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On the Stressing of Polygonal Tubes with Particular Reference to the Torsion of Tapered Tubes of Trapezoidal Section

By
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AERODYNAMIC SYMBOLS

1. GENERAL

m	Mass
t	Time
V	Resultant linear velocity
Ω	Resultant angular velocity
ρ	Density, σ relative density
ν	Kinematic coefficient of viscosity
R	Reynolds number, $R = lV/\nu$ (where l is a suitable linear dimension)

Normal temperature and pressure for aeronautical work are 15° C and 760 mm.

For air under these conditions $\begin{cases} \rho = 0\cdot002378 \text{ slug/cu. ft.} \\ \nu = 1\cdot56 \times 10^{-4} \text{ ft.}^2\text{/sec.} \end{cases}$

The slug is taken to be 32·2 lb.-mass.

α	Angle of incidence
ϵ	Angle of downwash
S	Area
b	Span
c	Chord
A	Aspect ratio, $A = b^2/S$
L	Lift, with coefficient $C_L = L/\frac{1}{2}\rho V^2 S$
D	Drag, with coefficient $C_D = D/\frac{1}{2}\rho V^2 S$
γ	Gliding angle, $\tan \gamma = D/L$
L	Rolling moment, with coefficient $C_l = L/\frac{1}{2}\rho V^2 bS$
M	Pitching moment, with coefficient $C_m = M/\frac{1}{2}\rho V^2 cS$
N	Yawing moment, with coefficient $C_n = N/\frac{1}{2}\rho V^2 bS$

2. AIRSCREWS

n	Revolutions per second
D	Diameter
J	V/nD
P	Power
T	Thrust, with coefficient $k_T = T/\rho n^2 D^4$
Q	Torque, with coefficient $k_Q = Q/\rho n^2 D^5$
η	Efficiency, $\eta = TV/P = Jk_T/2\pi k_Q$

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CONTENTS

Note.—Part II is complete in itself and the notation in this part differs slightly from the common notation of Parts III, IV and V.

Introduction and Summary.—A general method for stressing polygonal tubes is described and applied to the torsion of parallel and tapered tubes of rectangular and trapezoidal section. It is assumed that the shape of the tube is maintained by a limited number of frames. In treating parallel tubes deformation of these frames in their own planes is taken into account; the effect of this deformation is shown to be small, and in treating tapered tubes the frames are assumed to be rigid in their own planes. The method of stressing tapered tubes in torsion is applicable to any tube of trapezoidal section with one plane of symmetry, no matter how the dimensions may vary along the length of the tube; in particular the method is directly applicable to tubes having portions of their walls cut away. The successive stages in the computation are set out in tabular form and illustrated by worked examples, including cases with " cut-outs ".

The final stage in the computation involves the solution of a set of simultaneous equations equal in number to the number of frames, but these equations are of a special type, readily soluble by a straightforward process without danger of any serious loss of accuracy. The length of the computation is directly proportional to the number of frames, but it is demonstrated by examples that the stress distribution is affected only slightly by the addition of extra frames, so that in practice it should normally be permissible to ignore all but a few of the frames. In the special case of a conically tapered tube in which the wall thicknesses are uniform along the length of the tube, the results can be generalized to include the case of a tube with an infinite number of rigid frames. In this case the results obtained by the present method become identical with those obtained by Williams in R. & M. 1761[1] and by others using Williams's method.

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PART I

Stress Analysis of the General Polygonal Tube

I.1. *Introduction, Terminology and Basic Assumptions.*—This part describes a general method for stressing thin-walled polygonal tubes built up from a number of planar elements, which resist *distortion* in their own planes but which offer no resistance to *warping* out of their planes. A planar element transverse to the axis of the tube (that is, intersecting all the walls of the tube but not necessarily normal to any particular axis of the tube) is termed a *frame*; a planar element bounded between adjacent frames is termed a (*tube*) *wall*. A frame of which the warping is prevented or restricted by external constraint is termed a *bulkhead*. A longitudinal member along the line of intersection of adjacent tube walls between adjacent frames is termed a *boom*; a boom as a separate component is assumed to resist extension or compression, but to offer no resistance to flexure or torsion.

The general method is applicable to all loading cases, including both flexure and torsion; but the applications made in succeeding parts of the paper all concern cases of torsion, to which attention is therefore chiefly devoted. The method is directly applicable to cases in which portions of the tube walls are cut away.

I.2. *Principles of Method.*—(*a*) *Distribution of Stresses in the Tube Walls.*—Since a tube wall offers no resistance to warping, at its edges it can be subjected only to stresses in its own plane. Further, at the intersection of adjacent tube walls along a boom, any direct stresses normal to the edges of the walls would have a resultant, which could be resisted only by flexural stiffness of the boom itself. Since the flexural stiffness of the boom is assumed to be negligible, the direct stresses normal to the edges of the walls at the booms may also be neglected, so that at these edges the walls are subjected only to shear and to direct stress parallel to the edge. Following the ordinary simple theory of bending it is then permissible to assume* that *at any part* of the wall the stress system consists only of shear and direct stress in the radial direction through the apex of the wall (i.e., the point of intersection of the booms bounding the wall).

(*b*) *Separation of Shear and Bending: Equivalent Boom Areas.*—The systems of radial stress in the walls co-operate with the stresses in the booms in resisting bending of the tube as a whole and local bending associated with warping of the cross section. The analysis may therefore be considerably simplified by representing the resistance of the tube walls to direct (radial) loads and bending by appropriate additions to the areas of section of the booms themselves. The validity of this approximation is discussed below. By its adoption the general problem is reduced to that of a tube, in which the booms alone carry all the direct loads whilst the walls joining the booms carry all the shear loads;† at the frames shear loads are applied to the walls, at a bulkhead end loads are applied to the booms.

(*c*) *Shear and Bending Strains.*—The load in each boom due to the shear loads applied to the two adjacent walls can then be written down by inspection and the true boom stresses are found by dividing these boom loads by the equivalent boom areas of section. The bending deflections follow by the ordinary simple theory of bending (*see* Appendix). The effective shear loads in the walls, and the corresponding shear stresses and strains may also be computed by the rules stated in the Appendix.

(*d*) *Conditions for Continuity of the Tube Sections: Warping of the Tube Walls.*—The displacements of each pair of adjacent walls in their own planes must be reconciled at their common boom by warping each wall out of its own plane; by this means the distortion of any frame‡

* This assumption is exactly equivalent to the use of the simple theory of bending, and in this respect the taper of the wall, provided that it be not very great, has no significance. The simple theory of bending for a tapered beam is given in the Appendix, where the conditions which govern its validity are also discussed.

† Or strictly, the shear loads carried as shear stresses, because owing to the taper of the walls the boom loads themselves may have shear and even torsion components.

‡ For brevity "frame" is used here in place of "section of the tube at a frame": in the context the two terms are synonymous.

is completely described in terms of the distortion and warping of an adjacent frame and of the distortion of the tube walls between the two frames. Moreover, the rotations of sections of the walls, in the planes of the walls due to distortion and out of these planes due to warping, both resolved in planes normal to the frames, relate the warping of the two frames. The rotation, distortion and warping of the section of the tube at any frame may then be expressed in terms of the rotation, distortion and warping of an adjacent frame and of the shear and boom loads transmitted by the tube between the frames.

(*e*) *Elimination of the Shear Loads.*—The differences of the boom loads at the two ends of the bay between adjacent frames are linear functions of the shear loads in pairs of adjacent walls, and the torque transmitted is also a linear function of the shear and boom* loads. This system of relations is sufficient to enable all the shear loads to be expressed in terms of torque and boom loads, so that only the latter need be retained as dependent variables.

(*f*) *Conditions for Compatibility of Distortion at the Frames.*—The difference between the shear loads transmitted by a wall immediately on either side of a frame is the load transmitted between frame and wall. At any frame the difference between this system of " shear load differences " and the system of shears externally applied at that frame represents a distribution of internal load in the frame itself ; this " frame load system " must of course be self-equilibrated† and the distortion of the frame under this system of load must be consistent with the distortion of the section of the tube at that frame. In any N-sided tube each frame has N–3 degrees of freedom, and there are therefore N–3 conditions for compatibility of distortion of frame and tube.

(*g*) *Determination of the Boom Loads.*—Each frame introduces N dependent variables (the boom loads at the frame) but the complete system must be in equilibrium with the system of applied loads. The conditions for equilibrium with the applied torque and shear loads in the plane of the frame are already fulfilled, so that there are three further conditions establishing equilibrium with the resultant direct load normal to the frame and with two bending moments in planes normal to the frame. Thus N–3 dependent variables remain and the values of these are determined by the N–3 conditions for compatibility of frame distortions.

At whichever frame of the tube is regarded as the reference frame the total number of unknowns is 2N–3, namely N displacements of the walls in their own planes and N movements of the booms in the directions of their own lengths less 3 average values which represent merely rigid body displacements ; these unknowns are determined by N–3 " frame conditions " as previously and N " bulkhead conditions ", relating the movement of the end of each boom to the load in the boom.

I.3. *Effect of Cut-Outs.*—A cut-out in a wall of the tube may be regarded as a portion over which the thickness of the shear web tends to zero. The shear load over this portion must also be zero, but the " shear stress ", being the ratio of the shear load to the area of section of the web, may tend to a finite limit. Accordingly the shear strain over the cut-out may also be finite and it represents a new unknown which replaces the shear load over the cut-out.

Variation of the area of cross section of the booms past cut-outs introduces no special problem ; but if any boom be cut right away, the case may be treated in the same way. The bending moment and the moment of resistance, in both walls adjacent to the missing boom, both tend to zero, whilst the " curvatures " may remain finite ; the unknown " strain of the missing boom " replaces the boom load at one end of the cut-away, which is now of course zero. The boom load at the other end of the cut-away is also zero because the shears in the adjacent walls are zero.

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* The boom loads contribute directly to the torque only in tubes tapered unequally in different longitudinal sections : in a conically tapered tube all the boom loads pass through the apex of the tube and reduce to a single resultant force.

† The system of load applied at the frame has of course three resultant components, two shears and a torque.

I.4. *Validity of the Approximate Allowance for the Resistance of the Tube Walls to Direct Loads.*—By representing the resistance of the tube walls to direct loads by the addition of equivalent areas of section to the booms, the detail of the distribution of shear stress in the wall is misrepresented; but this fault may readily be corrected in the final results and is therefore no disadvantage. A more serious objection is that the area of section to be added to the boom may have any value between one-sixth and one-half the area of section of the wall, the lower value being appropriate if the wall is simply bent and the upper if it is simply extended or compressed. A close estimate of the appropriate ratio may always be made by reference to the nature of the loading condition; for instance, in cases of torsion, in which the tube walls are usually mainly bent, the value one-sixth is nearly always appropriate, whereas in cases of flexure of a rectangular section the appropriate value is one-half* for the flanges and one-sixth for the webs. In any case the error of the first estimate may always be judged by comparison with the final solution, and, when the areas of section of the actual booms are fairly big, the effect of errors in the estimates of the equivalent boom areas to represent the walls may be quite trifling.

I.5. *Applicability of General Method.*—The ease with which the general method may be applied depends mainly upon the number of sides in the tube and the disposition and stiffness of the frames; slight taper, in the sense that all the booms should be inclined at angles not greater than about 5°–10° to a common axis,† complicates the coefficients but does not otherwise affect the analysis; variation of the section along the length of the tube, including the extreme case of cut-outs, is also quite unimportant.

The examples treated in the succeeding parts of the paper are all cases of torsion of four-sided tubes, for which, corresponding to the single degree of freedom at each frame, there is only one unknown boom load at each frame. In Part II the method is applied to a symmetrical parallel tube of rectangular section with deformable frames; in Part III it is applied to a symmetrical tapered tube of rectangular section with rigid frames and Part IV extends the results of Part III to the case of a trapezoidal section with one plane of symmetry. The special case of a conically tapered tube having walls of uniform thickness along the span and with an infinitely close spacing of rigid frames is treated in Part V.

When the frames are deformable, each of the conditions for compatibility of frame distortions involves either three, four or five of the unknown boom loads. Solution of the final set of simultaneous equations for a large number of frames may therefore be a little laborious. When the frames are rigid each of the final equations involves at most three‡ of the boom loads and at the ends of the tube only two. In solving the final set of simultaneous equations therefore, each equation is used in turn to eliminate one boom load, and the complete solution is obtained by a series of steps equal in number to the number of the equations. The examples worked in Parts II and III illustrate this difference; in practice in cases of rigid frames the solution of the simultaneous equations should take only a few minutes.

The application of the general method to problems of flexure is worth while only in cases in which cut-outs or other concentrations of load render simpler treatment impossible. Unfortunately from this field the four-sided tube must be excluded, because with only four booms the method cannot represent "shear lag". The method should be useful in relation to tubes with six or more sides, but the complexity of the analysis is greatly increased by the additional unknown boom loads. This complexity is less serious if the frames may be regarded as rigid; but in practical examples, representative presumably of fuselages, it would be unwise to assume *ab initio* that the distortions of the frames could be disregarded.

I.6. *Illustrative Example of Stressing Procedure.*—An interesting example for the application of the method described is afforded by the case of a box of rectangular section with rigid end frames, each side being a symmetrical trapezium but adjacent sides being tapered in opposite directions.

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* Unless the flange buckles or is subject to marked shear lag effect, when values less than one-half would be appropriate.

† There is another sense in which the taper need not be slight: this is explained and illustrated by the example in § I.6 below.

‡ Except at a cut-out where two adjacent equations involve respectively two and four of the boom loads.

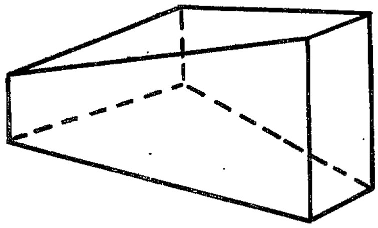

FIG. 1.

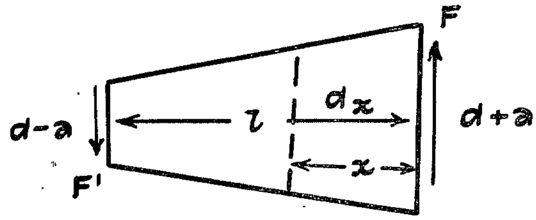

FIG. 2.

A sketch of the box is shown in Fig. 1, and the dimensions of each wall and the shear loads applied at the end frames are shown in Fig. 2. Taking moments about the point of intersection of the longitudinal edges of a wall (*see* Appendix), we have $\frac{F}{d-a} = \frac{F'}{d+a} = \lambda$, and by symmetry, the torque $T = F(d-a) + F'(d+a) = 2\lambda(d^2 + a^2)$.

The direct load P in each corner is $\frac{Fx}{d_x} - \frac{F'x}{2d - d_x}$ and

$d_x = d + a\left(1 - \frac{2x}{l}\right)$. Substituting for F, F′ and d_x.

$$P = \lambda l \left\{ \frac{(d-a)\frac{x}{l}}{d + a\left(1 - \frac{2x}{l}\right)} - \frac{(d+a)\frac{x}{l}}{d - a\left(1 - \frac{2x}{l}\right)} \right\}$$

or

$$\frac{P}{\lambda l} = \frac{-4da\frac{x}{l}\left(1 - \frac{x}{l}\right)}{d^2 - a^2\left(1 - \frac{2x}{l}\right)^2} = \frac{-4\frac{a}{d}u}{1 - \left(\frac{a}{d}\right)^2(1 - 4u)}$$

where $u = \frac{x}{l}\left(1 - \frac{x}{l}\right)$. The direct load is zero at both ends, and a maximum at $x = \frac{1}{2}l$ where $P = -\lambda l\frac{a}{d}$.

If the tube has no booms, the effective area carrying the load P is $dt/3$ where t is the wall thickness. The direct stress f is then

$\frac{3\lambda l}{dt} \cdot \frac{4\frac{a}{d}u}{1 - \left(\frac{a}{d}\right)^2(1 - 4u)}$ · The maximum shear stress q_m is about $\frac{\lambda(d+a)^*}{(d-a)t}$, so that the ratio of the maximum direct stress f_m to q_m is $\frac{f_m}{q_m} = \frac{3l}{d}\frac{\frac{a}{d}\left(1 - \frac{a}{d}\right)}{1 + \frac{a}{d}}$. This is a maximum when $a/d = (d-a)/(d+a) = \sqrt{2} - 1$, and is then $f_m/q_m = 3(3 - 2\sqrt{2})\frac{l}{d} = 0{\cdot}515\frac{l}{d}$. In the tube having walls of the shape shown in Fig. 3, the direct stress at A is about three times the shear stress at B.

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* No allowance is made for the variation of shear stress down the web, because at this section the wider walls act as heavy flanges. (*See* Appendix.)

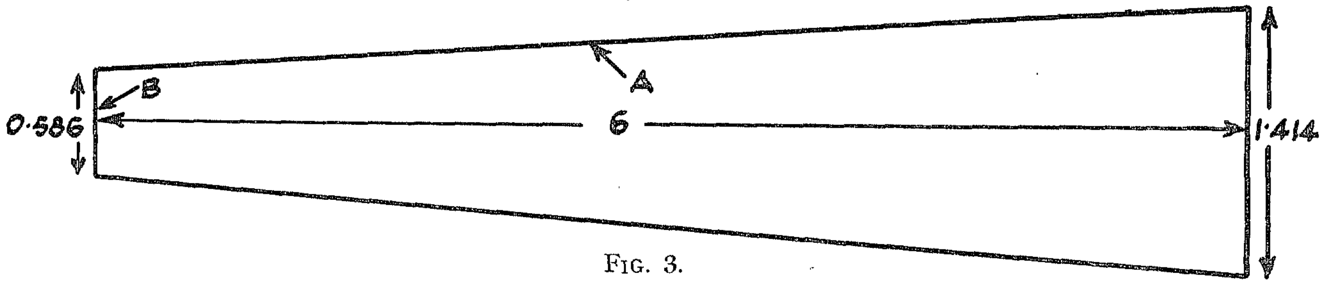

FIG. 3.

This example illustrates the two senses in which taper can be slight or marked. In the tube shown in Fig. 3, the taper, in the sense of inclination of booms to the axis of the tube, is certainly slight; but the effect of the taper in setting up direct stress is marked and actually grows more marked as the length of the tube is increased, although the inclination of the booms is thereby still further reduced. In the sense that $(d - a)/(d + a)$ differs considerably from unity, the taper is marked, and it is in this sense that the degree of taper should be judged. The addition of a rigid frame at the centre (square) section of the tube reduces the boom load to one-quarter of its value when no frame is fitted.

PART II

The Torsion of Symmetrical Parallel Tubes of Rectangular Cross Section with Deformable Frames

II.1. *Introduction.*—The effect on the torsion of rectangular tubes of axial constraint due to bulkheads has been treated by Williams in R. & M. 1619[2], R. & M. 1761[1] and succeeding papers on the assumption that all sections of the tube remain rectangular. Here the same problem is treated on the assumption that the shape of the tube is maintained only by a limited number of frames, not necessarily rigid. As would be expected, the latter method leads to lower estimates of the torsional rigidity; but the difference from the estimate by the method of R. & M. 1761 is rapidly reduced as the number of frames is increased, and in the limit when the number of frames is very large the present method becomes identical with that of R. & M. 1761.

The method is closely related to that developed by Ebner and described in N.A.C.A. Tech. Memo. No. 744; but it is here presented in a tabular form designed to simplify computation. By the use of a table of coefficients (Table 1) the set of simultaneous equations relating the boom loads along the span to the torques transmitted can be written down from the dimensions of the tube, no matter how the dimensions vary along the span. By the solution of these equations the boom loads are computed and the shear stresses and twist follow from simple relations. Complete solution of any specific numerical example should seldom take longer than one hour.

II.2. *Effect of Frame Load Systems in Causing Bending of the Tube Walls and Distortion of the Cross Sections.*—Any pair of couples applied as pairs of shear loads to the sides and to the top

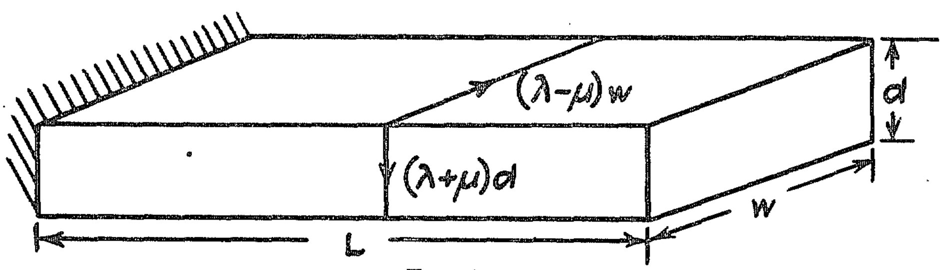

FIG. 4.

and bottom walls of the tube at a cross section perpendicular to the axis of the tube may be regarded as composed of a Batho torsion system and a frame load system. The Batho torsion system is represented by a coefficient μ, the load per unit length of edge ; this system, of course, causes only shear stresses and shear distortion. The frame load system is represented by a coefficient λ*, which again is the load per unit length of edge ; but in this system the two couples are equal and opposite and represent the self-equilibrated reactions from a frame or series of frames in pure shear. The frame load system causes both shear and direct stresses.

The resistance to direct load parallel to the axis of the tube being concentrated in booms at the corners, §I.4, it is clear that the two components of the frame load system make equal contributions to the boom load (the contributions of the two components of the Batho system are, of course, equal and opposite). Thus the boom loads P due to the complete λ-system are twice those due to either component separately.

In a tube of length L, width w, and depth d, supported at one end section from a rigid wall (frame bulkhead), let the shear force per unit length of edge in the rth bay due to frame load systems be λ_r and let the boom load at the outer end of this bay be P_r. Then, if the length of the bay is $\varrho_r L$, $P_{r-1} - P_r = 2\varrho_r L\lambda_r$.

If the area of section of the booms in the rth bay is A_r, the curvature of the side walls is $\dfrac{2P}{EA_r d}$ and $P = P_{r-1}\left(1 - \dfrac{x}{\varrho_r L}\right) + P_r \dfrac{x}{\varrho_r L}$, where x is measured outwards along the span from the r–1th frame. This curvature is $\dfrac{d^2y}{dx^2}$ where y is the downward deflection of the side wall ; but θ, the rotation of the side wall about the centre line of the tube, is $\dfrac{2y}{w}$. Therefore $\dfrac{d^2\theta}{dx^2} = \dfrac{2}{w}\dfrac{d^2y}{dx^2} = \dfrac{4P}{EA_r dw}$. From symmetry it is clear that the rotation of the top and bottom walls is the same but in the opposite sense. Therefore, if ϕ is that part of the shear distortion of the section of the tube due to bending of the tube walls,

$$\frac{d^2\phi}{dx^2} = \frac{8P}{EA_r dw} = \frac{8}{EA_r dw}\left\{P_{r-1} + (P_r - P_{r-1})\frac{x}{\varrho_r L}\right\}.$$

Integrating over the bay

$$\frac{d\phi}{dx} - \left(\frac{d\phi}{dx}\right)_{r-1} = \frac{8}{EA_r dw}\left\{P_{r-1}x + \tfrac{1}{2}(P_r - P_{r-1})\frac{x^2}{\varrho_r L}\right\},$$

$$\left(\frac{d\phi}{dx}\right)_r - \left(\frac{d\phi}{dx}\right)_{r-1} = \frac{4\varrho_r L}{EA_r dw}(P_r + P_{r-1}),$$

and

$$\frac{\phi_r - \phi_{r-1}}{\varrho_r L} = \frac{4\varrho_r L}{3EA_r dw}(P_r + 2P_{r-1}) + \left(\frac{d\phi}{dx}\right)_{r-1}.$$

Writing $\dfrac{A\varrho_r}{A_r} = \varrho_r'$, where A is some convenient standard value of A_r ;

$$\left(\frac{d\phi}{dx}\right)_{r-1} - \left(\frac{d\phi}{dx}\right)_{r-2} = \frac{4L}{EAdw}\varrho'_{r-1}(P_{r-1} + P_{r-2})$$

$$\left(\frac{d\phi}{dx}\right)_{r-2} - \left(\frac{d\phi}{dx}\right)_{r-3} = \frac{4L}{EAdw}\varrho'_{r-2}(P_{r-2} + P_{r-3})$$

.. ..

$$\left(\frac{d\phi}{dx}\right)_1 = \frac{4L}{EAdw}\varrho_1'(P_1 + P_0).$$

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* The λ–system is, of course, closely related to the Γ–system described by Williams in R. & M. 1619 ; but it has not the property of the Γ–system that its effect is zero at infinity.

