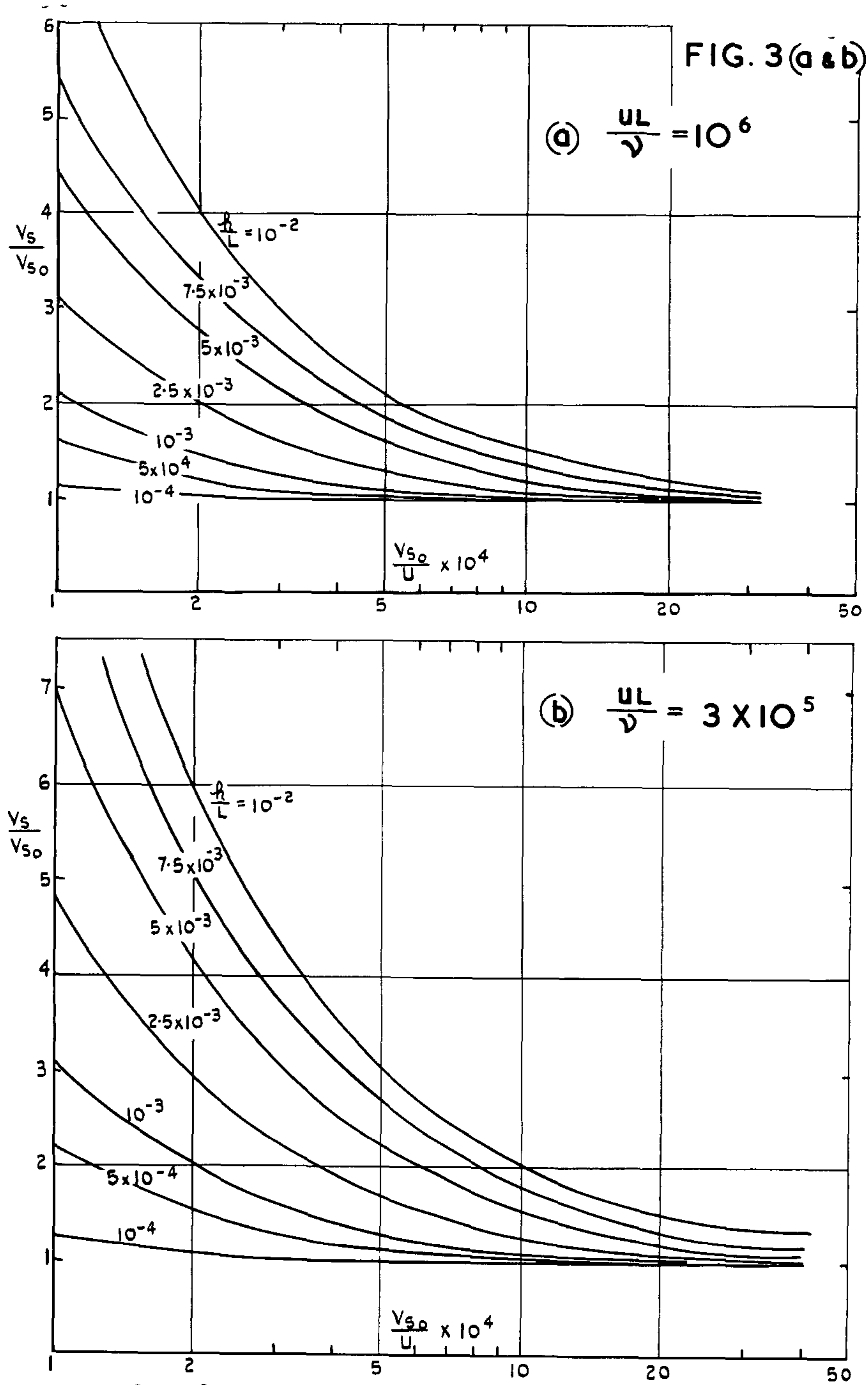
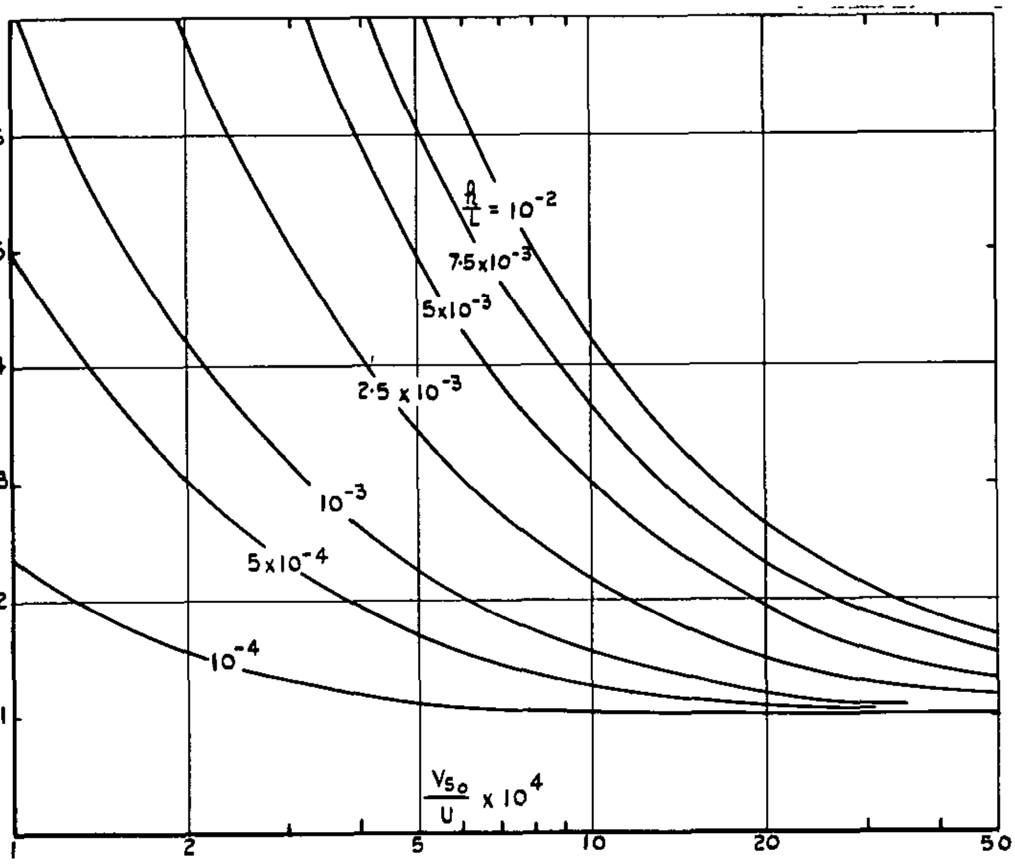
