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Summary and Conclusions.—The present paper discusses the effects on the performance of a contra-rotating airscrew pair of the oscillatory nature of the flow round the blades. The blade sections at a representative radius are developed into two infinite cascades in a plane, and the two-dimensional flow in this plane is discussed: for simplicity the blade sections are replaced by vortices with strengths equal to the circulations round the blades.

On this basis, it is shown that if the two screws are to absorb equal powers at equal rotational speeds, the mean circulations round the blades must be equal; however, this implies, for similar sections and equal chords, a coarser pitch setting for the front screw than for the rear. For this condition, the slipstream velocity has an oscillatory rotational component; its mean rotation is, however, zero.

In designing a contra-rotating airscrew pair, the most obvious way of assessing mean values for the local wind speed and direction is to imagine the number of blades to become infinite, while the blade settings and solidities are maintained; the slipstreams are then uniform. In the numerical example given it is shown that this method is quite good enough; although the local thrust variations are of the order of $\pm$ 20 per cent. from their mean values, the latter are less than $0 \cdot 5$ per cent. different from those given by the assumption of an infinite number of blades.

No account has been taken in the present paper of the vortices shed by the blades as the circulation changes; it may be anticipated that their effect will be to reduce the magnitude of the oscillatory variations in thrust, etc., to a degree depending on the value of the frequency parameter.

1. *Introduction.*—In a recent paper[1] the writer has pointed out that the angles of incidence of the blades of either airscrew of a contra-rotating pair vary periodically, owing to the passage of the blades through the velocity field resulting from the circulations round the blades of the other screw. In that paper, attention was confined to the possible effects of the oscillation in angle of incidence on stalled blade sections. In the present note, the effects in relation to unstalled sections are examined rather more fully.

2. *The Katzmayr Effect.*—The Katzmayr effect has been analysed by Cowley[2]. It may be recalled that Katzmayr showed that when an aerofoil is oscillated in a steady stream its drag increases; but when the aerofoil is held stationary in an oscillating stream, the " drag ", referred to axes fixed in the aerofoil, is reduced and may even become negative. Cowley showed that the reason for this is as follows: when the incidence increases from the mean, the lift force increases and at the same time develops a component directed forward along the mean incidence line. When the incidence decreases from the mean, the component of the lift force is directed backward, but since the lift is now reduced, the magnitude of the backward component is less than that of the previous forward component, and on the average there is a forward component opposing the drag force. The effect may be quite large; for example, in the case of one aerofoil of normal section and infinite aspect ratio (Section E of the family of airscrews[3]) the mean " drag " is zero at a mean " lift " coefficient of $0 \cdot 7$ when the incidence oscillates sinusoidally with an amplitude of about $\pm$ 4 deg.

In practice, it is of course usually the case (e.g. in flutter) that the aerofoil is oscillating in a steady wind, so that the drag increases. In the case of contra-rotating airscrews, however, it is the direction of the relative wind which changes, and therefore the Katzmayr effect might be expected to occur.

As a very simple illustration of the effect in relation to an airscrew, consider the blade section shown in Fig. 1. OA is normal and OB parallel to the plane of the airscrew disc ; the resultant wind has velocity W and makes an angle ϕ with OB. If the lift and drag coefficients of the section corresponding to the angle ϕ are C_L and C_D, then by resolution of the forces along OA and OB the thrust and torque grading for the screw, according to the formulae of the vortex theory, are given by

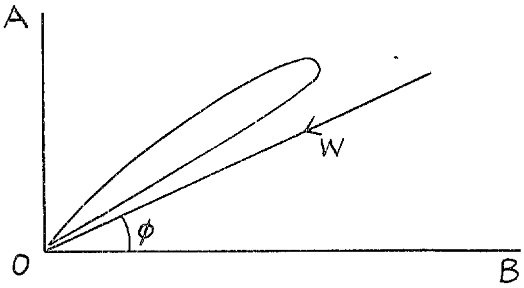

FIG. 1

$$\left.\begin{aligned} \frac{2}{\rho W^2 Nc}\frac{dT}{dr} &= C_L \cos\phi - C_D \sin\phi, \\ \frac{2}{\rho W^2 Ncr}\frac{dQ}{dr} &= C_L \sin\phi + C_D \cos\phi. \end{aligned}\right\} \quad \cdots \quad (1)$$

Suppose now that ϕ is increased by ε ; it will be assumed that W is unchanged in magnitude. The blade incidence is therefore decreased by ε ; if the slope of the curve of lift coefficient against incidence (assumed linear) is a_0, the new lift coefficient is $C_L - a_0\varepsilon$. If ε is not very large the change in C_D is unimportant, and we obtain in place of (1)

$$\left.\begin{aligned} \frac{2}{\rho W^2 Nc}\frac{dT}{dr} &= (C_L - a_0\varepsilon)\cos(\phi + \varepsilon) - C_D \sin(\phi + \varepsilon), \\ \frac{2}{\rho W^2 Ncr}\frac{dQ}{dr} &= (C_L - a_0\varepsilon)\sin(\phi + \varepsilon) + C_D \cos(\phi + \varepsilon). \end{aligned}\right\} \quad \cdots \quad (2)$$

Suppose that regimes of flow represented by ε positive and negative occur in alternation, changing abruptly from one to the other after equal time intervals. The average values of thrust and torque grading are then given by the arithmetic mean of equations (2) and the same equations with the sign of ε changed. To second order in ε these means are

$$\frac{2}{\rho W^2 Nc}\frac{dT}{dr} = C_L \cos\phi - C_D \sin\phi + \varepsilon^2 \left\{ a_0 \sin\phi - \tfrac{1}{2}(C_L \cos\phi - C_D \sin\phi) \right\}$$

$$\frac{2}{\rho W^2 Ncr}\frac{dQ}{dr} = C_L \sin\phi + C_D \cos\phi - \varepsilon^2 \left\{ a_0 \cos\phi + \tfrac{1}{2}(C_L \sin\phi + C_D \cos\phi) \right\}$$

If ε varies sinusoidally instead of abruptly, the coefficients of ε^2 in these equations are halved. However, under all operating conditions (except perhaps near static, when ϕ is small and C_L may be large) the expressions in braces are both positive ; the effect of the oscillation is therefore to increase the thrust grading while decreasing the torque grading.

On consideration it will be evident that the energy required to produce the apparent increase in efficiency is provided by the oscillating airstream ; and in the case of a contra-rotating pair of screws, this energy must be supplied by the other screw. On the whole, therefore, it is not to be expected that the oscillatory effects discussed above can give rise to an overall increase in efficiency* ; the work done against profile drag cannot be avoided. Nevertheless, the oscillations evidently affect the thrust and torque grading, and an investigation of the mutual interactions of the blades is therefore desirable.

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* There will, however, be a gain in efficiency due to reduction in slipstream rotation.

