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Summary.—A theory for the calculation of the aerodynamic forces acting on wings of finite span and any plan form is developed, and from it an approximate method which reduces the amount of numerical work is derived. Satisfactory agreement with the experimental evidence available is obtained.

CONTENTS

1. *Introduction*.—In an earlier report[1], a method is suggested for the calculation of the pressure distribution on a wing of any plan form in steady motion. The theory is now extended to include oscillatory motion. As it is based on vortex sheet theory, it is subject to the usual limitations. The amplitudes of the motion in flexure and torsion are assumed to be small, and the mean angle of incidence of the wing is small or zero.

The flow around the wing is reproduced by a linear combination of simple doublet distributions over the wing area and the wake. The normal induced velocity due to each doublet distribution can be calculated exactly. A suitable combination of simple doublet distributions can then be determined by collocation to satisfy the boundary conditions for any mode of oscillation. If the condition for tangential flow is satisfied at n points on the wing, n complex simultaneous equations will have to be solved to determine the arbitrary constants of the required total distribution of doublets. Unfortunately, however, the solution of large numbers of simultaneous equations with the computational aids now available is very laborious, and, in view of this, an approximate method is suggested which reduces the number of equations to be solved. This method has already been used to calculate the aerodynamic derivative coefficients for rectangular wings[2]. It is applied in this paper to the calculation of aerodynamic derivative coefficients for tapered wings. Satisfactory agreement between theory and experiment is obtained for both rectangular and tapered wings.

List of Symbols

OX, OY, OZ	co-ordinate axes
V	velocity of flow
$x = R(y) - 0{\cdot}5\, c(y) \cos\vartheta$	chordwise parameter
$y = l\eta$	spanwise parameter
$c(y)$	local chord
c_m	mean chord
c_0	root chord
l	reference section at $y = l$
s	semi-span
$z\ (= z'e^{ipt})$	downward displacement
$w\ (= We^{ipt})$	downward velocity
$\gamma\ (= \Gamma e^{ipt})$	bound vorticity
$\varepsilon\ (= Ee^{ipt})$	free vorticity
Φ	velocity potential
ϕ	acceleration potential
$\Phi_a - \Phi_b\ (= Ke^{ipt})$	circulation
$\rho V\gamma\ (= \bar{p}_b - \bar{p}_a)$	lift distribution
ρ	air density
$\bar{p}$	pressure
$p/2\pi$	frequency
$\omega = 2\omega' = pc/V$	local reduced frequency
$\omega_0 = pc_0/V$	reduced frequency parameter
$J_n(\omega'),\ H_0^{(2)}(\omega'),\ H_1^{(2)}(\omega')$	Bessel and Hankel Functions
$C(\omega') = A(\omega') - iB(\omega')$	$= H_1^{(2)}(\omega')/(H_1^{(2)}(\omega') + iH_0^{(2)}(\omega'))$
$\mu(\omega')$	$= \dfrac{\pi\omega'}{2}\,[H_0^{(2)}(\omega') - iH_1^{(2)}(\omega')]\, e^{i\omega'}$

List of Symbols—continued.

$$\Gamma_0' = 2\left[\operatorname{cosec}\vartheta - (1 - C)\cot\frac{\vartheta}{2}\right]$$

$$\Gamma_0'' = 2\left[\cot\frac{\vartheta}{2} - \operatorname{cosec}\vartheta + i\omega'\sin\vartheta\right]$$

$$\Gamma_1 = -2\sin\vartheta + \cot\frac{\vartheta}{2} + \frac{i\omega'}{2}\sin 2\vartheta + i\omega'\sin\vartheta$$

$$n \geqslant 2 \;.\;.\;.\; \Gamma_n = -2\sin n\vartheta + i\omega'\left[\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1}\right]$$

$$E_0' = -i\omega'.\exp(i\omega'\cos\vartheta)\int_0^{\vartheta}\Gamma_0'.\exp(-i\omega'\cos\vartheta)\sin\vartheta\,d\vartheta$$

$$E_0'(\pi) = -2\pi i\omega'/\mu$$

$$E_0'' = -2i\omega'\sin\vartheta$$

$$E_1 = -i\omega'\left[\frac{\sin 2\vartheta}{2} + \sin\vartheta\right]$$

$$n \geqslant 2 \;.\;.\;.\; E_n = -i\omega'\left[\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1}\right].$$

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2. *General Theory.*—The axes of co-ordinates OX, OY, OZ are taken as shown in Figs. 1 and 2. It is assumed that the leading and trailing edges, when the wing is in its mean position, lie in the plane $z = 0$. The wing itself is replaced by a thin sheet, and the z ordinate of any point x, y on the sheet is taken to be the mean of the ordinates for the upper and lower surfaces of the wing at x, y. The x, y co-ordinates satisfy the relation

$$x = R(y) - \frac{c(y)}{2}\cos\vartheta, \quad \ldots \quad (1)$$

where $\vartheta = 0, \pi/2, \pi$ define the leading edge, the mid-chord line and the trailing edge respectively. It is also assumed that the deviations of the mean surface from the plane $z = 0$ are small.

Let $V + u$, v, w be the velocity components of the disturbed motion of the surrounding air, where u, v, w are small compared to the velocity V of the undisturbed airstream. If second order terms are neglected, Euler's equations then give

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x},\; \frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y},\; \frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z}, \quad \ldots \quad (2)$$

where $\dfrac{d}{dt} \equiv \dfrac{\partial}{\partial t} + V\dfrac{\partial}{\partial x}$. On integration, (2) yields

$$\frac{d\Phi}{dt} = -\frac{p}{\rho} + F(t) = \phi, \quad \ldots \quad (3)$$

where Φ, ϕ are respectively the velocity potential and the acceleration potential. The discontinuity in the pressure field is related to the discontinuities* in Φ and ϕ by

$$\frac{d}{dt}(\Phi_a - \Phi_b) = -\frac{\bar{p}_a - \bar{p}_b}{\rho} = \phi_a - \phi_b . \qquad (4)$$

Since $\bar{p}_b - \bar{p}_a = \rho V\gamma$, where γ is the bound vorticity, and since $u_a - u_b \equiv \gamma + \varepsilon$ is the total vorticity at any point, equation (4) yields

$$\frac{\partial}{\partial t}(\Phi_a - \Phi_b) = -V\varepsilon, \qquad (5)$$

where ε represents the free vorticity distribution. As the wing is describing simple harmonic motion, let

$$\Phi_a - \Phi_b = Ke^{ipt},\ \gamma = \Gamma e^{ipt},\ \varepsilon = Ee^{ipt}.$$

Then substitution in (5) and differentiation with respect to x yield

$$ip\frac{\partial K}{\partial x} = ip(\Gamma + E) = -V\frac{\partial E}{\partial x} . \qquad (6)$$

On integration, (6) gives

$$\left.\begin{aligned} K(x) &= -\frac{VE}{ip} = \int_{x_l}^{x}(\Gamma + E)\,dx, \\ &= e^{-ipx/V}\int_{x_l}^{x}\Gamma e^{ipx/V}\,dx, \quad \ldots \quad x \leqslant x_t \\ &= e^{-ipx/V}\int_{x_l}^{x_t}\Gamma e^{ipx/V}\,dx, \quad \ldots \quad x \geqslant x_t \end{aligned}\right\} \qquad (7)$$

where x_l, x_t define the positions of the leading and trailing edges of the section y. It is clear from (7) that the value of K at any point corresponds to the total circulation forward of that point. In the wake, $Ke^{ipx/V}$ is constant along the direction of flow, but variable along the span. The discontinuity K in the amplitudes of the velocity potentials can be represented by a distribution of doublets of strength K over the wing and the wake. The induced velocity w ($\equiv We^{ipt}$) due to such a distribution is then given by

$$W(x_1, y_1, z_1) = \frac{1}{4\pi}\iint K\frac{\partial^2}{\partial z_1^{\,2}}\left(\frac{1}{r}\right)dx\,dy, \qquad (8)$$

where $r^2 = (x - x_1)^2 + (y - y_1)^2 + z_1^{\,2}$ and $z_1 \to 0$. Since the wing oscillations are of small amplitude the doublets can be assumed to lie in the plane $z = 0$, and K can be regarded as a function of x and y only. Let $z(\equiv z'(x, y)e^{ipt})$ denote the displacement of any point on the mean wing surface. Since the doublet distribution is such that the corresponding induced velocity is equal to the normal component of velocity of the wing at all points, the following condition must be satisfied :—

$$W = ipz' + V\frac{\partial z'}{\partial x} . \qquad (9)$$

* The suffices a, b refer to the flow above and below the surface respectively.

Let the bound vorticity distribution be represented by

$$\Gamma = V\left[(\Gamma_0' + \Gamma_0'')\sum_{m=1} C_{0m} A_m + \sum_{n=1} \Gamma_n \sum_{m=1} C_{nm} A_m\right], \quad \ldots \quad (10)$$

where Γ_0', Γ_0'', Γ_1, *etc.* are defined in the list of symbols, and where A_1, A_2, *etc.* are simple functions of the spanwise parameter. The corresponding doublet distribution is then represented by

$$K = V\left[(K_0' + K_0'')\sum_{m=1} C_{0m} A_m + \sum_{n=1} K_n \sum_{m=1} C_{nm} A_m\right], \quad \ldots \quad (11)$$

where

$$\left.\begin{aligned} K_0'(x) &= e^{-ipx/V}\int_{x_l}^{x} \Gamma_0' e^{ipx/V}\,dx, \quad \ldots\ldots \quad x < x_t \\ &= \frac{\pi c}{\mu} e^{-ip(x-x_t)/V}, \quad \ldots\ldots\ldots \quad x \geqslant x_t \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

and

$$\left.\begin{aligned} K_0'' &= c \sin\vartheta, \\ K_1 &= \frac{c}{2}\left(\sin\vartheta + \frac{\sin 2\vartheta}{2}\right), \\ n \geqslant 2 \ldots\ldots \quad K_n &= \frac{c}{2}\left(\frac{\sin(n+1)\vartheta}{n+1} - \frac{\sin(n-1)\vartheta}{n-1}\right) \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

It is shown in the Appendix that

$$K_0' = cS(\vartheta) = c.\exp(i\omega'\cos\vartheta)\left[X_0\vartheta + 2\sum(-i)^n X_n \frac{\sin n\vartheta}{n}\right],$$

where

$$X_n = CJ_n(\omega') - i(1-C)J_n'(\omega').$$

Values of $S(\vartheta)$ for various values of $\omega(=2\omega')$ are given in Table 1. It will be noted that K_0'', $K_1 \ldots K_n$ are independent of the frequency parameter. When $\omega = 0$, $K_0' = c\vartheta$ on the wing and $c\pi$ in the wake. If W_{nm} denotes the amplitude of the velocity induced by the doublet distribution $K_n A_m$ at the point x_1, y_1, z_1, then by (8)

$$W_{nm} = \frac{1}{4\pi}\iint K_n A_m \frac{\partial^2}{\partial z_1^2}\left(\frac{1}{r}\right) dx\,dy. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

The evaluation of integrals of this type is discussed in detail in the appendix to R. & M. 2145[1]. Values of W_{nm} for various values of n and m are given in Tables 3–5. The induced velocity W_{0m}' corresponding to $K_0' A_m$, which can be calculated as shown in the Appendix to this paper, is given to sufficient accuracy by Cicala's method (*see* §4). Hence the total induced velocity W can be expressed as

$$W = V\left[\sum_{m=1} C_{0m}(W_{0m}' + W_{0m}'') + \sum_{n=1}\sum_{m=1} C_{nm} W_{nm}\right] \quad \ldots \quad (15)$$

where C_{0m}, C_{1m}, *etc.* are arbitrary constants which can be determined to satisfy (9) at a number of points on the wing. When their values are known, the pressure distribution $\rho V\Gamma$ is given by (10) and the aerodynamic forces can then be calculated.

