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Frequency "Admittance" Curves for Coupled Engine Crankshaft (Torsional) and Contra-revolving Propeller (Flexural) Vibrations

By

J. Morris, B.A.

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Summary.—In this report the "admittance" method, for dealing with coupled vibrations of engine crankshaft propeller systems, is adapted to cover the case of contra-revolving propellers. The treatment is quite general in that the propellers may or may not be equal or may or may not revolve at equal speeds.

CONTENTS

1. *Equivalent Polar Moment of Inertia.*—Consider in the first instance the elementary system shown diagrammatically in Fig. 1. p is the polar moment of inertia of a rigid pulley attached to a shaft of torsional stiffness c and which is encastré at its other end. Let the pulley be in free torsional vibration and let $\omega/2\pi$ be the frequency and θ the amplitude. Then we have

$$p\omega^2\theta = c\theta \text{ or } p\omega^2 = c \,. \qquad (1)$$

Next suppose we have a system of two pulleys as in Fig. 2. The equations of motion will be

$$p_1\omega^2\theta_1 = c_1(\theta_1 - \theta), \qquad (2)$$

$$p\omega^2\theta = -c_1(\theta_1 - \theta) + c\theta \,; \qquad (3)$$

from which we derive

$$\left[p - \frac{c_1}{\omega^2} + \left(\frac{c_1^2}{p_1\omega^2}\right)\frac{1}{(\omega_1^2 - \omega^2)}\right]\omega^2 = c \,, \qquad (4)$$

where $\omega_1^2 = c_1/p_1$.

Thus
$$p_e = p - \frac{c_1}{\omega^2} + \left(\frac{c_1^2}{p_1\omega^2}\right)\frac{1}{(\omega_1^2 - \omega^2)} \qquad (5)$$

is the equivalent polar moment of inertia of the system $p_1, p,$ in free vibration and acting at p.

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To take the case of four masses p_1, p_2, p_3, p, the equations of motion are

$$p_1\omega^2\theta_1 = c_1\,(\theta_1 - \theta_2), \quad \ldots \quad (6)$$

$$p_2\omega^2\theta_2 = -c_1\,(\theta_1 - \theta_2) + c_2\,(\theta_2 - \theta_3), \quad \ldots \quad (7)$$

$$p_3\omega^2\theta_3 = -c_2\,(\theta_2 - \theta_3) + c_3\,(\theta_3 - \theta), \quad \ldots \quad (8)$$

$$p\omega^2\theta = -\,c_3\,(\theta_3 - \theta) + c\theta. \quad \ldots \quad (9)$$

Equation (6), (7), (8), may be written

$$(a_{11} - \omega^2)\,\phi_1 - a_{12}\phi_2 = 0, \quad \ldots \quad (10)$$

$$-\,a_{12}\phi_1 + (a_{22} - \omega^2)\,\phi_2 - a_{23}\phi_3 = 0, \quad \ldots \quad (11)$$

$$-\,a_{23}\phi_2 + (a_{33} - \omega^2)\,\phi_3 = (c_3/\sqrt{p_3})\theta, \quad \ldots \quad (12)$$

where $\phi_1 = \sqrt{p_1}\theta_1$, $\phi_2 = \sqrt{p_2}\theta_2$, $\phi_3 = \sqrt{p_3}\theta_3$;

$a_{11} = c_1/p_1$, $a_{22} = (c_1 + c_2)/p_2$, $a_{33} = (c_2 + c_3)/p_3$,

$a_{12} = c_1/\sqrt{p_1 p_2}$, $a_{23} = c_2/\sqrt{p_2 p_3}$.

The equations (10), (11), (12), are symmetrical and are readily dealt with by the escalator[1,2] method. Thus let ω_r^2 $(r = 1, 2, 3)$ be a root in ω^2 of the equations with $\theta = 0$, and let $(\phi_1)_r$, $(\phi_2)_r$, $(\phi_3)_r$, be the associated rectified modes, i.e. modes subject to the condition

$$(\phi_1)_r^2 + (\phi_2)_r^2 + (\phi_3)_r^2 = 1. \quad \ldots \quad (13)$$

Multiply (10), (11), (12), in turn by $(\phi_1)_r$, $(\phi_2)_r$, $(\phi_3)_r$, and add. We obtain

$$(\omega_r^2 - \omega^2)[\phi_1(\phi_1)_r + \phi_2(\phi_2)_r + \phi_3(\phi_3)_r] = (c_3/\sqrt{p_3})\,(\phi_3)_r\theta. \quad \ldots \quad (14)$$

From three such equations $(r = 1, 2, 3)$ we derive in virtue of the properties of the rectified modes,

$$\phi_3 = \frac{c_3}{\sqrt{p_3}}\left[\frac{(\phi_1)_3^2}{(\omega_1^2 - \omega^2)} + \frac{(\phi_2)_3^2}{(\omega_2^2 - \omega^2)} + \frac{(\phi_3)_3^2}{(\omega_3^2 - \omega^2)}\right]\theta\,. \quad \ldots \quad (15)$$

Now since $\phi_3 = \sqrt{p_3}\,\theta_3$, we obtain

$$c_3\,(\theta_3 - \theta) = \left[\frac{c_3^2}{p_3}\left\{\frac{(\phi_1)_3^2}{\omega_1^2 - \omega^2} + \frac{(\phi_2)_3^2}{\omega_2^2 - \omega^2} + \frac{(\phi_3)_3^2}{\omega_3^2 - \omega^2}\right\} - c_3\right]\theta\,. \quad \ldots \quad (16)$$

Hence using (16) in (9) we find $p_e\omega^2 = c$,

$$\text{where } p_e = p - \frac{c_3}{\omega^2} + \frac{c_3^2}{p_3\omega^2}\left[\frac{(\phi_1)_3^2}{(\omega_1^2 - \omega^2)} + \frac{(\phi_2)_3^2}{(\omega_2^2 - \omega^2)} + \frac{(\phi_3)_3^2}{(\omega_3^2 - \omega^2)}\right]; \quad \ldots \quad (17)$$

from which expression we infer the general formula for any number of pulleys. If in place of p in Fig. 1 we have a propeller the continuous mass of which is replaced by a series of discrete masses as in R. & M. 2011[3], then as there shown, the coupled flexural vibrations of the propeller blades and the torsional vibration about the shaft c are given by an equation of the form

$$\left[P_h^2 + n\sum_{r=1}^{r=2s}\frac{P_r^2\omega_r^2}{(\omega_r^2 - \omega^2)}\right]\omega^2 = c, \quad \ldots \quad (18)$$

where s is the number of discrete masses, $\omega/2\pi$ is a frequency, ω_1^2, ω_2^2, $\omega_3^2, \ldots \omega_{2s}^2$ correspond to the fixed root frequencies of the blades, P_h^2 is the polar moment of inertia of the hub of the propeller, P_1^2, P_2^2, $P_3^2 \ldots P_{2s}^2$ are quantities having the dimensions of moment of inertia and n is the number of blades.