3. *The Cascade Analogy.*—Consider a pair of contra-rotating screws each having N blades; take a cylinder coaxial with the screws and of radius r. It will be assumed that there is no component of velocity normal to the surface of this cylinder: this is the assumption of the vortex theory of airscrews, which has now been superseded; however, it is approximately true except when the rate of advance is high and the number of blades small. The performances of the blade sections at the surface of the cylinder can be determined by developing the cylinder into part of an infinite plane and considering the two-dimensional flow round the two infinite cascades into which the blade sections develop.

As an approximation, the blade sections will be replaced by vortices with strengths equal to the circulations round the sections. Consider the single infinite cascade of vortices part of which is shown in Fig. 2.

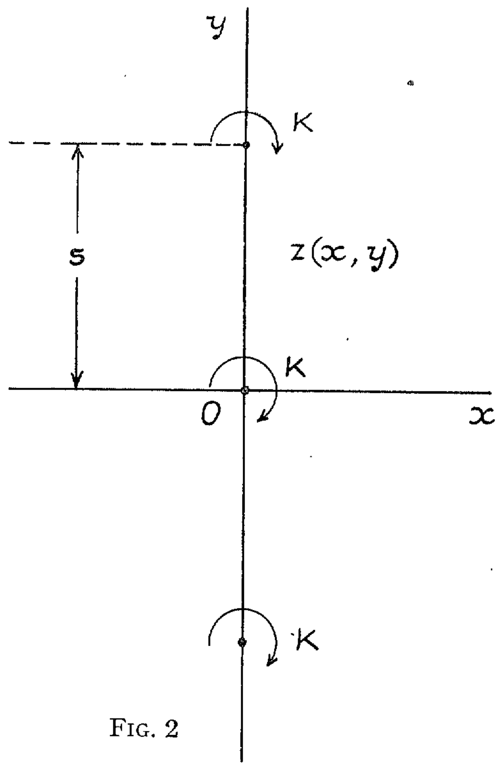

FIG. 2

Each vortex is of strength K (positive when the circulation is clockwise) and they are disposed in the z-plane at the points $z = 0$, $z = \pm is$, $z = \pm 2is, \ldots$, where $s = 2\pi r/N$. The velocity components at any point z are therefore given by

$$u - iv = \frac{iK}{2\pi}\left\{\frac{1}{z} + \frac{1}{z + is} + \frac{1}{z - is} + \ldots\right\} = \frac{iK}{2s}\coth\left(\frac{\pi z}{s}\right), \qquad \ldots \quad (3)$$

or

$$u = \frac{K \sin 2\pi\eta}{2s\,(\cosh 2\pi\xi - \cos 2\pi\eta)} \doteq \frac{K}{2s} f(\xi, \eta), \qquad \ldots \quad (4)$$

and

$$v = \frac{K \sinh 2\pi\xi}{2s\,(\cosh 2\pi\xi - \cos 2\pi\eta)} \equiv \frac{K}{2s} F(\xi, \eta), \qquad \ldots \quad (5)$$

where $\qquad \xi = x/s, \quad \eta = y/s.$

The local velocity components at any vortex due to the remaining vortices of the cascade are, by symmetry, zero. To check this, superpose on the system of Fig. 2 a vortex of strength $-K$ at the origin: the velocity components then become, by (3)

$$u - iv = \frac{iK}{2s}\coth\left(\frac{\pi z}{s}\right) - \frac{iK}{2\pi z} = \frac{iK}{2s}\left\{\coth\left(\frac{\pi z}{s}\right) - \left(\frac{s}{\pi z}\right)\right\}.$$

The limit of this expression as z becomes indefinitely small is zero.

4. *Instantaneous Velocity Components for the Double Cascade.*—In §§4–6 we shall consider only the conditions which exist at any given instant, so that the variation of conditions with time will not enter into the discussion.

The two cascades of vortices representing the blade sections of the contra-rotating pair are shown in Fig. 3. The distance between the planes of the airscrew discs is h; each cascade of

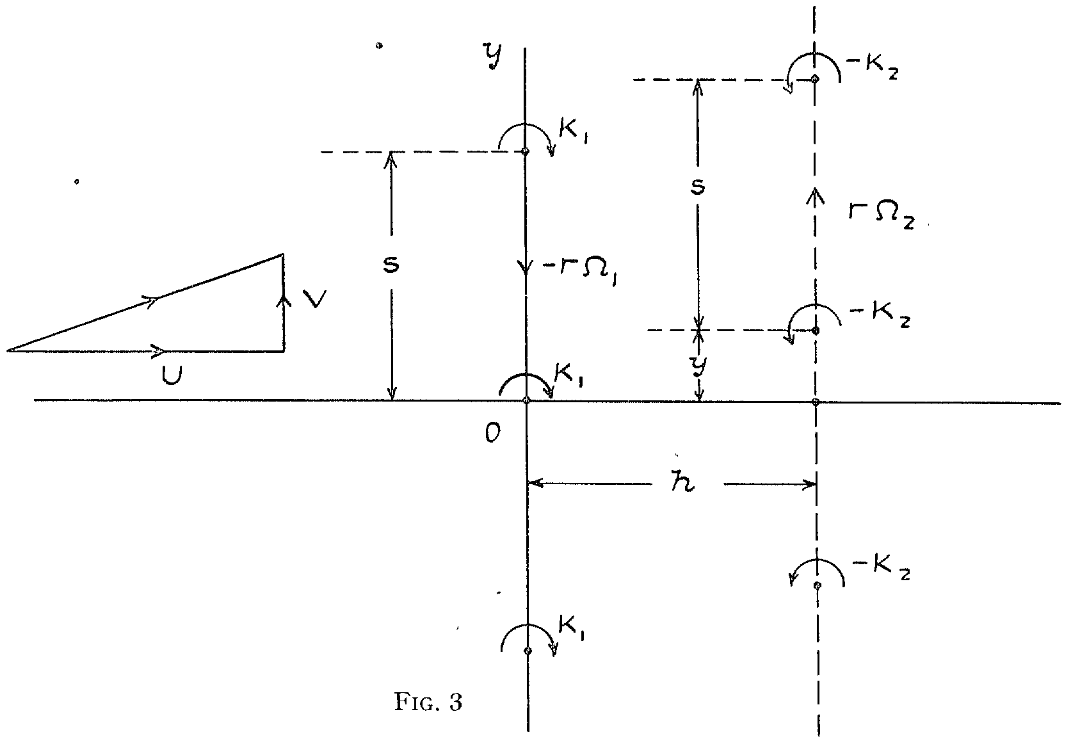

FIG. 3

vortices has the spacing s. At the given instant, take the y-axis along the front cascade, with one vortex at the origin; let one vortex of the back cascade be at the point (h, y). Suppose the vortices of the front cascade each to have strength K_1 and those of the rear cascade $-K_2$, and let the speeds of the two cascades in the positive direction of y be $-r\Omega_1$ and $r\Omega_2$, respectively.