Adding

$$\left(\frac{d\phi}{dx}\right)_{r-1} = \frac{4L}{EAdw}\left\{\varrho_1' P_0 + (\varrho_1' + \varrho_2') P_1 + (\varrho_2' + \varrho_3') P_2 + \ldots + \varrho_r'_{-1} P_{r-1}\right\}.$$

Hence, finally

$$\frac{\phi_r - \phi_{r-1}}{\varrho_r L} = \frac{4L}{3EAdw}\Big\{3\varrho_1' P_0 + 3(\varrho_1' + \varrho_2') P_1 + \ldots + 3(\varrho_r'_{-2} + \varrho_r'_{-1}) P_{r-2}$$
$$+ (3\varrho_r'_{-1} + 2\varrho_r') P_{r-1} + \varrho_r' P_r\Big\}.$$

II.3. *Shear Strain of Tube Walls.*—The deflection due to shear of a side wall of the tube over the *rth* bay is $\varrho_r L(\lambda_r + \mu_r)/Gt_r$, where t_r is the effective shear thickness of the side wall; μ_r is the shear force per unit length of edge due to the Batho load system. But $\lambda_r = (P_{r-1} - P_r)/2\varrho_r L$ and $\mu_r = T_r/2wd$, where T_r is the torque transmitted over the *rth* bay. The shear deflection of the side walls over the *rth* bay is therefore

$\{(P_{r-1} - P_r) + T_r\varrho_r L/wd\}/2Gt_r$, and the rotation due to shear is $\{(P_{r-1} - P_r) + T_r\varrho_r L/wd\}/Gwt_r$. Similarly the rotation of the top and bottom walls is $\{P_{r-1} - P_r) - T_r\varrho_r L/wd\}/Gds_r$ in the opposite sense where s_r is the effective shear thickness of the top or bottom wall. Therefore, if ψ is that part of the shear distortion of the cross section due to shear of the tube walls,

$$\frac{\psi_r - \psi_{r-1}}{\varrho_r L} = \frac{1}{GLds}\left\{\left(\frac{P_{r-1} - P_r}{\varrho_r}\right)\frac{s}{s_r}\left(1 + \frac{ds_r}{wt_r}\right) - \frac{T_r L}{wd}\frac{s}{s_r}\left(1 - \frac{ds_r}{wt_r}\right)\right\},$$

where s is any convenient standard value of s_r,

or
$$\frac{L}{Gdw^2 s}\left\{\frac{\alpha_r}{\varrho_r}(P_{r-1} - P_r) - \beta_r \frac{T_r}{L}\right\}.$$

where
$$\alpha_r = \frac{s}{s_r}\left(1 + \frac{ds_r}{wt_r}\right)\frac{w^2}{L^2} \text{ and } \beta_r = \frac{s}{s_r}\left(1 - \frac{ds_r}{wt_r}\right)\frac{w}{d}.$$

II.4. *Shear Distortion at Frames.*—If h_r is the effective shear thickness of the *rth* frame, the shear strain of the *rth* frame is

$$\frac{(\lambda_{r+1} - \lambda_r)}{Gh_r} = \frac{1}{2GLh_r}\left\{\frac{(P_r - P_{r+1})}{\varrho_{r+1}} - \frac{(P_{r-1} - P_r)}{\varrho_r}\right\},*$$

and the difference in shear distortion between the $(r-1)$th and rth frames divided by the distance between them is

$$\frac{1}{2G\varrho_r L^2}\Big[-\frac{P_{r+1}}{h_r\varrho_{r+1}} + P_r\left\{\frac{1}{h_r}\left(\frac{1}{\varrho_{r+1}} + \frac{1}{\varrho_r}\right) + \frac{1}{h_{r-1}\varrho_r}\right\}$$
$$- P_{r-1}\left\{\frac{1}{h_r\varrho_r} + \frac{1}{h_{r-1}}\left(\frac{1}{\varrho_r} + \frac{1}{\varrho_{r-1}}\right)\right\} + \frac{P_{r-2}}{h_{r-1}\varrho_{r-1}}\Big].$$

II.5. *Relations between Boom Loads and Torques in the Bays.*—The sum of the distortions of the tube section due to bending and shear of the tube walls must at each frame be identical with the shear strain of that frame, and the shear strain of the root frame being zero, this

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* Note the close correspondence to the second difference of the P's.

condition applies equally to the differences determined in §§II.2, II.3 and II.4. These identities then lead to the system of equations.

$$\frac{\alpha_r}{\varrho_r}(P_{r-1} - P_r) + \frac{\gamma_r}{\varrho_{r+1}} P_{r+1} - \left\{ \gamma_r\left(\frac{1}{\varrho_{r+1}} + \frac{1}{\varrho_r}\right) + \frac{\gamma_{r-1}}{\varrho_r}\right\} P_r$$

$$+ \left\{\frac{\gamma_r}{\varrho_r} + \gamma_{r-1}\left(\frac{1}{\varrho_r} + \frac{1}{\varrho_{r-1}}\right)\right\} P_{r-1} - \frac{\gamma_{r-1}}{\varrho_{r-1}} P_{r-2}$$

$$+ K\{3\varrho_1'P_0 + 3(\varrho_1' + \varrho_2')P_1 + \dots + 3(\varrho_r'{}_{-2} + \varrho_r'{}_{-1}) P_{r-2}$$

$$+ (3\varrho_r'{}_{-1} + 2\varrho_r')P_{r-1} + \varrho_r'P_r\}$$

$$= \beta_r \frac{T_r}{L}.$$

where $\gamma_r = \frac{s}{2h_r}\left(\frac{w}{L}\right)^3 \frac{d}{w}$ and $K = \frac{4Gws}{3EA}$.

This system of equations* is set out in tabular form in Table 1, where their structure is more easily seen. If any s_r or t_r be zero, the equation in which it occurs reduces to

$$P_{r-1} - P_r = \pm \varrho_r \frac{L^2}{wd} \frac{T_r}{L};$$

the remaining equations are not affected.

II.6. *Twist of the Tube.*—The true twist of the tube over the *rth* bay is

$$\frac{T_r\varrho_r L}{2Gwd} \frac{1}{ds_r}\left(1 + \frac{ds_r}{wt_r}\right) - \frac{P_{r-1} - P_r}{2Gds_r}\left(1 - \frac{ds_r}{wt_r}\right)$$

or

$$\frac{TL}{2Gwsd^2} \frac{s}{s_r}\left\{\frac{\varrho_r T_r}{T}\left(1 + \frac{ds_r}{wt_r}\right) - \frac{(P_{r-1} - P_r)L}{T} \frac{wd}{L^2}\left(1 - \frac{ds_r}{wt_r}\right)\right\},$$

where T is some convenient standard value of T_r. The total twist can be found by summation.

The rotation of the upper and lower surfaces differs from the true twist by half the shear strains of the frames; this difference is usually quite negligible.

If s_r is zero, the twist due to shear of the top or bottom surface over the cut-out can of course assume any value, so that in this bay the top and bottom walls cannot affect the distortion of the sides. Accordingly the twist in this bay, determined by the side walls along, is $\{(P_{r-1} - P_r) + T_r\varrho_r L/wd\}/Gwt_r$ plus half the expression for $\phi_r - \phi_{r-1}$ at the end of §II.2. Since $P_{r-1} - P_r = \varrho_r \frac{L^2}{wd} \frac{T_r}{L}$, the first term reduces to $\frac{2T_r\varrho_r L}{Gw^2dt_r}$, which is the rotation due to shear and the complete expression for the twist becomes

$$\frac{TL}{Gwsd^2} 2\varrho_r \frac{d}{w}\left[\frac{s}{t_r}\frac{T_r}{T} + \frac{K}{4}\frac{L}{T}\{3\varrho_1'P_0 + 3(\varrho_1' + \varrho_2')P_1 + \dots\right.$$

$$\left. + 3(\varrho_r'{}_{-2} + \varrho_r'{}_{-1}) P_{r-2} + (3\varrho_r'{}_{-1} + 2\varrho_r') P_{r-1} + \varrho_r'P_r\}\right].$$

---

* Table 1 would be considerably simplified by differencing in rows; but at the expense of the loss of the easy treatment of cases in which parts of the tube walls are cut away. Moreover although the difference $T_r - T_{r-1}$ is the torque applied at the *rth* frame, this difference results only if $\beta_r = \beta_{r-1}$, and in any case the transmitted torques T_r are needed for computation of twist and shear stress. *See* §II.8 and cf. Part IV.

II.7. *Application of Method.*—The computation is conveniently carried out in tabular form, of which four examples are given. Example 2 includes literally all the working necessary apart from additions or multiplications done mentally or by slide rule.

These examples are intended primarily to demonstrate the method ; but they have in fact been chosen from the series of tubes previously investigated by Allwright using Williams's method of R. & M. 1761, and it is of some interest to compare the results. Referring first to the tube without cut-out, we have the following comparative values of the twist under unit torque :

Allwright's investigation (infinite number of rigid frames)		0·3905
Present Paper	Frames at 0·1, 0·3, 0·75 and 1·0 of the span	0·3989
	Frames at 0·75 and 1·0 of the span	0·4077
	Rigid frames at 0·75 and 1·0 of the span	0·4076
	Frame at 0·75 only of the span	0·4294

For the tube with cut-out 0·2 span long with its centre line at 0·2 of the span (Example 2), Allwright gives 0·896 and the present paper 1·099. With the same cut-out now with centre line at 0·4 (Example 3), Allwright gives 0·924 and the present paper 1·153. For a cut-out 0·1 long with its centre line 0·15 from the root, Allwright gives 0·465 and the present paper 0·567.

In the case of the tube without cut-out the comparison is quite satisfactory ; but in the other cases the differences are rather larger than had been expected. However, all the differences are in the right direction, and the fourth example with eight frames more than halves the difference between the two twists for the 0·2 cut-out at 0·2 span.

II.8. *Examples—*

Table 1.—*Equations for the Determination of the Boom Loads from the Values of the Torques in the Bays.*

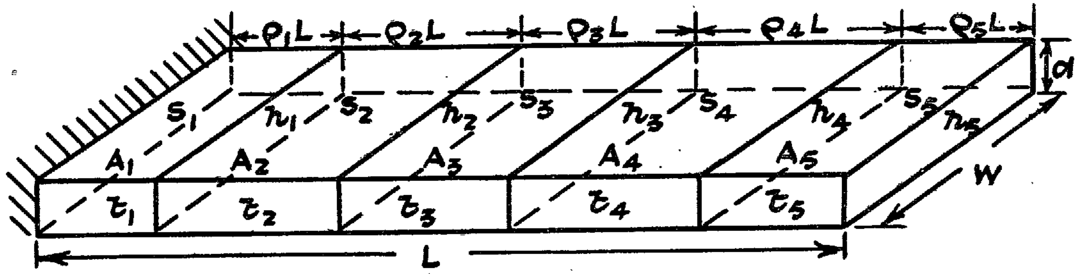

FIG. 5

P_0	P_1	P_2	P_3	P_4	
$2K\varrho_1' + \frac{(\alpha_1+\gamma_1)}{\varrho_1}$	$K\varrho_1' - \frac{\alpha_1+\gamma_1}{\varrho_1} - \frac{\gamma_1}{\varrho_2}$	$\frac{\gamma_1}{\varrho_2}$	—	—	$= \beta_1 \frac{T_1}{L}$
$3K\varrho_1' - \frac{\gamma_1}{\varrho_1}$	$K(3\varrho_1'+2\varrho_2')+\frac{\gamma_1}{\varrho_1}+\frac{\alpha_2+\gamma_1+\gamma_2}{\varrho_2}$	$K\varrho_2' - \frac{\alpha_2+\gamma_1+\gamma_2}{\varrho_2} - \frac{\gamma_2}{\varrho_3}$	$\frac{\gamma_2}{\varrho_3}$	—	$= \beta_2 \frac{T_2}{L}$
$3K\varrho_1'$	$3K(\varrho_1'+\varrho_2') - \frac{\gamma_2}{\varrho_2}$	$K(3\varrho_2'+2\varrho_3')+\frac{\gamma_2}{\varrho_2}+\frac{\alpha_3+\gamma_2+\gamma_3}{\varrho_3}$	$K\varrho_3' - \frac{\alpha_3+\gamma_2+\gamma_3}{\varrho_3} - \frac{\gamma_3}{\varrho_4}$	$\frac{\gamma_3}{\varrho_4}$	$= \beta_3 \frac{T_3}{L}$
$3K\varrho_1'$	$3K(\varrho_1'+\varrho_2')$	$3K(\varrho_2'+\varrho_3') - \frac{\gamma_3}{\varrho_3}$	$K(3\varrho_3'+2\varrho_4')+\frac{\gamma_3}{\varrho_3}+\frac{\alpha_4+\gamma_3+\gamma_4}{\varrho_4}$	$K\varrho_4' - \frac{\alpha_4+\gamma_3+\gamma_4}{\varrho_4} - \frac{\gamma_4}{\varrho_5}$	$= \beta_4 \frac{T_4}{L}$
$3K\varrho_1'$	$3K(\varrho_1'+\varrho_2')$	$3K(\varrho_2'+\varrho_3')$	$3K(\varrho_3'+\varrho_4') - \frac{\gamma_4}{\varrho_4}$	$K(3\varrho_4'+2\varrho_5')+\frac{\gamma_4}{\varrho_4}+\frac{\alpha_5+\gamma_4+\gamma_5}{\varrho_5}$	$= \beta_5 \frac{T_5}{L}$

Notes.—$\varrho_r' = \varrho_r \frac{A}{A_r}$, $K = \frac{4Gws}{3EA}$, $\alpha_r = \frac{s}{s_r}\left(1 + \frac{ds_r}{wt_r}\right)\frac{w^2}{L^2}$, $\beta_r = \frac{s}{s_r}\left(1 - \frac{ds_r}{wt_r}\right)\frac{w}{d}$, $\gamma_r = \frac{s}{2h_r}\left(\frac{w}{L}\right)^3 \frac{d}{w}$.

A and s are any convenient means of the values of A_r and s_r respectively. T_r is the torque transmitted over the rth bay, and P_{r-1} is the boom load at the inboard end of that bay.

The shear stresses in the rth bay are $\frac{T_r}{2wdt_r}\left\{1 + \frac{(P_{r-1}-P_r)L}{T_r}\,\frac{wd}{\varrho_r L^2}\right\}$ in the sides and $\frac{T_r}{2wds_r}\left\{1 - \frac{(P_{r-1}-P_r)L}{T_r}\,\frac{wd}{\varrho_r L^2}\right\}$ in top and bottom.

The twist over the rth bay is $\frac{T_r L}{2Gwsd^2}\,\frac{s}{s_r}\left\{\varrho_r\left(1 + \frac{ds_r}{wt_r}\right) - \frac{(P_{r-1}-P_r)L}{T_r}\,\frac{wd}{L^2}\left(1 - \frac{ds_r}{wt_r}\right)\right\}$

or $\frac{T_r L}{Gwsd^2}\, 2\varrho_r\, \frac{d}{w}\left[\frac{s}{t_r} + \frac{K}{4}\,\frac{L}{T_r}\left\{3\varrho_1' P_0 + 3(\varrho_1'+\varrho_2')P_1 + \ldots + 3(\varrho'_{r-2} + \varrho'_{r-1})P_{r-2} + (3\varrho'_{r-1} + 2\varrho_r')P_{r-1} + \varrho_r' P_r\right\}\right]$ if $s_r = 0$.

Example 1.—Plain Tube without Cut-out—No Added Booms.

$w/L = 0\cdot2$, $d/w = 0\cdot25$, $A = 1/6\,(ws + td) = ws/6\,(1 + td/ws) = ws/4$ in all bays.
Torque T applied at frame 3.

	Bay 1	Bay 2	Bay 3	Bay 4	Average Value	Notes
$\varrho_r' = \varrho_r$	0·1	0·2	0·45	0·25		Specified
A_r/ws	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	K = 2 (G/E = 3/8)
s_r/s	1	1	1	1	1	Specified
t_r/s	2	2	2	2		Specified
h_r/s	1	1	1	1		Specified
α_r	0·045	0·045	0·045	0·045		
β_r	3·5	3·5	3·5	3·5		
γ_r	0·001	0·001	0·001	0·001		
T_r/T	1	1	1	0		Specified

Table 1	P_0	P_1	P_2	P_3	T/L	
1st eqn.	0·860	−0·265	0·005	—	3·5 (3·500)	Check back shows accuracy maintained
2nd eqn.	0·590	1·645	0·163	0·002	3·5 (3·500)	
3rd eqn.	0·600	1·795	3·109	0·792	3·5 (3·500)	
4th eqn.	0·600	1·800	3·898	3·890	0 (0·000)	
1st $\times \dfrac{0\cdot6}{0\cdot86}$	0·600	−0·1849	0·0035	—	2·442	
2nd $\times \dfrac{0\cdot60}{0\cdot59}$	0·600	1·6729	0·1658	0·0022	3·559	
		0·005 1	0·789 157·8	3·098 619·6	−3·5 −700·0	
		1·9849 1	3·8945 1·9621	3·890 1·9598	−2·442 −1·2303	
		1·8578 1	0·1623 0·0874	0·0022 0·0012	1·117 0·6013	
			155·8 1	617·6 3·9640	−698·7 −4·4846	
			1·8747 1	1·9586 1·0448	−1·8316 −0·9770	
				2·9192 1	−3·5076 −1·2016	
Boom Loads	4·2465	0·5784	0·2784	−1·2016	× T/L	
$P_{r-1}-P_r$	3·6681	0·3000	1·4800	−1·2016	× T/L	
Shear stresses — Side	0·6834	0·5075	0·5164	−0·0240	× T/2*wds*	
Shear stresses — Top	0·6332	0·9850	0·9671	0·0481	× T/2*wds*	Twist at load
Twists	0·0402	0·1112	0·2466	0·0053	× $TL/Gwds^2$	0·3980

Example 2. Plain Tube with Cut-out—No Added Booms.

(As Example 1, but with top surface of Bay 2 removed.)
$w/\mathrm{L} = 0{\cdot}2$, $d/w = 0{\cdot}25$, $\mathrm{A} = ws/4$. Torque T applied at Frame 3.

	Bay 1	Bay 2	Bay 3	Bay 4	Average Value	Notes
ϱ_r	0·1	0·2	0·45	0·25		Specified
A_r/ws	$\frac{1}{4}$	1/12	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	K = 2 (G/E = 3/8)
ϱ_r'	0·1	0·6	0·45	0·25		
s_r/s	1	0	1	1	1	Specified with cut-out
t_r/s	2	2	2	2		Specified
h_r/s	1	1	1	1		Specified
α_r	0·045	—	0·045	0·045		
β_r	3·5	—	3·5	3·5		
γ_r	0·001	0·001	0·001	0·001		
T_r/T	1	1	1	0		Specified
Table 1	P_0	P_1	P_2	P_3	T/L	
1st eqn.	0·860	−0·265	0·005	—	3·5 (3·513)	Check back values in brackets show torque represented to $\frac{1}{2}$%.
2nd eqn.	—	1	−1	—	20 (20)	
3rd eqn.	0·600	4·195	5·510	0·792	3·5 (3·506)	
4th eqn.	0·600	4·200	6·298	3·890	0 (0·008)	
1st $\times \frac{0{\cdot}6}{0{\cdot}86}$	0·600	−0·1849	0·0035		2·442	
		0·005	0·788	3·098	−3·5	Diff. 4th—3rd.
		4·3849	6·2945	3·890	−2·442	Diff. 4th—1st $\times \frac{0{\cdot}6}{0{\cdot}86}$
			0·793	3·098	−3·6	Eliminating P_2 from 2nd
			10·6794	3·890	−90·140	
			0·9960	3·890	−4·520	
			9·6834		−85·62	
Boom Loads	7·574	11·158	−8·842	1·102	× T/L	
$\mathrm{P}_{r-1}-\mathrm{P}_r$	−3·584	20	−9·944	1·102	× T/L	
Shear stresses Side	0·321	1·000	0·389	0·022	× $\mathrm{T}/2wds$	
Shear stresses Top	1·358	0	1·221	−0·044	× $\mathrm{T}/2wds$	Twist at load
Twists	0·0674	0·7352	0·2966	−0·0048	× $\mathrm{TL}/\mathrm{G}wsd^2$	1·099

Example 3.—Plain Tube with Cut-out—No Added Booms.

$w/L = 0{\cdot}2$, $d/w = 0{\cdot}25$, $A = ws/4$. Torque T applied at Frame 3.

	Bay 1	Bay 2	Bay 3	Bay 4	Average Value	Notes
ϱ_r	0·3	0·2	0·25	0·25		
A_r/ws	$\frac{1}{4}$	1/12	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	K = 2 (G/E = 3/8)
s_r/s	1	0	1	1	1	
t_r/s	2	2	2	2		
h_r/s	1	1	1	1		
ϱ_r'	0·3	0·6	0·25	0·25		
α_r	0·045	—	0·045	0·045		
β_r	3·5	—	3·5	3·5		
γ_r	0·001	0·001	0·001	0·001		
T_r/T	1	1	1	0		
Table 1	P_0	P_1	P_2	P_3	T/L	
1st Bay	1·3533	0·4417	0·005	—	3·5 (3·500)	
2nd Bay	—	1	−1	—	20 (20)	
3rd Bay	1·800	5·395	4·793	0·308	3·5 (3·499)	
4th Bay	1·800	5·400	5·096	2·692	0 (−0·001)	
1st $\times \dfrac{1{\cdot}800}{1{\cdot}353}$	1·800	0·5875	0·0066	—	4·6563	
		0·005	0·303	2·384	−3·5	
		4·8125	5·0894	2·692	−4·6563	
			0·308 1	2·384 7·7403	−3·6 −11·6883	
			9·9019 1	2·692 0·2719	−100·906 −10·1906	
				7·4684	−1·4977	
Boom Loads	−0·596	+9·864	−10·136	−0·201	× T/L	
$P_{r-1} - P_r$	−10·460	20	−9·935	−0·201	× T/L	
Shear stresses: Side	0·326	1	0·301	−0·004	× $T/2wds$	
Shear stresses: Top	1·349	0	1·397	0·008	× $T/2wds$	Twist at Load
Twist	0·2145	0·7548	0·1841	0·0009	× $TL/Gwsd^2$	1·153

Example 4.—Plain Tube with Cut-out—No Added Booms.

(As Example 2, but with more and rigid frames.)

$w/\mathrm{L} = 0\cdot2$, $d/w = 0\cdot25$, $\mathrm{A} = ws/4$. Torque T applied at Frame 6.