We shall designate the expression on the left-hand side of (18) by $P(\omega^2).\ \omega^2$, $P(\omega^2)$ thereby being the equivalent polar moment of inertia of the propeller system in free vibration and regarded as acting at the point of attachment of the hub.

2. *Single Crankshaft with Two Contra-revolving Propellers.*—We consider first a case of a single crankshaft driving two propellers running on contra-revolving coaxial shafts as in Fig. 3.

Let $P_1(\omega^2)$, $P_2(\omega^2)$, be the equivalent polar moments of inertia of the two propellers which are not necessarily equal. If α_1, α_2, are the torsional angular amplitudes at the respective hubs, the torques in their respective shafts will be $P_1(\omega^2).\ \omega^2\ \alpha_1$, $P_2(\omega^2).\ \omega^2\ \alpha_2$ or say $f_1(\omega^2)\ \alpha_1$, $f_2(\omega^2)\ \alpha_2$.

Let $\qquad f_1(\omega^2) = c_1/r_1,\ f_2(\omega^2) = c_2/r_2\,;$

or $\qquad r_1 = c_1/f_1(\omega^2),\ r_2 = c_2/f_2(\omega^2)\,;$

then the torques in the propeller shafts can be expressed respectively as $(c_1/r_1)\ \alpha_1$, $(c_2/r_2)\ \alpha_2$. Considering the shaft c_1, let α_1' be the amplitude at the gear wheel end, where α_1' is measured in the opposite direction to α_1, then the torque in that shaft is also given by $c_1(\alpha_1 + \alpha_1')$.

Hence
$$(c_1/r_1)\ \alpha_1 = c_1(\alpha_1 + \alpha_1'), \qquad (19)$$
from which we derive
$$\alpha_1' = \alpha_1\left(\frac{1}{r_1} - 1\right). \qquad (20)$$
Thus
$$(c_1/r_1)\ \alpha_1 = [c_1/(1 - r_1)]\ \alpha_1'. \qquad (21)$$
Similarly
$$(c_2/r_2)\ \alpha_2 = [c_2/(1 - r_2)\ ]\ \alpha_2'. \qquad (22)$$

Coming now to the engine system, let θ_1, θ_2, θ_3, be the amplitudes of the masses p_1, p_2, p_3'. The kinetic energy of the engine masses is given by T, where

$$2T = p_1\dot\theta_1^2 + p_2\dot\theta_2^2 + (p_3' + np_3''\varrho_{p_3'p_3''}^2 + 2p_3'''\varrho_{p_3'p_3'''}^2)\dot\theta_3^2$$

in which $\varrho_{p_3'p_3''}$ is the gear ratio between p_3' and p_3'', $\varrho_{p_3'p_3'''}$ that between p_3' and p_3'''; and n is the number of gear wheels p_3''.

Hence
$$2T = p_1\dot\theta_1^2 + p_2\ \dot\theta_2^2 + p_3\dot\theta_3^2, \qquad (23)$$
where
$$p_3 = p_3' + n\varrho_{p_3'p_3''}^2\ p_3'' + 2\varrho_{p_3'p_3'''}^2\ p_3'''. \qquad (24)$$

We notice that angles at the gear wheels p_3''' which we have previously designated α_1', α_2' will each be $\varrho_{p_3'p_3'''}\ \theta_3$.

The strain energy of the system is given by V, where

$$2V = c_{12}(\theta_1 - \theta_2)^2 + c_{23}(\theta_2 - \theta_3)^2 + [c_1'/(1 - r_1) + c_2'/(1 - r_2)]\ \theta_3^2\,, \qquad (25)$$

where
$$c_1'/c_1 = c_2'/c_2 = \varrho_{p_3'p_3'''}^2. \qquad (26)$$

For the equations of motion, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} + \frac{\partial V}{\partial\theta} = 0\,, \qquad (27)$$

where θ is one of the co-ordinates.

Hence we derive the equations—

$$(c_{12} - p_1\omega^2)\,\theta_1 - c_{12}\theta_2 = 0\,, \quad \ldots \quad (28)$$

$$-\,c_{12}\theta_1 + (c_{12} + c_{23} - p_2\omega^2)\,\theta_2 - c_{23}\theta_3 = 0\,, \quad \ldots \quad (29)$$

$$-\,c_{23}\theta_2 + [\,c_{23} + c_1'/(1 - r_1) + c_2'/(1 - r_2) - p_3\omega^2\,]\,\theta_3 = 0\,, \quad \ldots \quad (30)$$

where $\omega/2\pi$ is frequency.

We may write these equations in symmetrical form as follows :—

Let $\sqrt{p_1}\,\theta_1 = \phi_1,\ \sqrt{p_2}\theta_2 = \phi_2,\ \sqrt{p_3}\theta_3 = \phi_3$;

$c_{12}/p_1 = a_{11},\ (c_{12} + c_{23})/p_2 = a_{22},\ c_{12}/\sqrt{p_1p_2} = a_{12},\ c_{23}/\sqrt{p_2p_3} = a_{23}.$

The equations (28), (29), (30), then become—

$$(a_{11} - \omega^2)\phi_1 - a_{12}\phi_2 = 0\,, \quad \ldots \quad (31)$$

$$-\,a_{12}\phi_1 + (a_{22} - \omega^2)\phi_2 - a_{23}\phi_3 = 0\,, \quad \ldots \quad (32)$$

$$-\,a_{23}\phi_2 + \left[\frac{1}{p_3}\left(c_{23} + \frac{c_1'}{1 - r_1} + \frac{c_2'}{1 - r_2}\right) - \omega^2\right]\phi_3 = 0\,. \quad \ldots \quad (33)$$

Keeping r_1, r_2, as parameters we may obtain a solution of equations (31), (32), (33), by the escalator method as follows :—

Consider the equation (31), (32), with $\phi_3 = 0$; i.e.