On the instantaneous velocity field due to the two cascades, superpose a uniform velocity having components U, V in the directions of x, y, respectively. The component U is the usual axial velocity component, and will be assumed to include the axial interference velocity. The component V can be determined from the condition that far in front of the two cascades, i.e. at $x = -\infty$, the total resultant flow must be purely real, since there is no rotation in the stream approaching an airscrew. The total imaginary component at $x = -\infty$ is, on use of (3),

$$0 = V + \frac{K_1}{2s} - \frac{K_2}{2s},$$

whence

$$V = \frac{K_2 - K_1}{2s}. \qquad \ldots \quad (6)$$

We may now write down the velocity components at any point of the field, and in particular, may evaluate the velocity components at any vortex due to all the remaining vortices of both cascades and to the uniform velocity. Thus at the point (h, y), there is no contribution from the other vortices of the second cascade, and there are contributions given by (4) and (5) due to the first cascade: there are also contributions U and V, where V is given by (6), from the uniform velocity. These are the total contributions to the velocity components relative to the fixed axes. Relative to the vortex itself, however, there is an additional component equal to the reversed translational velocity of the vortex, i.e. a contribution $-r\Omega_2$ in the positive direction of y. Hence if the components relative to the vortex are u_2, v_2

$$u_2 = U + \frac{K_1}{2s}f(\eta), \qquad (7)$$

$$v_2 = -r\Omega_2 + \frac{K_2 - K_1}{2s} - \frac{K_1}{2s}F(\eta), \qquad (8)$$

where $f(\eta)$, $F(\eta)$ are written for $f(\xi, \eta)$, $F(\xi, \eta)$ when ξ assumes the constant value h/s.

In a similar way, the velocity components relative to the vortices of the front cascade are readily shown to be

$$u_1 = U + \frac{K_2}{2s}f(\eta), \qquad (9)$$

$$v_1 = r\Omega_1 + \frac{K_2 - K_1}{2s} - \frac{K_2}{2s}F(\eta). \qquad (10)$$

If the resultant of u_1, v_1 is W_1 and of u_2, v_2 is W_2, and if these resultants make angles ϕ_1, ϕ_2 with the planes of the screws, then (see Fig. 4)

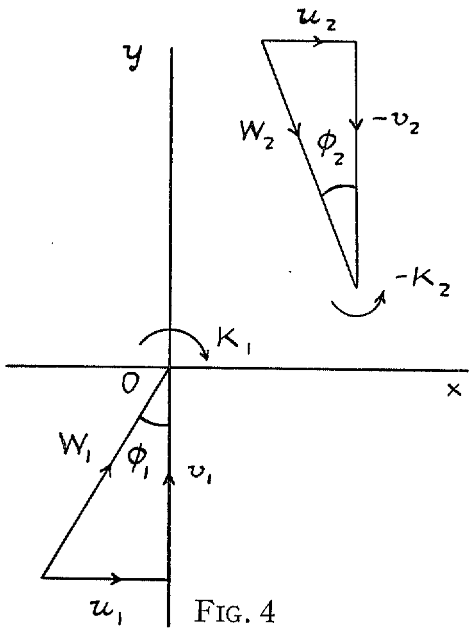

FIG. 4

$$u_1 = W_1 \sin \phi_1, \qquad (11)$$

$$v_1 = W_1 \cos \phi_1, \qquad (12)$$

and

$$u_2 = W_2 \sin \phi_2, \qquad (13)$$

$$-v_2 = W_2 \cos \phi_2, \qquad (14)$$

the negative sign being introduced in (14) so that ϕ_2, as is conventional, shall lie in the first quadrant.

5. *Instantaneous Thrust and Torque.*—The lift force on a vortex of strength K in a stream of velocity W is ρWK and is at right angles to the direction of W: the drag force is zero. Accordingly, the force components along the thrust axis due to the N vortices representing the sections of each screw give the thrust gradings as

$$\frac{dT_1}{dr} = \rho N W_1 K_1 \cos \phi_1$$

$$= \rho N K_1 v_1,$$

$$\frac{dT_2}{dr} = \rho N K_2 (- v_2),$$

on use of (12) and (14). Similarly, the torque gradings are

$$\frac{dQ_1}{dr} = \rho r N K_1 u_1,$$

$$\frac{dQ_2}{dr} = \rho r N K_2 u_2.$$

Substitution from equations (7) to (10) gives

$$\frac{1}{\rho N} \frac{dT_1}{dr} = r\Omega_1 K_1 + \frac{K_1(K_2 - K_1)}{2s} - \frac{K_1 K_2}{2s} F(\eta), \quad \ldots \quad (15)$$

$$\frac{1}{\rho N} \frac{dT_2}{dr} = r\Omega_2 K_2 - \frac{K_2(K_2 - K_1)}{2s} + \frac{K_1 K_2}{2s} F(\eta), \quad \ldots \quad (16)$$

and

$$\frac{1}{\rho r N} \frac{dQ_1}{dr} = UK_1 + \frac{K_1 K_2}{2s} f(\eta), \quad \ldots \quad (17)$$

$$\frac{1}{\rho r N} \frac{dQ_2}{dr} = UK_2 + \frac{K_1 K_2}{2s} f(\eta). \quad \ldots \quad (18)$$

Though the equations so far obtained relate only to instantaneous conditions, some interesting deductions may be made from them.

The individual thrust gradings due to each screw in the absence of the other are found by putting $K_2 = 0$ in (15) and $K_1 = 0$ in (16). The results are

$$\frac{1}{\rho N} \frac{dT_1}{dr} = r\Omega_1 K_1 - \frac{K_1^2}{2s},$$

$$\frac{1}{\rho N} \frac{dT_2}{dr} = r\Omega_2 K_2 - \frac{K_2^2}{2s},$$

and these equations are, as would be expected, independent of η. The second term on the right-hand side in the two expressions represents the loss in thrust due to slipstream rotation.

The total thrust grading due to the two screws in the presence of each other is obtained by adding (15) and (16); it is therefore given by

$$\frac{1}{\rho N} \frac{d(T_1 + T_2)}{dr} = r\Omega_1 K_1 + r\Omega_2 K_2 - \frac{(K_1 - K_2)^2}{2s}. \quad \ldots \quad (19)$$

This exceeds the sum of the individual components given above by $K_1 K_2 / s$; the increase is due to the reduction in slipstream rotation. The magnitude of the slipstream rotation is determined by the velocity parallel to Oy at $x = + \infty$: this is

$$V - \frac{K_1}{2s} + \frac{K_2}{2s} = \frac{K_2 - K_1}{s}, \quad \ldots \quad (20)$$

on use of (6). If $K_1 = K_2$, there is no slipstream rotation, while the efficiency of the system under consideration is evidently a maximum. Moreover, when $K_1 = K_2$ there is no torque reaction from the screws, as is evident when (18) is subtracted from (17).*

We may also compare the contra-rotating pair with a pair rotating in the same sense. In practice such a pair would be locked together, and would probably be identical and coplanar; however, the general result can be simply obtained by reversing the signs of both Ω_2 and K_2 in the foregoing discussion. As regards the thrust grading, this is, by (19)

$$\frac{1}{\rho N}\frac{d(T_1 + T_2)}{dr} = r\Omega_1 K_1 + r\Omega_2 K_2 - \frac{(K_1 + K_2)^2}{2s}. \qquad (21)$$

This thrust is less than the sum of the individual thrusts by the amount K_1K_2/s; the decrease is due to the increased slipstream rotation, the magnitude of which is determined by (20) as a velocity $-(K_1 + K_2)/s$ parallel to the planes of the screws.