	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Bay 6	Bay 7	Bay 8	
ϱ_r	0·05	0·05	0·20	0·15	0·15	0·15	0·15	0·10	
A_r/ws	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	K = 2
s_r/s	1	1	0	1	1	1	1	1	
t_r/s	2	2	2	2	2	2	2	2	
h_r/s	∞	∞	∞	∞	∞	∞	∞	∞	
ϱ_r'	0·05	0·05	0·60	0·15	0·15	0·15	0·15	0·10	
α_r	0·045	0·045	—	0·045	0·045	0·045	0·045	0·045	
β_r	3·5	3·5	—	3·5	3·5	3·5	3·5	3·5	
γ_r	0	0	0	0	0	0	0	0	
T_r/T	1	1	1	1	1	1	0	0	
Table 1	1·100	−0·800	—	—	—	—	—	—	3·5
	0·300	1·400	−0·800	—	—	—	—	—	3·5
	—	—	1	−1	—	—	—	—	20
	0·300	0·600	3·900	4·500	0	—	—	—	3·5
	0·300	0·600	3·900	4·500	1·800	0	—	—	3·5
	0·300	0·600	3·900	4·500	1·800	1·800	0	—	3·5
	0·300	0·600	3·900	4·500	1·800	1·800	1·800	0	0
	0·300	0·600	3·900	4·500	1·800	1·800	1·800	1·75	0
		1·78	−0·88						2·80
		−0·80	4·70	4·50					0
					1·800				0
						1·800			0
							1·800		−3·5
								1·75	0
P_{r-1}	8·1104	6·7769	10·3719	−9·6281	0	0	−1·9444	0	× T/L
$\mathrm{P}_{r-1}-\mathrm{P}_r$	1·3335	−3·5950	20	−9·6281	0	1·9444	−1·9444	0	× T/L
Twist	0·0223	0·0439	0·6237	0·1265	0·0844	0·0759	0·0085	0	× TL/Gwsd^2

Twist at Load = 0·977 TL/Gwsd^2

Shear load in	Frame 1	Frame 2	Frame 3	Frame 4	Frame 5	Frame 6	Frame 7	Frame 8	
per unit length of edge	−98·6	171·9	−164·2	64·2	13·0	−25·9	13·0	0	× T/2L

II.9. *Uniform Tube with Infinite Number of Rigid Frames.*—It is useful to demonstrate the relation of the treatment in this part to the treatment by Williams (R. & M. 1761), by considering the case of a uniform tube with an infinite number of rigid frames. In this case γ_r is zero, $\varrho_r' = \varrho_r = \varrho$, $\alpha_r = \alpha = \left(1 + \frac{ds}{wt}\right)\frac{w^2}{L^2}$ and $\beta_r = \beta = \left(1 - \frac{ds}{wt}\right)\frac{w}{d}$. The first differences of Table 1 (first row, second row minus the first, third minus the second, etc.) then reduce to

$$\left(2K\varrho + \frac{\alpha}{\varrho}\right) P_0 + \left(K\varrho - \frac{\alpha}{\varrho}\right) P_1 = \beta \frac{T_1}{L} \quad \ldots \quad (1)$$

$$\left(K\varrho - \frac{\alpha}{\varrho}\right) P_{r-2} + \left(4K\varrho + \frac{2\alpha}{\varrho}\right) P_{r-1} + \left(K\varrho - \frac{\alpha}{\varrho}\right) P_r = \beta \frac{T_r - T_{r-1}}{L}, \quad (2)$$

and

$$\left(K\varrho - \frac{\alpha}{\varrho}\right) P_{n-1} + \left(4K\varrho + \frac{2\alpha}{\varrho}\right) P_n = \beta \frac{T_n - T_{n-1}}{L} \quad \ldots \quad (3)$$

We may write

$$P_{r\pm 1} = P_r \pm \varrho L \left(\frac{dP}{dx}\right)_r + \frac{1}{2}(\varrho L)^2 \left(\frac{d^2P}{dx^2}\right)_r + \ldots,$$

and drop the suffixes, and

$$T_{r-1} = T_r - \varrho L \left(\frac{dT}{dx}\right)_r,$$

x being measured outwards from the root.

Substituting in (2) we have

$$6K\varrho P - \varrho\alpha L^2 \frac{d^2P}{dx^2} = \varrho\beta \frac{dT}{dx}, \quad \ldots \quad (4)$$

in (1)

$$3K\varrho P - \alpha L \frac{dP}{dx} - \tfrac{1}{2}\varrho\alpha L^2 \frac{d^2P}{dx^2} = \beta \frac{T_1}{L}, \quad \ldots \quad (5)$$

and in (3)

$$\left(5K\varrho + \frac{\alpha}{\varrho}\right) P + \alpha L \frac{dP}{dx} - \tfrac{1}{2}\varrho\alpha L^2 \frac{d^2P}{dx^2} = -\varrho\beta \frac{dT}{dx} \quad \ldots \quad (6)$$

to terms in ϱ^2.

Proceeding to the limit $\varrho \to 0$, we have

$$\frac{d^2P}{dx^2} - \frac{6K}{\alpha L^2} P + \frac{\beta}{\alpha L^2} \frac{dT}{dx} = 0 \quad \ldots \quad (7)$$

over the whole length of the tube with discontinuity $+ \beta T/\alpha L^2$* in dP/dx at any section where a concentrated torque T is applied. In addition, at the free end equation (6) requires $P = 0$.

The solution is of the form $P = H \cosh \mu x + J \sinh \mu x$, with adjustment of the constants to the end conditions, where

$$\mu^2 = \frac{6K}{\alpha L^2} = \frac{8Gst}{EA\,(wt + ds)}.$$

* The torque T_1 in equation (5) is really the reaction torque and should be regarded as a torque $- T_1$ at this section.

Converting to Williams's notation

$$A = \frac{I_1}{2b^2} + \frac{I_2}{2a^2} = \frac{1}{2a^2b^2}\,(a^2I_1 + b^2I_2),\ w = 2a,\ d = 2b,$$

$$t = \frac{1}{2Gbr_1} \text{ and } s = \frac{1}{2Gar_2}.$$

Making these substitutions,

$$\mu^2 = \frac{4a^2b^2/E}{(a^2I_1 + b^2I_2)\,(b^2r_1 + a^2r_2)}\text{, as given by Williams.}$$

This comparison demonstrates that the use of the conception of " effective boom area " is fully consistent with the treatment of R. & M. 1761.

An interesting special case arises out of equations (1) to (3) and is partially illustrated in Example 4. In a uniform tube if ϱ be chosen so that $\varrho^2 = \alpha/K$, boom loads occur only in the two bays on either sides of frames at which torques are applied, and the value of the boom load at the frame at which a torque T is applied is $-\dfrac{\beta T}{6\sqrt{K\alpha}}$, except at the root where the boom load is $\dfrac{\beta\,\Sigma T.}{3\sqrt{K\alpha}}$. The doubling of the boom load at the root is in a sense due to reflection ; the bulkhead effect of the root fixing could be reproduced by reflecting the tube in the root plane and twisting both ends in the same sense against the (double) reaction applied at the centre.

PART III

The Torsion of Tapered Tubes of Rectangular Section with Rigid Frames

III.1. *Introduction.*—In this part the general method of Part I is applied to the torsion of tapered tubes of rectangular section, using the rules for the bending of tapered beams derived in the Appendix. The tube is assumed to be symmetrical about two perpendicular planes through its axis, but the taper and the variation of section spanwise is entirely arbitrary.

III.2. *General Scheme of Present Method.*—The general scheme of the method is to establish systems of equations relating values of the boom loads directly to the values of the applied torques, and by solution of these equations under the conditions determined by the nature of the end fixings directly to compute the values of the boom loads. Associated with the torques (T's) are the twists (θ's), and similarly with the boom loads (P's) are associated the warpings (e's). At this stage it is not necessary specifically to define the conceptions of boom load and warping; it is sufficient to regard e as a numerical measure of the warping of the tube cross section in some prescribed form and P as a numerical measure of the corresponding system of axial loads. It is later assumed that plane sections of the tube walls remain plane but this is not necessary to the general argument,* and any other prescribed form of warping might be used. Similarly, the precise definition of the boom load P must be varied in sympathy with the prescription of the form of warping. The definition of P later adopted is consistent with the assumption that plane sections remain plane and is thus completely in accord with the general method of R. & M. 1761; in comparison with M_1 and M_2 of that paper, P here represents no more than a change of notation.

Considering any section of a tube between two cross sections 0 and 1, and using suffixes to distinguish between the two ends, it must be possible to establish relations in the forms :—

$$\theta_1 - \theta_0 = B_1T_1 + C_1P_0 + D_1P_1 \qquad (8)$$

$$- e_0 = C_1T_1 + H_1P_0 + J_1P_1 \qquad (9)$$

$$e_1 = D_1T_1 + J_1P_0 + K_1P_1 \qquad (10)$$

The warping e being measured in the same sense at each section, e_1 and e_0 are in effect of opposite sign, so that, if $P_0 = 1$ produces $e_1 = J_1$, $P_1 = 1$ must produce $e_0 = -J_1$. The other correspondences (C and D) also follow by the reciprocal theorem, but, of course, it is assumed that boom load and warping are so defined that multiplying constants are avoided. In consequence the forms of equations (8), (9) and (10) have to be modified by the introduction of geometrical constants multiplying $(\theta_1 - \theta_0)$, e_0, e_1 and T_1 (*see* §III.6 below).

If consideration is confined to a section of the tube between two rigid† frames, the values of the coefficients B_1 C_1, etc., may be computed directly; this computation is carried out for a doubly symmetrical but arbitrarily tapered tube of rectangular section in §III.4 below, and the forms of the coefficients are shown in §III.6 where their numerical evaluation is discussed.

By consideration of the next bay (1 to 2) (9) above gives $- e_1 = C_2T_2 + H_2P_1 + J_2P_2$. Eliminating e_1, and thus satisfying the condition of continuity from bay to bay, leads to the relation

$$J_1P_0 + (H_1 + K_2)\,P_1 + J_2P_2 + D_1T_1 + C_2T_2 = 0 \qquad (11)$$

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* Cf. The comparison between R. & M. 1619 and R. & M. 1761; where the conditions, under which the ordinary assumption that plane sections remain plane may lead to serious error, are explored.

† If the frames are not rigid, the forms for $\theta_1 - \theta_0$; e_0 and e_1 will include terms in the frame distortions. These can afterwards be eliminated by correlation with the frame loads, which are functions of the P's and T's; but the process is complicated and should seldom be necessary. The small effect of frame stiffness is demonstrated in Part II and by the example in §III 7.

This relation is one of a system equal in number to the number of bays, the forms of the first and last relations of the system being slightly modified by the end conditions, e.g., $P = 0$ at a free end, or $e = 0$ at a bulkhead. The values of the coefficients B_r, etc., having been computed, this system of simultaneous equations in the P's may at once be written down, and by their solution the values of the P's are determined. In general no difficulty or abnormal loss of accuracy is encountered in solving these equations.

The values of the P's having been determined, the twists follow from the expressions for $\theta_1 - \theta_0$, etc., and the shears in the tube walls and the frame loads may also be computed from simple relations to the P's and T's.

Neither variation of the thickness of the tube walls from bay to bay nor variation of the effective area of the booms causes any difficulty; they merely affect the values of the coefficients B_r, etc. In particular, if any tube wall be completely cut away over any bay, the same general method is still applicable. The shear force over the absent bay must be zero, but the shear strain takes the form 0/0 and is, of course, arbitrary. As a result, in place of the two relations of type (11) above, which include both the P's at either end of the absent wall, two other relations are found, one involving these two P's and these two only, and the other these two P's and the pair next on either side. Again the setting up and solution of the simultaneous equations in the P's is straightforward.

III.3. *Special Assumption and Range of Applicability of Present Treatment.*—The general method so far described could be developed for use for any four-sided tube, no matter how steeply tapered; but, since in practice the rate of taper is seldom likely to be great, it is worth while to make use of this restriction in order to simplify the derivation of the coefficients B_r, etc. The detailed development is therefore based on the assumption that the angles of inclination of the tube walls and tube edges to the tube axis are all so small that their cosines may be taken as unity; but sines of these angles cannot be disregarded. In order to simplify the presentation of the analysis it is here assumed that the tube is doubly symmetrical about its axis. The coefficients for a tube of trapezoidal section with one plane of symmetry are derived in Part IV. Apart from these two assumptions there is no restriction on the type of taper; the taper in adjacent walls need not be the same, nor need the taper be uniform from bay to bay. In cases of non-uniform taper from bay to bay the reactions at the frames necessary to maintain equilibrium of the booms are imposed on the frames; in exceptional cases the frames may as a result be liable to buckle in compression.

§III.4 below describes the derivation of the forms of the constants B_r, etc., and the establishment of the relations of type (11) in §III.2 above. The compatibility of the computed deformations is demonstrated in §III.5, and the forms of the constants B_r, etc., are summarized in §III.6, where the computation of these constants is discussed. The application of the method is illustrated by a worked example in §III.7. In this application the values of certain rather awkward integrals are required; a fairly satisfactory method for the computation of these integrals is described in §III.6. Apart from these integrals (tables of values of which might easily be prepared) §III.7 contains all the information necessary for the application of the method, and all the computation involved is straightforward arithmetic, involving no other reference to tables.

The case of uniform conical taper, which represents a special case of the general problem, is treated in Part V, where the results obtained are compared with results obtained by Williams in R. & M. 1761.

III.4. *Derivation of Coefficients.*—(a) *Deformation in a Bay.*—In this section, by consideration of the deformation of a single bay of a tapered tube between two rigid frames, formulae are obtained for the twist between the two ends of the bay and for the warping of each end section in terms of the boom loads at the two ends and the torque transmitted.

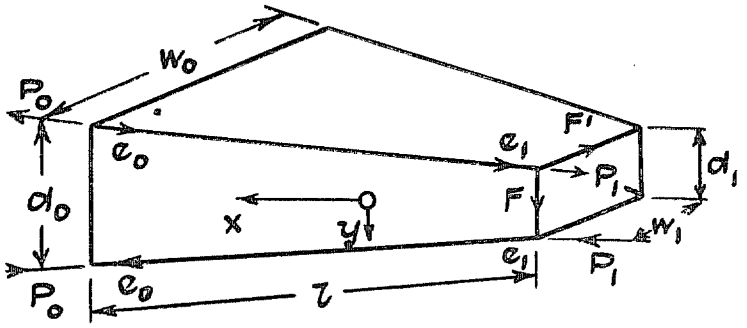

FIG. 6.

Notation as in Fig. 6

$$\frac{d_1}{d_0} = \alpha_1,\ \frac{w_1}{w_0} = \beta_1$$

$$(P_0 - P_1)/l = \frac{F}{d_0} + \frac{F'}{w_0}\,.$$

Torque transmitted

$$T = Fw_1 - F'd_1 + (d_0w_1 - d_1w_0)\frac{P_1}{l}\,.$$

$$\text{Hence } \frac{F}{d_0} = \left(\frac{T}{w_0d_0} + \alpha_1\frac{P_0}{l} - \beta_1\frac{P_1}{l}\right)\Big/(\alpha_1 + \beta_1) \quad \ldots \quad (12a)$$

$$\frac{F'}{w_0} = \left(-\frac{T}{w_0d_0} + \beta_1\frac{P_0}{l} - \alpha_1\frac{P_1}{l}\right)\Big/(\alpha_1 + \beta_1) \quad \ldots \quad (12b)$$

Boom load P at distance x from section 1 $= P_1 + Fx/d + F'x/w$, where $d\ (=\alpha d_0)$ and $w(=\beta w_0)$ are the depth and width at section x. Effective area of section of booms A $(=\gamma A_0)$ may vary over the length of the bay. The length l is presumed to be so large in comparison with the differences $d_0 - d_1$ and $w_0 - w_1$ that squares and higher powers of the ratios $(d_0 - d_1)/l$ and $(w_0 - w_1)/l$ are negligible. As a result distinction between the directions of the tube edges and of the tube axis is unnecessary, in so far as only cosines of the wall angles are involved; on the other hand sines of the wall angles are not negligible (*see* below and §III.5).

Deflection of front wall downwards $\delta - \delta_0$ | at section x, relative to original section
Deflection of top surface backwards $\varepsilon - \varepsilon_0$ | at frame 0, with appropriate suffixes at
Rotation of section $\theta - \theta_0$ | ends of bay.

$$\theta_1 - \theta_0 = 2(\delta_1 - \delta_0)/w_1 =$$

$$\frac{2}{w_1}\left[\frac{2}{E}\left\{P_1\int_l^o\int_l^x\frac{dxdx}{Ad} + F\int_l^o\int_l^x\frac{xdxdx}{Ad^2} + F'\int_l^o\int_l^x\frac{xdxdx}{Adw}\right\} + \frac{2e_0l}{d_0} + \frac{(d_0 + d_1)Fl}{2Gtd_0^{\,2}}\right] \quad \ldots \quad (13a)$$

$$= -2(\varepsilon_1 - \varepsilon_0)/d_1 =$$

$$-\frac{2}{d_1}\left[\frac{2}{E}\left\{P_1\int_l^o\int_l^x\frac{dxdx}{Aw} + F\int_l^o\int_l^x\frac{xdxdx}{Adw} + F'\int_l^o\int_l^x\frac{xdxdx}{Aw^2}\right\} + \frac{2e_0l}{w_0} + \frac{(w_0 + w_1)F'l}{2Gsw_0^{\,2}}\right] \ldots \quad (13b)$$

where t and s are the effective shear thicknesses of the sides and top and bottom respectively.

Eliminating e_0 by multiplying (13a) by w_1d_0/l^2, (13b) by w_0d_1/l^2 and adding

$$(\theta_1 - \theta_0)(\alpha_1 + \beta_1)\frac{w_0d_0}{l^2} = \frac{4l}{EA_0}\left\{\frac{P_1}{l}\int_1^o\int_1^u\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{dudu}{\gamma}\right.$$

$$\left. + \frac{F}{d_0}\int_1^o\int_1^u\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{ududu}{\alpha\gamma} + \frac{F'}{w_0}\int_1^o\int_1^u\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{ududu}{\beta\gamma}\right\}$$

$$+ (1 + \alpha_1)\frac{F}{Gtl} - (1 + \beta_1)\frac{F'}{Gsl}\,,\ \text{where } u = \frac{x}{l}.$$

or

$$(\theta_1 - \theta_0)(\alpha_1 + \beta_1)\frac{EA_0}{l} = \frac{4l^2}{w_0 d_0}\left\{I_1 \frac{P_1}{l} + I_2 \frac{F}{d_0} + I_3 \frac{F'}{w_0}\right\}$$

$$+ (1 + \alpha_1)\frac{EA_0}{Gtw_0}\frac{F}{d_0} - (1 + \beta_1)\frac{EA_0}{Gsd_0}\frac{F'}{w_0} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (14)$$

where

$$I_1 = \int_1^o \int_1^u \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{dudu}{\gamma}, \quad I_2 = \int_1^o \int_1^u \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{ududu}{\alpha\gamma},$$

$$\text{and } I_3 = \int_1^o \int_1^u \left(\frac{1}{\alpha} - \frac{I}{\beta}\right)\frac{ududu}{\beta\gamma}.$$

Substituting for F and F′ from (12a) and (12b)

$$\frac{w_0 d_0}{4l^2}(\theta_1 - \theta_0)(\alpha_1 + \beta_1)^2 EA_0$$

$$= \left[I_2 - I_3 + \frac{w_0 d_0}{4l^2}\left\{(1 + \alpha_1)\frac{EA_0}{Gtw_0} + (1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]\frac{Tl}{w_0 d_0}$$

$$+ \left[\alpha_1 I_2 + \beta_1 I_3 + \frac{w_0 d_0}{4l_2}\left\{\alpha_1(1 + \alpha_1)\frac{EA_0}{Gtw_0} - \beta_1(1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]P_0$$

$$+ \left[(\alpha_1 + \beta_1) I_1 - \beta_1 I_2 - \alpha_1 I_3\right.$$

$$\left. - \frac{w_0 d_0}{4l^2}\left\{\beta_1(1 + \alpha_1)\frac{EA_0}{Gtw_0} - \alpha_1(1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]P_1 \quad \ldots \quad (15)$$

Eliminating $(\theta_1 - \theta_0)$ by subtracting (13b) from (13a)

$$\frac{e_0}{l}(\alpha_1 + \beta_1)\frac{EA_0}{l} = -\left[I_4 \frac{P_1}{l} + I_5 \frac{F}{d_0} + I_6 \frac{F'}{w_0}\right.$$

$$\left. + \frac{w_0 d_0}{4l^2}\left\{\alpha_1(1 + \alpha_1)\frac{EA_0}{Gtw_0}\frac{F}{d_0} + \beta_1(1 + \beta_1)\frac{EA_0}{Gsd_0}\frac{F'}{w_0}\right\}\right] \quad \ldots \quad \ldots \quad (16)$$

where

$$I_4 = \int_1^o \int_1^u \left(\frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta}\right)\frac{dudu}{\gamma}, \quad I_5 = \int_1^o \int_1^u \left(\frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta}\right)\frac{ududu}{\alpha\gamma}$$

$$\text{and } I_6 = \int_1^o \int_1^u \left(\frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta}\right)\frac{ududu}{\beta\gamma}.$$

Substituting for F and F′ from (12a) and (12b)

$$\frac{e_0}{l}(\alpha_1 + \beta_1)^2 EA_0$$

$$= \left[I_5 - I_6 + \frac{w_0 d_0}{4l^2}\left\{\alpha_1(1 + \alpha_1)\frac{EA_0}{Gtw_0} - \beta_1(1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]\frac{Tl}{w_0 d_0}$$

$$+ \left[\alpha_1 I_6 + \beta_1 I_6 + \frac{w_0 d_0}{4l^2}\left\{\alpha_1^2(1 + \alpha_1)\frac{EA_0}{Gtw_0} + \beta_1^2(1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]P_0$$

$$+ \left[(\alpha_1 + \beta_1) I_4 - \beta_1 I_5 - \alpha_1 I_6\right.$$

$$\left. - \frac{w_0 d_0}{4l^2}\alpha_1\beta_1\left\{(1 + \alpha_1)\frac{EA_0}{Gtw_0} + (1 + \beta_1)\frac{EA_0}{Gsd_0}\right\}\right]P_1 \quad \ldots \quad (17,$$

For the warping e_1 at section 1, by considering rotations of sections of the front and top walls,

$$2\left(\frac{e_1}{d_1}-\frac{e_0}{d_0}\right) = -\frac{2}{\mathrm{E}}\left\{\mathrm{P}_1\int_l^0 \frac{dx}{\mathrm{A}d} + \mathrm{F}\int_l^0 \frac{xdx}{\mathrm{A}d^2} + \mathrm{F}'\int_l^0 \frac{xdx}{\mathrm{A}dw}\right\}$$

$$-\frac{(d_0{}^2-d_1{}^2)\mathrm{F}}{2\mathrm{G}td_0{}^2d_1} + (\theta_1-\theta_0)\frac{w_0-w_1}{2l} \quad \ldots \quad (18a)$$

and

$$2\left(\frac{e_1}{w_1}-\frac{e_0}{w_0}\right) = -\frac{2}{\mathrm{E}}\left\{\mathrm{P}_1\int_l^0 \frac{dx}{\mathrm{A}w} + \mathrm{F}\int_l^0 \frac{xdx}{\mathrm{A}dw} + \mathrm{F}'\int_l^0 \frac{xdx}{\mathrm{A}w^2}\right\}$$

$$-\frac{(w_0{}^2-w_1{}^2)\,\mathrm{F}'}{2\mathrm{G}sw_0{}^2w_1} - (\theta_1-\theta_0)\frac{d_0-d_1}{2l} \quad \ldots \quad (18b)$$

The terms in $(\theta_1 - \theta_0)$ arise from the inclination of the walls to the tube axis.