$$(a_{11} - \omega^2)\,\phi_1 - a_{12}\phi_2 \quad = 0\,, \quad \ldots \quad (34)$$

$$-\,a_{12}\phi_1 + (a_{22} - \omega^2)\,\phi_2 = 0\,. \quad \ldots \quad (35)$$

Let ω_1^2, ω_2^2, be the two roots in ω^2 of the appropriate frequency equation and let $(\phi_1)_1$, $(\phi_2)_1$; $(\phi_1)_2$, $(\phi_2)_2$, be the associated respective rectified modes, *i.e.* modes subject to the condition $(\phi_1)_s^2 + (\phi_2)_s^2 = 1$, $(s = 1, 2)$. The frequency equation in ω^2 for (31), (32), (33), may then be expressed as

$$a_{23}^2\left[\frac{(\phi_2)_1^2}{(\omega_1^2 - \omega^2)} + \frac{(\phi_2)_2^2}{(\omega_2^2 - \omega^2)}\right] = \frac{1}{p_3}\left[c_{23} + \frac{c_1'}{(1 - r_1)} + \frac{c_2'}{(1 - r_2)}\right] - \omega^2\,. \quad \ldots \quad (36)$$

Now (36) may be written $E\,(\omega^2) = y$, where

$$y = E\,(\omega^2) = a_{23}^2\left[\frac{(\phi_2)_1^2}{(\omega_1^2 - \omega^2)} + \frac{(\phi_2)_2^2}{(\omega_2^2 - \omega^2)}\right] - \frac{c_{23}}{p_3} + \omega^2\,, \quad \ldots \quad (37)$$

$$y = \frac{1}{p_3}\left[\frac{c_1'}{1 - r_1} + \frac{c_2'}{1 - r_2}\right]. \quad \ldots \quad (38)$$

But $f_1(\omega^2) = c_1/r_1$, $f_2(\omega^2) = c_2/r_2$. Hence y may be written—

$$y = \frac{1}{p_3}\left[\frac{f_1\,(\omega^2)c_1'}{f_1(\omega^2) - c_1} + \frac{f_2(\omega^2)c_2'}{f_2(\omega^2) - c_2}\right]. \quad \ldots \quad (39)$$

Thus we may, for convenience, plot the two equations (37), (39), in y and ω^2, and determine from the cuts of the resulting curves, the applicable values of ω^2 for the frequencies of the system comprising engine and propellers.

A simplification arises when the two propellers are equal in all respects. In such a case $f_1(\omega^2) = f_2(\omega^2)$ and, therefore, $c_2/r_2 = c_1/r_1$ or $r_2 = c_2 r_1/c_1$. In these circumstances we may use the single parameter $r = r_1$ as follows :—

$$\mathrm{E}\,(\omega^2) = \frac{1}{p_3}\left[\frac{c_1'}{1-r} + \frac{c_2'}{1-\dfrac{c_2 r}{c_1}}\right]^{*} \qquad \qquad (40)$$

and $f\,(\omega^2) = f_1\,(\omega^2) = c_1/r$. .. .. (41)

To take a numerical example in which the two propellers are equal, let the equations (28), (29), (30), with λ written for $\omega^2 \times 10^{-6}$ become in figures

$$(9{\cdot}47 - 7{\cdot}83\lambda)\,\theta_1 - 9{\cdot}47\theta_2 = 0, \qquad (42)$$

$$-\,9{\cdot}47\theta_1 + (14{\cdot}14 - 8{\cdot}24\lambda)\,\theta_2 - 4{\cdot}6\,\theta_3 = 0, \qquad (43)$$

$$-\,4{\cdot}67\theta_2 + \left[4{\cdot}67 + \frac{5{\cdot}44}{1-r} + \frac{0{\cdot}664}{1-0{\cdot}122r} - 4{\cdot}07\lambda\right]\theta_3 = 0, \qquad (44)$$

or in symmetrical form

$$(1{\cdot}209 - \lambda)\,\phi_1 - 1{\cdot}179\phi_2 = 0, \qquad (45)$$

$$-\,1{\cdot}179\phi_1 + (1{\cdot}716 - \lambda)\,\phi_2 - 0{\cdot}8064\phi_3 = 0, \qquad (46)$$

$$-\,0{\cdot}8064\phi_2 + \left[1{\cdot}147 + \frac{1{\cdot}337}{1-r} + \frac{0{\cdot}1631}{1-0{\cdot}122r} - \lambda\right]\phi_3 = 0. \qquad (47)$$

To solve these equations we first take the equations

$$(1{\cdot}209 - \lambda)\,\phi_1 - 1{\cdot}179\phi_2 = 0, \qquad (48)$$

$$-\,1{\cdot}179\phi_1 + (1{\cdot}716 - \lambda)\,\phi_2 = 0. \qquad (49)$$

The two roots in λ are found to be

$$\lambda_1 = 0{\cdot}2566, \qquad \lambda_2 = 2{\cdot}6684,$$

and the respective rectified modes are given by

$$(\phi_1)_1/(\phi_2)_1 = 1{\cdot}2379 = -\,(\phi_2)_2/(\phi_1)_2\,;$$

and since $(\phi_1)^2 + (\phi_2)_1^{\,2} = 1$, $(\phi_1)_2^{\,2} + (\phi_2)_2^{\,2} = 1$,

we find that $(\phi_1)_1^{\,2} = 0{\cdot}605118$, $(\phi_2)_1^{\,2} = 0{\cdot}394882$,

$(\phi_1)_2^{\,2} = 0{\cdot}394882$, $(\phi_2)_2^{\,2} = 0{\cdot}605118$.

Equation (36) thus becomes

$$\frac{0{\cdot}2568}{(0{\cdot}2566 - \lambda)} + \frac{0{\cdot}3935}{(2{\cdot}6684 - \lambda)} + \lambda - 1{\cdot}147$$

$$= \frac{1{\cdot}337}{(1-r)} + \frac{0{\cdot}1631}{(1-0{\cdot}122r)}. \qquad (50)$$

---

* We notice that since $c_1'/c_2' = c_1/c_2$, then if

$$z = c_1'/(1-r) + c_2'/\left(1 - \frac{c_2}{c_1}r\right),$$

$$r = \frac{c_1' + c_2'}{2c_2'} - \frac{c_1'}{z} \pm \sqrt{\frac{(c_1' - c_2')^2}{2c_2'} + \frac{c_1'^2}{z^2}}.$$

Now since $\lambda = \omega^2 \times 10^{-6}$, the corresponding frequency is given by

$$f = \omega/2\pi = 159 \cdot 15\sqrt{\lambda};$$

hence we may plot f against the parameter r and the resulting frequency curves are shown in Fig. 4. We notice that given any value of λ there will be two values of r both of which are applicable.