6. *Conditions for Equal Power Absorption.*—The most common operating condition for a pair of contra-rotating screws would probably be that for which the power input to each is the same. The rates of absorption of power by the sections are $\Omega_1 dQ_1/dr$ and $\Omega_2 dQ_2/dr$; if these are equal then by (17) and (18)

$$0 = U(\Omega_1 K_1 - \Omega_2 K_2) + (\Omega_1 - \Omega_2)\frac{K_1K_2}{2s}f(\eta). \qquad (22)$$

There is no unique solution of (22), but an obvious case which satisfies the equation is that for which

$$\Omega_1 = \Omega_2 = \Omega, \qquad (23)$$

$$K_1 = K_2 = K. \qquad (24)$$

These yield the advantages specified in §5; and in addition the engines deliver equal power at equal rotational speeds.

It should be remarked, however, that even when both conditions (23) and (24) are satisfied, v_1 and $-v_2$ are not equal, though u_1 and u_2 are (see equations (7) to (10)). It follows that the angles ϕ, the resultant speeds W, and the lift coefficients, are all unequal for the front and back blades. Substitution from (23) and (24) in (7) to (10) and use of (11) to (14) gives

$$\tan\phi_1 = \frac{2sU + Kf(\eta)}{2sr\Omega - KF(\eta)}, \qquad (25)$$

$$\tan\phi_2 = \frac{2sU + Kf(\eta)}{2sr\Omega + KF(\eta)}. \qquad (26)$$

Now for all values of η, $F(\eta)$ is positive; hence $\phi_1 > \phi_2$, also $W_2 > W_1$, so that for equal circulations and chord, $C_{L1} > C_{L2}$. For similar sections, this implies a larger incidence for the front screw as well as a larger angle ϕ, so that the front screw always requires a coarser pitch setting than the rear screw.

Again, though the torques are equal and opposite, the thrusts are unequal. Substitution from (23) and (24) in (15) and (16) gives

$$\frac{1}{\rho N}\frac{dT_1}{dr} = r\Omega K - \frac{K^2}{2s}F(\eta), \qquad (27)$$

$$\frac{1}{\rho N}\frac{dT_2}{dr} = r\Omega K + \frac{K^2}{2s}F(\eta), \qquad (28)$$

so that the thrust of the rear screw exceeds that of the front screw.

* The torque reaction is given by the difference, and not the sum, of (17) and (18), since each component is measured in the sense of rotation of its own screw.

7. *Conditions During Motion.*—The analysis of §§4–6 is concerned with the instantaneous conditions occurring during motion of the screws; and the results of §6, based on the conditions (23) and (24), only apply at the given instant. It is evidently possible to maintain the condition (23) throughout the motion, but the magnitudes of the circulations K_1 and K_2 depend on the mutual interactions of the blades, and would be expected to vary differently throughout the cycle $0 < \eta < 1$. We now proceed to examine this question.

If the pitch setting of a blade of the front screw, referred to its no-lift line, is θ_1, the incidence relative to this line is $\theta_1 - \phi_1$. The lift coefficient is accordingly

$$C_L = a_0 \sin (\theta_1 - \phi_1). \qquad (29)$$

In equation (29) the slope of the lift curve at no lift is written a_0; the theoretical value of a_0 for an isolated aerofoil is 2π. For aerofoils in cascade, this is modified[4], but in the case of airscrew blade sections the spacing is so wide that the theoretical value would be only very slightly less than 2π. However, in practice, for an isolated aerofoil under steady conditions, a_0 is only about 90 per cent. of the theoretical value. Moreover, it has been shown[1] that the frequency parameter of the oscillatory changes for a contra-rotating airscrew pair is of the order of unity, so that a reduction in a_0 of the order of 40 per cent. may be expected[5]: this reduction, however, would only apply to the variations from the mean incidence. For simplicity this complication will be omitted in the present paper, but its effect would probably be to reduce by 30–40 per cent. the magnitude of the oscillatory variations in thrust, etc., found in the present discussion.

Since the circulation round a section of the front screw is K_1, equation of the expressions for the lift force gives

$$\rho W_1 K_1 = \tfrac{1}{2}\rho W_1^2 c_1 C_L, \qquad (30)$$

which with (29) reduces to

$$K_1 = \frac{a_0 c_1}{2} W_1 \sin (\theta_1 - \phi_1). \qquad (31)$$

Similarly, for the sections of the rear screw,

$$K_2 = \frac{a_0 c_2}{2} W_2 \sin (\theta_2 - \phi_2). \qquad (32)$$

If we expand the sines in (31) and (32) and use equations (7) to (14) we obtain the following equations for K_1 and K_2:

$$A_1 K_1 + B_1 K_2 = C_1, \qquad (33)$$

$$A_2 K_2 + B_2 K_1 = C_2, \qquad (34)$$

where

$$A_1 = \frac{4s}{a_0 c_1} + \sin \theta_1, \qquad (35)$$

$$A_2 = \frac{4s}{a_0 c_2} + \sin \theta_2, \qquad (36)$$

$$B_1 = f \cos \theta_1 + (F - 1) \sin \theta_1, \qquad (37)$$

$$B_2 = f \cos \theta_2 - (F + 1) \sin \theta_2, \qquad (38)$$

$$C_1 = 2s(r\Omega_1 \sin \theta_1 - U \cos \theta_1), \qquad (39)$$

$$C_2 = 2s(r\Omega_2 \sin \theta_2 - U \cos \theta_2). \qquad (40)$$

In these equations the functional dependence of f and F on η has for convenience been omitted. The solutions of equations (33) and (34) are

$$K_1 = \frac{A_2C_1 - B_1C_2}{A_1A_2 - B_1B_2}, \qquad (41)$$

$$K_2 = \frac{A_1C_2 - B_2C_1}{A_1A_2 - B_1B_2}. \qquad (42)$$

Equations (41) and (42) determine the values of K_1 and K_2 for any given values of blade setting, rotational speed, etc., as functions of η, which defines the positions of the blades relative to each other. It will be seen that K_1 and K_2 depend on η through the quantities B_1 and B_2, which alone involve the functions f and F of η.

When tables or curves of K_1 and K_2 have been found as functions of η, these can be used in conjunction with equations (15) to (18) to determine the variation of thrust and torque grading with η.