3. *Lyon's Theory.*—In this theory [3, 4], the bound vorticity distribution is given in the same form as (10), but

$$\Gamma_0' \equiv 2\left[\operatorname{cosec}\vartheta - (1 - C)\cot\frac{\vartheta}{2}\right],$$

where C is here a function of y as well as ω. The vorticity distribution is represented by a system of superposed rectangular vortex sheets with constant vorticity strength in the spanwise direction. For any given value of ω, the function C is determined so that a vortex sheet extending from $-y$ to $+y$ with a chordwise total vorticity distribution $\Gamma_0' + E_0'$ gives zero induced velocity at the mid-chord mid-span point. This complication appears to be unnecessary, for (10) reduces to the form

$$\Gamma = B_0 \cot\frac{\vartheta}{2} + \sum_{n=1} B_n \sin n\vartheta,$$

where B_0, B_1 . . . B_n are functions of y and ω, whether C is a function of y or not. As the method is also based on rectangular vortex sheet representation of the vorticity distribution, it cannot be applied to the calculation of derivatives for tapered or swept-back wings without considerable modification.

4. *Cicala's Theory* [5, 6].—This method is in effect an extension of Glauert's lifting line theory to oscillatory motion. In two-dimensional vortex sheet theory, the normal induced velocities corresponding to K_0', K_0'', K_1 . . . K_n are 0, 1, $\frac{1}{2} + \cos\vartheta$, . . . , $\cos n\vartheta$ respectively, and Cicala assumes that the induced velocity W due to (10) can be expressed as

$$W = V\left[W_0' + \sum_{m=1} C_{0m}A_m + (\tfrac{1}{2} + \cos\vartheta)\sum_{m=1} C_{1m}A_m + \sum_{n=2}\cos n\vartheta \sum_{m=1} C_{nm}A_m\right], \quad .. \quad (16)$$

where

$$VW_0' = -\frac{1}{4\pi}\int_{-s}^{s}\left(\frac{1}{y - y_1} - \frac{ip}{V}F_0\right)\frac{\partial K}{\partial y}\,dy, \quad .. \quad .. \quad .. \quad (17)$$

and $K \equiv VK_0'(x_t)\sum_{m=1} C_{0m}A_m$. It should be noted that $K_0'(x_t)$, as defined by (12), is a function of ω. Cicala's function F_0 is given by

$$F_0(Y) = \int_0^\infty e^{-iX}\left(\frac{1}{X} + \frac{1}{Y} - \sqrt{\frac{1}{X^2} + \frac{1}{Y^2}}\right)dX, \quad .. \quad .. \quad .. \quad (18)$$

where $Y = \frac{p}{V}(y - y_1)$. When Y is negative $F_0(Y) = -F_0(-Y)$. The values of F_0 for a range of values of Y are given in R. & M. 2142[2]. When $p = 0$, as for steady motion, (17) reduces to the usual formula for the downwash induced by the circulation K. By the use of (9), (16) and (17), the arbitrary coefficients C_{0m}, C_{1m}, *etc.*, can be determined when the induced velocities W_{0m}' due to the spanwise distributions $K_0'(x_t)\,A_m$ have been calculated. However, it is shown in Table 3 that the assumption on which (16) is based is not justified for wings of aspect ratio 6, since the velocities corresponding to $K_0''\,A_m$, $K_1\,A_m$, $K_n\,A_m$ are not A_m, $A_m(\frac{1}{2} + \cos\vartheta)$, $A_m \cos n\vartheta$ respectively. For wings of large aspect ratios, however, this method might give fairly accurate results for low values of ω, since in the limiting case when $\omega = 0$ it reduces to Glauert's lifting line theory.

5. *Alternative Theory.*—In this method, which was first suggested in A.R.C. 4705[7], it is assumed that the induced velocities due to $K_0' A_m$, $K_0'' A_m$, $K_1 A_m$, $K_n A_m$ are proportional (but not equal) to the values 0, 1, $\frac{1}{2} + \cos\vartheta$, $\cos n\vartheta$ appropriate to the two-dimensional theory. On this basis, the velocity distributions are given approximately by

$$\left.\begin{aligned} &W_{0m}'(y,\vartheta) = W_{0m}'(y),\ W_{0m}''(y,\vartheta) = W_{0m}''(y), \\ &W_{1m}(y,\vartheta) = W_{1m}(y)(\tfrac{1}{2} + \cos\vartheta), \\ n \geqslant 2 \ldots\ &W_{nm}(y,\vartheta) = W_{nm}(y)\cos n\vartheta, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \quad (19)$$

where $W_{0m}'(y)$, $W_{0m}''(y)$, $W_{1m}(y)$, *etc.* are dependent on y but not on ϑ. They are chosen to satisfy (19) for a particular value of ϑ which need not necessarily be the same for each relation. Equations (9), (15) and (19) then give

$$\frac{i\omega z'}{c} + \frac{\partial z'}{\partial x} = \sum_{m=1} [C_{0m}W_{0m}(y) + C_{1m}W_{1m}(y)(\tfrac{1}{2} + \cos\vartheta) + \ldots + C_{nm}W_{nm}(y)\cos n\vartheta]. \quad \ldots \quad (20)$$

For the section $y = y_1$, (20) yields the relations

$$\sum_{m=1} C_{0m}W_{0m}(y_1) = \frac{1}{\pi}\int_0^\pi (1 - \cos\vartheta)\left(\frac{i\omega z'}{c} + \frac{\partial z'}{\partial x}\right)d\vartheta,$$

and

$$\sum_{m=1} C_{nm}W_{nm}(y_1) = \frac{2}{\pi}\int_0^\pi \left(\frac{i\omega z'}{c} + \frac{\partial z'}{\partial x}\right)\cos n\vartheta\, d\vartheta. \qquad \ldots \quad \ldots \quad \ldots \quad (21)$$

The values of the coefficients C_{nm} are determined to satisfy (21) at the required chordwise sections.

When $\omega_0 \to \infty$, $W_{0m}' \to 0$, and even for ω_0 values in the neighbourhood of $\omega_0 = 3$, the error introduced by the assumption that $W_0' = 0$ is not large. The velocity distributions W_{0m}'', W_{1m}, *etc.* are independent of ω_0, so that the assumption that W_{0m}' is zero simplifies the problem considerably as shown in §6. At low values of ω_0, however, the trailing vorticity effect represented by the W_{0m}' terms cannot be neglected.

Formulæ for the calculation of aerodynamic derivatives by this method are developed in the next section.

6. *Aerodynamic Derivative Coefficients.*—For simplicity, it is assumed that the mean wing surface is flat, and that each chordwise section is rigid in bending. Let the flexural and torsional modes of distortion be $f(\eta)$ and $F(\eta)$ respectively when referred to the line* $x = x_f(y)$. The amplitude of the displacement at any point x, y is then given by

$$z' = l\phi' f(\eta) + (x - x_f)\,\theta' F(\eta), \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (22)$$

where ϕ', θ' denote the angular displacements in flexure and torsion at the reference section $\eta = y/l = 1$. Alternatively, (22) can be expressed as

$$z' = l\phi' f + (\bar{m} - 0\cdot5\cos\vartheta)\,c(y)\,\theta' F, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

where $\bar{m}c(y) \equiv R(y) - x_f(y)$, and where $\bar{m}$ may also be a function of the spanwise parameter. On substitution for z', (20) yields after reduction and a comparison of the coefficients

$$\sum C_{0m}W_{0m}(y) = \frac{i\omega_0 l\phi' f}{c_0} + i\omega_0\left(\bar{m} + \frac{1}{4}\right)\frac{c}{c_0}\theta' F + \theta' F$$

and

$$\sum C_{1m}W_{1m}(y) = -\frac{i\omega_0 c}{2c_0}\theta' F. \qquad \ldots \quad \ldots \quad \ldots \quad (24)$$

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* This is usually taken to coincide with the flexural axis.

The terms W_{nm} for $n \geqslant 2$ are not required. Let

$$\left. \begin{aligned} C_{0m} &= \frac{i\omega l_0 \phi'}{c_0} E_{0m} + i\omega_0 \theta' G_{0m} + \theta' R_{0m} \\ \text{and} \qquad C_{1m} &= -\frac{i\omega_0 \theta'}{2} D_{1m} . \end{aligned} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad (25)$$

Then the conditions to be satisfied at various points along the span may be written as

$$\left. \begin{aligned} \sum E_{0m} W_{0m} &= f, \qquad \sum G_{0m} W_{0m} = \left(\overline{m} + \frac{1}{4}\right) \frac{c}{c_0} F, \\ \sum R_{0m} W_{0m} &= F, \qquad \sum D_{1m} W_{1m} = \frac{c}{c_0} F, \end{aligned} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad (26)$$

where E_{0m}, G_{0m}, R_{0m} and D_{1m} are arbitrary coefficients which can be determined by collocation. Since $W_{0m} = W_{0m}' + W_{0m}''$, where W_{0m}' is complex and a function of the frequency parameter, the complex coefficients E_{0m}, G_{0m} and R_{0m} will also vary with frequency. However, when $\omega_0 \rightarrow \infty$, $W_{0m}' \rightarrow 0$, and $W_{0m} \rightarrow W_{0m}''$ which is real and independent of frequency. The systems of equations in (26) would then be real instead of complex and more readily soluble. When E_{0m}, R_{0m}, *etc.* have been determined by collocation, the coefficients C_{0m} and C_{1m} can be calculated. The appropriate pressure distribution is then given by (10).

The amplitudes of the downward force and the pitching moment per unit span at $y = l\eta$ are given by

$$\left. \begin{aligned} \delta Z' &= -\rho V \int \Gamma \, dx = -\frac{\rho c V}{2} \int_0^\pi \Gamma \sin \vartheta d\vartheta, \\ \text{and} \qquad \delta M' &= -\rho V \int \Gamma (x - x_f) \, dx, \\ &= -\frac{\rho V c^2}{4} \int_0^\pi \Gamma (2\overline{m} - \cos \vartheta) \sin \vartheta \, d\vartheta \end{aligned} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad (27)$$

Let $C_0 \equiv \sum C_{0m} A_m$ and $C_1 \equiv \sum C_{1m} A_m$, and assume that

$$cA_m = s\left(\frac{y}{s}\right)^{m-1} \sqrt{1 - \frac{y^2}{s^2}} = sT_m . \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (28)$$

Then, on substitution for Γ from (10), (27) gives

$$\left. \begin{aligned} \delta Z' &= -\pi \rho c V^2 [\alpha C_0 + \beta C_1], \\ \delta M' &= -\pi \rho c^2 V^2 [\gamma C_0 + \delta C_1], \end{aligned} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

where

$$\left. \begin{aligned} \alpha &= C + \frac{i\omega}{4}, \qquad \gamma = \overline{m}\alpha - \frac{C}{4}, \\ \beta &= \frac{i\omega}{8}, \qquad 8\delta = i\omega\left(\overline{m} - \frac{1}{8}\right) - 1, \end{aligned} \right\} \qquad \ldots \quad \ldots \quad \ldots \quad (30)$$

and C is defined in the list of symbols.