Eliminating e_0* by multiplying (18a) by d_0 and (18b) by w_0 and subtracting

$$\frac{e_1}{l}(\alpha_1-\beta_1)\frac{\mathrm{EA}_0}{l} = \Bigg[\alpha_1\beta_1\left(\mathrm{I}_7\frac{\mathrm{P}_1}{l}+\mathrm{I}_8\frac{\mathrm{F}}{d_0}+\mathrm{I}_9\frac{\mathrm{F}'}{w_0}\right)$$

$$+\frac{w_0d_0}{4l^2}\left\{\beta_1(1-\alpha_1{}^2)\frac{\mathrm{EA}_0}{\mathrm{G}tw_0}\frac{\mathrm{F}}{d_0} - \alpha_1(1-\beta_1{}^2)\frac{\mathrm{EA}_0}{\mathrm{G}sd_0}\frac{\mathrm{F}'}{w_0}\right\}$$

$$-\frac{w_0d_0}{4l^2}(\theta_1-\theta_0)\,\alpha_1\beta_1(2-\alpha_1-\beta_1)\frac{\mathrm{EA}_0}{l}\Bigg]$$

where

$$\mathrm{I}_7 = \int_1^0\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{du}{\gamma},\ \mathrm{I}_8 = \int_1^0\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{udu}{\alpha\gamma} \text{ and } \mathrm{I}_9 = \int_1^0\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{udu}{\beta\gamma}.$$

Hence

$$\frac{e_1}{l}(\alpha_1+\beta_1)^2\,\mathrm{EA}_0 = \Bigg[\frac{\alpha_1\beta_1}{\alpha_1-\beta_1}\Big\{(\alpha_1+\beta_1)(\mathrm{I}_8-\mathrm{I}_9) - (2-\alpha_1-\beta_1)(\mathrm{I}_2-\mathrm{I}_3)\Big\}$$

$$-\frac{w_0d_0}{4l^2}\left\{\beta_1(1+\alpha_1)\frac{\mathrm{EA}_0}{\mathrm{G}tw_0} - \alpha_1(1+\beta_1)\frac{\mathrm{EA}_0}{\mathrm{G}sd_0}\right\}\Bigg]\frac{\mathrm{T}l}{w_0d_0}$$

$$+\alpha_1\beta_1\Bigg[\frac{\alpha_1+\beta_1}{\alpha_1-\beta_1}(\alpha_1\mathrm{I}_8+\beta_1\mathrm{I}_9) - \frac{2-\alpha_1-\beta_1}{\alpha_1-\beta_1}(\alpha_1\mathrm{I}_2-\beta_1\mathrm{I}_3)$$

$$-\frac{w_0d_0}{4l^2}\left\{(1+\alpha_1)\frac{\mathrm{EA}_0}{\mathrm{G}tw_0} + (1+\beta_1)\frac{\mathrm{EA}_0}{\mathrm{G}sd_0}\right\}\Bigg]\mathrm{P}_0$$

$$+\Bigg[\frac{\alpha_1\beta_1}{\alpha_1-\beta_1}\Big\{(\alpha_1+\beta_1)(\overline{\alpha_1+\beta_1}\,\mathrm{I}_7-\beta_1\mathrm{I}_8-\alpha_1\mathrm{I}_9)$$

$$-(2-\alpha_1-\beta_1)(\overline{\alpha_1+\beta_1}\,\mathrm{I}_1-\beta_1\mathrm{I}_2-\alpha_1\mathrm{I}_3)\Big\}$$

$$+\frac{w_0d_0}{4l^2}\left\{\beta_1{}^2(1+\alpha_1)\frac{\mathrm{EA}_0}{\mathrm{G}tw_0} + \alpha_1{}^2(1+\beta_1)\frac{\mathrm{EA}_0}{\mathrm{G}sd_0}\right\}\Bigg]\mathrm{P}_1 \quad \ldots \quad (19)$$

* By eliminating e_1 a further relation between e_0, $\theta_1 - \theta_0$, P_1, F and F′ may apparently be obtained; but, using equations (15) and (17), this relation proves to be an identity. This is demonstrated in §III. 5.

By the reciprocal theorem, the coefficient of T/w_0d_0 in (17), of T/w_0d_0 in (19) and of P_0/l in (19) must be identical with the coefficients of P_0/l in (15), P_1/l in (15) and P_1/l in (17) respectively. These identities are proved in §III.5 and the evaluation of the coefficients is discussed in §III.6. Here it is necessary to conclude only that equations (15) (17) and (19) may be generalized for the *rth* bay in the forms

$$EA\,\frac{l_r}{l}\cdot\frac{w_{r-1}d_{r-1}}{4l_r^2}\,(\theta_r-\theta_{r-1}) = B_r\,\frac{T_rl_r}{w_{r-1}d_{r-1}} + C_rP_{r-1} + D_rP_r \quad \text{(20)}$$

$$-EA\,\frac{e_{r-1}}{l} = C_r\,\frac{T_rl_r}{w_{r-1}d_{r-1}} + H_rP_{r-1} + J_rP_r \quad \text{(21)}$$

$$EA\,\frac{e_r}{l} = D_r\,\frac{T_rl_r}{w_{r-1}d_{r-1}} + J_rP_{r-1} + K_rP_r \quad \text{(22)}$$

A and l are convenient standard values of A_r and l_r, e.g., in the example worked in §III.7, A_0 is used for A and l is taken as 80 inches. The forms of the coefficients B_r, C_r, D_r, H_r, J_r and K_r are given in §III.6, section (6)

(*b*) *Deformation over a Cut-out.*—If s be zero, $F' = 0$ and

$$\beta_1P_0 - \alpha_1P_1 = \frac{Tl}{w_0d_0} \quad \text{(23)}$$

This is consistent with equations (15), (17) and (19); but regarded as formulae for $\theta_1-\theta_0$, e_0 and e_1 these equations become indeterminate in the form 0/0 Since F'/s may assume any value, equations (13b) and (18b) disappear, and the values of $\theta_1-\theta_0$ and e_1 are defined in terms of P_0, P_1, T and e_0 by equations (13a) and (18a) respectively. Taking into account equation (23), we again have two equations relating e_1, e_0, P_0, P_1 and T with a third relation defining $\theta_1-\theta_0$.

Using the notation of §III.6, namely,

$$\frac{w_0d_0}{4l^2} = R_1,\ (1+\alpha_1)\frac{EA_0}{Gtw_0} = \lambda_1,$$

$$\alpha = \alpha_1 + (1-\alpha_1)u,\ \beta = \beta_1 + (1-\beta_1)u,\ \gamma = \frac{A}{A_0},$$

$$\theta_1-\theta_0 = \Bigg[\left\{\int_0^1 \frac{\beta u\,du}{\alpha^2\gamma} + \frac{(\alpha_1-\beta_1)R_1\lambda_1}{\alpha_1}\right\}P_0 - \left\{\int_0^1 \frac{u\,(1-u)\,du}{\alpha^2\gamma} - \frac{R_1\lambda_1}{\alpha_1}\right\}\frac{Tl}{w_0d_0} + \frac{e_0}{l}EA_0\Bigg]\Big/R_1EA_0\beta_1 \quad \text{(24)}$$

$$\frac{e_1}{l} - \frac{\alpha_1}{\beta_1}\frac{e_0}{l} = \Bigg[\left\{\frac{\alpha_1}{\beta_1}\int_0^1 \frac{\beta^2du}{\alpha^2\gamma} + \frac{(\alpha_1-\beta_1)^2}{\alpha_1\beta_1}R_1\lambda_1\right\}P_0 - \left\{\frac{\alpha_1}{\beta_1}\int_0^1 \frac{\beta\,(1-u)\,du}{\alpha^2\gamma} - \frac{\alpha_1-\beta_1}{\alpha_1\beta_1}R_1\lambda_1\right\}\frac{Tl}{w_0d_0}\Bigg]\Big/EA_0 \quad \text{(25)}$$

Generalization by alteration of the suffixes follows as in §III.4 (*a*).

$$\beta_rP_{r-1} - \alpha_rP_r = \frac{T_rl_r}{w_{r-1}\,d_{r-1}} \quad \text{(23a)}$$

$$EA\,\frac{l_r}{l}\,R_r\,(\theta_r-\theta_{r-1}) = L_r\,\frac{T_rl_r}{w_{r-1}d_{r-1}} + M_rP_{r-1} + N_rEA\,\frac{e_{r-1}}{l} \quad \text{(24a)}$$

$$EA\left(\frac{e_r}{l} - \frac{\alpha_r}{\beta_r}\frac{e_{r-1}}{l}\right) = V_r\,\frac{T_rl_r}{w_{r-1}d_{r-1}} + W_rP_{r-1} \quad \text{(25a)}$$

The forms of L_r, M_r, N_r, V_r and W_r are given in § III. 6

(c) *Continuity from Bay to Bay.*—From (21) and (22) we have

$$-\text{EA}\,\frac{e_{r-1}}{l} = C_r\,\frac{T_r l_r}{w_{r-1}d_{r-1}} + H_r P_{r-1} + J_r P_r$$

and

$$\text{EA}\,\frac{e_{r-1}}{l} = D_{r-1}\,\frac{T_{r-1}l_{r-1}}{w_{r-2}d_{r-2}} + J_{r-1}\,P_{r-2} + K_{r-1}P_{r-1}.$$

Hence

$$J_{r-1}P_{r-2} + (H_r + K_{r-1})\,P_{r-1} + J_r P_r + D_{r-1}\,T_r{}'_{-1} + C_r T_r{}' = 0 \qquad \ldots \quad \ldots \quad \ldots \quad (26)$$

Where $T_r{}'$ is written for $\dfrac{T_r l_r}{w_{r-1}d_{r-1}}$.

If the root frame (frame 0) has no bulkhead stiffness

$$(K_1 + H_2)\,P_1 + J_2P_2 + D_1T_1{}' + C_2T_2{}' = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27a)$$

If the root frame is a rigid bulkhead $e_0 = 0$ and

$$H_1P_0 + J_1P_1 + C_1T_1{}' = 0 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27b)$$

If the root frame has a bulkhead stiffness S, so that $P_0 = e_0S$

$$\left(H_1 + \frac{1}{Sl_1}\right)P_0 + J_1P_1 + C_1T_1{}' = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (27c)$$

Similarly at the tip; but if the tip at the end of the *nth* bay is free, $P_n = 0$ and the last equation of the system (26) is

$$J_{n-1}\,P_{n-2} + (H_n + K_{n-1})\,P_{n-1} + D_{n-1}T_n{}'_{-1} + C_nT_n{}' = 0 \qquad \ldots \quad \ldots \quad (28)$$

Equations (26), (27) and (28) define all the P's in terms of the specified T's, when the θ's can be computed from (20).

(*d*) *Continuity over a Cut-out.*—If $s_r = 0$, the conditions over the *rth* bay are expressed by

$$\beta_rP_{r-1} - \alpha_rP_r = T_r{}' \ldots \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (29)$$

and

$$\text{EA}\left(\frac{e_r}{l} - \frac{\alpha_r}{\beta_r}\,\frac{e_{r-1}}{l}\right) = V_rT_r{}' + W_rP_{r-1}$$

From (22)

$$\text{EA}\,\frac{e_{r-1}}{l} = D_{r-1}T_r{}'_{-1} + J_{r-1}P_{r-2} + K_{r-1}P_{r-1}$$

and from (21)

$$-\text{EA}\frac{e_r}{l} = C_{r+1}T_r{}'_{+1} + H_{r+1}P_r + J_{r+1}P_{r+1}.$$

Eliminating e_{r-1} and e_r,

$$\frac{\alpha_r}{\beta_r}\,J_{r-1}P_{r-2} + \left(\frac{\alpha_r}{\beta_r}\,K_{r-1} + W_r\right)P_{r-1} + H_{r+1}P_r + J_{r+1}P_{r+1} + \frac{\alpha_r}{\beta_r}\,D_{r-1}T_r{}'_{-1}$$
$$+ V_rT_r{}' + C_{r+1}T_r{}'_{+1} = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

The two equations (29) and (30) replace the two equations of type (27) which would contain the coefficients B, C, etc., with suffix $_r$. If the bay next to the root is cut away, and if the root is held by a rigid bulkhead, $e_0 = 0$; equation (29) is unaltered and equation (30) also applies provided J_0 (i.e., J_{r-1}), K_0 and D_0 are taken as zero.

III.5. *Compatibility of Warping Distortions and Reciprocal Relations.*—Eliminating e_1 between equations (18a) and (18b) leads to the relation

$$\frac{e_0}{l}(\alpha_1-\beta_1)\frac{EA_0}{l}=\frac{P_1}{l}\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{du}{\gamma}+\frac{F}{d_0}\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{udu}{\alpha\gamma}$$

$$+\frac{F'}{w_0}\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{udu}{\beta\gamma}+\frac{w_0d_0}{4l^2}\left\{(1-\alpha_1^2)\frac{EA_0}{Gtw_0}\,\frac{F}{d_0}\right.$$

$$\left.-(1-\beta_1)\frac{EA_0}{Gsd_0}\,\frac{F'}{w_0}-(\alpha_1+\beta_1-2\alpha_1\beta_1)\frac{EA_0}{l}(\theta_1-\theta_0)\right\}.$$

Substituting for e_0 and $(\theta_0-\theta_1)$ from (16) and (14) respectively

$$\Big[(\alpha_1+\beta_1)\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{du}{\gamma}+(\alpha_1-\beta_1)\int_1^0\int_1^u\left(\frac{\alpha_1}{\alpha}+\frac{\beta_1}{\beta}\right)\frac{dudu}{\gamma}$$

$$-(\alpha_1+\beta_1-2\alpha_1\beta_1)\int_1^0\int_1^u\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{dudu}{\gamma}\Big]\frac{P_1}{l}$$

$$\Big[(\alpha_1+\beta_1)\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{udu}{\alpha\gamma}+(\alpha_1-\beta_1)\int_1^0\int_1^u\left(\frac{\alpha_1}{\alpha}+\frac{\beta_1}{\beta}\right)\frac{ududu}{\alpha\gamma}$$

$$-(\alpha_1+\beta_1-2\alpha_1\beta_1)\int_1^0\int_1^u\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{ududu}{\alpha\gamma}\Big]\frac{F}{d_0}$$

$$\Big[(\alpha_1+\beta_1)\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{udu}{\beta\gamma}+(\alpha_1-\beta_1)\int_1^0\int_1^u\left(\frac{\alpha_1}{\alpha}+\frac{\beta_1}{\beta}\right)\frac{ududu}{\beta\gamma}$$

$$-(\alpha_1+\beta_1-2\alpha_1\beta_1)\int_1^0\int_1^u\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)\frac{ududu}{\beta\gamma}\Big]\frac{F'}{w_0}$$

$$+\frac{w_0d_0}{4l^2}\left\{(\alpha_1+\beta_1)(1-\alpha_1^2)+(\alpha_1-\beta_1)\,\alpha_1\,(1+\alpha_1)-(\alpha_1+\beta_1-2\alpha_1\beta_1)(1+\alpha_1)\right\}\frac{EA_0}{Gtw_0}\,\frac{F}{d_0}$$

$$-\frac{w_0d_0}{4l^2}\left\{(\alpha_1+\beta_1)(1-\beta_1^2)-(\alpha_1-\beta_1)\,\beta_1\,(1+\beta_1)-(\alpha_1+\beta_1-2\alpha_1\beta_1)(1+\beta_1)\right\}\frac{EA_0}{Gsd_0}\,\frac{F'}{w_0}$$

should be identically zero.

The last two terms (the shear terms) obviously disappear. Each of the other three (the bending terms) may be put in the form

$$(\alpha_1+\beta_1)\left[\left\{\int_1^0\left(\frac{\alpha_1}{\alpha}-\frac{\beta_1}{\beta}\right)\frac{du}{\gamma}-\int_1^0\int_1^u\left(\frac{1-\alpha_1}{\alpha}-\frac{1-\beta_1}{\beta}\right)\frac{dudu}{\gamma}\right\}\frac{P_1}{l}\right]$$

with similar terms in $\frac{F}{d_0}$ and $\frac{F'}{w_0}$ with $\frac{u}{\alpha\gamma}$ and $\frac{u}{\beta\gamma}$ for $\frac{1}{\gamma}$ respectively.

Since $\alpha = \alpha_1 + (1 - \alpha_1)\, u$ and $\beta = \beta_1 + (1 - \beta_1)u$, all the coefficients take the common form $(\alpha_1^2 - \beta_1^2)\left(\int_1^o u\, f\, du + \int_1^o \int_1^u f\, du\, du\right)$ where f stands for $\frac{1}{\alpha\beta\gamma}, \frac{u}{\alpha^2\beta\gamma}$ or $\frac{u}{\alpha\beta^2\gamma}$ and may be regarded as an arbitrary function.

Integrating the first term by parts $\int_1^o u\, f\, du = \left[u \int_1^u f\, du \right]_1^o - \int_1^o \int_1^u f\, du du$, and the first term is zero because $u = 0$ at the upper limit and $\int_1^u f\ du = 0$ at the lower limit. Accordingly, $\int_1^o \int_1^u f\, du du = \int_o^1 u\, f\, du$ and the identity is satisfied whatever the form of f.

Using the same relation to transform all the double integrals I_1 to I_9 of §III. 4(a) to single integrals and then combining them in the forms occurring in equations (15) (17) and (19) of §III.4 (a), we have :

$$I_2 - I_3 = \int_o^1 \left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2 \frac{u^2 du}{\gamma} = I_{11}$$

$$\alpha_1 I_2 + \beta_1 I_3 = I_5 - I_6 = \int_o^1 \left(\frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{u^2 du}{\gamma} = I_{12}$$

$$\alpha_1 (I_1 - I_3) + \beta_1 (I_1 - I_2) = \frac{\alpha_1\beta_1}{\alpha_1 - \beta_1} \{(\alpha_1 + \beta_1)(I_8 - I_9)$$

$$- (2 - \alpha_1 - \beta_1)(I_2 - I_3)\} = \alpha_1\beta_1 \int_o^1 \left(\frac{1}{\alpha^2} - \frac{1}{\beta^2}\right) \frac{u\,(1 - u)\, du}{\gamma} = I_{13}$$

$$\alpha_1 I_5 + \beta_1 I_6 = \int_o^1 \left(\frac{\alpha_1}{\alpha} + \frac{\beta_1}{\beta}\right)^2 \frac{u^2 du}{\gamma} = I_{22}$$

$$\alpha_1 (I_4 - I_6) + \beta_1 (I_4 - I_5) = \frac{\alpha_1\beta_1}{\alpha_1 - \beta_1} \{(\alpha_1 + \beta_1)(\alpha_1 I_8 + \beta_1 I_9)$$

$$- (2 - \alpha_1 - \beta_1)(\alpha_1 I_2 + \beta_1 I_5)\}$$

$$= \alpha_1\beta_1 \int_o^1 \left(\frac{\alpha_1}{a} + \frac{\beta_1}{\beta}\right)\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)\frac{u(1 - u)\, du}{\gamma} = I_{23}$$

$$\frac{\alpha_1\beta_1}{\alpha_1 - \beta_1} [(\alpha_1 + \beta_1)\{\alpha_1 (I_7 - I_9) + \beta_1 (I_7 - I_8)\}$$

$$- (2 - \alpha_1 - \beta_1)\{\alpha_1 (I_1 - I_3) + \beta_1 (I_2 - I_3)\}]$$

$$= \alpha_1^2\beta_1^2 \int_o^1 \left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^2 \frac{(1 - u)^2 du}{\gamma} = I_{33}.$$

III.6. *Computation of the Geometrical Constants of Deformation.*—From §III.4(*a*), using the results of §III.5, the coefficients in equations (20), (21), (22) and subsequently, are

$$B_r = \{I_{11} + R\,(\lambda + \mu)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\cdot\frac{l}{l_r}$$

$$C_r = \{I_{12} + R\,(\alpha\lambda - \beta\mu)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\,\frac{l}{l_r}\,.$$

$$D_r = \{I_{13} + R\,(\alpha\mu - \beta\lambda)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\,\frac{l}{l_r}$$

$$H_r = \{I_{22} + R\,(\alpha^2\lambda + \beta^2\mu)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\,\frac{l}{l_r}$$

$$J_r = \{I_{23} - R\alpha\beta\,(\lambda + \mu)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\,\frac{l}{l_r}$$

$$K_r = \{I_{33} + R\,(\alpha^2\mu + \beta^2\lambda)\}_r / (\alpha_r + \beta_r)^2\,\frac{A_{r-1}}{A}\,\frac{l}{l_r}$$

where a suffix outside a bracket applies to all the terms within the bracket and where

$$R_r = w_{r-1}d_{r-1}/4l_r^2$$

$$\lambda_r = (1 + \alpha_r)EA_{r-1}/Gt_r w_{r-1}\ (\alpha_r = d_r/d_{r-1})$$

$$\mu_r = (1 + \beta_r)\,EA_{r-1}/Gs_r d_{r-1}\ (\beta_r = w_r/w_{r-1})$$

and I_{11}, etc., have the forms given at the end of §III.5.

From §III.4(*b*), using the same notation

$$L_r = (I_1' + O_1')_r \Big/ \frac{A_{r-1}}{A}\cdot\frac{l}{l_r}$$

$$M_r = (I_2' + O_2')_r \Big/ \frac{A_{r-1}}{A}\cdot\frac{l}{l_r}$$

$$N_r = \frac{1}{\beta_r}$$

$$V_r = (I_3' + O_2')_r \Big/ \frac{A_{r-1}}{A}\cdot\frac{l}{l_r}$$

$$W_r = (I_4' + O_3')_r \Big/ \frac{A_{r-1}}{A}\cdot\frac{l}{l_r}$$

where a suffix outside a bracket applies to both the terms within the bracket and where

$$O_{1r}' = R_r\lambda_r/\alpha_r\beta_r,\quad O_{2r}' = (\alpha_r - \beta_r)\,O_{1r}',\quad O_{3r}' = (\alpha_r - \beta_r)\,O_{2r}'$$

$$I_{1r}' = -\frac{1}{\beta_r}\int_0^1 \frac{u(1-u)du}{\alpha^2\gamma},\qquad I_{2r}' = \frac{1}{\beta_r}\int_0^1 \frac{\beta u du}{\alpha^2\gamma}$$

$$I_{3r}' = -\frac{\alpha_r}{\beta_r}\int_0^1 \frac{\beta(1-u)du}{\alpha^2\gamma},\ \text{and}\ I_{4r}' = \frac{\alpha_r}{\beta_r}\int_0^1 \frac{\beta^2 du}{\alpha^2\gamma}$$

($\alpha = \alpha_r + (1 - \alpha_r)u$, etc., as before).

Given the numerical values of I_{11}, etc., the evaluation of the coefficients B, C, etc., is straightforward ; but the evaluation of I_{11}, etc., presents some practical difficulties. All the integrals are derived from the three integrals

$$\int_0^1 \frac{du}{\alpha\beta\gamma}, \int_0^1 \frac{udu}{\alpha\beta\gamma} \text{ and } \int_0^1 \frac{u^2du}{\alpha\beta\gamma},$$

and their special forms when $\alpha = \beta$; but these forms are cumbersome* and tedious to evaluate. A more serious objection to evaluation of I_{11}, etc., by this means is that, when α_1 is nearly equal to β_1 (or either α_1 or β_1 nearly equal to γ_1), the sum of the two terms in $\log_e\alpha_1$ and $\log_e\beta_1$ is much smaller than either term separately, so that the accuracy attainable by the use of tables is insufficient, and approximation by expansion in series becomes necessary. In these circumstances it is often preferable to avoid the log forms entirely, by expansion of the basic function in series.

If $\quad F = \frac{1}{\alpha\beta\gamma} \ldots$, then $\frac{(-1)^n}{n!}\frac{d^nF}{dx^n} = FQ_n\left(\frac{1-\alpha_1}{\alpha}, \frac{1-\beta_1}{\beta}, \frac{1-\gamma_1}{\gamma}, \ldots\right)$

where Q_n ( ) stands for the sum of all possible products

$$\left(\frac{1-\alpha_1}{\alpha}\right)^r\left(\frac{1-\beta_1}{\beta}\right)^s\ldots$$

under the condition $r + s + \ldots = n$. This lemma may be proved by induction.

Assuming that $(A + Bu + Cu^2)/\alpha\beta\gamma = a_0 + a_1 u + a_2 u^2 + \ldots$, by successful differentiation and substitution of $u = 0$ in each differential,

$$\left.\begin{aligned}
\alpha_1\beta_1\gamma_1 a_0 &= A \\
\alpha_1\beta_1\gamma_1 a_1 &= B - AQ_1(p, q, r) \\
\alpha_1\beta_1\gamma_1 a_2 &= C - BQ_1(p, q, r) + AQ_2(p, q, r) \\
\alpha_1\beta_1\gamma_1 a_3 &= -CQ_1(p, q, r) + BQ_2(p, q, r) - AQ_3(p, q, r) \\
&\quad \text{etc.}
\end{aligned}\right| \quad
\begin{aligned}
\text{where } p &= \frac{1-\alpha_1}{\alpha_1} \\
q &= \frac{1-\beta_1}{\beta_1} \\
r &= \frac{1-\gamma_1}{\gamma_1}
\end{aligned}$$

Then

$$\int_0^1 \frac{A + Bu + Cu^2}{\alpha\beta\gamma} du = \frac{1}{\alpha_1\beta_1\gamma_1}\left\{A\left(Q_0 - \frac{1}{2}Q_1 + \frac{1}{3}Q_2 - \ldots\ldots\right)\right.$$
$$+ B\left(\frac{1}{2}Q_0 - \frac{1}{3}Q_1 + \frac{1}{4}Q_2 - \ldots\ldots\right)$$
$$\left. + C\left(\frac{1}{3}Q_0 - \frac{1}{4}Q_1 + \frac{1}{5}Q_2 - \ldots\ldots\right)\right\}.$$

* For instance :

$$\int_0^1 \frac{u^2\,du}{\alpha\beta\gamma} = \frac{\alpha_1^2\log_e\alpha_1}{(1-\alpha_1)(\gamma_1-\alpha_1)(\alpha_1-\beta_1)} + \frac{\beta_1^2\log_e\beta_1}{(1-\beta_1)(\alpha_1-\beta_1)(\beta_1-\gamma_1)} + \frac{\gamma_1^2\log_e\gamma_1}{(1-\gamma_1)(\beta_1-\gamma_1)(\gamma_1-\alpha_1)}$$

It may be shown that

$$Q_n = \frac{p^{n+2}}{(p-q)(p-r)} + \frac{q^{n+2}}{(q-r)(q-p)} + \frac{r^{n+2}}{(r-p)(r-q)} .$$

Substitution for the Q's in this form and summation of the resulting series leads to the ordinary log forms for the integral; but this form for Q_n reintroduces the computational difficulty when two of the set p, q, r are nearly equal. In deriving the Q's therefore, it is preferable to use the recurrence formula

$$Q_{n+1}(p, q, r) = p\, Q_n(p, q, r) + q\, Q_n(q, r) + r^{n+1}.$$

The computation is readily carried out in tabular form (*see* example below). This method of computation has the additional advantage that it is still applicable when $p = q$, so that all the integrals required can be computed by the same method.