We may remark that, to determine the performance of either screw in the absence of the other, it is sufficient to imagine the chord of the other blade to become indefinitely small. Thus, if c_2 is made indefinitely small, the quantity A_2 tends to infinity. Equation (42) then shows that K_2 tends to zero, while (41) reduces to

$$K_1 = C_1/A_1, \qquad (43)$$

which is, as would be expected, independent of η.

Equations (41) and (42) also show that it is not possible to maintain equality of circulation throughout the cycle $0 < \eta < 1$. The conclusions of §6 accordingly require further examination. It is, however, evident at once that, since the circulations are not always equal, slipstream rotation cannot be always absent. If the mean torques on the screws are equal and opposite, the mean slipstream rotation must evidently be zero, but it is now apparent that in this case the angular velocity of the slipstream will oscillate about its mean value of zero. It is interesting to note that as a result there will probably be a true Katzmayr effect on the wing roots and tail surfaces of an aircraft with contra-rotating screws: these surfaces will tend to extract any oscillatory energy from the slipstream. It is also conceivable that breakaway on wing root surfaces which might be present in a steady stream would be removed by the known unstalling effect of oscillatory motion[1].

8. *Conditions for Equal Mean Power Absorption.*—We now proceed to examine afresh the conclusion of §6, that for the two screws to absorb equal powers the front screw requires to be set at a coarser pitch than the rear screw. This conclusion of course implies equal rotational speeds and chords; we shall therefore use equation (23), and shall write c for c_1 and c_2.

The condition for equal mean power absorption is evidently that the time integrals of the rates of power absorption shall be equal over a complete cycle. Since the speeds of rotation are constant with time, the time integral may be replaced by a space integral, so that the condition is

$$\int_0^1 \Omega \frac{dQ_1}{dr}\, d\eta = \int_0^1 \Omega \frac{dQ_2}{dr}\, d\eta.$$

On use of (17) and (18), this equation reduces simply to

$$\int_0^1 K_1 d\eta = \int_0^1 K_2 d\eta, \qquad (44)$$

which is the generalisation of equation (24). We may note that since the mean powers are equal and the rotational speeds are equal, the mean torques are equal, so that the mean slipstream rotation is zero.

We may regard the condition (44) as fixing some relation between the quantities defined by equations (35) to (40). In practice this relation would usually be between the blade settings θ_1 and θ_2. In a practical application of the formulae, the dimensions of the screws and Ω and U would be regarded as fixed, and (41) and (42) would then be used to obtain curves of K_1 and K_2 as functions of η for a range of values of θ_1 and θ_2; corresponding pairs of values of θ_1 and θ_2 would then be chosen from the curves of K_1 and K_2 which satisfied (44).

To obtain more general insight into the question of blade settings, however, we shall introduce a simplification into equations (41) and (42). Suppose both blade settings (which are referred to the no-lift lines) to be reduced until the thrust on each vanishes; there is then no circulation round either blade and no interaction between them. Equations (33) and (34) then show that $C_1 = C_2 = 0$, so that, by (39) and (40) (and on use of (23)) both blade settings become equal to ϕ_0, where

$$W_0 \sin \phi_0 = U, \qquad (45)$$

$$W_0 \cos \phi_0 = r\Omega, \qquad (46)$$

and W_0 is the resultant of U and $r\Omega$.

For the general settings, let

$$\theta_1 = \phi_0 + \delta\theta_1, \qquad (47)$$

$$\theta_2 = \phi_0 + \delta\theta_2, \qquad (48)$$

and let $\delta\theta_1$ and $\delta\theta_2$ be small. If these equations are used to substitute for θ_1 and θ_2 in equations (35) to (40) then to a first approximation it is found that

$$K_1 = \frac{C(A\,\delta\theta_1 - B_1\delta\theta_2)}{A^2 - B_1B_2}, \qquad (49)$$

$$K_2 = \frac{C(A\,\delta\theta_2 - B_2\delta\theta_1)}{A^2 - B_1B_2}, \qquad (50)$$

where

$$A = \sin \phi_0 + \frac{4s}{a_0c}, \qquad (51)$$

$$B_1 = f \cos \phi_0 - \sin \phi_0 + F \sin \phi_0, \qquad (52)$$

$$B_2 = f \cos \phi_0 - \sin \phi_0 - F \sin \phi_0, \qquad (53)$$

$$C = 2sW_0. \qquad (54)$$

If now we substitute K_1 and K_2 from (49) and (50) in the condition (44) and rearrange, we obtain

$$\delta\theta_1 \int_0^1 \frac{A + B_2}{A^2 - B_1B_2}\, d\eta = \delta\theta_2 \int_0^1 \frac{A + B_1}{A^2 - B_1B_2}\, d\eta. \qquad (55)$$

On inspection of these equations it will be evident that in all practical cases $A^2 - B_1B_2$ is positive and that $A + B_1$ and $A + B_2$ are also positive, while

$$A + B_1 > A + B_2$$

for all values of η, since F is positive. It follows that the integral on the right of (55) is greater than that on the left; and hence

$$\delta\theta_1 > \delta\theta_2. \qquad (56)$$

This supports the conclusion of §6.

We may evaluate the integrals in (55) approximately as follows. We first note that in view of (3), (4) and (5)

$$\int_0^1 (F - if)\, d\eta = \int_0^1 \coth\left(\frac{\pi z}{s}\right) d\eta = \frac{1}{i\pi}\int_{\pi\xi}^{\pi\xi + i\pi} \coth\left(\frac{\pi z}{s}\right) d\left(\frac{\pi z}{s}\right),$$

since ξ is constant. This gives

$$\int_0^1 (F - if)\, d\eta = \frac{1}{i\pi}\log\left(\frac{\sinh(\pi\xi + i\pi)}{\sinh \pi\xi}\right) = \frac{1}{i\pi}\log e^{i\pi} = 1.$$

Hence

$$\int_0^1 F d\eta = 1, \qquad \text{(57)}$$

$$\int_0^1 f d\eta = 0, \qquad \text{(58)}$$

independently of the value of ξ. It follows that the mean values of B_1 and B_2 in (52) and (53) are 0 and $-2 \sin \phi_0$, respectively; while from the forms of F and f it is evident that the variations of B_1 and B_2 from these means will not be more than a unit or two unless ξ is very small. On the other hand, since s/c is usually large, A^2 will be of the order of 10^2; accordingly we may without much error replace B_1 and B_2 in the denominators of the integrals in (55) by their mean values. The equation then gives at once

$$\frac{\delta\theta_1}{\delta\theta_2} = \frac{\theta_1 - \phi_0}{\theta_2 - \phi_0} = \frac{4s + a_0 c \sin \phi_0}{4s - a_0 c \sin \phi_0}, \qquad \text{(59)}$$

in confirmation of (56). This simple equation gives fair agreement with the more exact results obtained by the method indicated earlier.