If

$$e_0 \equiv \sum E_{0m}T_m, \quad g_0 \equiv \sum G_{0m}T_m, \quad r_0 \equiv \sum R_{0m}T_m, \quad d_1 \equiv \sum D_{1m}T_m,$$

then by (25)

$$\left.\begin{aligned} \sum C_{0m}T_m &= \frac{i\omega_0 l\phi' e_0}{c_0} + i\omega_0 g_0\theta' + r_0\theta' \\ \text{and} \qquad \sum C_{1m}T_m &= -\frac{i\omega_0 d_1\theta'}{2}. \end{aligned}\right\} \qquad \ldots \quad (31)$$

The local force and moment are given by

$$\delta Z' = -\pi\rho s V^2\left\{\alpha\left[\frac{i\omega_0 l e_0\phi'}{c_0} + (r_0 + i\omega_0 g_0)\theta'\right] - \frac{i\omega_0\beta d_1\theta'}{2}\right\}$$

and

$$\delta M' = -\pi\rho c s V^2\left\{\gamma\left[\frac{i\omega_0 l e_0\phi'}{c_0} + (r_0 + i\omega_0 g_0)\theta'\right] - \frac{i\omega_0\delta d_0\theta'}{2}\right\}. \qquad (32)$$

Since α, β, γ and δ are functions of the local frequency parameter $\omega(\equiv pc/V)$, they will vary along the span, except when c = const. as for a rectangular wing. The amplitudes of the flexural and torsional moments at the reference section are given by the integrals

$$\left.\begin{aligned} L' &= \int_0^{s/l} \delta Z' l^2 f d\eta \\ \text{and} \qquad M' &= \int_0^{s/l} \delta M' l F d\eta. \end{aligned}\right\} \qquad \ldots \quad (33)$$

If $\vartheta' = c_0\theta'/l$, the aerodynamic moments can be expressed in the form

$$\left.\begin{aligned} -\frac{L'}{\rho V^2 l^3} &= L_{12}\phi' + L_{34}\vartheta' \\ \text{and} \qquad -\frac{M'}{\rho V^2 l^2 c_0} &= M_{12}\phi' + M_{34}\vartheta', \end{aligned}\right\} \qquad \ldots \quad (34)$$

where the airload coefficients $L_{12} \equiv L_1 + iL_2$, *etc.* are given by

$$\left.\begin{aligned} L_{12} &= \frac{\pi i\omega_0 s}{c_0}\int_0^{s/l} e_0\alpha f d\eta, \\ L_{34} &= \frac{\pi s}{c_0}\int_0^{s/l}\left\{\alpha\,[r_0 + i\omega_0 g_0] - \frac{i\omega_0\beta d_1}{2}\right\} f d\eta, \\ M_{12} &= \frac{\pi i\omega_0 s}{c_0}\int_0^{s/l}\gamma e_0\left(\frac{c}{c_0}\right) F d\eta, \\ \text{and} \qquad M_{34} &= \frac{\pi s}{c_0}\int_0^{s/l}\left\{\gamma\,[r_0 + i\omega_0 g_0] - \frac{i\omega_0\delta d_1}{2}\right\}\left(\frac{c}{c_0}\right) F d\eta. \end{aligned}\right\} \qquad \ldots \quad (35)$$

As $\omega_0 \to \infty$, let $e_0 \to \bar{e}_0$, $g_0 \to \bar{g}_0$, $r_0 \to \bar{r}_0$, and write $\bar{d}_1$ for d_1 which is independent of ω_0. Then, in the limit, (35) yields

$$\left.\begin{aligned}\hat{a}_1 &= -\frac{\bar{L}_1}{\omega_0^2} = \frac{\pi s}{4c_0}\int_0^{s/l}\left(\frac{c}{c_0}\right)\bar{e}_0 f d\eta, \\ \hat{a}_3 &= -\frac{\bar{L}_3}{\omega_0^2} = \frac{\pi s}{4c_0}\int_0^{s/l}\left(\bar{g}_0 - \frac{\bar{d}_1}{4}\right)\left(\frac{c}{c_0}\right) f d\eta, \\ \hat{g}_1 &= -\frac{\bar{M}_1}{\omega_0^2} = \frac{\pi s}{4c_0}\int_0^{s/l}\bar{m}\left(\frac{c}{c_0}\right)^2 \bar{e}_0 F d\eta, \\ \hat{g}_3 &= -\frac{\bar{M}_3}{\omega_0^2} = \frac{\pi s}{4c_0}\int_0^{s/l}\left[\overline{m g}_0 - \frac{1}{4}\left(\bar{m} - \frac{1}{8}\right)\bar{d}_1\right]\left(\frac{c}{c_0}\right)^2 F d\eta,\end{aligned}\right\} \quad \ldots \quad \ldots \quad (36)$$

which are the formulæ for the aerodynamic inertia coefficients in still air. Derivative coefficients of the classical type are then given by

$$\left.\begin{aligned}L_{12} - \bar{L}_1 &= c_1 + i\omega_0 b_1, \\ L_{34} - \bar{L}_3 &= c_3 + i\omega_0 b_3, \\ M_{12} - \bar{M}_1 &= k_1 + i\omega_0 j_1, \\ M_{34} - \bar{M}_3 &= k_3 + i\omega_0 j_3,\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

where c_1, b_1, *etc.* correspond to the derivatives used in R. & M. 1782[8]. The corresponding fundamental derivative coefficients λ_ϕ', $\lambda_{\dot\phi}'$, μ_θ', $\mu_{\dot\theta}'$, *etc.* are such that

$$\left.\begin{aligned}c_1 &= \lambda_\phi'\int_0^{s/l} f^2 d\eta \quad , & b_1 &= \lambda_{\dot\phi}'\int_0^{s/l}\frac{c}{c_0} f^2 d\eta \\ c_3 &= \lambda_\theta'\int_0^{s/l}\frac{c}{c_0} fF d\eta \quad , & b_3 &= \lambda_{\dot\theta}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^2 fF d\eta \\ k_1 &= \mu_\phi'\int_0^{s/l}\frac{c}{c_0} fF d\eta \quad , & j_1 &= \mu_{\dot\phi}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^2 fF d\eta \\ k_3 &= \mu_\theta'\int_0^{s/l}\left(\frac{c}{c_0}\right)^2 F^2 d\eta, & j_3 &= \mu_{\dot\theta}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^3 F^2 d\eta\end{aligned}\right\} \quad \ldots \quad \ldots \quad (38)$$

Similarly, the fundamental aerodynamic inertia coefficients $\lambda_{\ddot\phi}'$, $\lambda_{\ddot\theta}'$, $\mu_{\ddot\phi}'$, $\mu_{\ddot\theta}'$ are given by

$$\begin{aligned}\hat{a}_1 &= \lambda_{\ddot\phi}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^2 f^2 d\eta, & \hat{a}_3 &= \lambda_{\ddot\theta}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^3 fF d\eta \\ \hat{g}_1 &= \mu_{\ddot\phi}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^3 fF d\eta, & \hat{g}_3 &= \mu_{\ddot\theta}'\int_0^{s/l}\left(\frac{c}{c_0}\right)^4 F^2 d\eta.\end{aligned}$$

By the use of (37) and (38) the values of the fundamental derivatives can be calculated when the airload coefficients are known. The values of λ_ϕ', $\lambda_{\dot\phi}'$, *etc.* refer to the reference point of the reference section. If this is at a distance hc behind the leading edge, the fundamental derivative coefficients* referred to the leading-edge point of the reference section will be given by the usual formulæ

$$\begin{aligned}\lambda_\phi &= \lambda_\phi', \quad \lambda_\theta = \lambda_\theta' + h\lambda_\phi', \quad \mu_\phi = \mu_\phi' + h\lambda_\phi', \\ \mu_\theta &= \mu_\theta' + h(\mu_\phi' + \lambda_\theta') + h^2\lambda_\phi',\end{aligned}$$

and so on.

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*λ_ϕ, $\lambda_{\dot\phi}$, *etc.*, correspond to the fundamental derivative coefficients of R. & M. 1782[8].

7. *Numerical Applications.*—The aerodynamic forces acting on a symmetrically tapered wing A and a tapered wing B with its $0 \cdot 3$ axis at right angles to the root chord are considered. Both wings are of aspect ratio 6 about (exact value $= 5 \cdot 84$) and have the same taper ratio. The local chord c is defined in terms of the root chord c_0 and the spanwise co-ordinate as stated in Fig. 3. In the calculations the reference section is assumed to be at the wing tip ($l = s$).

The modes considered are (i) $f = \eta$, $F = \eta^{1 \cdot 5}$, and (ii) $f = \eta^2$, $F = \eta$; and they are referred to the mid-chord axis of wing A and the $0 \cdot 3c$ axis of wing B. Both symmetrical (S) and anti-symmetrical (A–S) motions are considered.

The calculations are based on the method suggested in §5. In the first instance, fundamental aerodynamical derivative coefficients are deduced for the case when $W_{0m}' = 0$ for a range of values of ω_0. The values obtained are given in Table 6, and $\omega_0\lambda_{\dot{\phi}}$, $\omega_0\mu_{\dot{\theta}}$ and μ_θ plotted for comparison with various experimental results in Figs. 4–7. By use of these curves as guides, more accurate values of the coefficients corresponding to the case when W_{0m}' terms are included can be estimated when the true values are known for two or more values of ω_0 in the practical range. In steady motion, W_{0m}' corresponds to the velocity induced by the trailing vortices and cannot be neglected. For oscillatory motion, however, the induced velocity due to the vorticity in the wake is not as important, and for large values of ω_0 it can be neglected altogether.

The values of $\lambda_{\dot{\phi}}$ were determined with particular care, and it is believed that the values obtained would not differ appreciably from the values which would be given by the exact theory of §2. The validity of the assumptions made in §5 is confirmed by the insensitivity of the results to the values of ϑ_1 chosen in (19). For instance, the values of $\lambda_{\dot{\phi}}$ obtained are practically the same whether the normal velocity conditions defined by (26) are satisfied along $0 \cdot 3c$ or along $0 \cdot 75c$ of wing B. They are also the same when the values of W_{0m}'' along mid-chord of the symmetrically tapered wing A, which are easier to calculate, are used instead of those corresponding to $0 \cdot 3c$ of wing B (*see* Table 7).

Values of the remaining derivatives are also given in Table 6. They are applicable for wings of thin symmetrical sections. In general, however, they might well be dependent to some extent on thickness and shape.

The aerodynamic inertia coefficients $\lambda_{\ddot{\phi}}$, *etc.* for the modes considered are approximately $0 \cdot 755$ of the values given by two-dimensional theory as compared with $0 \cdot 744$ for a rectangular wing and $0 \cdot 87$ for an elliptic wing of the same span and area[9, 10].

8. *Experimental Comparisons.*—Values of the fundamental flexural damping derivative coefficient $\lambda_{\dot{\phi}}$ for rectangular and tapered wings have been measured recently by Bratt[11]. Experimental values for the fundamental torsional damping derivative coefficient $\mu_{\dot{\theta}}'$ for a rigid aerofoil oscillating about its $0 \cdot 5c$ axis have also been determined in the Compressed Air Tunnel[12]. These results are in good agreement with the theoretical values of this paper as shown in Figs. 4—7. Experimental values obtained by Jones and Lambourne[13] from tests carried out on wing B with distortion modes $f = \eta$, $F = \eta^{1 \cdot 5}$ (Mode I) are somewhat higher than the theoretical values. This is not surprising as the experimental results include the apparatus damping and frictional effects present in the system.

Further confirmation of the theoretical values for $\lambda_{\dot{\phi}}$, *etc.* for rectangular wings as given in R. & M. 2142[2] can be deduced from the experimental results of R. & M. 1155[14]. It has now been established that the modes of the model wing used were as shown in Fig. 8. Both the flexural mode and the torsional mode can be represented approximately by $f = \eta^{7/6} = F$. No theoretical values of the fundamental derivative coefficients corresponding to these modes are available, but they should not differ much from the values given in R. & M. 2142[2] for the modes $f = \eta = F$, as derivative values are not very sensitive to mode variations. From the derivative values given in R. & M. 1155[14], the following leading-edge fundamental damping derivative coefficients were deduced :—

$$\lambda_{\dot{\phi}} = 1 \cdot 13, \ \lambda_{\dot{\theta}} = 1 \cdot 08, \ \mu_{\dot{\phi}} = 0 \cdot 35, \ \mu_{\dot{\theta}} = 0 \cdot 56.$$

They correspond to a mean frequency parameter value of $\omega = 0 \cdot 7$. The corresponding theoretical results from R. & M. 2142[2] for $\omega = \frac{2}{3}$ are

$$\lambda_{\dot{\phi}} = 1 \cdot 24, \ \lambda_{\dot{\theta}} = 1 \cdot 12, \ \mu_{\dot{\phi}} = 0 \cdot 30, \ \mu_{\dot{\theta}} = 0 \cdot 54.$$

The experimental value of $\lambda_{\dot{\phi}}$ is lower than the theoretical value, as one would expect, since for a parabolic flexural mode, $\lambda_{\dot{\phi}} = 1 \cdot 01$. In R. & M. 1155[14] it is stated that $L_{\dot{\theta}}$ and $M_{\dot{\phi}}$ were difficult to measure, but that $L_{\dot{\theta}} + M_{\dot{\phi}}$ could be measured with reasonable accuracy. The corresponding mean experimental values of $\lambda_{\dot{\theta}} + \mu_{\dot{\phi}}$ is $1 \cdot 43$ as compared with the theoretical value of $1 \cdot 42$. The values of L_{θ} and M_{θ} were measured statically, and they give $\lambda_{\theta} = 1 \cdot 48_5$ and $\mu_{\theta} = 0 \cdot 34$ as compared with the theoretical values $\lambda_{\theta} = 1 \cdot 45$ and $\mu_{\theta} = 0 \cdot 36$ respectively.