If any of α_1 β_1 or γ_1 is less than $\frac{1}{2}$ the corresponding ratio p, q or r is greater than unity. In this case the Q's themselves form a divergent series, and the recurrence formula above is useless. If α_1 β_1 and γ_1 are all greater than $\frac{1}{2}$, the Q series are absolutely convergent; but the convergence may be very slow. In general, if α_1 β_1 and γ_1 are all greater than $0{\cdot}7$, the Q series are usable, whilst for lower values the log forms are preferable. One difficult case remains, that when two of α_1 β_1 and γ_1 are small and nearly equal.

The computation of I_{11}, etc., has been discussed at some length, because the practical value of the method proposed is greatly reduced, if the evaluation of these integrals is too laborious. If the method proposed were to be adopted for general use, the logical course would be to prepare tables of I_{11}, etc., in terms of α_1, β_1 and γ_1.

Example.

$$\left.\begin{array}{l} \alpha_1 = 0{\cdot}778, \quad \beta_1 = 0{\cdot}833, \quad \gamma_1 = 0{\cdot}750 \\ p = 0{\cdot}2857, \; q = 0{\cdot}2000, \; r = 0{\cdot}3333 \end{array}\right\} (Q_0 = 1).$$

n	$pQ_{n-1}(p,q,r)$	$qQ_{n-1}(q,r)$	r^n	$Q_n(q,r)$	$Q_n(p,q,r)$	1·0000	0·5000	0·3333
1	2857	2000	3333	5333	8190	−0·4095	−0·2730	−0·2047
2	2340	1067	1111	2178	4518	+0·1506	+0·1129	+0·0904
3	1291	436	370	806	2097	−0·0524	−0·0420	−0·0349
4	599	161	123	284	883	+0·0177	+0·0147	+0·0126
5	252	57	41	98	350	−0·0058	−0·0050	−0·0044
6	100	20	14	34	134	+0·0019	+0·0017	+0·0015
7	38	7	5	12	50	−0·0006	−0·0006	−0·0005
8	14	2	2	4	18	+0·0002	+0·0002	+0·0002
9	5	1	1	2	7	−0·0001	−0·0001	−0·0001
						0·7020A	+0·3088B	+0·1934C

$$\int_0^1 \frac{A + Bu + Cu^2}{\alpha\beta\gamma}\, du = \frac{0{\cdot}7020A}{0{\cdot}778} + \frac{0{\cdot}3088B}{0{\cdot}833} + \frac{0{\cdot}1934C}{0{\cdot}750}$$

$$= 1{\cdot}4441A + 0{\cdot}6352B + 0{\cdot}3979C$$

n	$pQ_{n-1}(p,p,r)$	$pQ_{n-1}(p,r)$	r^n	$Q_n(p,r)$	$Q_n(p,p,r)$	1·0000	0·5000	0·3333
1	2857	2857	3333	6190	9047	−0·4523	−0·3016	−0·2262
2	2585	1769	1111	2880	5465	+0·1822	+0·1366	+0·1093
3	1561	823	370	1193	2754	−0·0688	−0·0551	−0·0459
4	787	341	123	564	1351	+0·0270	+0·0225	+0·0193
5	386	175	41	216	602	−0·0100	−0·0086	−0·0075
6	172	62	14	76	248	+0·0035	+0·0031	+0·0028
7	71	22	5	27	98	−0·0012	−0·0011	−0·0010
8	28	8	2	10	38	+0·0004	+0·0004	+0·0003
9	11	3	1	4	15	−0·0001	−0·0001	−0·0001
						0·6807A	+0·2961B	+0·1843C

$$1{\cdot}5003A + 0{\cdot}6526B + 0{\cdot}4062C = \frac{0{\cdot}6807A + 0{\cdot}2961B + 0{\cdot}1843C}{(0{\cdot}778)^2 \times 0{\cdot}750}$$

n	$qQ_{n-1}(q,q,r)$	$qQ_{n-1}(q,r)$	r^n	$Q_n(q,r)$	$Q_n(q,q,r)$			
						1·0000	0·5000	0·3333
1	2000	2000	3333	5333	7333	−0·3667	−0·2444	−0·1833
2	1467	1067	1111	2178	3644	+0·1215	+0·0911	+0·0729
3	729	436	370	806	1535	−0·0384	−0·0307	−0·0256
4	307	161	123	284	591	+0·0118	+0·0098	+0·0082
5	118	57	41	98	216	−0·0036	−0·0031	−0·0027
6	43	20	14	34	77	+0·0011	+0·0010	+0·0009
7	15	7	5	12	27	−0·0003	−0·0003	−0·0003
8	5	2	2	4	9	+0·0001	+0·0001	+0·0001
						0·7255A	+0·3235B	+0·2035C

$$1\cdot3930A + 0\cdot6211B + 0\cdot3907C = \frac{0\cdot7255A + 0\cdot3235B + 0\cdot2035C}{(0\cdot833)^2 \times 0\cdot750}$$

$$I_{11} = 0\cdot4062 + 0\cdot3907 - 2 \times 0\cdot3979 \quad \ldots \quad = \underline{0\cdot0011}$$

$$I_{12} = 0\cdot4062 \times 0\cdot778 - 0\cdot3907 \times 0\cdot833 + 0\cdot3979\,(0\cdot833 - 0\cdot788) \quad \ldots \quad = \underline{0\cdot0125}$$

$$I_{13} = 0\cdot778 \times 0\cdot833 \left\{(0\cdot6526 - 0\cdot4062) - (0\cdot6211 - 0\cdot3907)\right\} \quad \ldots \quad = \underline{0\cdot0104}$$

$$I_{22} = 0\cdot4062 \times (0\cdot778)^2 + 0\cdot3907 \times (0\cdot833)^2 + 2 \times 0\cdot3979 \times 0\cdot778 \times 0\cdot833 \quad \ldots \quad = \underline{1\cdot0327}$$

$$I_{23} = 0\cdot778 \times 0\cdot833 \left\{(0\cdot6526 - 0\cdot4062) \times 0\cdot778 + (0\cdot6211 - 0\cdot3907) \times 0\cdot833 + (0\cdot6352 - 0\cdot3979) \times (0\cdot833 + 0\cdot778)\right\} \quad \ldots \quad = \underline{0\cdot4965}$$

$$I_{33} = (0\cdot778 \times 0\cdot833)^2 \left\{1\cdot5003 - 2 \times 0\cdot6526 + 0\cdot4062 + 1\cdot3930 - 2 \times 0\cdot6211 + 0\cdot3907 + 2\,(1\cdot4441 - 2 \times 0\cdot6352 + 0\cdot3979)\right\} \quad \ldots \quad = \underline{0\cdot9603}$$

III.7. *Summary and Examples.*—This section summarizes the conclusions of §§III.4–6 and presents them in the form and order in which they must be used in numerical computation. The application is illustrated by a worked example.

Whilst the present section contains all the information necessary for use of the method, for explanation of the formulae reference should be made to §§III. 4, 5 and 6.

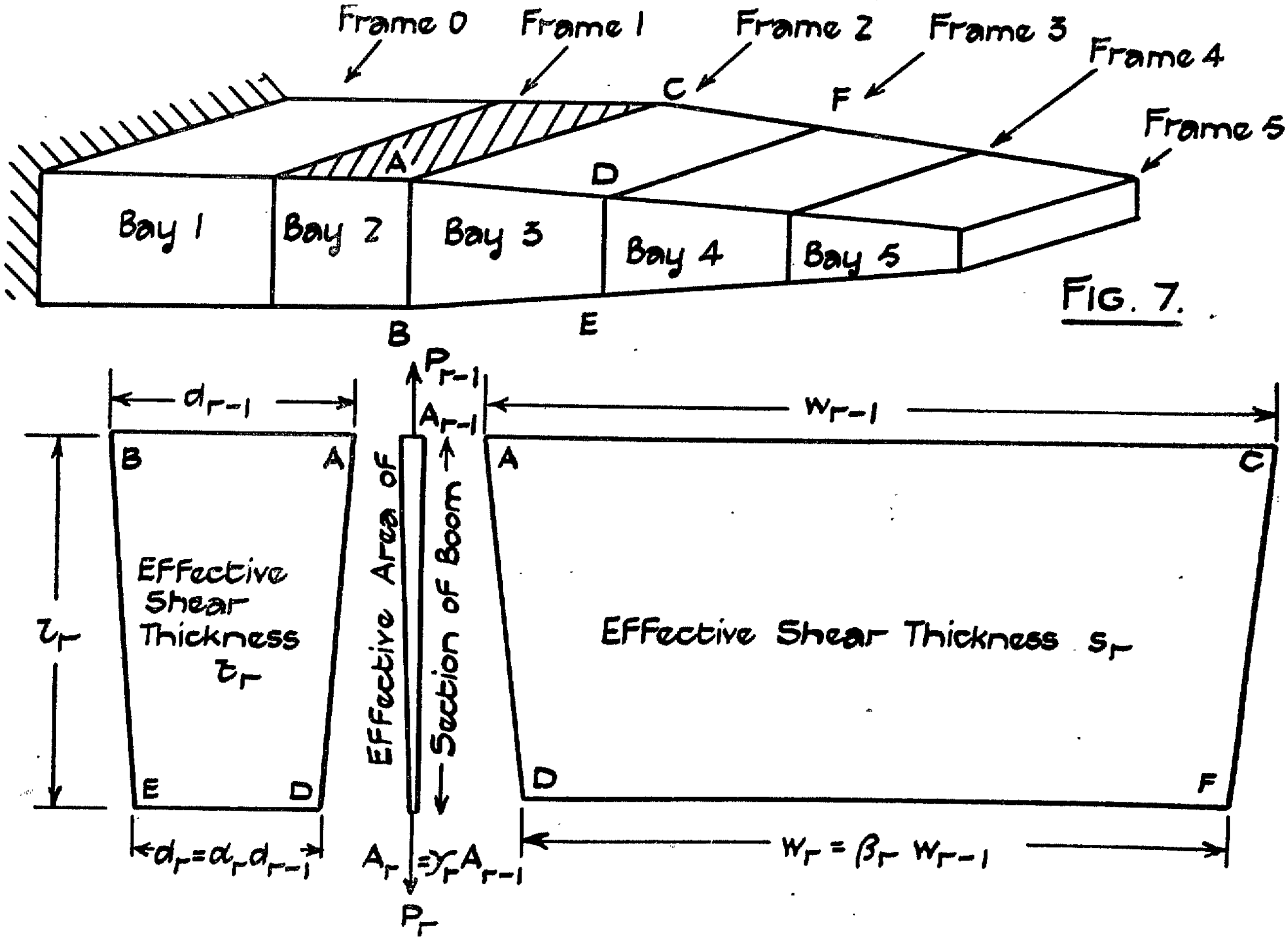

FIG. 7.

A $=$ actual area of section of boom $+ 1/6\ (dt + ws)$.

T_r $=$ total torque transmitted over *rth* bay.

e_r $=$ warping movement of point D away from A parallel to axis of tube.*

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* No distinction need be made between the direction of the axis of the tube and the direction of its edges (*see* §III 4).

(*a*) *Tube with Cut-out.*

Specification (dimensions in inches — torques in 1,000 lb.-inches).

Frame or Bay No.		0	1	2	3	4	5	Notes
d_r	—	18	18	18	14	10	6	
w_r	—	60	60	60	50	40	30	
A_r	—	6·0	6·0	6·0	4·5	3·0	2·0	$A = 6{\cdot}0$
—	l_r	—	80	50	80	80	80	$l = 80$
—	t_r	—	0·30	0·30	0·30	0·20	0·15	Top or bottom surface cut away over Bay 2.
—	s_r	—	0·030	0	0·030	0·020	0·020	
—	T_r	—	135	92	61	35	7	

Primary Ratios	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Notes
$\alpha_r = d_r/d_{r-1}$	1	1	0·778	0·714	0·600	
$\beta_r = w_r/w_{r-1}$	1	1	0·833	0·800	0·750	
$\gamma_r = A_r/A_{r-1}$	1	1	0·750	0·667	0·667	
$R_r = \dfrac{w_{r-1}d_{r-1}}{4l_r^2}$	0·0422	0·1080	0·0422	0·0273	0·0156	
$\lambda_r = \dfrac{(1+\alpha_r)EA_{r-1}}{Gt_r\,w_{r-1}}$	1·733	1·733	1·541	2·006	2·080	$\dfrac{E}{G} = 2{\cdot}6$
$\mu_r = \dfrac{(1+\beta_r)EA_{r-1}}{Gs_r d_{r-1}}$	57·78	infinity	52·96	75·21	68·25	
$l_r/w_{r-1}d_{r-1}$	0·0741	0·0463	0·0741	0·1143	0·2000	

Secondary Ratios	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Notes
$O_{11} = R_r(\lambda_r+\mu_r)$	2·511	*See* Notes	2·300	2·108	1·097	Over cut-out
$O_{12} = R_r(\alpha_r\lambda_r-\beta_r\mu_r)$	−2·365	$O_1'=0\cdot1872$	−1·812	−1·604	−0·779	$O_1' = R_2\lambda_2/\alpha_2\beta_2$
$O_{13} = R_r(\alpha_r\mu_r-\beta_r\lambda_r)$	2·365	$O_2'=0$	1·684	1·423	0·614	$O_2' = (\alpha_2-\beta_2)O_1'$
$O_{22} = R_r(\alpha_r^2\lambda_r+\beta_r^2\mu_r)$	2·511	$O_3'=0$	1·591	1·342	0·611	$O_3'=(\alpha_2-\beta_2)O_2'$
$O_{23} = -R\alpha_r\beta_r(\lambda_r+\mu_r)$	−2·511	—	−1·491	−1·205	−0·493	
$O_{33} = R_r(\alpha_r^2\mu_r+\beta_r^2\lambda_r)$	2·511	—	1·397	1·083	0·401	
$I_{11} = \int_0^1\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2\frac{u^2du}{\gamma}$	0	—	0·001	0·001	0·004	$\alpha = \alpha_r+(1-\alpha_r)u$
$I_{12} = \int_0^1\left(\frac{\alpha_r}{\alpha}+\frac{\beta_r}{\beta}\right)\left(\frac{1}{\alpha}-\frac{1}{\beta}\right)^2\frac{udu}{\gamma}$	0	—	0·012	0·017	0·030	$\beta = \beta_r+(1-\beta_r)u$
$I_{13} = \alpha_r\beta_r\int_0^1\left(\frac{1}{\alpha^2}-\frac{1}{\beta^2}\right)\frac{u(1-u)du}{\gamma}$	0	—	0·010	0·018	0·029	$\gamma=\alpha_r+(1-\gamma_r)u$ Values computed as in §III.6.
$I_{22} = \int_0^1\left(\frac{\alpha_r}{\alpha}+\frac{\beta_r}{\beta}\right)^2\frac{u^2du}{\gamma}$	1·333	—	1·033	0·965	0·809	
$I_{23} = \alpha_r\beta_r\int_0^1\left(\frac{\alpha_r}{\alpha}+\frac{\beta_r}{\beta}\right)\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\frac{u(1-u)du}{\gamma}$	0·667	—	0·496	0·463	0·363	
$I_{33} = \alpha_r^2\beta_r^2\int_0^1\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^2\frac{(1-u)^2du}{\gamma}$	1·333	*See* Notes	0·960	0·893	0·675	
$I_1' = -\frac{1}{\beta_r}\int_0^1\frac{u(1-u)du}{\alpha^2\gamma}$	—	−0·167	—	—	—	Over cut-out
$I_2' = \frac{1}{\beta_r}\int_0^1\frac{\beta udu}{\alpha^2\gamma}$	—	0·500	—	—	—	I_1', etc., replace I_{11}, etc.

Secondary Ratios	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Notes
$I_3' = -\dfrac{\alpha_r}{\beta_r}\int_0^1 \dfrac{\beta(1-u)du}{\alpha^2\gamma}$	—	−0·500	—	—	—	—
$I_4' = \dfrac{\alpha_r}{\beta_r}\int_0^1 \dfrac{\beta^2 du}{\alpha^2\gamma}$	—	1·000	—	—	—	—
$(\alpha_r + \beta_r)^2$	4	—	2·596	2·293	1·822	—
A_{r-1}/A	1	1	1	0·75	0·5	—
l/l_r	1	1·6	1	1	1	—
$(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}$	4	1·6*	2·596	1·720	0·911	$^*\dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}$ only

Coefficients	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	
$B_r = \dfrac{(I_{11} + O_{11})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	0·628	$L_2 = 0·012$	0·886	1·226	1·208	$L_2 = \dfrac{(O_1' + I_1')}{A_1 l / A l_2}$
$C_r = \dfrac{(I_{12} + O_{12})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	−0·591	$M_2 = 0·312$	−0·693	−0·923	−0·822	$M_2 = \dfrac{(O_2' + I_2')}{A_1 l / A l_2}$
$D_r = \dfrac{(I_{13} + O_{13})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	0·591	$N_2 = 1·000$	0·653	0·838	0·701	$N_2 = 1/\beta_2$
$H_r = \dfrac{(I_{22} + O_{22})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	0·961	$V_2 = 0·312$	1·011	1·341	1·560	$V_2 = \dfrac{(O_2' + I_3')}{A_1 l / A l_2}$
$J_r = \dfrac{(I_{23} + O_{23})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	−0·461	$W_2 = 0·625$	−0·383	−0·431	−0·143	$W_2 = \dfrac{(O_3' + I_4')}{A_1 l / A l_2}$
$K_r = \dfrac{(I_{33} + O_{33})_r}{(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \cdot \dfrac{l}{l_r}}$	0·961	—	0·908	1·149	1·181	—
$T_r' = \dfrac{T_r l_r}{w_{r-1} d_{r-1}}$	10	4·26	4·52	4·00	1·40	—

Equations of Equilibrium	P_0	P_1	P_2	P_3	P_4	Torques	Notes Refs. to § III.4
Forms of Equations: Each row of terms added and equated to zero.	H_1	J_1	—	—	—	C_1T_1'	Equation (27b)
	—	β_2	$-\alpha_2$	—	—	$-T_2'$	Equation (29)
	$\frac{\alpha_2}{\beta_2} J_1$	$\frac{\alpha_2}{\beta_2} K_1+W_2$	H_3	J_3	—	$\frac{\alpha_2}{\beta_2} D_1T_1'+V_2T_2'+C_3T_3'$	Equation (30)
	—	—	J_3	H_4+K_3	J_4	$D_3T_3'+C_4T_4'$	Equation (26)
	—	—	—	J_4	H_5+K_4	$D_4T_4'+C_5T_5'$	Equation (28)

Equations of Equilibrium		P_0	P_1	P_2	P_3	P_4	Torques	Notes
Equations for Numerical Case in form above.	(1)	0·961	−0·461	—	—	—	−5·91	−5·9100
	(2)	—	1	−1	—	—	−4·26	−4·2600
	(3)	−0·461	1·586	1·011	−0·383	—	1·45	1·4500
	(4)	—	—	−0·383	2·249	−0·431	−0·74	−0·7398
	(5)	—	—	—	−0·431	2·709	2·20	2·1999
From (3) and (1) ..	(6)	—	1·36486	1·011	−0·383	—	−1·38507	Check back above shows accuracy maintained.
From (6) and (2) ..	(7)	—	—	2·37586	−0·383	—	4·42923	
From (7) and (4) ..	(8)	—	—	—	2·18726	−0·431	−0·02599	
From (8) and (5) ..	(9)	—	—	—	—	2·62407	2·19488	
Boom Loads (P_{r-1})		7·2873	2·3711	−1·8889	−0·1530	−0·8364	—	×1,000 lb.
Twists		Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Total	
B_rT_r'		6·280	0·051*	4·005	4·904	1·691	—	$*L_2T_2'$ in Bay 2
C_rP_{r-1}		−4·307	0·740*	1·309	0·141	0·687	—	$*M_2P_1$ in Bay 2
D_rP_r		1·401	4·829*	−0·100	−0·701	—	—	$*N_2EA\frac{e_1}{l}$ in Bay 2
Sum $\doteq EA\frac{l_r}{l}R_r(\theta_r-\theta_{r-1})$ ..		3·374	5·620	5·214	4·344	2·378	—	$=N_2(J_1P_0+K_1P_1+D_1T_1')$
$EA\frac{l_r}{l}R_r$		2·532	4·050	2·532	1·638	0·936	—	$\times 10^6$ lb.
$\theta_r-\theta_{r-1}$		1·332	1·388	2·059	2·652	2·541	9·972	$\times 10^{-3}$ radians
Twist per inch		1·665	2·776	2·572	3·315	3·176	2·695	$\times 10^{-5}$ radians

Shear Stresses	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Total	
$(\alpha_r + \beta_r) \dfrac{F_r l_r}{d_{r-1}} = T_r' + \alpha_r P_{r-1} - \beta_r P_r$	14·916	8·520	3·178	4·560	0·898	—	×1,000 lb.
$(\alpha_r + \beta_r) l_r t_r$	48	30	38·67	24·23	16·20	—	in.²
Shear Stress in Sides ($F_r/d_{r-1}t_r$)	310	280	80	190	55	—	lb./in.²
$(\alpha_r + \beta_r) \dfrac{F_r' l_r}{w_{r-1}} = -T_r' + \beta_r P_{r-1} - \alpha_r P_r$	− 5·084	0	− 5·975	− 3·525	− 2·027	—	×1,000 lb.
$(\alpha_r + \beta_r) l_r s_r$	4·8	0	3·867	2·423	2·160	—	in.²
Shear Stress in Top or Bottom ($F_r'/w_{r-1}s_r$)	1060	—	1550	1400	940	—	lb./in.²
Boom Stresses (P_{r-1}/A_{r-1}) ..	1220	400	−310	−30	−280	—	lb./in.² tension.

Note.—The boom stress quoted is that in the top front boom of the sketch, the applied torques being anticlockwise.