The complementary propeller frequency curves will be given by the plot of the appropriate equation $f(\omega^2) = c_1/r$, which will be of the form given by (18), and by means of which we similarly plot frequency against the parameter r.

3. *Two Coupled Crankshafts with Two Contra-revolving Propellers.*—Referring to Fig. 5 the p's represent polar moments of inertia of appropriate gear wheels; the c_{rs}'s represent torsional stiffness of appropriate shafts—thus $c_{8,10}$ represents the torsional stiffness of shaft between the gear wheels p_8 and p_{10}; the ϱ_{rs}'s represent appropriate gear ratios—thus $\varrho_{7,11}$ represents the gear ratio, between the gear wheels p_7 and p_{11}; and the $\varrho\theta$'s represent appropriate angular displacements. Both crankshafts are equal and their dynamic systems are as shown in Fig. 6.

The equations of motion for the crankshaft system are :—

$$p_1\omega^2\theta_1 = c_{12}(\theta_1 - \theta_2), \quad \ldots \quad (51)$$

$$p_2\omega^2\theta_2 = -c_{12}(\theta_1 - \theta_2) + c_{23}(\theta_2 - \theta_3), \quad \ldots \quad (52)$$

$$p_3\omega^2\theta_3 = -c_{23}(\theta_2 - \theta_3) + c_{34}(\theta_3 - \theta_4), \quad \ldots \quad (53)$$

$$p_4\omega^2\theta_4 = -c_{34}(\theta_3 - \theta_4) + c_{45}(\theta_4 - \theta_5), \quad \ldots \quad (54)$$

$$p_5\omega^2\theta_5 = -c_{45}(\theta_4 - \theta_5) + c_{56}(\theta_5 - \theta_6), \quad \ldots \quad (55)$$

$$p_6\omega^2\theta_6 = -c_{56}(\theta_5 - \theta_6) + c_{67}(\theta_6 - \theta_7), \quad \ldots \quad (56)$$

where $\omega/2\pi$ is frequency.

Case I—Node at the Gears.—For this case we put $\theta_7 = 0$ in (56). The solution of equations (51) to (56), with $\theta_7 = 0$, will then give the frequencies and associated modes. Let all the p's be equal and let $c_{12} = c_{23}$, $c_{45} = c_{56} = 8 \cdot 591 \times 10^6$; $c_{34} = 7 \cdot 88 \times 10^6$ and $c_{67} = 8 \cdot 695 \times 10^6$ lb. in. rad.

Let the equal p's be designated p and the equal c's as c; and let $c_{34} = r_{34}c$, $c_{67} = r_{67}c$, the r's thus being pure ratios. Let also

$$p\omega^2/c = 2(1 - \cos\alpha) = 4\sin^2\tfrac{1}{2}\alpha.$$

In these circumstances the frequency equation is given by (*see* Strength of Shafts in Vibration[4], page 127)

$$\left[\left(\frac{1}{r_{n,n+1}} - 2\right)\cos(n - \tfrac{1}{2})\alpha - \frac{1}{r_{n,n+1}}\cos(n + \tfrac{1}{2})\alpha\right]\left[\left(\frac{1}{r_{2n,2n+1}} - 1\right)\cos(n - \tfrac{1}{2})\alpha - \frac{1}{r_{2n,2n+1}}\cos(n + \tfrac{1}{2})\alpha\right] - \cos^2\tfrac{1}{2}\alpha = 0, \quad \ldots \quad (A)$$

where $2n$ is the number of equal p's.

The relative appropriate modes will be given by

$$\left.\begin{aligned} \theta_s\,(s = 1 \text{ to } n) &= \frac{\cos(n + \frac{1}{2})\alpha - (1 - r_{2n,2n+1})\cos(n - \frac{1}{2})\alpha}{\sin\alpha\sin n\alpha}\cos(s - \tfrac{1}{2})\alpha \\ \theta_s\,(s = n + 1 \text{ to } 2n) &= [\sin(2n - s + 1)\alpha - (1 - r_{2n,2n+1})\sin(2n - s)\alpha]/\sin\alpha \end{aligned}\right\} \quad . \quad (B)$$

In the particular case $n = 3$, $1/r_{34} = 1\cdot091$, $1/r_{67'} = 0\cdot988$; with these c values we find

r	1	2	3	4	5	6
α_r	13° 55′	41° 11′	69° 5′	95° 33′	124° 35′	147° 28′
$\omega_r^2/10^6$	1·34	11·27	29·4	50·6	71·4	84·1

But in view of the fact that the two unequal c's are not far removed from the equal c's, we shall have a close approximation if we take all the c's equal to their average value viz. $c_a = 8\cdot49 \times 10^6$. In these circumstances the frequency equation becomes

$$\cos\,(2n + \tfrac{1}{2})\,\alpha = 0,$$

and the frequencies are given by $\omega/2\pi$, where

$$p\omega 2/c_a = 2\,(1 - \cos\alpha) = 4\sin^2\tfrac{1}{2}\,\alpha.$$

We thus obtain :—

r	1	2	3	4	5	6
α_r	13° 51′	41° 33′	69° 15′	96° 57′	124° 39′	152° 11′
$\omega_r^2/10^6$	1·31	11·32	29·1	50·55	70·6	84·9

Comparing this table with the corresponding one given previously, we notice that the greatest error is 2 per cent. in the square of a frequency, which means 1 per cent. in frequency and this only with the fundamental.

The relative modes in the " equal " case are given by

$$\cos\tfrac{1}{2}\alpha : \cos 1\tfrac{1}{2}\alpha ; \cos 2\tfrac{1}{2}\alpha \ldots\ldots\ldots : \cos\,(2n - \tfrac{1}{2})\,\alpha$$

and these will be of the same order of approximation to their actual values as the frequencies were found to be in the particular case. We find that the fundamental frequency is 184 c.p.s. on the more accurate basis and 182 c.p.s. by the " equal " approximation.