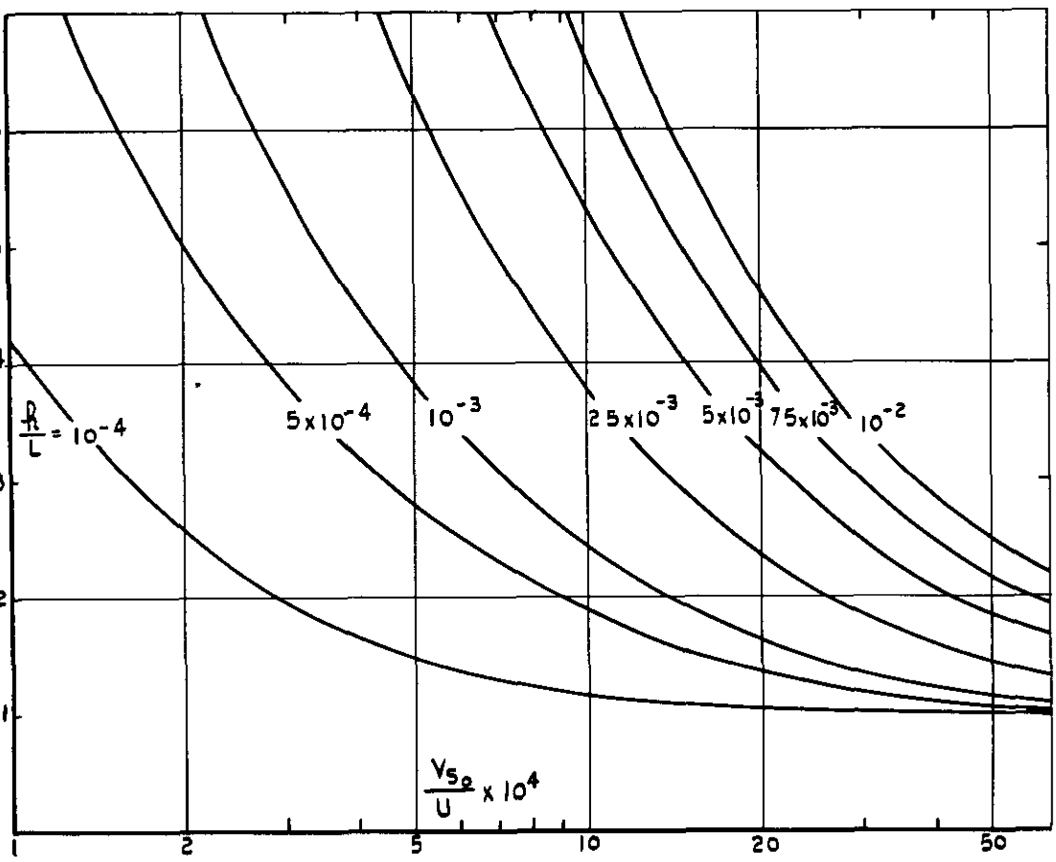