(*b*) *Tube as in previous case, but without cut-out,* $l_2/w_1 d_1 = 0{\cdot}0463$ $(\alpha_2 + \beta_2)^2 A_1/A\ l/l_2 = 6{\cdot}4$

$$\text{For Bay 2 in this case} \left\{ \begin{array}{llll} \alpha_2 = 1 & O_{11} = 6{\cdot}427 & I_{11} = 0 & B_2 = 1{\cdot}004 \\ \beta_2 = 1 & O_{12} = -6{\cdot}053 & I_{12} = 0 & C_2 = -0{\cdot}948 \\ \gamma_2 = 1 & O_{13} = 6{\cdot}053 & I_{13} = 0 & D_2 = 0{\cdot}948 \\ R_2 = 0{\cdot}1080 & O_{22} = 6{\cdot}427 & I_{22} = 1{\cdot}333 & H_2 = 1{\cdot}212 \\ \lambda_2 = 1{\cdot}733 & O_{23} = -6{\cdot}427 & I_{23} = 0{\cdot}667 & J_2 = -0{\cdot}900 \\ \mu_2 = 57{\cdot}78 & O_{33} = 6{\cdot}427 & I_{33} = 1{\cdot}333 & K_2 = 1{\cdot}212 \end{array} \right\} \text{Other coefficients as before}$$

Equations of Equilibrium	P_0	P_1	P_2	P_3	P_4	Torques	Notes Refs. to §III.4
Forms of Equations : Each row of terms added and equated to zero.	H_1	J_1	—	—	—	C_1T_1'	Equation (27b)
	J_1	H_2+K_1	J_2	—	—	$D_1T_1'+C_2T_2'$	Equation (26)
	—	J_2	H_3+K_2	J_3	—	$D_2T_2'+C_3T_3'$	
	—	—	J_3	H_4+K_3	J_4	$D_3T_3'+C_4T_4'$	
	—	—	—	J_4	H_5+K_4	$D_4T_4'+C_5T_5'$	Equation (28)
Equation for Numerical case in form above. (1)	0·961	—0·461	—	—	—	—5·910	
(2)	—0·461	2·173	—0·900	—	—	1·872	
(3)	—	—0·900	2·223	—0·383	—	0·906	
(4)	—	—	—0·383	2·249	—0·431	—0·740	
(5)	—	—	—	—0·431	2·709	2·200	
From (1) and (2) .. .. (6)	—	1·95185	—0·900	—	—	—0·96309	
From (6) and (3) .. .. (7)	—	—	1·80800	—0·383	—	0·46192	
From (7) and (4) .. .. (8)	—	—	—	2·16787	—0·431	—0·64215	
From (8) and (5) .. .. (9)	—	—	—	—	2·62331	2·07233	
Boom Loads	6·3366	0·3892	—0·2260	0·1391	—0·7900	—	× 1,000 lb.

Twists	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Total	
B_rT_r'	6·280	4·262	4·005	4·904	1·691	—	
C_rP_{r-1}	—3·745	—0·368	0·156	—0·128	0·650	—	
D_rP_r	0·230	—0·214	0·091	—0·661	—	—	
Sum $= EA\frac{l_r}{l}R_r(\theta_r-\theta_{r-1})$ ..	2·765	3·680	4·252	4·115	2·341	—	$E=10\times10^6$ lb./in.^{2}
$EA\frac{l_r}{l}R_r$	2·532	4·050	2·532	1·638	0·936	—	$\times 10^6$ lb.
$\theta_r-\theta_{r-1}$	1·092	0·909	1·680	2·512	2·501	8·694	$\times 10^{-3}$ radians
Twist per inch	1·365	1·818	2·100	3·140	3·126	2·350	$\times 10^{-5}$ radians

Shear Stresses	Bay 1	Bay 2	Bay 3	Bay 4	Bay 5	Total	
$(\alpha_r+\beta_r)\frac{F_rl_r}{d_{r-1}} = T_r'+\alpha_rP_{r-1}-\beta_rP_r$	15·978	4·875	4·228	4·731	0·926		$\times$ 1,000 lb.
$(\alpha_r+\beta_r)l_rt_r$	48	30	38·67	24·23	16·20		in.^{2}
Shear Stress (sides)	330	160	110	190	60		lb./in.^{2}
$(\alpha_r\beta_r)\frac{F_r'\,l_r}{w_{r-1}} = -T_r'+\beta_rP_{r-1}-\alpha_rP_r$	— 4·022	— 3·645	— 4·816	— 3·325	— 1·993		$\times$ 1,000 lb.
$(\alpha_r+\beta_r)l_rt_r$	4·8	3·0	3·867	2·423	2·160		in.^{2}
Shear Stress (top or bottom) ..	840	1210	1240	1370	920		lb./in.^{2}
Boom Stresses	1060	60	—40	30	—260		lb./in.^{2} tension

(*c*) *Tube with Two Additional Frames.*—As a further example, the same tube with two additional frames, one midway between frames 0 and 1 and the other midway between frames 2 and 3, has been analysed. The effect of the added frames is summarized in the following comparisons.

With no Cut-out

Bay No.		1	12	2	23	3	4	5	Notes
Boom loads	7 frames	6·37	2·79	0·45	−0·24	0·14	0·15	−0·79	At inner ends of bays.
	5 frames	6·37	—	0·39	−0·23	—	0·14	−0·79	
Twist per inch	7 frames	1·01	1·73	1·79	1·88	2·30	3·13	3·12	Mean 2·344
	5 frames	1·36		1·82	2·10		3·14	3·13	Mean 2·350

With Cut-out over Bay 2

Bay No.		1	12	2	23	3	4	5	Notes
Boom loads	7 frames	7·31	3·95	2·34	−1·92	−0·74	−0·27	−0·86	At inner ends of bays.
	5 frames	7·29	—	2·37	−1·89	—	−0·15	−0·84	
Twist per inch	7 frames	1·14	2·16	2·67	1·64	2·65	3·38	3·20	Mean 2·584
	5 frames	1·66		2·78	2·57		3·31	3·18	Mean 2·695

In the absence of a cut-out the effect of the added frames is quite negligible. When the cover over bay 2 is cut away, the intermediate frame inboard towards the root still has practically no effect, but the intermediate frame outboard of the cut-out slightly stiffens the tube.

PART IV

Torsion of Tapered Tubes of Trapezoidal Section with Rigid Frames

IV.1. *Introduction.*—In this part the method of Part III is extended to the case of a tube of trapezoidal section. In this case also the twist over a bay between two rigid frames and the warping at each frame are related to the torque and boom loads transmitted over the bay by linear relations like equations (8), (9) and (10). The conceptions of boom load and warping again need not be closely defined; the boom load P is merely a quantitative measure of the stress system tending to cause warping and e, the warping at the boom, is the *corresponding* measure of the warping distortion. In the present part, the analysis is carried only to expression of the constants B, C, etc., in equations (8), (9) and (10); thereafter analysis would follow exactly the same lines as in Part III.

IV.2. *Distribution of Stresses in the General Four-Sided Tube in Pure Torsion.*—In Fig. 8, ABCD is a section of a tube formed by four planes, OADR, OBCR, PBAS and PCDS, which intersect in opposite pairs in OR and PS; the "ridge lines" OR and PS are not necessarily co-planar. At the section ABCD of the tube, the sides AB, BC, etc., are subjected to shears along the sides and to direct stresses acting radially through the ridge lines OR and PS. Since all the radial loads pass through the ridge line PS, they may be represented by a single force through any point on this line and a moment in a plane containing the line, e.g. by a force through P and a couple about an axis normal to PS; and if the section ABCD is free from end load, the force through P must lie in the plane ABCD. If there is no moment on the plane ABCD, the couple in any plane containing PS must be zero, so that all the direct loads must reduce to a single force through P in the plane ABCD. Similarly they must reduce also to a single force through O, so the resultant of all the direct loads is a single force directed along OP; let the magnitude of this force be X.

If there is no shear across the section ABCD, the torque in this plane is $F_1.PJ - F_3.PE$, where PJ and PE are perpendicular to OAD and OBC respectively, or it is $F_4.OG - F_2.OH$ where OG and OH are perpendicular to PCD and PBA respectively. Therefore, we may assume that $F_1 = \mu DA + \lambda PE$, $F_3 = \mu BC + \lambda PJ$, $F_2 = \mu AB + \nu OG$ and $F_4 = \mu CD + \nu OH$, when torque $= 2\mu \times$ area ABCD. Since OJP and OEP are right angles OJEP is concyclic and the angle EOP = the angle EJP; but the triangle PJE is similar to the triangle ONM where ON represents the component λPJ of F_3 and OM the component λPE of F_1. Therefore MN is parallel to OP, and the resultant of the λ-components of F_1 and F_3 is a force λJE directed along OP. Similarly the resultant of ν-components of F_2 and F_4 is a force νGH in the same line. Hence for equilibrium

$$\lambda JE + \nu GH + X = 0 \qquad .. \qquad (31)$$

μ is the Batho torsion system, λ or ν is the frame load system (cf. §II.2). The one remaining parameter λ or ν is determined by variation of the end loads along the length of the tube.

The distribution of direct (radial) stresses is most easily discussed on the basis of boom loads; as explained in Part I this is merely a convenient convention, which does not vitiate the generality of the argument.

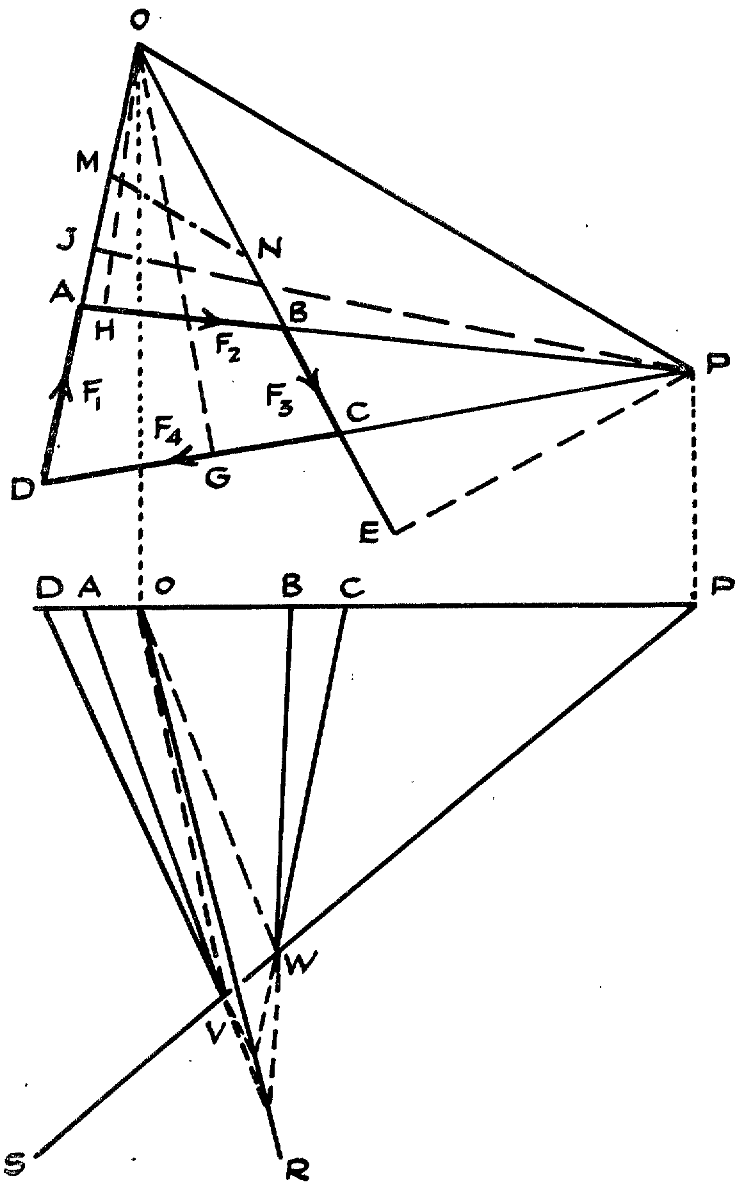


FIG. 8

In any triangle ABD (Fig. 9), forces AB.CD along BA* and AD.BC along DA are in equilibrium with a force AC.BD along AC, where C is any point in BD between B and D.

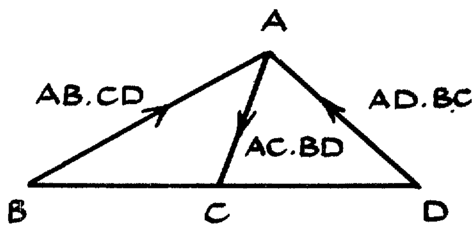


FIG. 9

Using this lemma, a force OP.VW along OP (Fig. 8) may be replaced by OV.WP along VO and OW.VP along OW; and a force OV.AD along VO may be replaced by DV.AO along DV and AV.DO along VA. Similarly for OW.BC along OW, so that a force OP.VW. BC.AD along OP may be replaced by the system.

DV.WP.AO.BC along DV
AV.WP.DO.BC along VA
CW.VP.BO.AD along WC
BW.VP.CO.AD along BW

*Here AB.CD represents only the magnitude of the force and its direction and *sense* are denoted by "along BA". The same convention is used throughout this section.

Since VP and WP are proportional to the lengths p_1 and p_2 of the perpendiculars from V and W respectively on the plane ABCD, a force OP $\left(\frac{1}{p_2} - \frac{1}{p_1}\right)$ BC.AD along OP may be replaced by forces along VD, VA, WC and WB having components P_D, P_A, etc., normal to the plane ABCD equal to $-$ OA.BC, OD.BC, OB.AD, $-$ OC.AD respectively. This result may be put in the alternative form

$$\frac{P_A}{\Delta ABCD} = \frac{-P_B}{\Delta CDA} = \frac{P_C}{\Delta DAB} = \frac{-P_D}{\Delta ABC} \quad \ldots \quad (32)$$

in which form the result is otherwise obvious by taking moments about AC and BD. The resultant X of these boom loads is then $\frac{1}{2}$OP $\left(\frac{1}{p_2} - \frac{1}{p_1}\right)(P_A - P_B - P_C + P_D)$ along OP; there is of course an alternative form for X in terms of the perpendicular distances from the plane ABCD of the intersections of the line OR with the tube walls PBAS and PCDS, but no specially simple form seems to result from combination of the two.

Finally the shear forces F_1, etc., are related to the variations of the boom loads along the length of the tube. If the boom loads at a section A′B′C′D′ parallel to ABCD and distant l from it are P_A', P_B', etc.,

$$\frac{P_A'-P_A}{l} = \frac{F_1}{A'D'} - \frac{F_2}{B'A'} = \mu\left(\frac{AD}{A'D'} - \frac{BA}{B'A'}\right) + \lambda\frac{PE}{A'D'} - \nu\frac{OG}{B'A'} \quad \ldots \quad (33a)$$

$$\frac{P_B'-P_B}{l} = \frac{F_2}{B'A'} - \frac{F_3}{C'B'} = \mu\left(\frac{BA}{B'A'} - \frac{CB}{C'B'}\right) + \nu\frac{OG}{B'A'} - \lambda\frac{PJ}{C'B'} \quad \ldots \quad (33b)$$

$$\frac{P_C'-P_C}{l} = \frac{F_3}{C'B'} - \frac{F_4}{D'C'} = \mu\left(\frac{CB}{C'B'} - \frac{DC}{D'C'}\right) + \lambda\frac{PJ}{C'B'} - \nu\frac{OH}{D'C'} \quad \ldots \quad (33c)$$

$$\frac{P_D'-P_D}{l} = \frac{F_4}{D'C'} - \frac{F_1}{A'D'} = \mu\left(\frac{DC}{D'C'} - \frac{AD}{A'D'}\right) + \nu\frac{OH}{D'C'} - \lambda\frac{PE}{A'D'} \quad \ldots \quad (33d)$$

Since the ratios P_A/P_B, P_A'/P_B', etc., are determined by the geometry of the tube and by the condition of pure torsion, these four relations must of course reduce to one only. One of the inherent indentities, that corresponding to the condition $P_A + P_B + P_C + P_D = O$ at both sections is obvious; the other two corresponding to the conditions for no bending moment on the planes ABCD and A′B′C′D′ are not easy to demonstrate. However, since $P_A - P_B - P_C + P_D = 2Xp_1p_2/OP\ (p_1 - p_2)$ the four equations may be summarized in the form

$$\frac{X'\ (p_1 + l)\ (p_2 + l)}{O'P'\ (p_1 - p_2)l} - \frac{Xp_1p_2}{OP\ (p_1 - p_2)l} = \mu\left(\frac{DC}{D'C'} - \frac{BA}{B'A'}\right) + \nu\left(\frac{OH}{D'C'} - \frac{OG}{B'A'}\right)^* \quad \ldots \quad (34)$$

and this relation together with the relations torque $= 2\mu \times$ area ABCD and $\lambda DE + \nu GH + X = O$ enable λ, μ and ν to be expressed in terms of torque, X and X′.

* As before there is an alternative form involving μ and λ.

IV.3. *Shear and Direct Loads in a Tube of Trapezoidal Section.*—In this case

$$AD = d_1,\ BC = d_1'\ AB = CD = w_1,\ JE = w_1,\ PJ = \frac{w_1 d_1}{d_1 - d_1'},$$

$$A'D' = d_0,\ B'C' = d_0',\ A'B' = C'D' = w_0,\ GH = \frac{d_1 - d_1'}{w_1}\ OG$$

$$\frac{p_1}{l} = \frac{d_1}{d_0 - d_1},\quad \frac{p_2}{l} = \frac{d_1'}{d_0' - d_1'}$$

$$\frac{P_A}{d_1'} = \frac{-P_B}{d_1} = \frac{P_C}{d_1} = \frac{-P_D}{d_1'} = \frac{P_1}{d_1'}\ \text{(Definition of } P_1)$$

$$X = P_1 d_1 \left(\frac{1}{p_2} - \frac{1}{p_1}\right)^* = \frac{P_1 d_1}{l}\left(\frac{d_0'-d_1'}{d_1'} - \frac{d_0-d_1}{d_1}\right) = \frac{P_1}{l d_1'}(d_1 d_0' - d_0 d_1')$$

$$\frac{P_A'}{d_0'} = \frac{-P_B'}{d_0} = \frac{P_C'}{d_0'} = \frac{-P_D}{d_0} = \frac{P_0}{d_0'}\ \text{(Definition of } P_0)$$

$$F_1 = \mu d_1 + \lambda \frac{d_1' w_1}{d_1 - d_1'},\quad F_3 = \mu d_1' + \frac{\lambda d_1 w_1}{d_1 - d_1'},\quad F_2 = F_4 = \mu w_1 + K\dagger$$

Then

$$\lambda JE + \nu GH + X = \lambda w_1 + \frac{d_1 - d_1'}{w_1} K + \frac{P_1}{l d_1'}(d_1 d_0' - d_0 d_1') = O,$$

(this condition may of course be found otherwise by resolving all forces on the plane ABCD parallel to AD).

$$\mu = \frac{T}{w_1 (d_1 + d_1')} \qquad \text{(35)}$$

where T is the torque.

$$\frac{P_A' - P_A}{l} + \frac{P_B' - P_B}{l} = \frac{P_0}{l}\left(1 - \frac{d_0}{d_0'}\right) - \frac{P_1}{l}\left(1 - \frac{d_1}{d_1'}\right) = \mu\left(\frac{d_1}{d_0} - \frac{d_1'}{d_0'}\right)$$

$$+ \frac{\lambda w_1}{d_1 - d_1'}\left(\frac{d_1'}{d_0} - \frac{d_1}{d_0'}\right),\ \text{from equations (33}a\text{) and (33}b\text{)}$$

Writing

$$\alpha_1 = \frac{d_1}{d_0},\ \beta_1 = \frac{w_1}{w_0},\ \eta_0 = \frac{d_0'}{d_0},\ \eta_1 = \frac{d_1'}{d_1}\ddagger,\ \text{and } T' = \frac{Tl}{w_0 d_0},$$

these relations may be expressed in the forms

$$\mu l = \frac{T'}{\beta_1 \alpha_1 (1 + \eta_1)} \qquad \text{(35a)}$$

$$\frac{\lambda\, w_1\, l}{d_1 - d_1'} = \left\{\frac{T'(\beta_1 - \alpha_1)}{\beta_1 \alpha_1 (1 + \eta_1)} + P_0 - \frac{P_1 \beta_1 \eta_0}{\alpha_1 \eta_1}\right\} \Big/ (\beta_1 + \alpha_1 \eta_1) \qquad \text{(36)}$$

---

* By resolving boom loads parallel to OP i.e. AD.

† K represents νOG because OG is infinite, but νOG remains finite.

‡ α_1, β_1, η_0 and η_1 are interrelated in the form $\alpha_1 (1 - \eta_1) \equiv \beta_1 (1 - \eta_0)$ but it is inconvenient to discard any one of these parameters. In the derivations of the forms for $\mu\, l$, etc., and subsequently frequent use is made of the identity in the form $\alpha_1 + \beta_1 \eta_0 = \alpha_1 \eta_1 + \beta_1$. Occasionally also it is convenient to use $\alpha_1' = \frac{d_1'}{d_0'} = \frac{\eta_1 \alpha_1}{\eta_0}$.

and

$$\frac{Kl}{w_1} = \left\{ \frac{T'(\alpha_1 - \beta_1)}{\alpha_1 (1 + \eta_1)} - \beta_1 P_0 + \alpha_1 P_1 \right\} \Big/ \beta_1 (\beta_1 + \alpha_1 \eta_1) \quad \ldots \quad \ldots \quad (37)$$

whence the shear forces are given by

$$(\beta_1 + \alpha_1\eta_1) \frac{F_1 l}{d_0} = T' + \alpha_1\eta_1 P_0 - \beta_1\eta_0 P_1 \quad \ldots \quad (38a)$$

$$(\beta_1 + \alpha_1\eta_1) \frac{F_3 l}{d_0'} = T' + \left(\frac{\alpha_1}{\eta_0}\right) P_0 - \left(\frac{\beta_1}{\eta_1}\right) P_1 \quad \ldots \quad (38b)$$

$$(\beta_1 + \alpha_1\eta_1) \frac{F_2 l}{w_0} = T' - \beta_1 P_0 + \alpha_1 P_1 \ldots (\text{and } F_4 = F_2) \quad \ldots \quad (38c)$$

IV.4. *Deformation of Walls, Warping and Twist.*

Deflection of AD downwards δ ,, ,, BC upwards δ' ,, ,, AB or CD sidewards ε } all relative to original plane of A′B′C′D′, i.e., deflections at frame 1 relative to original plane of frame 0.

Then twist over the bay 01 $= (\delta + \delta')/w_1 = -2\varepsilon \Big/ \left(\frac{d_1\delta' + d_1'\delta}{\delta + \delta'}\right)$, the centre of rotation being determined by the value of the ratio δ'/δ. The relation between δ, δ' and ε is more conveniently stated in the form $d_1'\delta + d_1\delta' + 2w_1\varepsilon = 0$.

From §III.4 (*a*), equations (13a) and (13b), using the results of §III.5,

$$\delta = \frac{2l^2}{EA_0 d_0} \left\{ P_1 \int \frac{u\,du}{\alpha\gamma} + \frac{F_1 l}{d_0} \int \frac{u^2 du}{\alpha^2\gamma} - \frac{F_2 l}{w_0} \int \frac{u^2 du}{\alpha\beta\gamma} \right\} + \frac{2e_0 l}{d_0} + \frac{(1 + \alpha_1)}{2Gt} \frac{F_1 l}{d_0} \quad \ldots \quad (39a)$$

$$\delta' = \frac{2\varrho l^2}{EA_0 d_0'} \left\{ \frac{P_1}{\eta_1} \int \frac{u\,du}{\alpha'\gamma'} + \frac{F_3 l}{d_0'} \int \frac{u^2 du}{\alpha'^2\gamma'} - \frac{F_2 l}{w_0} \int \frac{u^2 du}{\alpha'\beta\gamma'} \right\} + \frac{2e_0' l}{d_0'} + \frac{(1 + \alpha_1')}{2Gt'} \frac{F_3 l}{d_0'} \quad \ldots \quad (39b)$$

$$\varepsilon = \frac{l^2}{EA_0 w_0} \left\{ P_1 \int \frac{u\,du}{\beta\gamma} + \frac{\varrho P_1}{\eta_1} \int \frac{u\,du}{\beta\gamma'} + \frac{F_1 l}{d_0} \int \frac{u^2 du}{\alpha\beta\gamma} + \frac{\varrho F_3 l}{d_0'} \int \frac{u^2 du}{\alpha'\beta\gamma'} - \frac{F_2 l}{w_0} \int \left(\frac{1}{\gamma} + \frac{\varrho}{\gamma'}\right) \frac{u^2 du}{\beta^2} \right\}$$
$$+ \frac{(e_0 + e_0')l}{w_0} - \frac{(1 + \beta_1)}{2Gs} \frac{F_2 l}{w_0} \quad \ldots \quad (39c)$$

where $\alpha = \alpha_1 + (1 - \alpha_1)u$, $\beta = \beta_1 + (1 - \beta_1)u$ and similarly for γ, α' and γ', $\gamma_1\delta = A_1/A_0$, $\gamma_1' = A_1'/A_0'$ and $\varrho = A_0'/A_0$, A and A′ being the effective boom areas of section, assumed equal at top and bottom. All the integrals are between the limits 0 and 1. e_0 and $-e_0'$ are the longitudinal movements at frame 0 of the booms at A and B respectively. G is the effective shear modulus of the walls and t, t' and s are the thicknesses of the walls.*

Terms in e_0 and e_0' in the expression $d_1'\delta + d_1\delta' + 2w_1\varepsilon$ are proportional to

$$\alpha_1\eta_1 e_0 + \left(\frac{\alpha_1}{\eta_0}\right) e_0' + \beta_1 (e_0 + e_0') = (\alpha_1\eta_1 + \beta_1) \left(e_0 + \frac{e_0'}{\eta_0} \right)$$

Similarly in the expression $(\delta + \delta')/w_1$ for the twist, e_0 and e_0' appear only in the combination $e_0 + e_0'/\eta_0$, and this expression represents the effective warping corresponding to P_0.†

* The tube being symmetrical about the mid-plane between AB and CD, A, e and s are all equal top and bottom.