In Ref. 15 it is pointed out that the stability derivative $\mathrm{L}_p = 2L_{\dot{\phi}}$ approximately. On the basis of this assumption it can be deduced that the non-dimensional stability derivative coefficient

$$l_p = \frac{L_{\dot{\phi}}}{\rho V s^3 c_m} = -\lambda_{\dot{\phi}} \int_0^1 \frac{c}{c_m} \eta^2 \, d\eta,$$

where c_m is the mean chord and $\eta = y/s$. If the local chord is defined by $c = c_0 (1 - \beta\eta)$, it is readily proved that

$$-l_p = \lambda_{\dot{\phi}} \frac{(1 - 0 \cdot 75\beta)}{3 (1 - 0 \cdot 5\beta)} = \frac{\lambda_{\dot{\phi}}}{3} \left(\frac{c_r}{c_m}\right),$$

where c_r is the chord at $0 \cdot 75s$. By the use of measured values of l_p, the fundamental coefficient $\lambda_{\dot{\phi}}$ can then be estimated. The experimental results given in Ref. 16 yield a value of $l_p = -0 \cdot 44$ for a rectangular wing of aspect ratio 6, and give $l_p = -0 \cdot 46$ for a tapered wing ($c_t = 0 \cdot 5c_0$) of the same span and aspect ratio. Hence, for a rectangular wing, $\lambda_{\dot{\phi}} = 1 \cdot 32$, and $\lambda_{\phi} = 1 \cdot 66$ for the tapered wing. These values correspond to low values of ω_0, and are in fair agreement with the theoretical and experimental values given in Figs. 4 and 5.

Since the values of l_p for the tapered wing and its equivalent rectangular wing are nearly equal, it can be assumed without serious error that $l_p = -0 \cdot 45$ for both wings. This leads to the relation

$$\lambda_{\dot{\phi}} (R) = \lambda_{\dot{\phi}} (T) \left(\frac{c_r}{c_m}\right),$$

where $\lambda_{\dot{\phi}}(R)$ and $\lambda_{\dot{\phi}}(T)$ refer to the rectangular wing and the tapered wing respectively. It is interesting to note that Bratt's mean experimental values $\lambda_{\dot{\phi}}(R) = 1 \cdot 15$ and $\lambda_{\dot{\phi}}(T) = 1 \cdot 35$, which correspond to values of ω_0 in the practical range, also satisfy this relation approximately. For wing B, $\beta = 10/21$, and $c_r/c_m = 0 \cdot 844$, and if the value $\lambda_{\dot{\phi}}(R) = 1 \cdot 15$ is assumed, the above relation gives $\lambda_{\dot{\phi}}(T) = 1 \cdot 36$.

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APPENDIX

Evaluation of Integrals.—(i) *Doublet Distribution* K_0'.—On substitution for x in (12), the following expressions for K_0' are obtained :—

$$\left.\begin{aligned} K_0' &= \frac{c}{2} \exp (i\omega' \cos \vartheta) \int_0^\vartheta \Gamma_0' \exp (-i\omega' \cos \vartheta) \sin \vartheta \, d\vartheta, \\ &= c \exp (i\omega' \cos \vartheta) \int_0^\vartheta \left[\operatorname{cosec} \vartheta - (1 - C) \cot \frac{\vartheta}{2}\right] \exp (-i\omega' \cos \vartheta) \sin \vartheta \, d\vartheta, \\ &= c \exp (i\omega' \cos \vartheta) \int_0^\vartheta [C - (1 - C) \cos \vartheta] \exp (-i\omega' \cos \vartheta) \, d\vartheta, \end{aligned}\right\} \quad (39)$$

where $\omega' = \dfrac{pc}{2V} = \dfrac{\omega}{2}$. Furthermore,

$$\exp (-i\omega' \cos \vartheta) = J_0 (\omega') + 2\sum_{n=1}^{\infty} (- i)^n J_n(\omega') \cos n\vartheta, \quad (40)$$

and by differentiation with respect to ω' this gives

$$\cos \vartheta \exp (-i\omega' \cos \vartheta) = i [J_0'(\omega') + 2\sum_{n=1}^{\infty} (- i)^n J_n'(\omega') \cos n\vartheta], \quad (41)$$

where $J_n' = \dfrac{\partial J_n(\omega')}{\partial \omega'}$. It is then readily deduced that

$$\int_0^\vartheta [C - (1 - C) \cos \vartheta] \exp (-i\omega' \cos \vartheta) \, d\vartheta = X_0 \vartheta + 2\sum_{n=1}^{\infty} (-i)^n X_n \frac{\sin n\vartheta}{n}, \quad (42)$$

where

$$X_n = C J_n (\omega') - i (1 - C) J_n' (\omega')$$

and

$$2J_n' = J_{n-1} - J_{n+1}.$$

If

$$S(\vartheta) = \exp (i\omega' \cos\vartheta) \left[X_0 \vartheta + 2 \sum_{n=1}^{\infty} (- i)^n X_n \frac{\sin n\vartheta}{n}\right], \quad (43)$$

equations (39) and (42) then give

$$K_0' = cS (\vartheta). \quad (44)$$

At the trailing edge, $\vartheta = \pi$, and

$$K_0' = ce^{-i\omega'} X_0 \pi = \frac{\pi c}{\mu}, \quad (45)$$

where $\mu = e^{i\omega'}/X_0$. But, by definition,

$$\begin{aligned} X_0 &= C J_0 + i (1 - C) J_1 \\ &= \frac{H_1^{(2)} J_0 - H_0^{(2)} J_1}{H_1^{(2)} + i H_0^{(2)}} = \frac{2i}{\pi \omega' (H_1^{(2)} + i H_0^{(2)})}, \end{aligned} \quad (46)$$

so that $\mu = \frac{\pi\omega'}{2} e^{i\omega'} (H_0^{(2)} - iH_1^{(2)})$. In the wake, $\Gamma_0' = 0$,

and
$$K_0' = e^{-ipx/V} \int_{x_l}^{x_t} \Gamma_0' e^{ipx/V} dx,$$
$$= \frac{\pi c}{\mu} e^{-ip(x - x_t)/V}. \qquad \ldots \quad (47)$$

When $\omega' = 0$ as for steady motion, (44) and (47) reduce to $K_0' = c\vartheta$ over the wing and $K_0' = \pi c$ in the wake respectively.

(ii) *Calculation of* W_{0m}'.—The normal induced velocity W_{0m}' is given by

$$W_{0m}' = \frac{\partial \Phi_{0m}'}{\partial z_1} = \iint \frac{K_0' A_m}{4\pi} \frac{\partial^2}{\partial z_1^2} \left(\frac{1}{r}\right) dx\, dy, \qquad \ldots \quad (48)$$

where the integral extends over the area between the leading edge of the wing and infinity downstream, and $z_1 \to 0$. Firstly, consider the integral

$$\left.\begin{aligned} 4\pi\Phi_{0m}' &= - z_1 \iint K_0' A_m \frac{dx\, dy}{r^3} \\ &= - z_1 \int_{-s}^{s} cA_m \int_{x_l}^{x_t} \frac{S(\vartheta)\, dx\, dy}{r^3} \\ &\quad - z_1 \int_{-s}^{s} \frac{\pi c A_m}{\mu} \int_{x_t}^{\infty} \frac{e^{-ip(x - x_t)/V}}{r^3} dx\, dy. \end{aligned}\right\} \qquad \ldots \quad (49)$$

Since $\exp(i\omega' \cos\vartheta) = J_0(\omega') + 2 \sum_{r=1}^{\infty} i^r J_r(\omega') \cos r\vartheta$ and $\frac{\pi - \vartheta}{2} = \sum_{n=1}^{\infty} \frac{\sin n\vartheta}{n}$, the function S defined by (43) can be expressed as

$$S(\vartheta) = J_0 \left[X_0\vartheta + 2 \sum_{n=1}^{\infty} (-i)^n X_n \frac{\sin n\vartheta}{n}\right]$$
$$+ 2 \sum_{r=1}^{\infty} i^r J_r \cos r\vartheta \left[X_0\pi - 2 \sum_{n=1}^{\infty} (X_0 - (-i)^n X_n) \frac{\sin n\vartheta}{n}\right]. \quad (50)$$

It is then evident that

$$\int_{x_l}^{x_t} \frac{S(\vartheta)\, dx}{r^3} \equiv \int_0^{\pi} \frac{cS(\vartheta) \sin\vartheta\, d\vartheta}{2r^3} \qquad \ldots \quad (51)$$

can be expressed in terms of integrals of the following types :—

$$\int_0^{\pi} \frac{\vartheta \sin\vartheta\, d\vartheta}{D^3}, \quad \int_0^{\pi} \frac{\sin n\vartheta \sin\vartheta\, d\vartheta}{D^3}, \quad \int_0^{\pi} \frac{\cos n\vartheta \sin\vartheta\, d\vartheta}{D^3},$$

where

$$\left.\begin{aligned} D^2 &= (a - \cos\vartheta)^2 + b^2 = 4r^2/c^2, \\ ac &= 2\,[R(y) - R(y)_1] + c_1 \cos\vartheta_1, \\ bc &= 2\sqrt{(y - y_1)^2 + z_1^2}. \end{aligned}\right\} \quad \ldots \quad (52)$$

and

Integrals of the first two types are expressed in terms of elliptic integrals in Ref. 1. The third type of integral can be evaluated by means of reduction formulæ. If

$$I_r \equiv \int_0^\pi \frac{\cos^r\vartheta \sin\vartheta \, d\vartheta}{D^3}, \quad \ldots \quad (53)$$

then it can be deduced that

$$I_0 = \frac{1}{b^2}\left[\frac{a+1}{[(a+1)^2+b^2]^{1/2}} - \frac{a-1}{[(a-1)^2+b^2]^{1/2}}\right],$$

$$I_1 = aI_0 + \frac{1}{[(a+1)^2+b^2]^{1/2}} - \frac{1}{[(a-1)^2+b^2]^{1/2}},$$

$$I_2 = (a-1)\,I_1 - aI_0 + \log_e \frac{a+1+[(a+1)^2+b^2]^{1/2}}{a-1+[(a-1)^2+b^2]^{1/2}},$$

and

$$rI_r = [2a\,(r-1) - a]\,I_{r-1} - (r-1)\,(a^2+b^2)\,I_{r-2} - \frac{(-1)^{r-1}}{[(a+1)^2+b^2]^{1/2}} + \frac{1}{[(a-1)^2+b^2]^{1/2}}.$$

Since

$$\cos n\vartheta = 2^{n-1}\cos^n\vartheta - \frac{n\,.\,2^{n-3}}{1\,!}\cos^{n-2}\vartheta + \frac{n\,.\,n-3\,.\,2^{n-5}}{2\,!}\cos^{n-4}\vartheta - etc.,$$

the required integral can be expressed in the form

$$\int_0^\pi \frac{\cos n\vartheta \sin\vartheta\, d\vartheta}{D^3} = 2^{n-1}\,I_n - \frac{n\,.\,2^{n-3}}{1\,!}\,I_{n-2} + etc. \quad \ldots \quad (54)$$

By the use of the preceding relations and the results of Ref. 1 the integral defined by (51) is given to sufficient accuracy by a finite number of terms of the series expansion for $S(\vartheta)$ for a value of ω' in the practical range $(0 < \omega' \leqslant 2)$.

Next consider

$$\int_{x_t}^\infty \frac{e^{-ip(x-x_t)/V}}{r^3}\,dx = e^{ip(x_t - x_1)/V}\int_0^\infty \frac{e^{-ipX/V}\,dX}{(X^2+(y-y_1)^2+z_1^2)^{3/2}} - \int_{x_1}^{x_t} \frac{e^{-ip(x-x_t)/V}\,dx}{r^3}, \quad \ldots \quad (55)$$

where $X \equiv x - x_1$.