FIG. 3 (a & b) EXTRA SUCTION REQUIRED FOR STABILITY ON A WAVY SURFACE.

FIG. 3 (c & d).




(c). $\frac{UL}{\nu} = 3 \times 10^4$.




(d). $\frac{UL}{\nu} = 6 \times 10^3$.

FIG. 3 (c & d). EXTRA SUCTION REQUIRED FOR STABILITY ON A WAVY SURFACE.

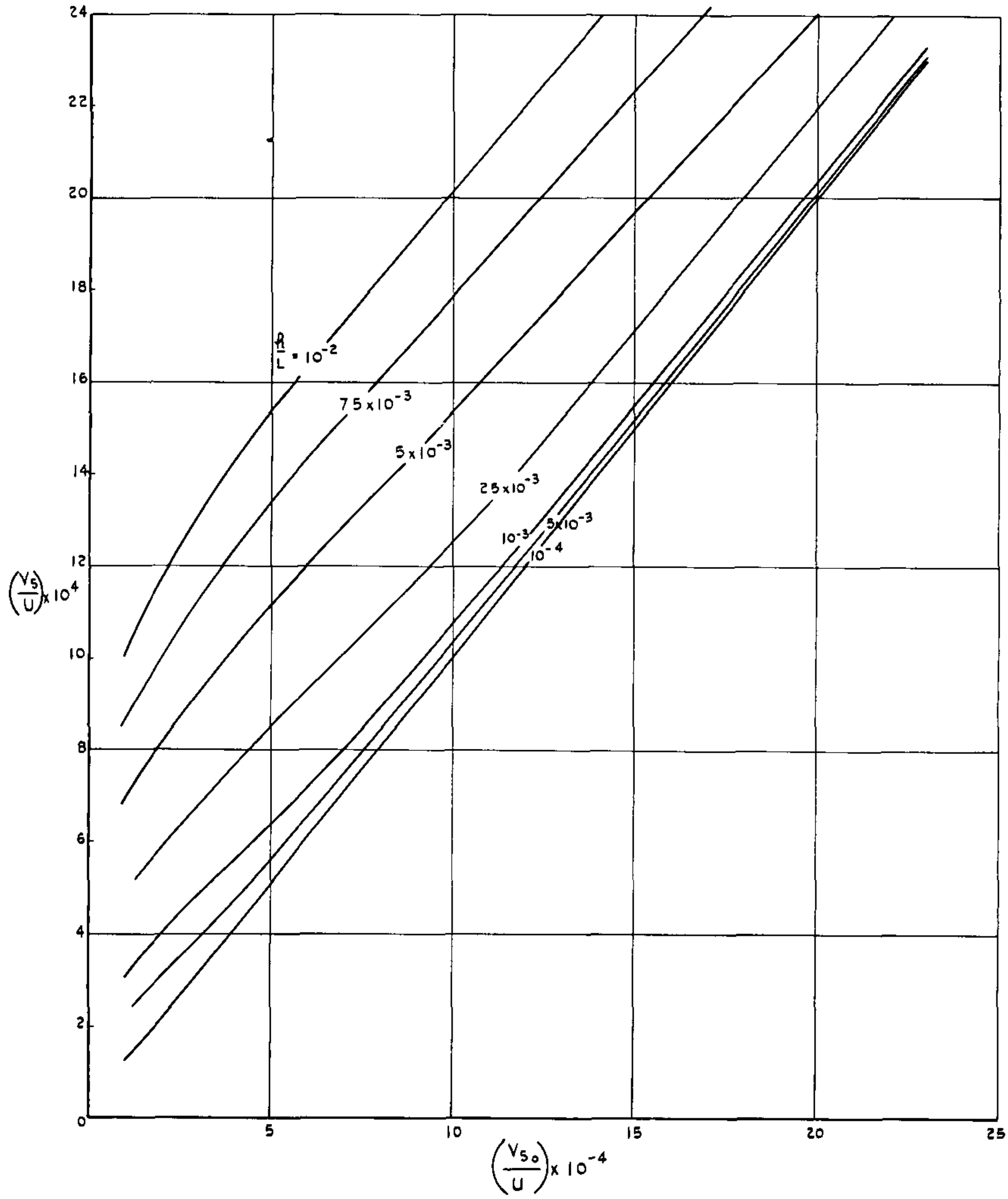



FIG. 3 (e). $\frac{V_s}{U}$ AGAINST $\frac{V_{s_0}}{U}$ FOR $\frac{UL}{\nu} = 3 \times 10^5$.