Case II—Engines Taken Separately.—We next consider the case of the two engines not being geared together, auxiliary gears being neglected. Taking the bottom engine in Fig. 5, we have, in addition to equations (51) to (56) the equations derived from the energy terms involving θ_7, θ_8, viz. :—

$$\tfrac{1}{2}\,(p_7 + 2\,\varrho_{78}{}^2\,p_8)\,\dot\theta_7{}^2 + \tfrac{1}{2}\,(2\varrho_{78}{}^2\,p_{10} + \varrho_{7,11}{}^2\,p_{11})\,\dot\theta_8{}^2$$
$$+ \tfrac{1}{2}\,c_{67}\,(\theta_6 - \theta_7)^2 + \tfrac{1}{2} \times 2\varrho_{78}{}^2\,c_{8,10}\,(\theta_7 - \theta_8)^2 + \tfrac{1}{2}\frac{c_0}{(1-r)}\,\varrho_{7,11}{}^2\,\theta_8{}^2,$$

in which a node has been assumed at a point on the propeller shaft at a torsional stiffness c_0/r from the hub, c_0 being the torsional stiffness of the appropriate propeller shaft. From these energy terms we derive the equations :—

$$- c_{67}\theta_6 + [\,c_{67} + 2\varrho_{78}{}^2\,c_{8,10} - (p_7 + 2\varrho_{78}{}^2\,p_8)\,\omega^2\,]\,\theta_7 - 2\varrho_{78}{}^2\,c_{8,10}\,\theta_8 = 0 \quad .. \qquad (57)$$

$$- \varrho_{78}{}^2\,c_{8,10}\,\theta_7 + [\,\varrho_{78}{}^2\,c_{8,10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,c_0\,/\,(1-r) - (\varrho_{78}{}^2\,p_{10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,p_{11})\,\omega^2\,]\,\theta_8 = 0. \quad .. \qquad (58)$$

Reverting now to equations (51) to (56), with the p's and c's equal, we have in the general case of n masses :—

$$(1 - p\omega^2/c)\,\theta_1 - \theta_2 = 0\,, \qquad (59)$$

$$-\theta_s + (2 - p\omega^2/c)\,\theta_{s+1} - \theta_{s+2} = 0 \quad (s = 1 \text{ to } n-2)\,, \qquad (60)$$

$$-\theta_{n-1} + (2 - p\omega^2/c)\,\theta_n = \theta_{n+1} = \phi, \text{ say}\,. \qquad (61)$$

As a solution of (59), (60), (61), we assume

$$\theta_s = A_n \cos(s-1)\,\alpha + B_n \sin(s-1)\,\alpha,$$

where $p\omega^2/c = 2\,(1 - \cos\alpha) = 4\sin^2\tfrac{1}{2}\alpha$; then from (59)

$$A_n\,(2\cos\alpha - 1) - (A_n\cos\alpha + B_n\sin\alpha) = 0,$$

or $B_n = -A_n \tan\tfrac{1}{2}\alpha$.

Thus $\theta_s = A_n \cos(s - \tfrac{1}{2})\alpha/\cos\tfrac{1}{2}\alpha$. (62)

From (61) we find $A_n \cos(n + \tfrac{1}{2})\,\alpha/\cos\tfrac{1}{2}\alpha = \phi$;

so that $A_n = \phi \cos\dfrac{\alpha}{2}\,/\cos(n + \tfrac{1}{2})\,\alpha.$

Hence $\theta_s = \phi\cos(s - \tfrac{1}{2})\,\alpha/\cos(n + \tfrac{1}{2})\,\alpha.$ (63)

Thus in the particular case where $n = 6$ and $\phi = \theta_7$, we have

$$\theta_6 = \theta_7 \cos 5\tfrac{1}{2}\,\alpha/\cos 6\tfrac{1}{2}\,\alpha. \qquad (64)$$

Now using (58) and (64) in (57), we obtain the frequency equation

$$-c_{67}\cos 5\tfrac{1}{2}\,\alpha/\cos 6\tfrac{1}{2}\,\alpha + [\,c_{67} + 2\,\varrho_{78}{}^2\,c_{8,10} - (p_7 + 2\,\varrho_{78}{}^2 p_8)\,\omega^2\,]$$
$$-\,2\,\varrho_{78}{}^4\,c_{8,10}{}^2\,/\,[\,\varrho_{78}{}^2\,c_{8,10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,c_0/(1-r) - (\varrho_{78}{}^2\,p_{10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,p_{11})\,\omega^2\,] = 0. \qquad (65)$$

Taking numerical values, we have

$p = 145{\cdot}7$ lb. in.^{2} $\qquad \varrho_{78} = 18/31, \qquad \varrho_{78}{}^2 = 0{\cdot}33715,$

$p_7 = 19{\cdot}2 \qquad \varrho_{7,11} = 0{\cdot}216, \qquad \varrho_{7,11}{}^2 = 0{\cdot}0467,$

$p_8 = 98{\cdot}2 \qquad p_7 + 2\,\varrho_{78}{}^2\,p_8 = 85{\cdot}216 = 0{\cdot}585p,$

$p_{10} = 17{\cdot}9 \qquad \varrho_{78}{}^2\,p_{10} + \tfrac{1}{2}\varrho_{7,11}{}^2\,p_{11} = 14{\cdot}56 = 0{\cdot}1p\,;$

$p_{11} = 365$

$c_{67} = c_a = 8{\cdot}49 \times 10^6$ lb. in./rad., $c_{8,10} = 8{\cdot}81 \times 10^6,$

$\varrho_{78}{}^2\,c_{8,10} = 2{\cdot}97 \times 10^6 = 0{\cdot}35c_a,\ 2\varrho_{78}{}^4\,c_{8,10}{}^2 = 0{\cdot}245\,c_a{}^2,$

$c_{67} + 2\varrho_{78}{}^2\,c_{8,10} = 14{\cdot}43 \times 10^6 = 1{\cdot}696$

$\tfrac{1}{2}\varrho_{7,11}{}^2\,c_0 = \tfrac{1}{2} \times 0{\cdot}0467 \times 15{\cdot}36 \times 10^6 = 0{\cdot}0422c_a\,.$

Thus, since $\omega^2 = 2c_a\,(1 - \cos\alpha)/p$, (65) becomes

$$-\cos 5\tfrac{1}{2}\,\alpha/\cos 6\tfrac{1}{2}\,\alpha + [\,1{\cdot}896 - 0{\cdot}585 \times 2\,(1 - \cos\alpha)\,]$$
$$= 0{\cdot}245/\,[\,0{\cdot}35 + 0{\cdot}0422/(1-r) - 0{\cdot}1 \times 2\,(1 - \cos\alpha)\,]\,, \qquad (66)$$

which may be written $0{\cdot}15 + 0{\cdot}2\cos\alpha + 0{\cdot}0422/(1-r)$

$$= 1/\,[\,2{\cdot}147 + 0{\cdot}69\cos\alpha - 4{\cdot}07\sin\alpha\tan 6\tfrac{1}{2}\,\alpha\,]\,, \qquad (67)$$

in which frequency $f = \dfrac{1}{\pi}\sqrt{\dfrac{c_a}{p}}\sin\tfrac{1}{2}\,\alpha = 1507\sin\tfrac{1}{2}\,\alpha$ in c.p.s.