The second integral can be expressed as

$$\frac{4}{c^2}\int_{\vartheta_1}^{\pi}\frac{\exp[i\omega'(1+\cos\vartheta)]\sin\vartheta\,d\vartheta}{D^3} = \frac{4e^{i\omega'}}{c^2}\int_{\vartheta_1}^{\pi}\frac{\left[J_0+2\sum_{1}^{\infty} i^n J_n\cos n\vartheta\right]\sin\vartheta\,d\vartheta}{D^3}, \quad \ldots \quad (56)$$

which can be integrated in the same way as the integral of the third type already considered. The first integral in (55) can be expressed in terms of Bessel Functions. Let $x \equiv pX/V$, and $\alpha^2 \equiv p^2[(y-y_1)^2+z_1^2)]/V^2$. Then

$$\int_0^{\infty}\frac{e^{-ipX/V}\,dX}{[X^2+(y-y_1)^2+z_1^2]^{1/2}} = \int_0^{\infty}\frac{e^{-ix}\,dx}{(x^2+\alpha^2)^{1/2}}, = K_0(\alpha) - iT_0(\alpha), \quad \ldots \quad \ldots \quad (57)$$

where $K_0(\alpha)$ denotes the usual modified Bessel Function of the second kind. The function $T_0(\alpha)$ has been discussed in the Appendix to Ref. 17. There it is shown that

$$T_0(\alpha) = \frac{\pi}{2}I_0(\alpha) - I_0(\alpha)\int_0^{\alpha}K_0(\alpha)\,d\alpha + K_0(\alpha)\int_0^{\alpha}I_0(\alpha)\,d\alpha,$$

$$= \frac{\pi}{2}I_0(\alpha) - \frac{\alpha}{1^2} - \frac{\alpha^3}{1^2 3^2} - \frac{\alpha^5}{1^2 3^2 5^2} - etc., \quad \ldots \quad \ldots \quad (58)$$

where $I_0(\alpha)$ is the modified Bessel Function of the first kind. The asymptotic form of $T_0(\alpha)$ is given by

$$T_0(\alpha) = \frac{1}{\alpha} + \frac{1^2}{\alpha^3} + \frac{1^2 3^2}{\alpha^5} + \frac{1^2 3^2 5^2}{\alpha^7} + etc. \quad \ldots \quad \ldots \quad \ldots \quad (59)$$

By differentiation with respect to α, (57) yields

$$\int_0^{\infty}\frac{e^{-ix}\,dx}{(x^2+\alpha^2)^{3/2}} = \frac{K_1(\alpha) - iT_1(\alpha)}{\alpha}, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (60)$$

where $K_1 = -\dfrac{\partial K_0}{\partial\alpha}$ and $T_1 = -\dfrac{\partial T_0}{\partial\alpha}$.

From the preceding formulæ, it is evident that Φ_{0m}' can be expressed in the form

$$\Phi_{0m}' = z_1\int_{-s}^{s} f(y-y_1, z_1)\,dy, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (61)$$

where $f \to \infty$ when $z_1 = 0$ and $y \to y_1$. The corresponding induced velocity distribution is given by

$$W_{0m}' = \int_{-s}^{s}\left(f + z_1\frac{\partial f}{\partial z_1}\right)dy = \int_{-s}^{s} g\,dy,$$

where g has a singularity at $y = y_1$ when $z_1 = 0$. The above integral can be evaluated by the method given in Appendix II of Ref. 1.

The calculation of W_{0m}' by the above method would be very laborious. Fortunately, however, the values of W_{0m}' are relatively small in comparison to W_{0m}'', and they can be calculated to sufficient accuracy by Cicala's method. The values of W_{0m}' given in Table 2 were calculated by the latter method with μ assumed to be constant and equal to its mean value along the span. This is equivalent to the assumption that the tapered wing can be replaced by a rectangular wing of the same span and area. The calculation of W_{0m}' for a rectangular wing is described in detail in Ref. 2.

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16	M. Knight and C. J. Wenzinger	Rolling Moments due to Rolling and Yaw for Four Wing Models in Rotation. N.A.C.A. Report No. 379, 1931.
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TABLE 1

Values of $S(\vartheta)$ $(\equiv K_0'/c)$

$\frac{12\vartheta}{\pi}$	$\omega = 0{\cdot}4$	$\omega = 0{\cdot}8$	$\omega = 1{\cdot}2$	$\omega = 1{\cdot}6$	$\omega = 2{\cdot}0$
0	0	0	0	0	0
1	$0{\cdot}1195-0{\cdot}0985i$	$0{\cdot}0660-0{\cdot}0865i$	$0{\cdot}0418-0{\cdot}0725i$	$0{\cdot}0294-0{\cdot}0616i$	$0{\cdot}0227-0{\cdot}0534i$
2	$0{\cdot}2410-0{\cdot}1975i$	$0{\cdot}1335-0{\cdot}17375i$	$0{\cdot}0852-0{\cdot}1458i$	$0{\cdot}05925-0{\cdot}1240i$	$0{\cdot}0445-0{\cdot}1073i$
3	$0{\cdot}3675-0{\cdot}2955i$	$0{\cdot}2055-0{\cdot}2620i$	$0{\cdot}1313-0{\cdot}2205i$	$0{\cdot}09125-0{\cdot}1876i$	$0{\cdot}0670-0{\cdot}1622i$
4	$0{\cdot}5000-0{\cdot}3930i$	$0{\cdot}2840-0{\cdot}3515i$	$0{\cdot}1833-0{\cdot}2978i$	$0{\cdot}1280-0{\cdot}2540i$	$0{\cdot}0939-0{\cdot}2197i$
5	$0{\cdot}6425-0{\cdot}4885i$	$0{\cdot}37175-0{\cdot}44275i$	$0{\cdot}2438-0{\cdot}3785i$	$0{\cdot}1725-0{\cdot}3251i$	$0{\cdot}1285-0{\cdot}2827i$
6	$0{\cdot}7955-0{\cdot}5815i$	$0{\cdot}4720-0{\cdot}53525i$	$0{\cdot}3158-0{\cdot}4628i$	$0{\cdot}22775-0{\cdot}40175i$	$0{\cdot}1725-0{\cdot}3530i$
7	$0{\cdot}9630-0{\cdot}6700i$	$0{\cdot}58825-0{\cdot}6275i$	$0{\cdot}4033-0{\cdot}5503i$	$0{\cdot}29675-0{\cdot}48325i$	$0{\cdot}2284-0{\cdot}4294i$
8	$1{\cdot}1465-0{\cdot}7515i$	$0{\cdot}7245-0{\cdot}7175i$	$0{\cdot}5110-0{\cdot}6383i$	$0{\cdot}3850-0{\cdot}5674i$	$0{\cdot}3029-0{\cdot}5091i$
9	$1{\cdot}3475-0{\cdot}8220i$	$0{\cdot}8850-0{\cdot}8005i$	$0{\cdot}6448-0{\cdot}7233i$	$0{\cdot}50025-0{\cdot}6510i$	$0{\cdot}4044-0{\cdot}5899i$
10	$1{\cdot}5675-0{\cdot}8780i$	$1{\cdot}07325-0{\cdot}87125i$	$0{\cdot}8112-0{\cdot}7992i$	$0{\cdot}65075-0{\cdot}7286i$	$0{\cdot}5427-0{\cdot}6677i$
11	$1{\cdot}8055-0{\cdot}9160i$	$1{\cdot}2915-0{\cdot}91675i$	$1{\cdot}0147-0{\cdot}8558i$	$0{\cdot}84275-0{\cdot}7894i$	$0{\cdot}7252-0{\cdot}7316i$
12	$2{\cdot}0605-0{\cdot}9295i$	$1{\cdot}53925-0{\cdot}9405i$	$1{\cdot}2575-0{\cdot}8788i$	$1{\cdot}08075-0{\cdot}8155i$	$0{\cdot}9587-0{\cdot}7608i$

TABLE 2

Values of W_{0m}' for Wings* of Aspect Ratio 6

sp/V	η_1 \ m	1	2	3	4	5	6
1	0	0·1727−0·2972i	0	−0·2001+0·1696i	0	−0·0395+0·0471i	0
	0·2	0·1795−0·2978i	0·1028−0·1323i	−0·1626+0·1298i	−0·0821+0·0524i	−0·0663+0·0625i	−0·0281+0·218i
	0·4	0·2023−0·2991i	0·2239−0·2665i	−0·0447+0·0095i	−0·0911+0·0403i	−0·1091+0·0764i	−0·0798+0·0514i
	0·6	0·2489−0·2990i	0·3589−0·3908i	+0·1727−0·1870i	+0·0464−0·0918i	−0·0466−0·0015i	−0·0869+0·0287i
	0·8	0·3438−0·2842i	0·5653−0·5001i	0·5398−0·4419i	0·4700−0·3791i	+0·3683−0·2950i	+0·2711−0·2244i
	1·0	0·8384+0·1125i	1·224 −0·2068i	1·490 −0·3463i	1·723 −0·4859i	1·924 −0·5836i	2·108 −0·6828i
2	0	0·0363−0·1960i	0	−0·0963+0·1580i	0	−0·0108+0·0388i	0
	0·2	0·0411−0·1993i	0·0321−0·1009i	−0·0820+0·1245i	−0·0487+0·0560i	−0·0284+0·0558i	−0·0140+0·0217i
	0·4	0·0584−0·2104i	0·0796−0·2086i	−0·0337+0·0219i	−0·0625+0·0540i	−0·0620+0·0772i	−0·0470+0·0555i
	0·6	0·0979−0·2315i	0·1434−0·3186i	+0·0692−0·1559i	−0·0019−0·0619i	−0·0444+0·0131i	−0·0669+0·0444i
	0·8	0·1929−0·2590i	0·2843−0·4413i	0·2864−0·4112i	+0·2523−0·3400i	+0·1969−0·2691i	+0·1415−0·1950i
	1·0	0·9193+0·1152i	1·121 −0·1640i	1·300 −0·3211i	1·453 −0·4602i	1·594 −0·5689i	1·723 −0·6720i
4	0	−0·0109−0·0968i	0	−0·0239+0·1044i	0	+0·0038+0·0198i	0
	0·2	−0·0092−0·0989i	−0·0018−0·0555i	−0·0230+0·0857i	−0·0187+0·0425i	−0·0044+0·0348i	−0·0043+0·01220i
	0·4	−0·0046−0·1077i	+0·0062−0·1182i	−0·0172+0·0250i	−0·0277+0·0491i	−0·0233+0·0589i	−0·0167+0·0430i
	0·6	+0·0153−0·1372i	0·0213−0·1899i	+0·0074−0·0939i	−0·0173−0·0229i	−0·0297+0·0237i	−0·0363+0·0484i
	0·8	0·0752−0·1951i	0·0905−0·3005i	0·0959−0·2965i	+0·0944−0·2269i	+0·0715−0·1822i	+0·0509−0·1189i
	1·0	0·9814+0·0860i	1·064 −0·1000i	1·158 −0·2354i	1·242 −0·3488i	1·325 −0·4446i	1·401 −0·5348i
6	0	−0·0161−0·0599i	0	−0·0041+0·0719i	0	0·0057 +0·0114i	0
	0·2	−0·0148−0·0602i	−0·0087−0·0364i	−0·0056+0·0606i	−0·0093+0·0302i	0·0012 +0·0233i	−0·0015+0·0080i
	0·4	−0·0077−0·0624i	−0·0081−0·0772i	−0·0071+0·0226i	−0·0130+0·0381i	−0·0088 +0·0444i	−0·0061+0·0313i
	0·6	−0·0031−0·0902i	−0·0054−0·1236i	−0·0056−0·0610i	−0·0116−0·0044i	−0·01790+0·0235i	−0·0168+0·0458i
	0·8	+0·0284−0·1515i	+0·0268−0·2133i	+0·0292−0·2202i	+0·0460−0·1482i	+0·0313 −0·1237i	+0·0279−0·0653i
	1·0	1·005 +0·0617i	1·049 −0·0698i	1·107 −0·1835i	1·161 −0·2756i	1·216 −0·3566i	1·269 −0·4347i

* For rectangular and tapered wings of the same span and generalized aspect ratio.