9. *A Comparison between Oscillatory and Steady Motion Cases.*—In the discussion of the Katzmayr effect in §2 it was assumed that the incidence was the only variable. In the case of a contra-rotating airscrew pair, the local velocity is also a variable; and from the form of the equations it is evident that the analytical determination of the mean incidence and speed presents considerable difficulty. The simplest way of assessing mean values for these quantities, and the way which would probably be used in determining blade settings for a contra-rotating pair, is to imagine the number of blades to become indefinitely large, while the solidity remains constant. The airscrews then become sheets of vorticity, and the velocities on each side become independent of circumferential displacement.

In the cascade analogy, it is evident that, instead of reducing chord and spacing indefinitely, the same result may be achieved by increasing indefinitely the distance h between the cascades. In the present discussion we shall assume that the chords and rotational speeds of the two cascades are respectively equal, and that, when h is indefinitely large, the blade settings are such that there are equal circulations K_0 round each: the power inputs are then equal (see §6).

Equations (4) and (5) show that when ξ becomes indefinitely large, f becomes zero and F unity; these values agree with (57) and (58). On substitution in (37) and (38) we obtain

$$B_1 = 0, \qquad \text{(60)}$$

$$B_2 = -2 \sin \theta_2 \equiv -S, \qquad \text{(61)}$$

and when these values are substituted in (41) and (42) these equations readily yield

$$A_1 K_0 = C_1,$$

or, in full,

$$\left(\frac{4s}{a_0 c} + \sin \theta_1\right) K_0 = 2s(r\Omega \sin \theta_1 - U \cos \theta_1), \qquad \text{(62)}$$

$$\left.\begin{aligned} &\text{and} && (A_2 - S)\,K_0 = C_2, \\ &\text{or} && \left(\frac{4s}{a_0 c} - \sin\theta_2\right) K_0 = 2s(r\Omega \sin\theta_2 - U\cos\theta_2). \end{aligned}\right| \qquad \ldots \quad \ldots \quad \ldots \quad (63)$$

Equation (62) is the same as (43), which defines the circulation round the front screw blades when the rear screw is removed. Equations (62) and (63), in the present instance, respectively define θ_1 and θ_2 for a given value of K_0.

When θ_1 and θ_2 have been fixed in this way, the distance h may be made to assume its actual finite value; in equations (35) to (40), A_1, A_2, C_1 and C_2 are then given fixed quantities, while B_1 and B_2 are given functions of η. Accordingly, K_1 and K_2 are given as functions of η by (41) and (42), and may be compared with the steady value K_0 they assume when h is infinite; while the differences between K_0 and the integrals of K_1 and K_2 over the range 0 to 1 of η may be regarded as the "effect" of the oscillatory character of the motion. When K_1 and K_2 have been determined as functions of η, the oscillatory variations in thrust and torque grading may be found from equations (15) to (18).

In view of (62) and (63), the equations (33) and (34) become

$$A_1K_1 + B_1K_2 = A_1K_0, \qquad \ldots \quad (64)$$

$$A_2K_2 + B_2K_1 = (A_2 - S)\,K_0, \qquad \ldots \quad (65)$$

which yield

$$\frac{K_1}{K_0} = \frac{A_1A_2 - B_1(A_2 - S)}{A_1A_2 - B_1B_2}, \qquad \ldots \quad (66)$$

$$\frac{K_2}{K_0} = \frac{A_1A_2 - (B_2 + S)A_1}{A_1A_2 - B_1B_2}, \qquad \ldots \quad (67)$$

which are rather more convenient for computation than (41) and (42).

We may obtain general insight into the effect of the oscillatory character of the motion by considering the case where h is large but finite. By equations (3) to (5)

$$\begin{aligned} F - if &= \coth \pi\,(\xi + i\eta) \\ &= \frac{1 + e^{-2\pi(\xi + i\eta)}}{1 - e^{-2\pi(\xi + i\eta)}}. \end{aligned} \qquad \ldots \quad (68)$$

If h is large, we may write

$$e^{-2\pi\xi} = \varepsilon, \qquad \ldots \quad (69)$$

where ε is small; to the second order in ε, (68) then becomes

$$F - if = 1 + 2\varepsilon e^{-2\pi i\eta} + 2\varepsilon^2 e^{-4\pi i\eta},$$

or

$$F - 1 = 2\varepsilon \cos 2\pi\eta + 2\varepsilon^2 \cos 4\pi\eta, \qquad \ldots \quad (70)$$

$$f = 2\varepsilon \sin 2\pi\eta + 2\varepsilon^2 \sin 4\pi\eta. \qquad \ldots \quad (71)$$

We shall neglect powers of ε higher than the second in the following analysis. Equations (37) and (38) give, in view of (70) and (71),

$$B_1 = 2\varepsilon \sin(2\pi\eta + \theta_1) + 2\varepsilon^2 \sin(4\pi\eta + \theta_1), \qquad \ldots \quad (72)$$

$$B_2 = -S + 2\varepsilon \sin(2\pi\eta - \theta_2) + 2\varepsilon^2 \sin(4\pi\eta - \theta_2). \qquad \ldots \quad (73)$$

Write

$$K_1 = K_0(1 + a_1\varepsilon + b_1\varepsilon^2), \qquad \ldots \quad (74)$$

$$K_2 = K_0(1 + a_2\varepsilon + b_2\varepsilon^2). \qquad \ldots \quad (75)$$

We now substitute from (72) to (75) in equations (64) and (65) and expand to second order in ε ; equation of the coefficients of the powers of ε in the two equations yields four equations, the solutions of which are given by

$$\tfrac{1}{2}A_1A_2a_1 = -A_2 \sin(2\pi\eta + \theta_1), \qquad (76)$$

$$\tfrac{1}{2}A_1A_2a_2 = -A_1 \sin(2\pi\eta - \theta_2) - S\sin(2\pi\eta + \theta_1), \qquad (77)$$

$$\tfrac{1}{2}A_1^2A_2^2b_1 = -A_1A_2^2 \sin(4\pi\eta + \theta_1) + A_1A_2\Big\{\cos(\theta_1 + \theta_2) - \cos(4\pi\eta + \theta_1 - \theta_2)\Big\} + A_2S\Big\{1 - \cos(4\pi\eta + 2\theta_1)\Big\}, \qquad (78)$$

$$\tfrac{1}{2}A_1^2A_2^2b_2 = -A_1^2A_2 \sin(4\pi\eta - \theta_2) - A_1A_2S\sin(4\pi\eta + \theta_1) + A_1(A_2 + S)\Big\{\cos(\theta_1 + \theta_2) - \cos(4\pi\eta + \theta_1 - \theta_2)\Big\} + S^2\Big\{1 - \cos(4\pi\eta + 2\theta_1)\Big\}. \qquad (79)$$