For the top engine we have

$$p_7' = 19{\cdot}2 \text{ lb. in.}^2, \qquad c_{8,10}' = 52{\cdot}64 \times 10^6 \text{ lb./in./rad.},$$
$$p_8' = 104{\cdot}2, \qquad p_7' + 2\varrho_{78}{}^2 p_8' = 89{\cdot}462 = 0{\cdot}614\,p,$$
$$p_{10}' = 14{\cdot}1, \qquad \varrho_{78}{}^2 p_{10}' + \tfrac{1}{2}\varrho_{7,11}{}^2 p_{11}' = 13{\cdot}498 = 0{\cdot}0926p,$$
$$p_{11}' = 374{\cdot}5,$$
$$\varrho_{78}{}^2\, c_{8,10}' = 17{\cdot}748 \times 10^6 = 2{\cdot}09\, c_a, \quad 2\varrho_{78}{}^4\, c_{8,10}'^2 = 8{\cdot}74 c_a{}^2,$$
$$c_{67} + 2\varrho_{78}{}^2\, c_{8,10}' = 43{\cdot}986 \times 10^6 = 5{\cdot}18 c_a,$$
$$\tfrac{1}{2}\varrho_{7,11}{}^2\, c_a = \tfrac{1}{2} \times 0{\cdot}0467 \times 4{\cdot}163 \times 10^6 = 0{\cdot}0114 c_a.$$

Thus (65) appropriately modified becomes

$$-\cos 5\tfrac{1}{2}\,\alpha/\cos 6\tfrac{1}{2}\,\alpha + [\,5{\cdot}18 - 0{\cdot}614 \times 2\,(1 - \cos\alpha)\,]$$
$$= 8{\cdot}74/[\,2{\cdot}09 + 0{\cdot}0114/(1 - r) - 0{\cdot}0926 \times 2\,(1 - \cos\alpha\,], \qquad (68)$$

which may be written

$$1{\cdot}905 + 0{\cdot}185 \cos\alpha + 0{\cdot}0114/(1 - r)$$
$$= 1/[\,0{\cdot}452 + 0{\cdot}0261 \cos\alpha - 0{\cdot}114 \sin\alpha \tan 6\tfrac{1}{2}\,\alpha\,]. \qquad (69)$$

We may thus plot the "admittance" lines by selecting various values of α (corresponding to appropriate frequencies) and ascertaining the associated r's.

For the modes we have

(1) for both engines

$$\theta_s\ (s = 1 \text{ to } 6) = \theta_7 \cos\,(s - \tfrac{1}{2})\,\alpha/\cos 6\tfrac{1}{2}\,\alpha\,; \qquad (70)$$

(2) for the bottom engine

$$\theta_8 = \theta_7/[\,0{\cdot}429 + 0{\cdot}571 \cos\alpha + 0{\cdot}121/(1 - r)\,]\,; \qquad (71)$$

(3) for the top engine

$$\theta_8' = \theta_7/[\,0{\cdot}909 + 0{\cdot}0855 \cos\alpha + 0{\cdot}00545/(1 - r)\,]. \qquad (72)$$

Case III—*The Two Engines in Combination.*—In this case we have, neglecting auxiliary gears, for the energy terms involving θ_7, θ_8, θ_8',

$$\tfrac{1}{2}[\,(p_7 + p_7') + 2\varrho_{78}{}^2\,(p_8 + p_8')\,]\,\dot\theta_7{}^2,$$
$$+ \tfrac{1}{2}\,(2\varrho_{78}{}^2 p_{10} + \varrho_{7,11}{}^2 p_{11})\,\dot\theta_8{}^2 + \tfrac{1}{2}\,(2\varrho_{78}{}^2 p_{10}' + \varrho_{7,11}{}^2 p_{11}')\,\dot\theta_8'^2$$
$$+ \tfrac{1}{2} \times 2c_{67}\,(\theta_6 - \theta_7)^2 + \tfrac{1}{2} \times 2\,\varrho_{78}{}^2\,[\,c_{8,10}(\theta_7 - \theta_8)^2 + c_{8,10}'\,(\theta_7 - \theta_8')^2\,]$$
$$+ \tfrac{1}{2}\varrho_{7,11}{}^2\,\frac{c_0}{(1 - r)}\,\theta_8{}^2 + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,\frac{c_i}{\left(1 - \dfrac{c_i}{c_0}\,r\right)}\,\theta_8'^2, \qquad (73)$$

in which we assume nodes in each propeller shaft for which the torsional stiffness of the shaft between the propeller hub and node is c_0/r in each case, the propellers being assumed dynamically equal. From (73) we derive the equations

$$-c_{67}\theta_6 + [\,c_{67} + \varrho_{78}{}^2\,(c_{8,10} + c_{8,10}') - \{\tfrac{1}{2}\,(p_7 + p_7') + \varrho_{78}{}^2\,(p_8 + p_8')\}\,\omega^2\,]\,\theta_7$$
$$- \varrho_{78}{}^2(c_{8,10}\theta_8 + c_{8,10}'\theta_8') = 0, \qquad (74)$$

$$-\varrho_{78}{}^2 c_{8,10}\theta_7 + [\,\varrho_{78}{}^2\, c_{8,10} + \tfrac{1}{2}\varrho_{7,11}{}^2\, c_0/(1 - r) - (\varrho_{78}{}^2 p_{10} + \tfrac{1}{2}\varrho_{7,11}{}^2 p_{11})\,\omega^2\,]\,\theta_8 = 0, \qquad (75)$$

$$-\varrho_{78}{}^2 c_{8,10}'\theta_7 + \left[\varrho_{78}{}^2\, c_{8,10}' + \tfrac{1}{2}\varrho_{7,11}{}^2\, c_i\Big/\left(1 - \frac{c_i}{c_0}\,r\right) - (\varrho_{78}{}^2 p_{10}' + \tfrac{1}{2}\varrho_{7,11}{}^2 p_{11}')\,\omega^2\right]\theta_8' = 0. \qquad (76)$$