TABLE 2—*continued.*

sp/V	η_1 \ m	7	8	9	10	11
1	0	−0·0179+0·0240*i*	0	−0·0106+0·0151*i*	0	−0·0072+0·0106*i*
	0·2	−0·0268+0·0301*i*	−0·0122+0·0107*i*	−0·0144+0·0186*i*	−0·0071+0·0067*i*	−0·0090+0·0132*i*
	0·4	−0·0668+0·0523*i*	−0·0443+0·0321*i*	−0·0371+0·0322*i*	−0·0248+0·0197*i*	−0·0223+0·0211*i*
	0·6	−0·1065+0·0554*i*	−0·1049+0·0553*i*	−0·0989+0·0592*i*	−0·0865+0·0508*i*	−0·0759+0·0486*i*
	0·8	+0·1804−0·1574*i*	+0·1034−0·1053*i*	+0·0399−0·0597*i*	−0·0109−0·0264*i*	−0·0495+0·0019*i*
	1·0	2·276 −0·7612*i*	2·434 −0·8414*i*	2·581 −0·9085*i*	+2·722 −0·9772*i*	+2·854 −1·037*i*
2	0	−0·0034+0·0188*i*	0	−0·0016+0·0116*i*	0	−0·0009+0·0080*i*
	0·2	−0·0086+0·0252*i*	−0·0051+0·0102*i*	−0·0034+0·0152*i*	−0·0027+0·0064*i*	−0·0011+0·0107*i*
	0·4	−0·0349+0·0505*i*	−0·0238+0·0333*i*	−0·0175+0·0299*i*	−0·0121+0·0199*i*	−0·0095+0·0190*i*
	0·6	−0·0720+0·0631*i*	−0·0703+0·0644*i*	−0·0627+0·0633*i*	−0·0551+0·0563*i*	−0·0463+0·0507*i*
	0·8	+0·0881−0·1354*i*	+0·0425−0·0817*i*	+0·0044−0·0409*i*	−0·0260−0·0070*i*	−0·0490+0·0179*i*
	1·0	1·844 −0·7583*i*	1·957 −0·8431*i*	2·064 −0·9165*i*	+2·166 −0·9898*i*	+2·263 −1·055*i*
4	0	0·0030+0·0089*i*	0	0·0022+0·0054*i*	0	0·0016+0·0037*i*
	0·2	0·0011+0·0138*i*	−0·0020+0·0031*i*	+0·0021+0·0083*i*	−0·0017+0·0008*i*	+0·0027+0·0058*i*
	0·4	−0·0108+0·0360*i*	−0·0064+0·0240*i*	−0·0038+0·0198*i*	−0·0020+0·0134*i*	−0·0012+0·0117*i*
	0·6	−0·0356+0·0563*i*	−0·0331+0·0581*i*	−0·0281+0·0528*i*	−0·0237+0·0483*i*	−0·0190+0·0408*i*
	0·8	+0·0270−0·0786*i*	+0·0067−0·0346*i*	−0·0106−0·0069*i*	−0·0252+0·0199*i*	−0·0352+0·0363*i*
	1·0	1·475 −0·6112*i*	1·546 −0·6877*i*	+1·613 −0·7527*i*	+1·678 −0·8199*i*	+1·741 −0·8772*i*
6	0	0·0038+0·0051*i*	0	0·0028+0·0032*i*	0	0·0022+0·0023*i*
	0·2	+0·0025+0·0085*i*	−0·0002+0·0024*i*	+0·0028+0·0055*i*	0·0001+0·0016*i*	0·0032+0·0038*i*
	0·4	−0·0031+0·0256*i*	−0·0013+0·0164*i*	−0·0004+0·0132*i*	+0·0004+0·0085*i*	+0·0003+0·0073*i*
	0·6	−0·0179+0·0457*i*	−0·0135+0·0502*i*	−0·0127+0·0412*i*	−0·0083+0·0406*i*	−0·0076+0·0309*i*
	0·8	+0·0122−0·0419*i*	+0·0043−0·0028*i*	−0·0073+0·0127*i*	−0·0140+0·0364*i*	−0·0211+0·0443*i*
	1·0	1·320 −0·4995*i*	1·370 −0·5678*i*	+1·418 −0·6229*i*	+1·465 −0·6838*i*	+1·511 −0·7326*i*

(73294)

TABLE 3

Values* of W_{0m}'' along the Mid-chord Axis of Wing A

η_1 \ m	1	2	3	4	5	6	7	8	9	10	11
0	2·446	0	−0·1059	0	−0·0139	0	−0·0062	0	−0·0041	0	−0·0033
0·2	2·516	0·5504	+0·0155	−0·0269	−0·0263	−0·0119	−0·0100	−0·0045	−0·0054	−0·0025	−0·0038
0·4	2·630	1·137	0·3912	+0·1106	+0·0073	−0·0176	−0·0230	−0·0180	−0·0154	−0·0107	−0·0095
0·6	2·600	1·677	0·9896	0·5586	0·2912	+0·1385	+0·0505	+0·0059	−0·0177	−0·0261	−0·0297
0·8	2·286	1·981	1·638	1·328	1·054	0·8235	0·6302	0·4738	+0·3463	+0·2459	+0·1656
1·0	1·316	1·546	1·727	1·893	2·043	2·185	2·317	2·443	2·563	2·679	2·789

Values of W_{0m}'' along $0 \cdot 3c$ and $0 \cdot 75c$ of Wing B

	Values along $0 \cdot 3c$ axis						Values along $0 \cdot 75c$ axis					
η_1 \ m	1	3	5	7	9	11	1	3	5	7	9	11
0	2·444	−0·0986	−0·0134	−0·0060	−0·0040	−0·0028	2·849	−0·0907	−0·0127	−0·0058	−0·0039	−0·0028
0·2	2·546	+0·0226	−0·0245	−0·0096	−0·0053	−0·0037	2·518	+0·0327	−0·0224	−0·0092	−0·0050	−0·0035
0·4	2·635	0·3933	+0·0117	−0·0208	−0·0145	−0·0091	2·619	0·4068	+0·0190	−0·0180	−0·0132	−0·0084
0·6	2·603	0·9871	0·2954	+0·0562	−0·0130	−0·0262	2·587	0·9993	0·3100	+0·0672	−0·0058	−0·0218
0·8	2·299	1·636	1·053	0·6352	+0·3553	+0·1775	2·263	1·626	1·059	0·6480	+0·3718	+0·1945
1·0	1·239	1·618	1·907	2·157	2·380	2·585	1·112	1·483	1·762	2·002	2·217	2·414

* These values correspond to the aspect ratio 5·84.

TABLE 4

Values of W_{1m} along the $0 \cdot 3c$ Axis of Wing B

η_1 \ m	1	3	5	7	9	11
0	2·147	−0·0581	−0·0072	−0·0033	−0·0023	−0·0017
0·2	2·249	+0·0438	−0·0128	−0·0053	−0·0030	−0·0021
0·4	2·339	0·3593	+0·0270	−0·0089	−0·0078	−0·0051
0·6	2·309	0·8605	0·2734	+0·0680	+0·0044	−0·0114
0·8	2·029	1·394	0·8955	0·5482	0·3185	+0·1722
1·0	0·9941	1·230	1·416	1·580	1·727	1·864

TABLE 5

Values of W_{0m} ($\equiv W_{0m}' + W_{0m}''$) for Steady Motion*

η_1 \ m	1	3	5	7	9	11
0	3·230	−0·4913	−0·1116	−0·0551	−0·0347	−0·0243
0·2	3·331	−0·2759	−0·1635	−0·0732	−0·0428	−0·0293
0·4	3·420	+0·3777	−0·1745	−0·1448	−0·0880	−0·0558
0·6	3·388	1·443	+0·2820	−0·0969	−0·1698	−0·1518
0·8	3·084	2·751	1·810	+1·035	+0·4946	+0·1511
1·0	2·024	3·582	4·558	5·348	6·031	6·644

* Values along $0 \cdot 3c$ axis of wing B due to circulation $\pi s \eta^{m-1} \sqrt{1 - \eta^2}$.

TABLE 6

Fundamental Derivative Coefficients Referred to the Leading Edge.
Mode I :— $f = \eta$, $F = \eta^{1\cdot 5}$; Mode II :— $f = \eta^2$, $F = \eta$

Case	ω_0	λ_ϕ	$\lambda_{\dot\phi}$	λ_θ	$\lambda_{\dot\theta}$	μ_ϕ	$\mu_{\dot\phi}$	μ_θ	$\mu_{\dot\theta}$	Remarks
1	0·45	0·116	1·808	1·691	0·460	0·025	0·414	0·370	0·385	Wing A, Mode II—Symmetrical. Reference axis at mid-chord ($\overline{m} = 0$). Values of W_{0m}'' calculated along the mid-chord axis and W_{0m}' neglected. $W_{1m} = W_{0m}''$. (*See* Ref. 1.)
	0·9	0·227	1·559	1·547	0·985	0·050	0·357	0·349	0·505	
	1·8	0·357	1·353	1·458	1·244	0·079	0·309	0·371	0·566	
	2·7	0·424	1·272	1·435	1·309	0·095	0·294	0·435	0·583	
		$\lambda_{\ddot\phi} = 0\cdot595$.		$\lambda_{\ddot\theta} = 0\cdot297_5$.		$\mu_{\ddot\phi} = 0\cdot297_5$.		$\mu_{\ddot\theta} = 0\cdot1675$.		
2	0·45	0·116	1·808	1·688	0·396	0·028	0·444	0·411	0·390	Wing B, Mode II—Symmetrical. Reference axis at $0\cdot3c$ behind the leading edge. Values of W_{0m}'' along mid-chord of Wing A used and W_{0m}' neglected. $W_{1m} = W_{0m}''$.
	0·9	0·227	1·559	1·541	0·929	0·055	0·383	0·375	0·521	
	1·8	0·357	1·353	1·447	1·196	0·087	0·333	0·352	0·587	
	2·7	0·424	1·272	1·420	1·265	0·104	0·312	0·345	0·605	
		$\lambda_{\ddot\phi} = 0\cdot595$.		$\lambda_{\ddot\theta} = 0\cdot281$.		$\mu_{\ddot\phi} = 0\cdot309$.		$\mu_{\ddot\theta} = 0\cdot166$.		
3	0·45	0·116	1·810	1·976	0·424	0·025	0·419	0·467	0·404	Wing A, Mode II—Antisymmetrical. Reference axis at mid-chord ($\overline{m} = 0$). Values of W_{0m}'' calculated along the mid-chord axis and W_{0m}' neglected. $W_{1m} = W_{0m}''$. (*See* Ref. 1.)
	0·9	0·228	1·561	1·803	1·044	0·050	0·362	0·426	0·557	
	1·8	1·358	1·355	1·690	1·357	0·080	0·313	0·397	0·638	
	2·7	0·425	1·274	1·658	1·438	0·096	0·294	0·389	0·660	
		$\lambda_{\ddot\phi} = 0\cdot594$.		$\lambda_{\ddot\theta} = 0\cdot297$.		$\mu_{\ddot\phi} = 0\cdot297$.		$\mu_{\ddot\theta} = 0\cdot169$.		
4	0·9	0·095	1·266	1·428	1·283	0·028	0·267	0·321	0·580	As for Case 3 with W_{0m}' terms included.
	2·7	0·342	1·110	1·431	1·360	0·104	0·228	0·373	0·656	
5	0·45	0·122	1·746	1·968	0·436	0·031	0·437	0·492	0·417	Wing B, Mode 1—Symmetrical. Reference axis at $0\cdot3c$ behind the leading edge. Values of W_{0m}'' for mid-chord axis of Wing A used and W_{0m}' terms neglected. $W_{1m} = W_{0m}''$.
	0·9	0·234	1·503	1·801	1·052	0·058	0·376	0·450	0·571	
	1·8	0·358	1·310	1·689	1·362	0·089	0·328	0·422	0·649	
	2·7	0·420	1·237	1·657	1·443	0·105	0·309	0·414	0·669	
		$\lambda_{\ddot\phi} = 0\cdot593$.		$\lambda_{\ddot\theta} = 0\cdot303$.		$\mu_{\ddot\phi} = 0\cdot296$.		$\mu_{\ddot\theta} = 0\cdot171$.		
6	0·45	0·121	1·732	1·980	0·430	0·030	0·433	0·495	0·416	As for Case 5, but with W_{0m}'' values along $0\cdot3c$ axis of Wing B used and W_{0m} terms neglected. $W_{1m} = W_{0m}''$.
	0·9	0·232	1·492	1·807	1·048	0·058	0·373	0·452	0·571	
	1·8	0·355	1·300	1·694	1·360	0·089	0·325	0·424	0·649	
	2·7	0·417	1·228	1·662	1·442	0·104	0·307	0·415	0·669	
		$\lambda_{\ddot\phi} = 0\cdot588$.		$\lambda_{\ddot\theta} = 0\cdot302$.		$\mu_{\ddot\phi} = 0\cdot294$.		$\mu_{\ddot\theta} = 0\cdot170_5$.		
7	2·7	0·420	1·237	1·661	1·443	0·105	0·309	0·415	0·653	As for Case 5 with W_{0m}'' values along $0\cdot75c$ and W_{1m} values along $0\cdot3c$ used ; W_{0m}' neglected.
8	0·9	0·141	1·336	1·506	1·250	0·039	0·329	0·371	0·594	As for Case 5 with W_{0m}' terms included.
	2·7	0·359	1·150	1·471	1·398	0·102	0·274	0·362	0·638	
9	2·7	0·360	1·138	1·466	1·390	0·109	0·270	0·363	0·618	As for Case 7 with W_{0m}' terms included.
10	0·9	0·181	1·527	1·756	1·516	0·062	0·370	0·437	0·721	Wing B, Mode I—Symmetrical. Reference axis at $0\cdot3c$ behind the leading edge. *Cicala's method* used.
	2·7	0·697	1·357	1·892	1·663	0·265	0·320	0·502	0·770	