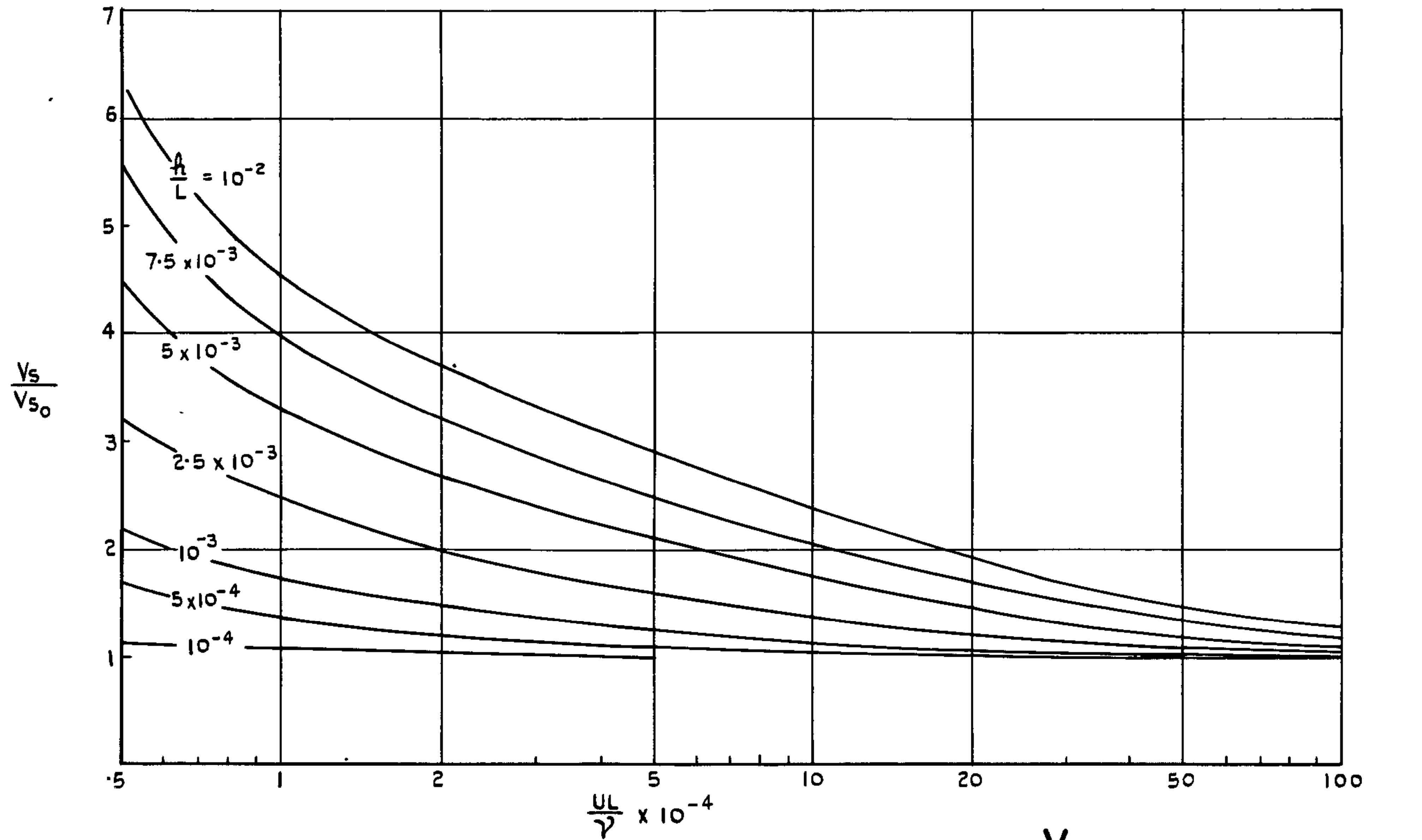



FIG. 4. REQUIRED INCREASE IN SUCTION, FOR $\frac{V_{S_0}}{U} = {\cdot}0014$, AS FUNCTION OF $\frac{h}{L}$ AND $\frac{UL}{\nu}$

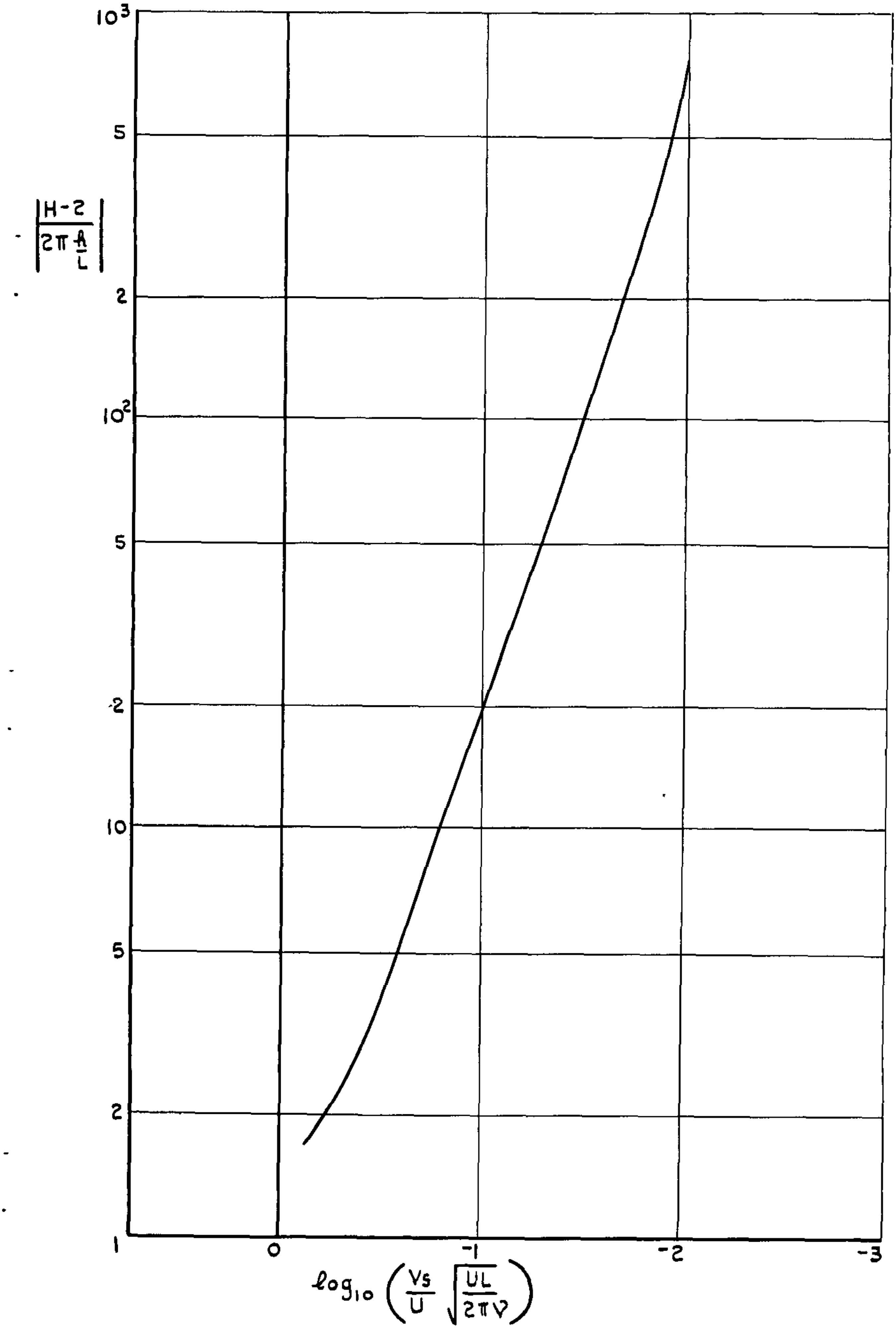



FIG. 5. VARIATION IN FORM PARAMETER ON A WAVY SURFACE WITH SUCTION.