Equations (74) to (79) define the way in which K_1 and K_2 vary with η. We may use these equations to find how the mean values for a complete cycle depend on ε ; from (74) and (75)

$$\frac{1}{K_0}\int_0^1 K_1 d\eta = 1 + \varepsilon\int_0^1 a_1 d\eta + \varepsilon^2\int_0^1 b_1 d\eta, \qquad (80)$$

$$\frac{1}{K_0}\int_0^1 K_2 d\eta = 1 + \varepsilon\int_0^1 a_2 d\eta + \varepsilon^2\int_0^1 b_2 d\eta. \qquad (81)$$

Equations (76) and (77) show that the coefficients of ε in (80) and (81) both vanish ; the difference between K_0 and the mean values of K_1 and K_2 is thus of the second order in ε. We find

$$\frac{1}{K_0}\int_0^1 K_1 d\eta = 1 + \frac{2\varepsilon^2}{A_1^2A_2^2}\Big\{A_1A_2\cos(\theta_1 + \theta_2) + A_2S\Big\}, \qquad (82)$$

$$\frac{1}{K_0}\int_0^1 K_2 d\eta = 1 + \frac{2\varepsilon^2}{A_1^2A_2^2}\Big\{A_1(A_2 + S)\cos(\theta_1 + \theta_2) + S^2\Big\}. \qquad (83)$$

The coefficients of ε^2 in these equations are evidently small quantities ; accordingly it may be anticipated that the mean values of the circulations will differ only very slightly from the steady value of K_0.

Except for the radii at which the sum of the blade angles considerably exceeds 90 deg., both coefficients of ε^2 in (82) and (83) are positive. We may accordingly conclude that the effect of the oscillations is to increase the mean circulations. The increases, however, are not equal for the front and back screws.

By substitution from (74) and (75) in equations (15) to (18), and use of (76) to (79), the variation and mean values of thrust and torque may be obtained also. It will be sufficient to remark here that the changes in mean thrust and torque grading are also of the second order in ε.

10. *A Comment on the Theory.*—In the analysis of the present paper, it has been assumed that the strength of a vortex in a perfect fluid can change. This, of course, is not strictly valid ; however, if it is assumed, it must also be assumed that vorticity is shed when the circulation changes ; the total strength of the vortices shed in a given time being equal to the change in circulation. In the present theory, no account has been taken of the velocities induced by the shed vortices ; however, if the variations from the mean circulation are small,

the velocities induced by the shed vortices will also be small. It follows that the present theory can only be expected to apply when the distance between the screws is sufficiently large for the variations in circulation to be reasonably small compared with the mean circulation.

It was remarked in §7 that the magnitude of the oscillatory variations in thrust and torque would probably be reduced by the frequency parameter effect. This effect results from the velocities induced by the vortices shed by an aerofoil in oscillatory motion; it therefore seems possible that the effects due to the shed vortices may be represented with sufficient accuracy by a reduction in the amplitudes of the forces corresponding to the given frequency parameter.

11. *A Numerical Example.*—The numerical values assumed for the present illustration are based on data applicable to a particular contra-rotating airscrew pair. The conditions assumed correspond very roughly to cruising at 240 m.p.h. at 15,000 ft. The numerical values adopted are

$$\begin{aligned} N &= 3 \\ c &= 0\cdot 7 \text{ ft.} \\ h &= 0\cdot 75 \text{ ft.} \\ r &= 4 \text{ ft.} \\ s &= 2\pi r/N = 8\cdot 379 \text{ ft.} \\ 2\pi\xi &= Nh/r = 0\cdot 5625 \\ U &= 360 \text{ ft./sec.} \\ r\Omega &= 540 \text{ ft./sec.} \\ a_0 &= 5\cdot 6. \end{aligned}$$

The value of r chosen is very nearly $0\cdot 7$ of the tip radius of the airscrew pair, which is 11 ft. 6 in. in diameter. The rotational speed corresponds to 2,400 engine r.p.m. nearly.

We shall commence by a determination of the blade angles θ_1 and θ_2 (referred to the no lift lines) at $r = 4$ ft., on the assumption mentioned at the beginning of §9, namely, that each airscrew can be treated as a sheet of vorticity, so that there are no oscillatory effects. The blade angles are then defined by (62) and (63). In these formulae we shall assume

$$K_0 = 100 \text{ ft.}^2\text{/sec.}$$

a value which gives the cruising lift coefficient of the blades as about $0\cdot 45$ at $r = 4$ ft. This lift coefficient is low, but the total solidity of the screws is correspondingly high; and the thrust of the screws, computed from the formula

$$T = 0\cdot 8 \left[r \frac{dT}{dr} \right]_{r\,=\,0.7R} ,$$

is found to be of the correct order of magnitude for the value of K_0 chosen.

With $K_0 = 100$, equations (62) and (63) give

$$\theta_1 = 38\cdot 53 \text{ deg.},$$

$$\theta_2 = 37\cdot 87 \text{ deg.},$$

so that the difference in blade angle* is $0\cdot 66$ deg.

* It should be remarked that the theoretical difference in blade angle is based on the assumption that the streamlines lie on the surface of a cylinder. In practice this is probably not true, with the result that $\theta_1 - \theta_2$ at a given radius r may differ appreciably from the theoretical value.

Retaining these blade angles, we shall now compute the variations with η of K_1 and K_2 as determined by equations (41) and (42). As a preliminary the variation of the functions $f(\eta)$ and $F(\eta)$ will be found. For the given data these become, by (4) and (5)

$$f(\eta) = \frac{\sin 2\pi\eta}{1 \cdot 1624 - \cos 2\pi\eta},$$

$$F(\eta) = \frac{0 \cdot 5926}{1 \cdot 1624 - \cos 2\pi\eta},$$

and graphs of these functions are given in Fig. 5.

By means of (37) and (38) the variation of B_1 and B_2 with η is next found, and then (41) and (42) determine the circulations K_1 and K_2. Curves of these quantities are also given in Fig. 5. It will be seen that the variations from $K_0 = 100$ are of the order of $\pm$ 20 per cent. Mean values of K_1 and K_2 have been found by Simpson's rule, with ordinates spaced 1/48 apart where the variation is rapid. They are

$$\int_0^1 K_1 d\eta = 100 \cdot 45 \text{ ft.}^2/\text{sec.},$$

$$\int_0^1 K_2 d\eta = 100 \cdot 33 \text{ ft.}^2/\text{sec.},$$

and it will be seen that the difference from K_0 is trifling. It may be remarked here that the accuracy of this application of Simpson's rule may be estimated by using it to determine a mean value of $F(\eta)$, ordinates of which are known at the same abscissae as for K_1 and K_2. The mean value should be unity; Simpson's rule gives in the present instance

$$\int_0^1 F(\eta)\, d\eta = 1 \cdot 00055.$$

The formulae (82) and (83) also lead to the conclusion that the differences between K_0 and the mean values of K_1 and K_2 are trifling, although the value of ε given by (69) is too high for any accuracy to be attached to the results. Equations (82) and (83) give

$$\frac{1}{K_0}\int_0^1 K_1 d\eta = 1 \cdot 0028,$$

$$\frac{1}{K_0}\int_0^1 K_2 d\eta = 1 \cdot 0022.$$

We may conclude that, so far as blade settings and overall performance of a contra-rotating pair are concerned, it is quite accurate enough to assume that the screws can each be replaced by a sheet of vorticity as suggested in §9.