TABLE 7

Values of $\lambda_{\dot{\phi}}$ for Tapered Wings. Parabolic Mode in Flexure ; $f = \eta^2$

ω_0	0·45	0·9	1·8	2·7	Remarks
$\lambda_{\dot{\phi}}$	1·808	1·559	1·353	1·272	Wing A, Mode II, S ; $W_{0m}' = 0$. (*See* Case I of Table 6.)
	1·810	1·561	1·355	1·274	Wing A, Mode II, A-S ; $W_{0m}' = 0$. (*See* Case 3 of Table 6.)
	—	1·266	—	1·110	Wing A, Mode II, A-S ; $W_{0m}' \neq 0$. (*See* Case 3 of Table 6.)
	—	1·290	—	1·118	Wing B, Mode II, S ; $W_{0m}' \neq 0$. W_{0m}'' values along $0·3c$ axis used.

S ≡ symmetrical motion ; A-S ≡ antisymmetrical motion.

Linear Mode in Flexure, $f = \eta$; Symmetrical Motion Only

ω_0	0·45	0·9	1·8	2·7	Remarks
$\lambda_{\dot{\phi}}$	1·746	1·503	1·310	1·237	Wing B, $W_{0m}' = 0$, and W_{0m}'' values along $0·5c$ of Wing A used. (*See* Case 5 of Table 6.)
	1·732	1·492	1·300	1·228	As above with W_{0m}'' values at $0·3c$ of Wing B used. (*See* Case 6 of Table 6.)
	—	—	—	1·237	As above with W_{0m}'' values at $0·75c$ of Wing B used. (*See* Case 7 of Table 6.)
	—	1·336	—	1·150	Wing B, $W_{0m}' \neq 0$, and W_{0m}'' values along $0·5c$ of Wing A used. (*See* Case 5 of Table 6.)
	—	1·331	—	1·151	Wing B, $W_{0m}' \neq 0$, and W_{0m}'' values along $0·3c$ of Wing B used.
	—	—	—	1·138	Wing B, $W_{0m}' \neq 0$, and W_{0m}'' values along $0·75c$ axis of Wing B used.
*	—	1·527	—	1·357	Cicala's method ; Wing B, reference axis at $0·3c$.

* In Cicala's method the values of W_{0m}'' corresponding to $K_0'' A_m$ are assumed to be given by A_m. (*See* § 4.)

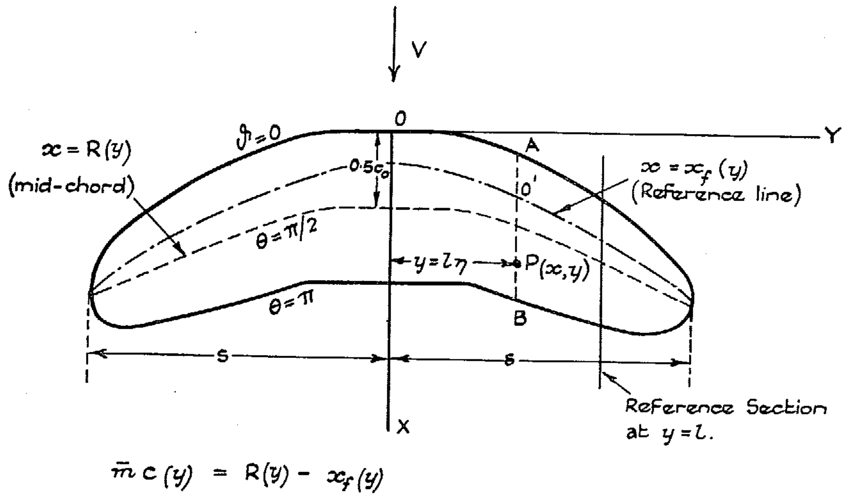


FIG. 1.

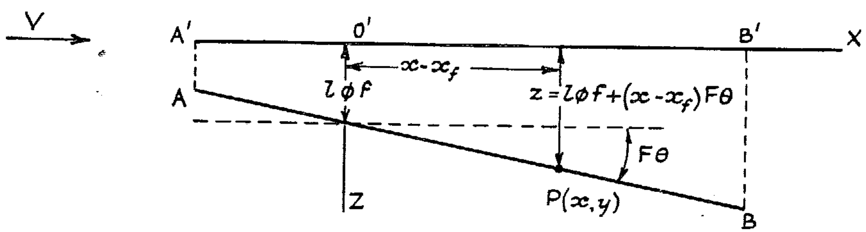
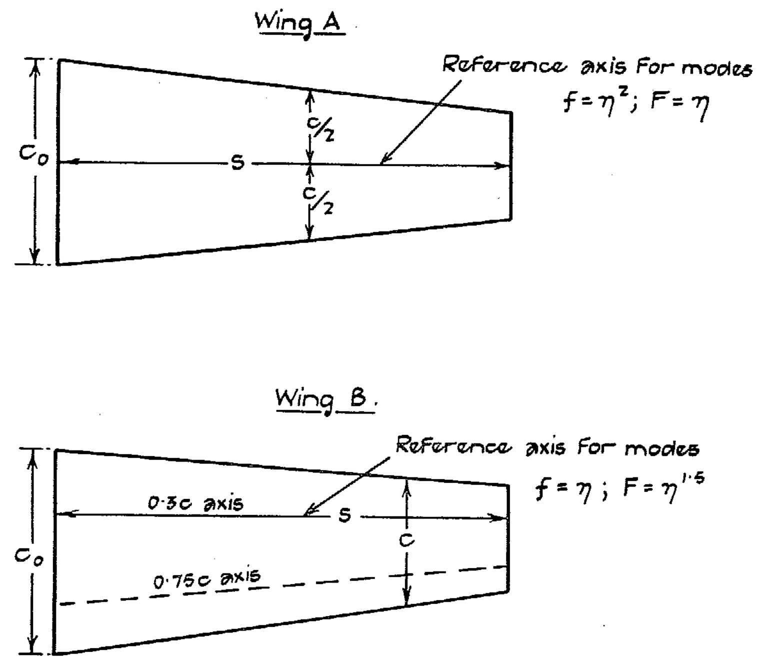


FIG. 2.



$c = c_0\left(1 - \frac{10\eta}{21}\right)$; $\eta = y/s$; $l = s$

$s = 4.5$ Ft; $c_0 = 2.025$ Ft.

Aspect ratio 6 (exact value 5.84)

FIG. 3.

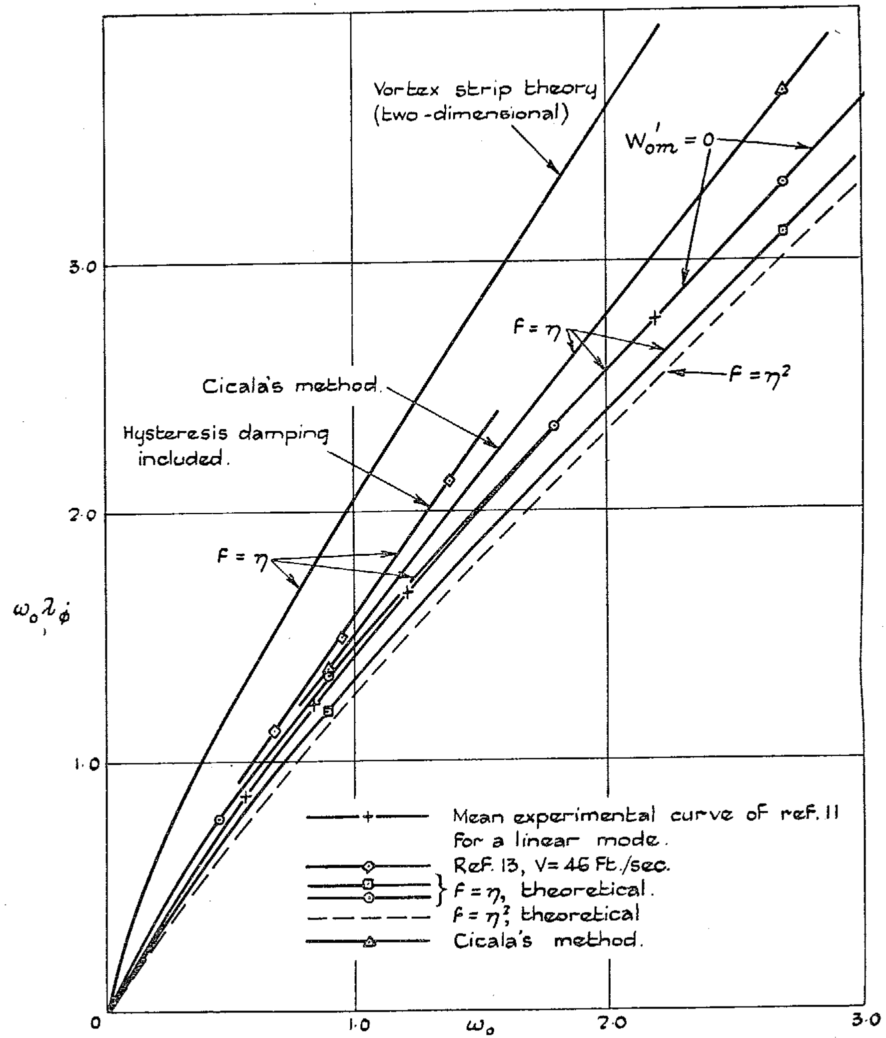


FIG. 4.—Values of $\omega_0 \lambda_{\dot{\phi}}$ for Tapered Wing B.