FIG. 6.

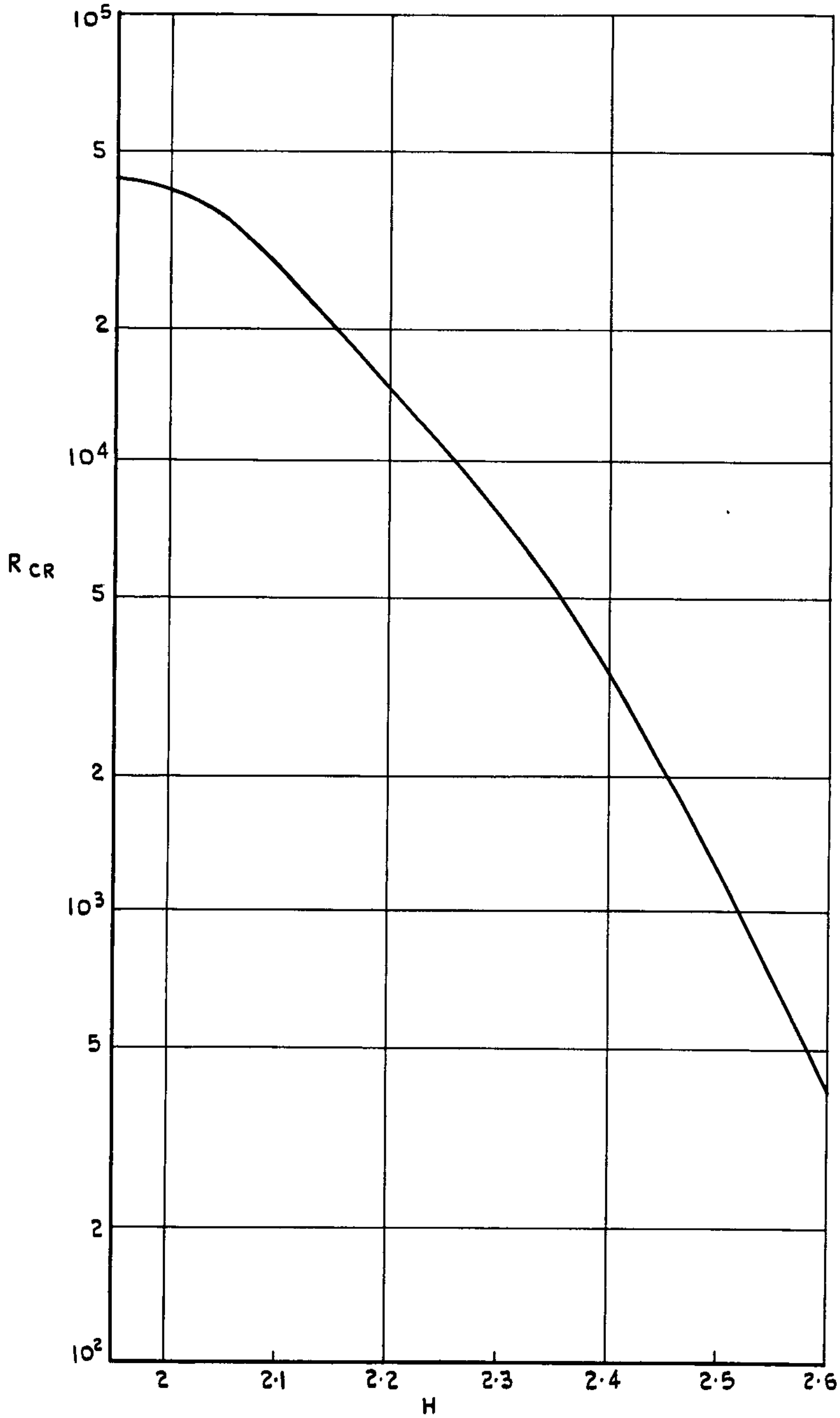



FIG. 6. VARIATION OF $(R_{\delta^*})_{crit.}$ WITH H FOR SUCTION PROFILES.

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