Using (64), (75), (76), in (74), we obtain the frequency equation

$$- c_{67} \cos 5\tfrac{1}{2}\,\alpha/\cos 6\tfrac{1}{2}\,\alpha + [\,c_{67} + \varrho_{78}{}^2\,(c_{8,10} + c_{8,10}{}') - \{\,\tfrac{1}{2}\,(p_7 + p_7{}') + \varrho_{78}{}^2\,(p_8 + p_8{}')\,\}\,\omega^2\,]$$
$$- \varrho_{78}{}^4\,c_{8,10}{}^2 / [\,\varrho_{78}{}^2\,c_{8,10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,c_0/(1-r) - (\varrho_{78}{}^2\,p_{10} + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,p_{11})\,\omega^2\,]$$
$$-\varrho_{78}{}^4\,c_{8,10}{}'^2 \Big/ \Big[\,\varrho_{78}{}^2\,c_{8,10}{}' + \tfrac{1}{2}\,\varrho_{7,11}{}^2\,c_i \Big/ \Big(1 - \frac{c_i}{c_0}\,r\Big) - (\varrho_{78}{}^2\,p_{10}{}' + \tfrac{1}{2}\,\varrho_{11}{}^2\,p_{11}{}')\,\omega^2\,\Big] = 0. \quad .. \quad (77)$$

Taking numerical values we have

$$c_{67} = c_a,\; c_{67} + \varrho_{78}{}^2\,(c_{8,10} + c_{8,10}{}') = 3\cdot 44 c_a,\; c_i/c_0 = 0\cdot 271,$$

$\tfrac{1}{2}\,(p_7 + p_7{}') + \varrho_{78}{}^2\,(p_8 + p_8{}') = 0\cdot 6p$, all the other quantities being already known. With these values (77) becomes

$$- \cos 5\tfrac{1}{2}\alpha/\cos 6\tfrac{1}{2}\,\alpha + 3\cdot 44 - 0\cdot 6 \times 2\,(1 - \cos\alpha)$$
$$= 0\cdot 1225/[\,0\cdot 15 + 0\cdot 2\cos\alpha + 0\cdot 0422/(1-r)\,]$$
$$+ 4\cdot 37/[\,1\cdot 905 + 0\cdot 185\cos\alpha + 0\cdot 0114/(1 - 0\cdot 271 r)\,], \quad .. \quad .. \quad .. \quad (78)$$

or $2\cdot 24 + 0\cdot 2\cos\alpha - \sin\alpha\tan 6\tfrac{1}{2}\,\alpha = 1/[\,1\cdot 224 + 1\cdot 63\cos\alpha + 0\cdot 346/(1-r)\,]$

$$+ 1/[\,0\cdot 435 + 0\cdot 0423\cos\alpha + 0\cdot 00261/(1 - 0\cdot 271\,r)\,] \ldots \quad .. \quad .. \quad .. \quad (79)$$

For we have the modes

$$\theta_s\;(s = 1 \text{ to } 6) = \theta_7\cos(s - \tfrac{1}{2})\,\alpha/\cos 6\tfrac{1}{2}\,\alpha, \quad .. \quad .. \quad .. \quad .. \quad .. \quad (80)$$
$$\theta_8 = \theta_7/[\,0\cdot 429 + 0\cdot 571\cos\alpha + 0\cdot 121/(1-r)\,], \quad .. \quad .. \quad .. \quad .. \quad .. \quad (81)$$
$$\theta_8{}' = \theta_7/[\,0\cdot 909 + 0\cdot 0835\cos\alpha + 0\cdot 00545/(1 - 0\cdot 271\,r)\,]. \quad .. \quad .. \quad .. \quad (82)$$

We notice from the form of (79) that for any given α there are two values of r, so that having chosen an α value we have a quadratic equation in r to solve for the appropriate r's, both of which are applicable. It may also be noticed that the frequencies for which the trigonometrical solution is valid must be such that these α's do not exceed π ; i.e. in our particular case for frequencies not exceeding 1,507 c.p.s.

To gain some idea of values we consider the case when $r = 0$, i.e. for two rigid propellers of infinite moments of inertia. In such a case we find that the lowest value of α to satisfy (79) with $r = 0$ is $4\tfrac{1}{2}$ deg. approx., which corresponds to a fundamental frequency of 1507 sin $2\tfrac{1}{4}$ deg. = 59 c.p.s. approx. For the associated modes we find from (80), (81) and (82), that

$$\theta_1/\theta_7 = 1\cdot 145, \quad \theta_2/\theta_7 = 1\cdot 138, \quad \theta_3/\theta_7 = 1\cdot 124, \quad \theta_4/\theta_7 = 1\cdot 103,$$
$$\theta_5/\theta_7 = 1\cdot 075, \quad \theta_6/\theta_7 = 1\cdot 041, \quad \theta_8/\theta_7 = 0\cdot 8926, \quad \theta_8{}'/\theta_7 = 1\cdot 003\,.$$

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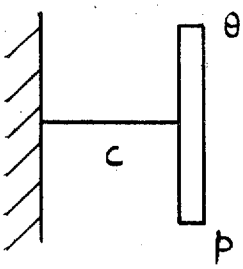


Fig. 1.

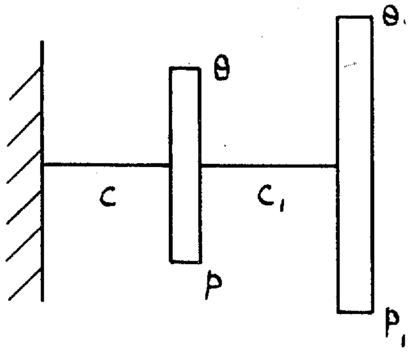


Fig. 2.