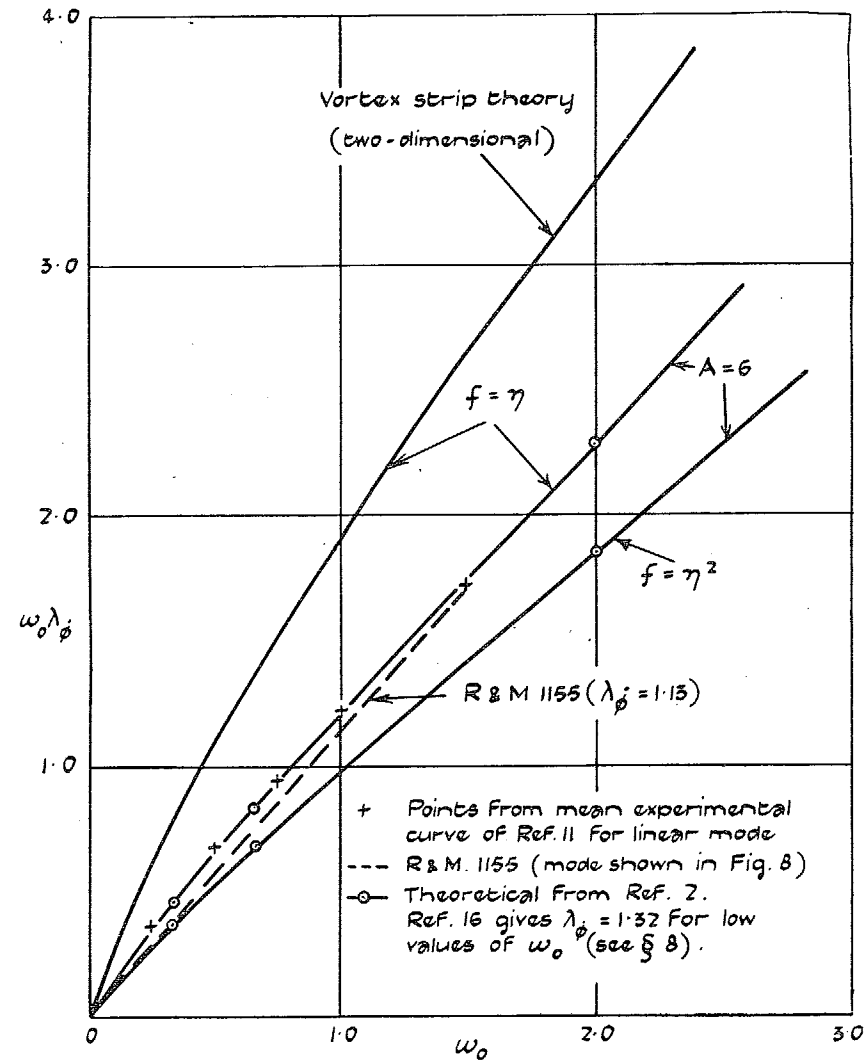


FIG. 5.—Values of $\omega_0 \lambda_{\dot{\phi}}$ for Rectangular Wings.

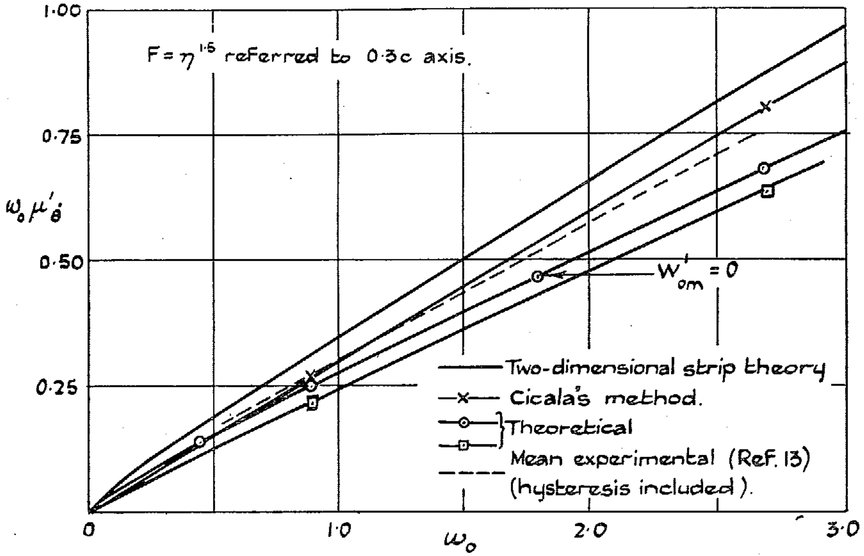


FIG. 6a.

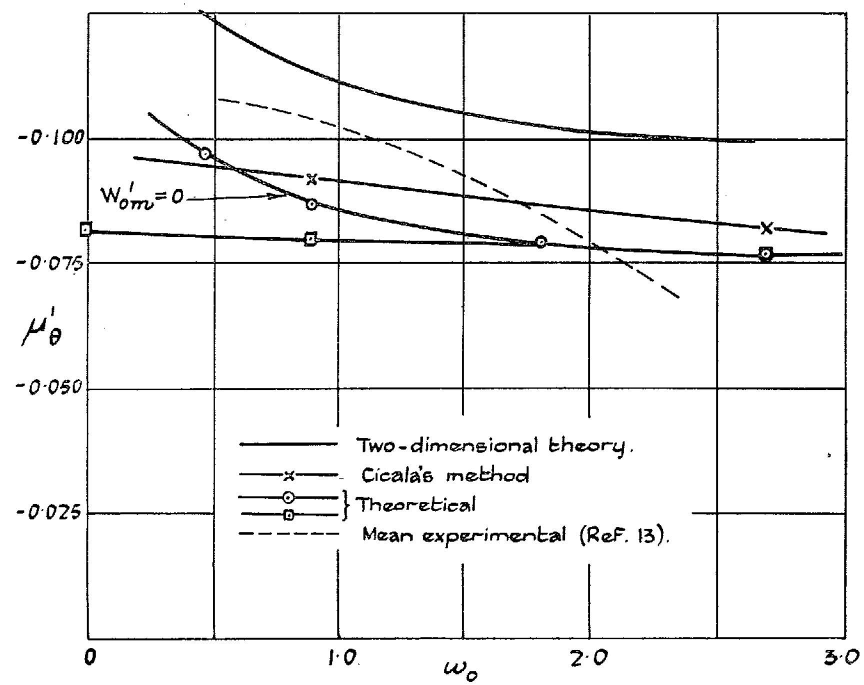


FIG. 6b.—Values of μ_θ' and $\omega_0\mu_{\dot\theta}'$ for Wing B referred to the $0 \cdot 3c$ Axis.

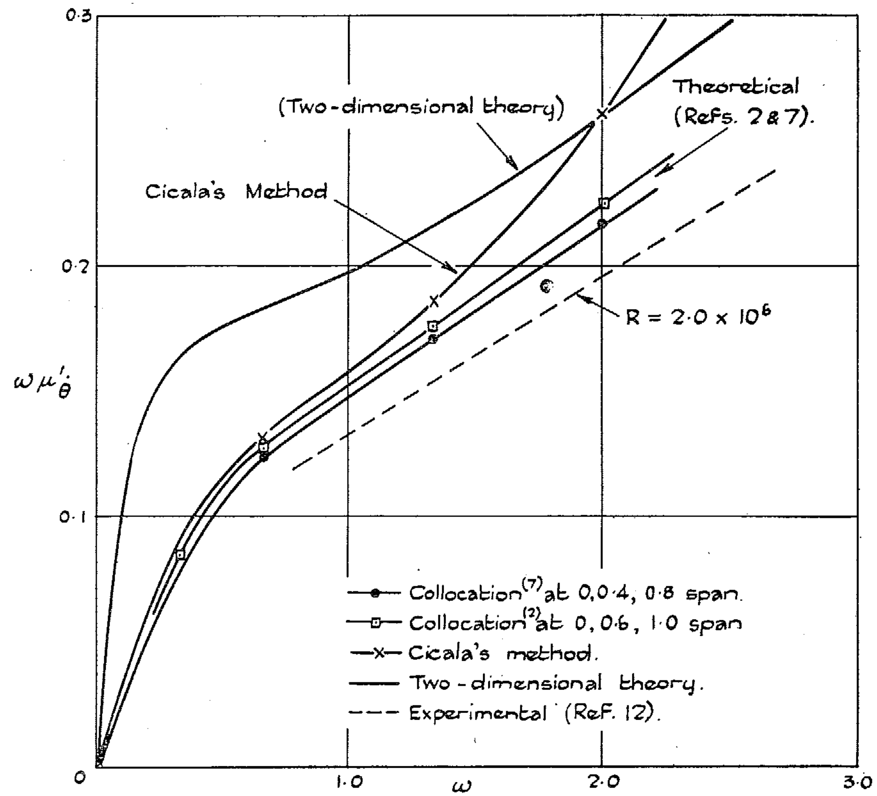


FIG. 7.—Values of $\omega\mu_{\dot\theta}'$ for a Rectangular Aerofoil ($A = 6$) oscillating about the $0 \cdot 5c$ Axis.

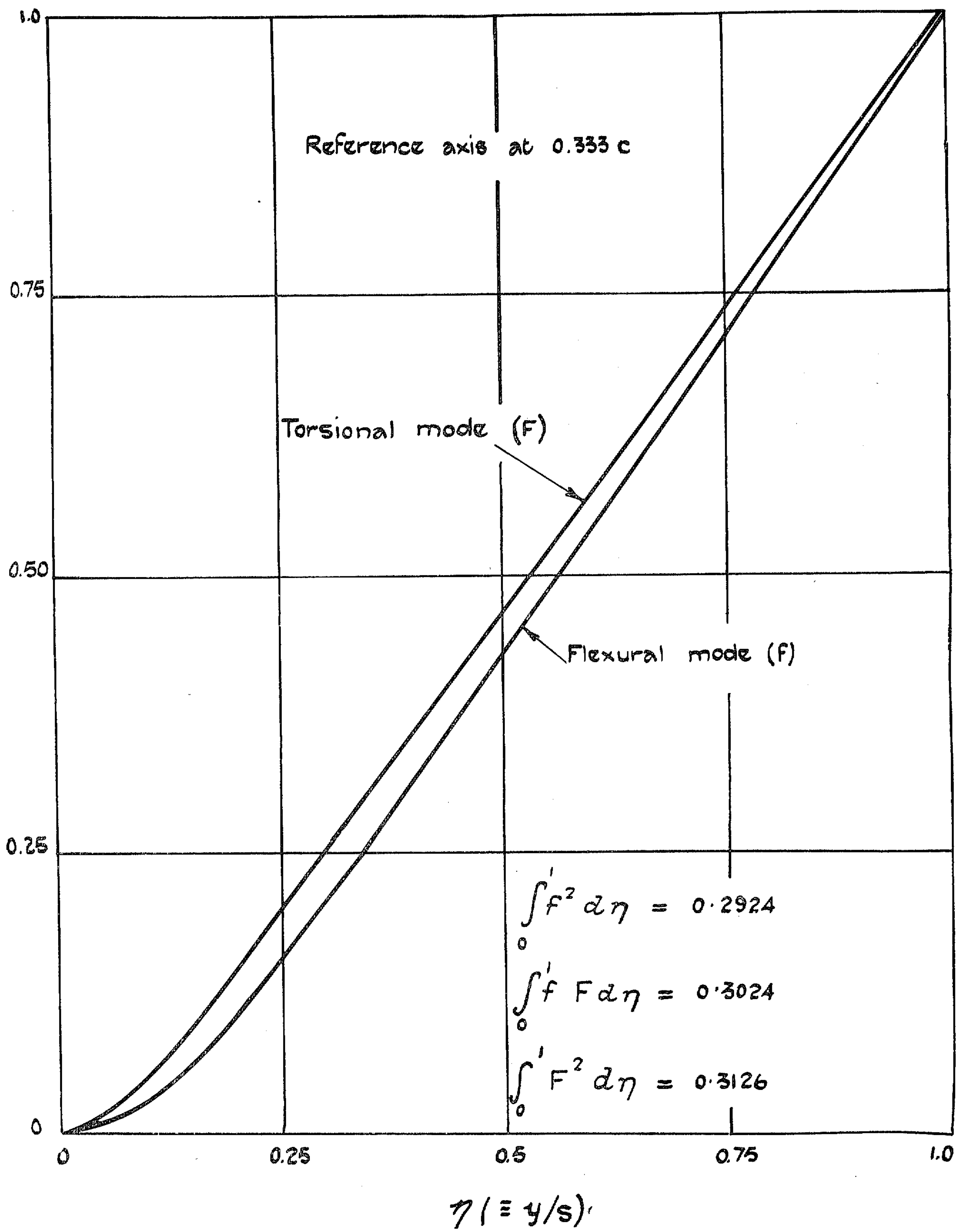


FIG. 8.—Modes of Deflection for Model Wing of R. & M. 1155.

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