† The energy associated with warping is proportional to $P_0 e_0 + \frac{P_0}{\eta_0} e_0'$.

By substitution for the F's in terms of T′, P_0 and P_1 from equations (38) in equations (39) and then by substitution for δ, δ' and ε in the relation $d_1'\delta + d_1\delta' + 2w_1\,\varepsilon = 0$, it may be shown that

$$-\frac{EA_0}{l}(\alpha_1\eta_1+\beta_1)^2\left(e_0+\frac{e_0'}{\eta_0}\right)=$$
$$[I_{12}+\varrho I_{12}'+R_1(\alpha_1\eta_1\lambda_1+\alpha_1\lambda_1'-2\beta_1\mu_1)]\,T'$$
$$+\left[I_{22}+\varrho I_{22}'+R_1\left\{\alpha_1^2\eta_1^2\lambda_1+\left(\frac{\alpha_1^2}{\eta_0}\right)\lambda_1'+2\beta_1^2\mu_1\right\}\right]P_0$$
$$+\left[I_{23}+\varrho I_{23}'-R_1\alpha_1\beta_1\left(\eta_0\eta_1\lambda_1+\frac{\lambda_1'}{\eta_1}+2\mu_1\right)\right]P_1{}^{*} \qquad \ldots \quad \ldots \qquad (40)$$

Using this result the twist $\theta_1 - \theta_0 = (\delta + \delta')/w_1$ is given by $2EA_0R_1\,(\alpha_1\eta_1 + \beta_1)^2\,(\theta_1 - \theta_0) =$

$$\left[I_{11}+\varrho I_{11}'+R_1(\lambda_1+\eta_0\lambda_1'+2\mu_1)\right]T'$$
$$+\left[I_{12}+\varrho I_{12}'+R_1(\alpha_1\eta_1\lambda_1+\alpha_1\lambda_1'-2\beta_1\mu_1)\right]P_0$$
$$+\left[I_{13}+\varrho I_{13}'-R_1\left(\beta_1\eta_0\lambda_1+\beta_1\eta_0\frac{\lambda_1'}{\eta_1}-2\alpha_1\mu_1\right)\right]P_1 \qquad \ldots \quad \ldots \qquad (41)$$

Also from §III.4 (*a*) using equation (18a)

$$\frac{EA_0}{l}(e_1-\alpha_1e_0)=\alpha_1\left(P_1\int\frac{du}{\alpha\gamma}+\frac{F_1l}{d_0}\int\frac{u\,du}{\alpha^2\gamma}-\frac{F_2l}{w_0}\int\frac{u\,du}{\alpha\beta\gamma}\right)$$
$$-(1-\alpha_1^2)\frac{EA_0d_0}{4Gtl^2}\,\frac{F_1l}{d_0}+\frac{w_0d_0}{4l^2}EA_0\alpha_1(1-\beta_1)(\theta_1-\theta_0)$$

and similarly $\frac{EA_0}{l}(e_1'-\alpha_1'e_0') = \frac{EA_0}{l}\left(e_1' - \frac{\eta_1\alpha_1e_0'}{\eta_0}\right)$ = a similar function so that

$\frac{EA_0}{l}\left\{e_1+\frac{e_1'}{\eta_1}-\alpha_1\left(e_0+\frac{e_0'}{\eta_0}\right)\right\}$ may be expressed in terms of T′, P_0 and P_1.

Using the expressions for $e_0 + \frac{e_0'}{\eta_0}$ and $(\theta_1 - \theta_0)$, it may be shown that

$$\frac{EA_0}{l}(\alpha_1\eta_1+\beta_1)^2\left(e_1+\frac{e_1'}{\eta_1}\right)=\left[I_{13}+\ldots,\text{etc.}\right]T'+\left[I_{23}+\ldots,\text{etc.}\right]P_0$$
$$+\left[I_{33}+\varrho I_{33}'+R_1\left\{\beta_1^2\eta_0^2\lambda_1+\left(\frac{\beta_1^2\eta_0}{\eta_1^2}\right)\lambda_1'+2\alpha_1^2\mu_1\right\}\right]P_1 \qquad \ldots \quad \ldots \quad \ldots \qquad (42)$$

All these results are summarized below, where the forms of I_{11}, etc., are given.

IV.5. *Summary of Conclusions.*—For the trapezoidal section therefore

$$EA\frac{l_r}{l}R_r(\theta_r-\theta_{r-1})=B_rT_r'+C_rP_{r-1}+D_rP_r \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (43)$$

$$-\frac{EA}{2l}\left(e_{r-1}+\frac{e'_{r-1}}{\eta_{r-1}}\right)=C_rT_r'+H_rP_{r-1}+J_rP_r \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (44)$$

$$\frac{EA}{2l}\left(e_r+\frac{e_r'}{\eta_r}\right)=D_rT_r'+J_rP_{r-1}+K_rP_r \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (45)$$

These relations replace equations (20), (21) and (22) at the end of §III.6 (*a*) from which the only changes are the substitution of $\frac{1}{2}\left(e_r + \frac{e_r'}{\eta_r}\right)$ for e_r and similarly for e_{r-1}. As before A and l are convenient standard values of A_r and l_r.

* λ_1 and μ_1 used in this and subsequent expressions are mere geometrical constants; they are not related to λ and μ in §§IV.2 and 3.

In these equations for the trapezoidal section

$$B_r = \left\{ (I_{11} + \varrho I_{11}')_r + R_r\,(\lambda_r + \eta_{r-1}\lambda_r' + 2\mu_r) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

$$C_r = \left\{ (I_{12} + \varrho I_{12}')_r + R_r\,(\alpha_r\eta_r\lambda_r + \alpha_r\lambda_r' - 2\beta_r\mu_r) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

$$D_r = \left\{ (I_{13} + \varrho I_{13}')_r - R_r\left(\beta_r\eta_{r-1}\lambda_r + \beta_r\eta_{r-1}\frac{\lambda_r'}{\eta_r} - 2\alpha_r\mu_r\right) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

$$H_r = \left\{ (I_{22} + \varrho I_{22}')_r + R_r\left(\alpha_r^2\eta_r^2\lambda_r + \alpha_r^2\frac{\lambda_r'}{\eta_{r-1}} + 2\beta_r^2\mu_r\right) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

$$J_r = \left\{ (I_{23} + \varrho I_{23}')_r - R_r\alpha_r\beta_r\left(\eta_{r-1}\eta_r\lambda_r + \frac{\lambda_r'}{\eta_r} + 2\mu_r\right) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

$$K_r = \left\{ (I_{33} + \varrho I_{33}')_r + R_r\left(\beta_r^2\eta_{r-1}^2\lambda_r + \beta_r^2\eta_{r-1}\frac{\lambda_r'}{\eta_r^2} + 2\alpha_r^2\mu_r\right) \right\} \Big/ 2(\alpha_r\eta_r + \beta_r)^2 \frac{A_{r-1}}{A}\frac{l}{l_r}$$

where

$$\alpha_r = \frac{d_r}{d_{r-1}},\ \alpha_r' = \frac{d_r'}{d_{r-1}'},\ \beta_r = \frac{w_r}{w_{r-1}},\ \eta_r = \frac{d_r'}{d_r},\ \eta_{r-1} = \frac{d_{r-1}'}{d_{r-1}},\ R_r = \frac{w_{r-1}d_{r-1}}{4l_r^2},$$

$$\lambda_r = (1 + \alpha_r)\frac{EA_{r-1}}{Gt_rw_{r-1}},\ \ \lambda_r' = (1 + \alpha_r')\frac{EA_{r-1}}{Gt_r'w_{r-1}},\ \ \mu_r = (1 + \beta_r)\frac{EA_{r-1}}{Gs_rd_{r-1}}.$$

$$\gamma_r = \frac{A_r}{A_{r-1}},\ \gamma_r' = \frac{A_r'}{A_{r-1}'},\ \varrho_r = \frac{A_{r-1}}{A_{r-1}'}$$

and

$$(I_{11})_r = \int\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)^2\frac{u^2du}{\gamma},\quad (I_{11}')_r = \int\left(\frac{1}{\alpha'} - \frac{1}{\beta}\right)^2\frac{u^2du}{\gamma'}$$

$$(I_{12})_r = \int\left(\frac{\alpha_r\eta_r}{\alpha} + \frac{\beta_r}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{u^2du}{\gamma},\quad (I_{12}')_r = \int\left(\frac{\alpha_r'}{\eta_r\alpha'} + \frac{\beta_r}{\beta}\right)\left(\frac{1}{\alpha'} - \frac{1}{\beta}\right)\frac{u^2du}{\gamma'}$$

$$(I_{13})_r = \alpha_r\beta_r\int\left(\frac{\eta_{r-1}}{\alpha} + \frac{1}{\beta}\right)\left(\frac{1}{\alpha} - \frac{1}{\beta}\right)\frac{u(1-u)du}{\gamma},$$

$$(I_{13}')_r = \alpha_r\beta_r\int\left(\frac{1}{\eta_{r-1}\alpha'} + \frac{1}{\beta}\right)\left(\frac{1}{\alpha'} - \frac{1}{\beta}\right)\frac{u(1-u)du}{\gamma'},$$

$$(I_{22})_r = \int\left(\frac{\alpha_r\eta_r}{\alpha} + \frac{\beta_r}{\beta}\right)^2\frac{u^2du}{\gamma},\quad (I_{22}')_r = \int\left(\frac{\alpha_r'}{\eta_r\alpha'} + \frac{\beta_r}{\beta}\right)^2\frac{u^2du}{\gamma'}$$

$$(I_{23})_r = \alpha_r\beta_r\int\left(\frac{\alpha_r\eta_r}{\alpha} + \frac{\beta_r}{\beta}\right)\left(\frac{\eta_{r-1}}{\alpha} + \frac{1}{\beta}\right)\frac{u(1-u)du}{\gamma},$$

$$(I_{23}')_r = \alpha_r\beta_r\int\left(\frac{\alpha_r'}{\eta_r\alpha'} + \frac{\beta_r}{\beta}\right)\left(\frac{1}{\eta_{r-1}\alpha'} + \frac{1}{\beta}\right)\frac{u\,(1-u)^2du}{\gamma'}$$

$$(I_{33})_r = \alpha_r^2\beta_r^2\int\left(\frac{\eta_{r-1}}{\alpha} + \frac{1}{\beta}\right)^2\frac{(1-u)^2du}{\gamma},$$

$$(I_{33}')_r = \alpha_r^2\beta_r^2\int\left(\frac{1}{\eta_{r-1}\alpha'} + \frac{1}{\beta}\right)^2\frac{(1-u)^2du}{\gamma'},$$

all the integrals being between the limits 0 and 1.

In all other respects the analysis of the trapezoidal section follows exactly the same lines as that for the rectangular section.

PART V

Conically Tapered Tube of Rectangular Section

In a conically tapered tube $w_{r-1}/w_0 = d_{r-1}/d_0 = r/r_0$ where r is the distance from the apex of the tube. Assume also that $A_{r-1}/A_0 = r/r_0$: if no actual booms are fitted $A_{r-1} = 1/6\ (td_{r-1} + sw_{r-1})$ and the condition is satisfied provided that t and s are constant. If now l_r is made also proportional to $r (l_r = rz$ say where z is a constant), all the coefficients B_r, etc., become invariant with r. From §III.4, we have

$$\alpha_r = d_r/d_{r-1} = (r - rz)/r = 1 - z = \beta_r = \gamma_r$$

similarly from §III.6

$$R_r = w_0 d_0/4r_0^2 z^2,\ I_{11} = I_{12} = I_{13} = 0$$

$$\lambda_r = (2 - z)\ EA_0/Gtw_0,\ I_{22} = 4(1 - z)^2 \left(\frac{1}{3} + \frac{1}{4}z + \frac{1}{5}z^2 + \ldots.\right)$$

$$\mu_r = (2 - z)\ EA_0/Gsd_0,\ I_{23} = 4\ (1 - z)^2 \left(\frac{1}{2.3} + \frac{z}{3.4} + \frac{z^2}{4.5} + \ldots.\right)$$

$$I_{33} = 8\ (1 - z)^2 \left(\frac{1}{1.2.3} + \frac{z}{2.3.4} + \frac{z^2}{3.4.5} + \ldots.\right)$$

and taking l as $r_0 z$ and A as A_0, $(\alpha_r + \beta_r)^2 \dfrac{A_{r-1}}{A} \dfrac{l}{l_r} = 4\ (1 - z)^2$.

Then, dropping suffixes, which are no longer necessary,

$$H + K = \left(\frac{2}{3} + \frac{1}{3}z + \frac{7}{30}z^2 + \ldots.\right) + \frac{1}{8z^2}(2 - z)\frac{EA_0}{Gr_0^2}\left(\frac{d_0}{t} + \frac{w_0}{s}\right) \qquad (46)$$

$$J = \left(\frac{1}{6} + \frac{1}{12}z + \frac{1}{20}z^2 + \ldots.\right) - \text{half term above} \qquad (47)$$

$$-C = D = \frac{1}{16z^2}\ \frac{2 - z}{1 - z}\ \frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s} - \frac{d_0}{t}\right) \qquad (48)$$

and

$$T_n' \left(= \frac{T_n l_r}{w_{r-1} d_{r-1}}\right) = \frac{r_0^2 z}{w_0 d_0}\ \frac{T_n}{r}\ (T_n \text{ being the torque transmitted at section } r) \qquad (49)$$

From §III.4 (*c*) (equation 26)

$$JP_{n-1} + (H + K)P_n + JP_{n+1} = D\ (T_{n+1}' - T_n') \qquad (50)$$

But

$$P_{n\pm1} = P_n \mp rz\left(\frac{dP}{dr}\right)_n + \tfrac{1}{2}\ (rz)^2\left(\frac{d^2P}{dr^2}\right)_n^{*} \qquad (51)$$

and

$$T_{n+1} = T_n - rz\left(\frac{dT}{dr}\right)_n \qquad (52)$$

From (49) and (52)

$$T_{n+1}' - T_n' = \frac{r_0^2 z}{w_0 d_0}\left(\frac{T_{n+1}}{r(1 - z)} - \frac{T_n}{r}\right) = \frac{r_0^2 z}{w_0 d_0 r(1 - z)}\left\{T_n - rz\left(\frac{dT}{dr}\right)_n - T_n\ (1 - z)\right\}$$

$$= \frac{r_0^2 z^2}{w_0 d_0\ (1 - z)}\left(\frac{T}{r} - \frac{dT}{dr}\right) \qquad (53)$$

---

* Negative signs because direction of n increasing is that of r decreasing.

Then using (46), (47), (48), (51) and (53) in (50) and dropping terms in z^2 and higher powers

$$\left\{\frac{1}{6}\left(1+\frac{z}{2}\right)-\frac{2-z}{16z^2}\,\frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s}+\frac{d_0}{t}\right)\right\}\left(2P+r^2z^2\frac{d^2P}{dr^2}\right)$$
$$+\left\{\frac{2}{3}\left(1+\frac{z}{2}\right)+\frac{2-z}{8z^2}\,\frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s}+\frac{d_0}{t}\right)\right\}P$$
$$=\frac{1}{16z^2}\,\frac{2-z}{1-z}\,\frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s}-\frac{d_0}{t}\right)\frac{r_0^2z^2}{w_0d_0\,(1-z)}\left(\frac{T}{r}-\frac{dT}{dr}\right) \quad \ldots \quad (54)$$

or

$$P\left(1+\frac{z}{2}\right)-\frac{2-z}{16}\,\frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s}+\frac{d_0}{t}\right)r^2\frac{d^2P}{dr^2}$$
$$=\frac{2-z}{16\,(1-z)^2}\,\frac{EA_0}{Gw_0d_0}\left(\frac{w_0}{s}-\frac{d_0}{t}\right)\left(\frac{T}{r}-\frac{dT}{dr}\right) \quad \ldots \quad \ldots \quad \ldots \quad (54a)$$

In the limit when $z \rightarrow 0$

$$\frac{EA_0}{Gr_0^2}\left(\frac{w_0}{s}+\frac{d_0}{t}\right)r^2\frac{d^2P}{dr^2}-8P-\frac{EA_0}{Gw_0d_0}\left(\frac{w_0}{s}-\frac{d_0}{t}\right)\left(\frac{dT}{dr}-\frac{T}{r}\right)=0 \quad \ldots \quad \ldots \quad (55)$$

When r and r_0 tend to infinity, putting $r_0 - r = x$, this reduces to

$$\frac{EA_0}{G}\left(\frac{w}{s}+\frac{d}{t}\right)\frac{d^2P}{dx^2}-8P+\frac{EA_0}{Gwd}\left(\frac{w}{s}-\frac{d}{t}\right)\frac{dT}{dx}=0 \quad \ldots \quad \ldots \quad \ldots \quad (55a)$$

agreeing with equation (7) of §II.9.

When $\frac{w_0}{d_0}=\frac{s}{t}$, P is independent of T, and torsion produces no warping of the tube cross sections. Also when T varies in proportion to r, the shear force in each wall is independent of r and there are no boom loads.*

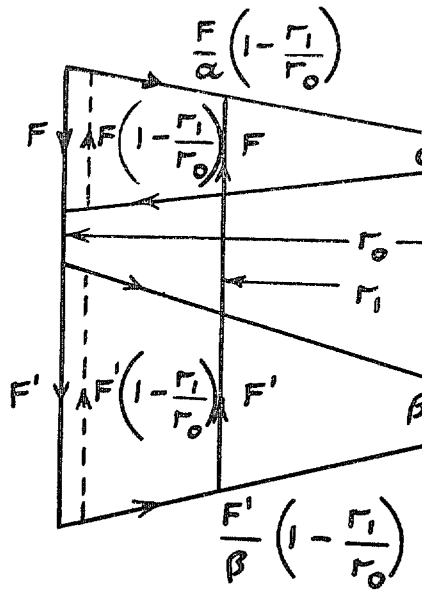


FIG. 10

* This special case perhaps needs further explanation. The diagram, Fig. 10, represents one short bay of the tube from just inboard one frame (r_1) to just inboard the next frame (r_0). If F is the shear load on one tube wall just inboard of the frame at r_1, by moments about the apex of the tube, the shear force just *outboard* of the frame at r_0 must be Fr_1/r_0, because the only other forces on the wall between these two sections are those along the top and bottom edges, and these forces pass through the apex of the wall. Therefore the frame at r_0 must apply a force

$$F\left(1-\frac{r_1}{r_0}\right)$$

to bring the total shear just inboard of the frame back to F. Then, resolving forces vertically, the forces along the sloping edges of this wall must be

$$\frac{F}{\alpha}\left(1-\frac{r_1}{r_0}\right),$$

where α is the apical angle of the wall (α small). Then change of bending moment in the wall between r_1 and r_0 is

$$\frac{F}{\alpha}\left(1-\frac{r_1}{r_0}\right)r_0\alpha - F\,(r_0 - r_1)$$

and is zero. Of course F/α must equal F'/β, that is F and F' must result from equal loads per unit length of edge—the Batho system. It is interesting to note that the forces on the frame are such that the frame itself is not sheared; this implies, of course, that the sections of the tube do not tend to distort. Provided that the forces

$$F\left(1-\frac{r_1}{r_0}\right)$$

are properly applied, the frames may be omitted without altering the conditions. Naturally, the reaction torque, wherever it be applied, still gives rise to boom loads.

In the general tapered tube, held at the root and subjected to a torque T_0 at the tip

$$P = \frac{Y}{X-2}\,\frac{T_0}{r} + Vr^n + Wr^{1-n} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (56)$$

where

$$n = \tfrac{1}{2} + \sqrt{X + \tfrac{1}{4}}$$

$$X = 8Gr_0^2/EA_0\left(\frac{w_0}{s} + \frac{d_0}{t}\right)$$

$$Y = r_0^2\left(\frac{w_0}{s} - \frac{d_0}{t}\right)\Big/\, w_0d_0\left(\frac{w_0}{s} + \frac{d_0}{t}\right)$$

and V and W are constants to be determined from the end conditions. From §III.4 (c) (equation (27b)), writing

$$P_1 = P_0 - rz\left(\frac{dP}{dr}\right)_0 + \ldots\ldots$$

and

$$T_1' = \frac{r_0^2 z}{w_0 d_0}\,\frac{T_1}{r}$$

terms in $\dfrac{1}{z}$ give $r\,\dfrac{dP}{dr} = \dfrac{r_0^2}{w_0d_0}\,\dfrac{\dfrac{w_0}{s} - \dfrac{d_0}{t}}{\dfrac{w_0}{s} + \dfrac{d_0}{t}}\,\dfrac{T}{r} = Y\,\dfrac{T}{r}$ at $r = r_0$*

Hence

$$-\frac{Y}{X-2}\,\frac{T_0}{r_0} + n\,V\,r_0^n + (1-n)Wr_0^{1-n} = +\,Y\,\frac{T_0}{r_0}$$

and, since $P = 0$ at $r = r_1$

$$\frac{Y}{X-2}\,\frac{T_0}{r_1} + Vr_1^n + Wr_1^{1-n} = 0.$$

In many cases, X is large and n is much greater than unity. In that case nVr_0^n and Wr_1^{1-n} are comparable in magnitude to T_0/r_0, and Vr_1^n and $(1-n)Wr_0^{1-n}$ are negligible in comparison. Therefore approximately

$$Pr = \frac{Y}{X-2}\,T_0\left\{1 + \frac{X-1}{n}\left(\frac{r}{r_0}\right)^{n+1} - \left(\frac{r_1}{r}\right)^{n-2}\right\}. \qquad \ldots \quad \ldots \quad \ldots \qquad (57)$$

Over the greater part of the length of the tube

$$Pr = \frac{Y}{X-2}\,T_0$$

and is constant ; at the free end P_r falls abruptly to zero and at the fixed end it rises abruptly to

$$1 + \frac{X-1}{n}$$

times its general level. Referring to equation (54a) it will be seen that the variation of the equation of equilibrium with z is only slight. Accordingly, variation of the boom loads as the

*Sign corresponds with that in §II.9 because r here is measured in opposite direction to x in that section. Y/r^2 corresponds to $\beta/\alpha L^2$ in that part.

average frame spacing is varied, is only slight. This is illustrated in Fig. 11 which shows the variation of boom load along the length of the most highly tapered tube treated by Williams in R. & M. 1761, §4, p. 20, *et seq.* The full continuous line shows the distribution calculated from equation (57) for this case, whilst the two broken lines show the distribution computed with four and six frames respectively ; the detailed computation for the latter two cases is shown in the attached sheets. It will be seen that the differences of boom load resulting from variation of the number of frames are quite minor.