From the values of K_1 and K_2 at each given value of η, values of u_1, v_1, u_2 and v_2 were calculated, and these were used to determine the curves of variation with η of the thrust grading at $r = 4$ given in Fig. 6. These also vary by about $\pm$ 20 per cent. from their means; the means, found by Simpson's rule, are given by

$$\frac{1}{\rho N}\int_0^1 \frac{dT_1}{dr}\, d\eta = 53 \cdot 64 \times 10^3$$

$$\frac{1}{\rho N}\int_0^1 \frac{dT_2}{dr}\, d\eta = 54 \cdot 77 \times 10^3$$

and these also are only slightly different from the values $53 \cdot 40 \times 10^3$ and $54 \cdot 60 \times 10^3$ obtained on the " vortex sheet " assumption. It will be observed that the thrust on the back screw (assuming it to be proportional to the thrust grading at $r = 4$) is only just over 2 per cent. in excess of that on the front. The difference would, however, increase at lower rates of advance, that is, for the climb and static conditions.

The angular oscillation in the slipstream at a large distance behind the airscrew pair is, by equation (20) determined by the angle

$$\Delta = \tan^{-1}\left(\frac{K_2 - K_1}{sU}\right),$$

for the particular radius considered. In the present example Δ varies from $+ 0 \cdot 60$ deg. to $- 0 \cdot 18$ deg. For comparison, an equivalent single six-bladed screw working under the same conditions would give the constant value $\Delta = 3 \cdot 8$ deg. It will be noticed that for lower rates of advance and higher lift coefficients, the values of Δ given above would be correspondingly increased.

Finally, curves of ϕ_1 and ϕ_2, obtained from the components of the local velocity, are plotted in Fig. 6. It will be seen that the range of variation, which is the range of variation of incidence, is of the order of $\pm$ 1 deg.

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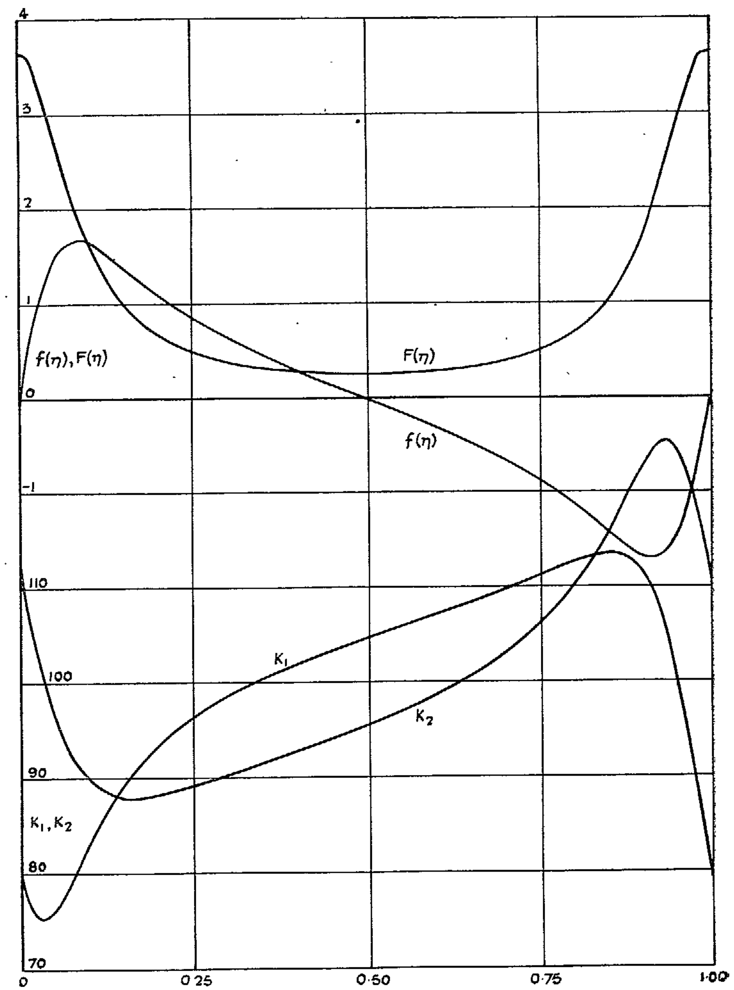

FIG. 5

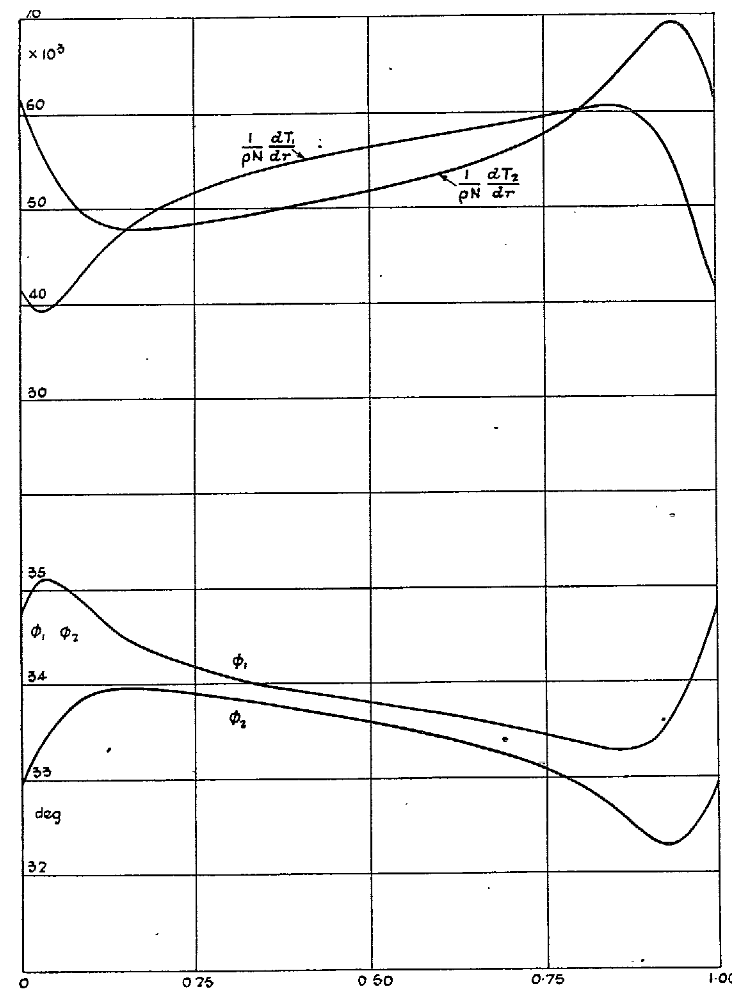

FIG. 6

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