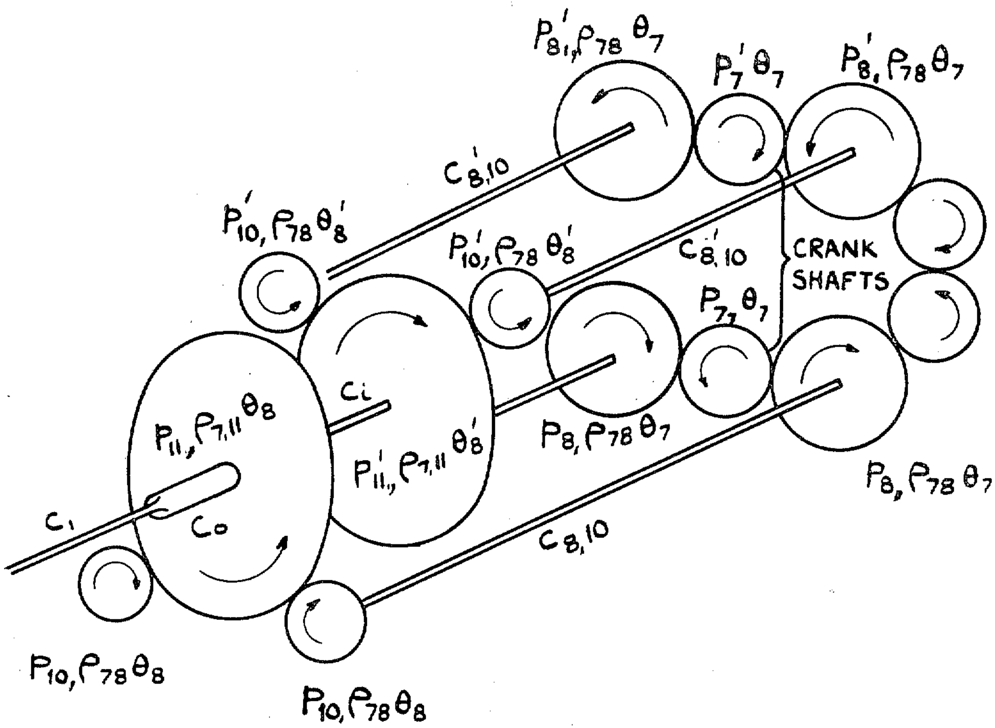
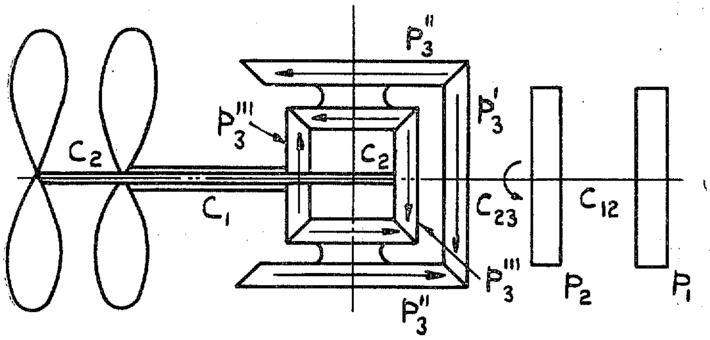


Fig. 5.



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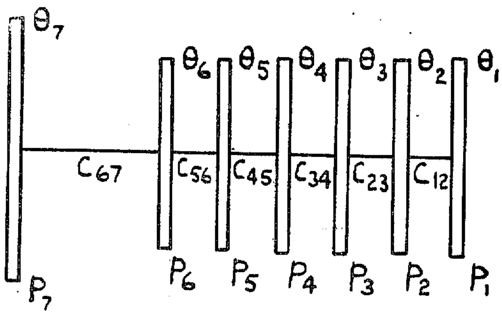

Fig. 6.

Engine Contra-revolving Propeller System. Equivalent Dynamic Data.

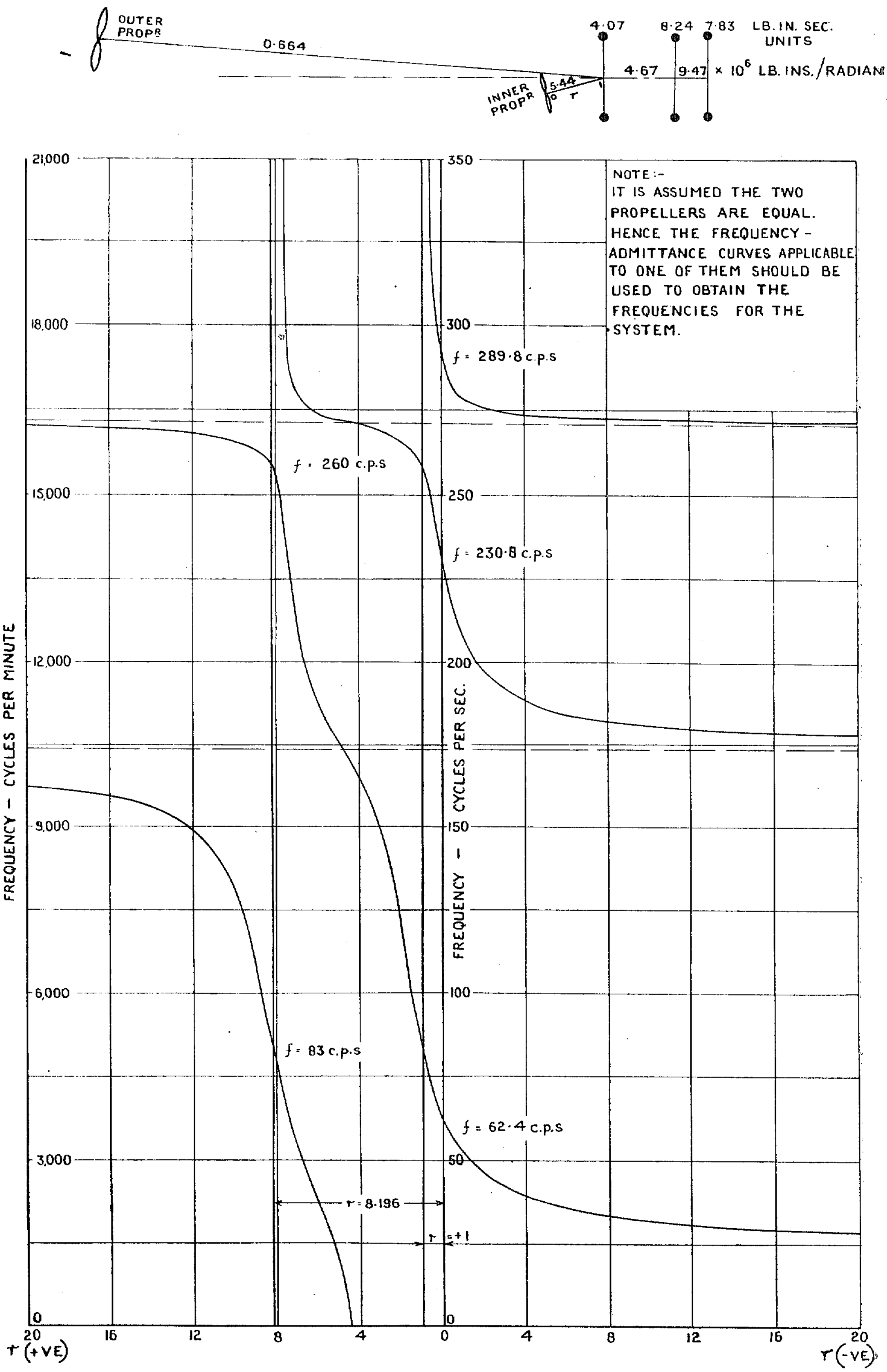

FIG. 4. Engine Frequency Admittance Curves

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