On Fig. 11, the distribution of boom load computed by Williams in R. & M. 1761 is also shown. This distribution is similar in form to that found here but indicates considerably greater values of the boom loads, particularly near the tip of the tube. These differences are to be expected, in view of the different method of computation used in R. & M. 1761. Detailed comparison of the two analyses is difficult ; but the following points are worth noting. Equation (100) of R. & M. 1761 corresponds to the relation

$$\mathrm{T} = \mathrm{F}w_1 - \mathrm{F}'d_1 + (d_0 w_1 - d_1 w_0)\,\frac{\mathrm{P}_1}{l}$$

in §III.4(a), but since d/w is constant, the term in P_1 disappears. In accordance with this correspondence

$$\left.\begin{aligned} &\mathrm{F} \text{ in the present paper} = 2ak\dot{\mathrm{S}} - \frac{d\mathrm{M}_1}{dx} - \frac{\mathrm{M}_1}{l-x} \\ &\text{and} \\ &\mathrm{F}' \text{ in the present paper} = -\,2a\mathrm{S} + \frac{d\mathrm{M}_2}{dx} + \frac{\mathrm{M}_2}{l-x} \end{aligned}\right\} \text{Notation of R. \& M. 1761.}$$

In R. & M. 1761 the shear force across the tube wall is taken as

$$\frac{d\mathrm{M}_1}{dx} - 2\,ak\mathrm{S} \text{ (equation (98)), that is } \mathrm{F} + \frac{\mathrm{M}_1}{l-x},$$

whereas in the present paper it is taken as F in accordance with the conclusions of the Appendix. This difference is associated with the treatment in R. & M. 1761 of the bending moment as resulting from a system of parallel forces, whereas in the present paper a radial system is assumed. Moreover in R. & M. 1761 the rotation of sections of the tube walls as a result of shear distortion is disregarded. These separate differences should be at least partially self-compensating ; but Fig. 11 shows that their combined effect may still leave considerable discrepancies.

Conically Tapered Tube with Four Frames.

Example taken from R. & M. 1761, § 4.18, taking $l = 135$ inches.

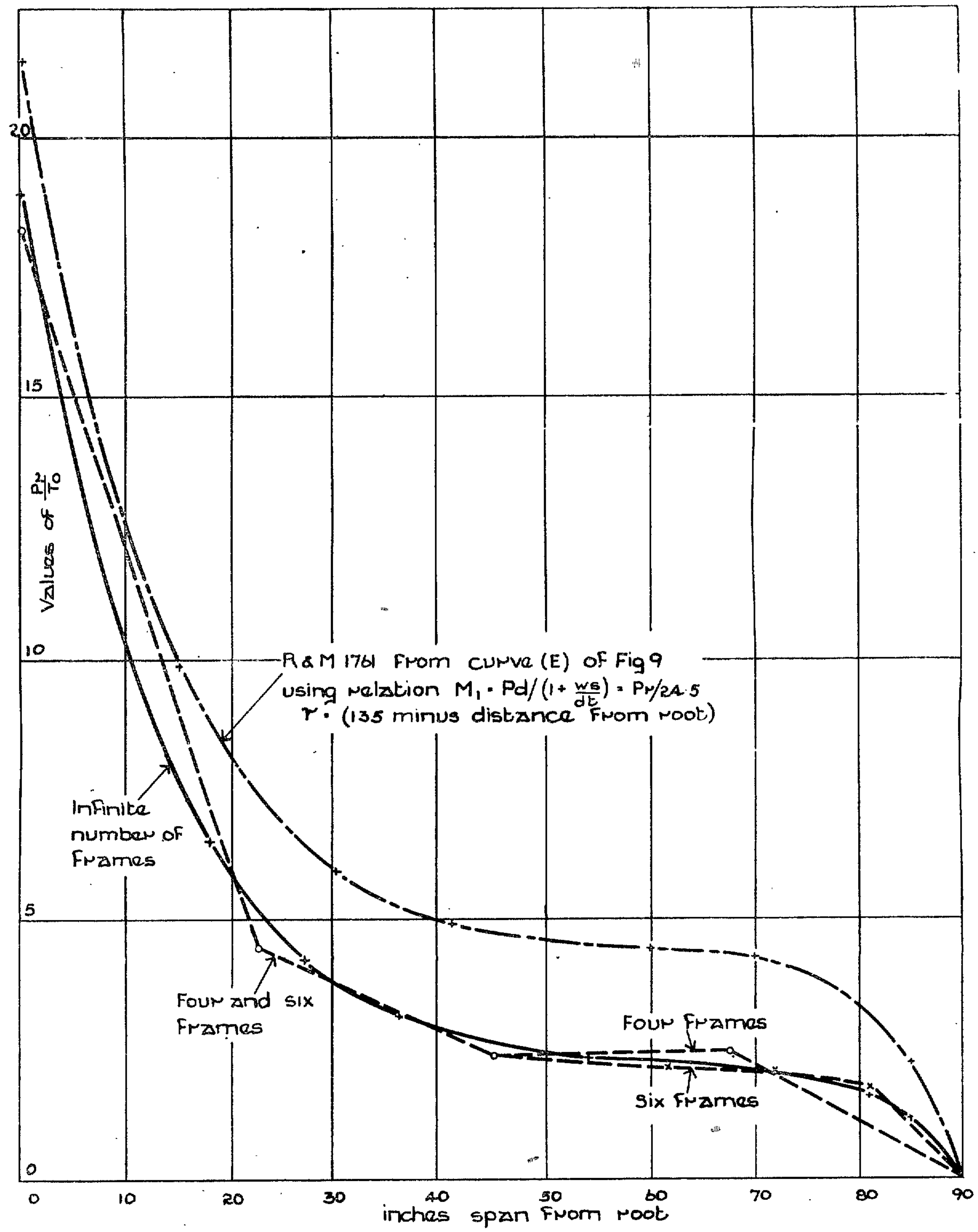
Specification (dimensions in inches : unit torque applied at tip)

Frame or Bay No.	0	1	2	3	4		
d_r	6	5	4	3	2	—	—
w_r	20	$20\times\frac{5}{6}$	$20\times\frac{2}{3}$	10	$20\times\frac{1}{3}$	—	—
A_r	1·631	same $\times\frac{5}{6}$	same $\times\frac{2}{3}$	same $\times\frac{1}{2}$	same $\times\frac{1}{3}$	$A = 1{\cdot}631$	—
l_r	—	22·5	22·5	22·5	22·5	$l = 22{\cdot}5$	—
t_r	—	1·5	1·5	1·5	1·5	—	—
s_r	—	0·0394	0·0394	0·0394	0·0394	—	—
T_r	—	1	1	1	1	—	Torque applied at tip.
$\alpha_r = \beta_r = \gamma_r = d_r/d_{r-1}$	—	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{3}{4}$	$\frac{2}{3}$	—	—
$R_r = w_{r-1}d_{r-1}/4l_r^2$ ..	—	0·05926	0·04116	0·02634	$0{\cdot}01481_5$	—	—
$\lambda_r = (1+\alpha_r)EA_{r-1}/Gt_rw_{r-1}$	—	0·2592	0·2545	0·2475	0·2357	—	$\left.\right\} \dfrac{E}{G} = 2{\cdot}6$
$\mu_r = (1+\beta_r)EA_{r-1}/GS_rd_{r-1}$	—	31·304	30·735	29·881	28·458	—	
$l_r/w_{r-1}d_{r-1}$	—	0·1875	0·2700	0·4219	0·7500	—	—
$O_{11} = R_r(\lambda_r+\mu_r)$ ..	—	1·8704	1·2762	0·7982	0·4327	—	—
$-O_{12} = O_{13} = R_r\alpha_r(\mu_r-\lambda_r)$ ..	—	1·5330	1·0032	0·5854	0·2787	—	—
$O_{22} = -O_{23} = O_{33} = R_r\alpha_r^2(\lambda_r+\mu_r)$	—	1·2989	0·8168	0·4490	0·1923	—	—
I_{22}	—	1·0594	1·0058	0·9261	0·7954	—	$I_{11} = I_{12} = I_{13} = 0$
I_{23}	—	0·5006	0·4753	0·4305	0·3586	—	—
I_{33}	—	0·9672	0·8996	0·8022	0·6498	—	—
$B_r = (I_{11}+O_{11})_r/(\alpha_r+\beta_r)^2 \dfrac{A_{r-1}}{A}\dfrac{l}{l_r}$	—	+0·6733	+0·5982	+0·5321	+0·4868	—	$A = 1{\cdot}631$
$C_r = (I_{12}+O_{12})_r/4\alpha_r^2 \dfrac{A_{r-1}}{A}$	—	−0·5519	−0·4702	−0·3903	−0·3135	—	—
$D_r = (I_{13}+O_{13})_r/4\alpha_r^2 \dfrac{A_{r-1}}{A}$	—	+0·5519	+0·4702	+0·3903	+0·3135	—	—
$H_r = (I_{22}+O_{22})_r/4\alpha_r^2 \dfrac{A_{r-1}}{A}$	—	+0·8490	+0·8542	+0·9167	+1·1112	—	—
$J_r = (I_{23}+O_{23})_r/4\alpha_r^2 \dfrac{A_{r-1}}{A}$	—	−0·2874	−0·1601	−0·0123	+0·1871	—	—
$K_r = (I_{33}+O_{33})_r/4\alpha_r^2 \dfrac{A_{r-1}}{A}$	—	+0·8158	+0·8045	+0·8341	+0·9474	—	—
$T_r' = T_rl_r/w_{r-1}d_{r-1}$ ..	—	0·1875	0·2700	0·4219	0·7500	—	—
	—	+0·8490	−0·2874	—	—	−0·10348	—
	—	−0·2874	+1·6700	−0·1601	—	−0·02347	—
	—	—	−0·1601	+1·7212	−0·0123	−0·03773	—
	—	—	—	−0·0123	+1·9453	−0·07044	—
	—	—	+1·572710	−0·1601	—	−0·058500	—
	—	—	—	+1·704900	−0·0123	−0·043686	—
	—	—	—	—	+1·945211	−0·070755	—
Boom Loads..	—	+0·13537	+0·03984	$+0{\cdot}02588_5$	+0·03637	—	lb./lb.-in. torque

Conically Tapered Tube with Six Frames.
Same Tube as in previous Example.

$t_r = 1{\cdot}5$, $s_r = 0{\cdot}0394$, $T_r = 1$

Frame or Bay No.	0	1	2	3	4	5	6	Notes
$d_r = 0{\cdot}3w_r$	6	5	4	3·2	2·8	2·4	2·0	$A_r = \frac{1{\cdot}631}{6} d_r$
l_r	—	22·5	22·5	18	9	9	9	—
$\alpha_r = \beta_r = \gamma_r = d_r/d_{r-1}$	—	$\frac{5}{6}$	$\frac{4}{5}$	$\frac{4}{5}$	$\frac{7}{8}$	$\frac{6}{7}$	$\frac{5}{6}$	—
$R_r = w_{r-1}d_{r-1}/4l_r^2$	—	0·05926	0·04116	0·04116	0·10535	0·08066	0·05926	—
$\lambda_r = (1 + \alpha_r)EA_{r-1}/Gt_rw_{r-1}$ ..	—	0·2592	0·2545	0·2545	0·2651	0·2626	0·2592	—
$\mu_r = (1 + \alpha_r)EA_{r-1}/Gs_rd_{r-1}$ ..	—	31·304	30·735	30·735	32·016	31·710	31·304	—
$O_{11} = R_r(\lambda_r + \mu_r)$	—	1·8704	1·2762	1·2762	3·4008	2·5789	1·8704	—
$-O_{12} = O_{13} = R_r\alpha_r(\beta_r - \lambda_r)$	—	1·5330	1·0032	1·0032	2·9269	2·1742	1·5330	—
$O_{22} = -O_{23} = O_{33} = R_r\alpha_r^2(\lambda_r + \mu_r)$	—	1·2989	0·8168	0·8168	2·6037	1·8947	1·2989	—
I_{22}	—	1·0594	1·0058	1·0058	1·1273	1·0982	1·0594	$I_{11} = I_{12} = I_{13} = 0$
I_{23}	—	0·5006	0·4753	0·4753	0·5458	0·5280	0·5006	—
I_{33}	—	0·9672	0·8996	0·8996	1·0547	1·0204	0·9672	—
$B_r = (I_{11} + O_{11})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	+0·6733	+0·5982	+0·5982	+0·8328	+0·7522	+0·6733	$l = 22{\cdot}5$, $A = 1{\cdot}631$
$C_r = (I_{12} + O_{12})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	−0·5519	−0·4702	−0·4702	−0·7168	−0·6341	−0·5519	—
$D_r = (I_{13} + O_{13})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	+0·5519	+0·4702	+0·4702	+0·7168	+0·6341	+0·5519	—
$H_r = (I_{22} + O_{22})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	+0·8490	+0·8542	+0·8542	+0·9137	+0·8729	+0·8490	—
$J_r = (I_{23} + O_{23})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	−0·2874	−0·1601	−0·1601	−0·5040	−0·3986	−0·2874	—
$K_r = (I_{33} + O_{33})/4\alpha_r^2 \; \frac{A_{r-1}}{A} \; \frac{l}{l_r}$	—	+0·8158	+0·8045	+0·8045	+0·8959	+0·8502	+0·8158	—
$T_r' = T_rl_r/w_{r-1}d_{r-1}$	—	0·1875	0·2700	0·3375	0·2637	0·3444	0·4687	—
	—	+0·8490	−0·2874	—	—	—	—	−0·10348
	—	−0·2874	+1·6700	−0·1601	—	—	—	−0·02347
	—	—	−0·1601	+1·6587	−0·1601	—	—	−0·03174
	—	—	—	−0·1601	+1·7182	−0·5040	—	−0·03033
	—	—	—	—	−0·5040	+1·7688	−0·3986	−0·02936
	—	—	—	—	—	−0·3986	+1·6992	−0·04032
	—	·	+1·57271	−0·1601	—	—	—	−0·058500
	—	—	—	+1·64240	−0·1601	—	—	−0·037695
	—	—	—	—	+1·70259	−0·5040	—	−0·034004
	—	—	—	—	—	+1·61961	−0·3986	−0·039426
	—	—	—	—	—	—	+1·60110	−0·050023
Boom Loads	—	+0·13537	+0·03983	+0·02584	+0·02943	+0·03202	+0·03124	lb./lb.-in. torque



Ratios of (Boom Load x radius) to Torque for tube 90 inches long, 20 inches wide and 6 in. deep at the root tapering to a point in 135 inches, with cover thickness 0·0394 inches (across width) and spar thickness 1·5 inches (down depth) – no actual booms fitted.

FIG. 11.

APPENDIX

The Simple Theory of Bending and Shear for Tapered Beams

A.1. *Introduction.*—The ordinary simple theory of bending as commonly used by engineers is exact only in certain special cases ; but its use in other cases, to which it is not strictly applicable, is common practice and the practice is justified by experience. In stressing *parallel* tubes treatment of the warping distortions by the simple theory of bending for a parallel beam yields results in good agreement with experimental results. For stressing tapered tubes, therefore it is deemed sufficient to establish for tapered beams a comparable simple theory of bending.

A.2. *Basic Analysis.*—The analysis here presented has been previously developed in more detail by Atkin[3] in " Aircraft Engineering ", November and December, 1938. It is repeated here in order to demonstrate the basis of the simple rules for stressing slightly tapered beams, of which use is made in Parts III and IV of the paper.

Using cylindrical co-ordinates, with pole at the apex of the beam, the stress distribution in a beam of uniform (unit) width has to satisfy the equations of equilibrium.

$$\frac{\delta\,(r.\overline{rr})}{\delta r} + \frac{\delta \overline{r\theta}}{\delta \theta} - \overline{\theta\theta} = 0 \text{ and } \frac{\delta\,(r^2\,\overline{r\theta})}{\delta r} + r\,\frac{\delta \overline{\theta\theta}}{\delta \theta} = 0 \quad \ldots \quad (58)$$

Solutions of these equations are

$$\overline{rr} = \frac{f'\,(\theta)}{r^2} + \frac{\varphi'\,(\theta)}{r},\ \overline{\theta\theta} = 0 \text{ and } \overline{r\theta} = \frac{f\,(\theta)}{r^2} \quad \ldots \quad (59)$$

where $f(\theta)$ and $\varphi\,(\theta)$ are arbitrary functions of θ.

The relations between stress and strain

$$e_{rr} = (\overline{rr} - \sigma\overline{\theta\theta})/E,\ e_{\theta\theta} = (\overline{\theta\theta} - \sigma\overline{rr})/E \text{ and } e_{r\theta} = \frac{\overline{r\theta}}{G} \quad \ldots \quad (60)$$

and the conditions for compatibility of strains

$$e_{rr} = \frac{\delta u}{\delta r},\ e_{\theta\theta} = \frac{\delta v}{r\delta\theta} + \frac{u}{r} \text{ and } e_{r\theta} = \frac{\delta u}{r\delta\theta} + r\,\frac{\delta}{\delta r}\left(\frac{v}{r}\right) \quad \ldots \quad (61)$$

lead to the formulae

$$Eu = -\frac{f'\,(\theta)}{r} + \varphi'(\theta)\log r + \psi'\,(\theta) \quad \ldots \quad (62a)$$

$$Ev = (1-\sigma)\,\frac{f(\theta)}{r} - (\sigma + \log r)\,\varphi\,(\theta) - \varphi\,(\theta) + F\,(r) \quad \ldots \quad (62b)$$

and the condition

$$-\left[f''\,(\theta) + \left\{2\,(1-\sigma) + \frac{E}{G}\right\} f\,(\theta)\right]\frac{1}{r} + \left\{\varphi''\,(\theta) + \varphi\,(\theta)\right\}\log r$$

$$+\ \psi''\,(\theta) + \psi\,(\theta) - (1-\sigma)\,\varphi\,(\theta) + rF'(r) - F(r) = 0 \quad \ldots \quad (63)$$

Assuming that E/G $= 2\,(1 + \sigma)$, this condition is satisfied by

$$\left.\begin{aligned} f(\theta) &= A + B\cos 2\,\theta + C\sin 2\theta \\ \varphi(\theta) &= L\cos\theta + M\sin\theta \\ \psi(\theta) &= P\cos\theta + Q\sin\theta + \frac{1-\sigma}{2}\,\theta\,(L\sin\theta + M\cos\theta) \\ \text{and}\quad F(r) &= Kr - \frac{2A}{r}. \end{aligned}\right\}\ \begin{array}{c}\text{where A, B, C, L,}\\ \text{M, P, Q, and K}\\ \text{are constants.}\end{array}$$

When the stresses become

$$\overline{rr} = \frac{(-2B\sin 2\theta + 2C\cos 2\theta)}{r^2} + \frac{(-L\sin\theta + M\cos\theta)}{r} \quad \text{(64a)}$$

$$\overline{\theta\theta} = 0 \quad \text{(64b)}$$

$$\overline{r\theta} = \frac{A + B\cos 2\,\theta + C\sin 2\theta}{r^2} \quad \text{(64c)}$$

and the displacements U and V parallel and perpendicular to the line $\theta = 0$ are given by

$$\begin{aligned} EU &= \frac{3-\sigma}{2r}(B\sin\theta - C\cos\theta) + \frac{1+\sigma}{2r}(B\sin 3\,\theta - C\cos 3\theta + 2A\sin\theta) \\ &\quad - Kr\sin\theta + Q + M\log r + \frac{1+\sigma}{4}(L\sin 2\,\theta - M\cos 2\,\theta) - \frac{1-3\sigma}{4}M \\ &\quad + \frac{1-\alpha}{2}L\theta \end{aligned} \quad \text{(65a)}$$

and

$$\begin{aligned} EV &= \frac{3-\sigma}{2r}(B\cos\theta + C\sin\theta) - \frac{1+\sigma}{2r}(B\cos 3\theta + C\sin 3\theta + 2A\cos\theta) \\ &\quad + Kr\cos\theta - P - L\log r - \frac{1+\sigma}{4}(L\cos 2\,\theta + M\sin 2\,\theta) + \frac{1-3\sigma}{4}L \\ &\quad + \frac{1-\sigma}{2}M\theta \end{aligned} \quad \text{(65b)}$$

and the rotation ξ of the plane section through $(r, \pm\alpha)$ is given by

$$\begin{aligned} E\xi &= \frac{3-\sigma}{2r^2}B + \frac{1+\sigma}{2r^2}\Big\{B\cdot(3 - 4\sin^2\alpha) + 2A\Big\} - K + \frac{1+\sigma}{2r}L\cos\alpha \\ &\quad + \frac{1-\sigma}{2r}L\frac{\alpha}{\sin\alpha}. \end{aligned} \quad \text{(66)}$$

From equations (65) and (66) it will be seen that P and Q represent body translations of the whole beam and that K represents rotation of the whole beam about its pole. The C and M terms represent the effects of loads along the axis of the beam and these terms will not be further considered.

Then for the deflection V_0 of the centre line $(\theta = 0)$ of the beam

$$EV_0 = \frac{2B}{r} - \frac{1+\sigma}{r}(A + B) + Kr - L\log r + \text{const.} \quad \text{(67)}$$

$$E\frac{\delta V_0}{\delta r} = -\frac{2B}{r^2} + \frac{1+\sigma}{r^2}(A + B) + K - \frac{L}{r} \quad \text{(68)}$$

$$E\frac{\delta^2 V_0}{\delta r^2} = \frac{4B}{r^3} + \frac{L}{r^2} - \frac{2(1+\sigma)}{r^3}(A + B). \quad \text{(69)}$$

The difference $e_1 - e_2$ between the radial strains at $(r, \pm\alpha)$ divided by the distance $(d = 2r \sin \alpha)$ between these points is

$$\frac{e_1 - e_2}{d} = \frac{\widehat{rr}_\alpha - \widehat{rr}_{-\alpha}}{\mathrm{E}d} = -\frac{1}{\mathrm{E}}\left(\frac{4\mathrm{B}\cos\alpha}{r^3} + \frac{\mathrm{L}}{r^2}\right) \quad \ldots \quad (70)$$

and the shear stress q_0 (at $\theta = 0$) $= (\mathrm{A} + \mathrm{B})/r^2$. .. .. (71)

If α be so small that α^2 is negligible in comparison with unity, equation (66) reduces to

$$\mathrm{E}\xi = \frac{2\mathrm{B}}{r^2} + \frac{\mathrm{L}}{r} + \frac{(1+\sigma)(\mathrm{A}+\mathrm{B})}{r^2} - \mathrm{K} \quad \ldots \quad (72)$$

equation (67) to

$$\mathrm{EV}_0 = \frac{2\mathrm{B}}{r} - \mathrm{L}\log r + \mathrm{K}r - \frac{(1+\sigma)(\mathrm{A}+\mathrm{B})}{r} + \text{const.} \quad \ldots \quad (73)$$

and equation (70) to

$$\frac{(e_1 - e_2)}{d} = -\frac{1}{\mathrm{E}}\left(\frac{4\mathrm{B}}{r^3} + \frac{\mathrm{L}}{r^2}\right). \quad \ldots \quad (74)$$

Then choosing the constants in equations (72) and (73) so that $\xi = 0$ and $\mathrm{V}_0 = 0$ at $r = r_0$

$$\mathrm{E}\xi = -\int_{r_0}^{r}\left(\frac{4\mathrm{B}}{r^3} + \frac{\mathrm{L}}{r^2}\right)dr - 2(1+\sigma)(\mathrm{A}+\mathrm{B})\int_{r_0}^{r}\frac{dr}{r^3} \quad \ldots \quad (75)$$

and

$$\mathrm{EV}_0 = -\int_{r_0}^{r}\left(\frac{2\mathrm{B}}{r^2} + \frac{\mathrm{L}}{r}\right)dr + \int_{r_0}^{r}\left(\frac{2\mathrm{B}}{r_0^2} + \frac{\mathrm{L}}{r_0}\right)dr$$

$$+\, 2(1+\sigma)(\mathrm{A}+\mathrm{B})\, r\int_{r_0}^{r}\frac{dr}{r^3} \quad \ldots$$

or

$$\int_{r_0}^{r}\int_{r_0}^{r}\left(\frac{4\mathrm{B}}{r^3} + \frac{\mathrm{L}}{r^2}\right)drdr + 2(1+\sigma)(\mathrm{A}+\mathrm{B})\, r\int_{r_0}^{r}\frac{dr}{r^3}. \quad \ldots \quad (76)$$

Hence, using equations (71) and (74)

$$\xi = \int_{r_0}^{r}\frac{e_1 - e_2}{d}\,dr - \frac{q_0 r^2}{\mathrm{G}}\int_{r_0}^{r}\frac{dr}{r^3} \quad \ldots \quad (77)$$

and

$$\mathrm{V}_0 = -\int_{r_0}^{r}\int_{r_0}^{r}\frac{e_1 - e_2}{d}\,drdr + \frac{q_0 r^3}{\mathrm{G}}\int_{r_0}^{r}\frac{dr}{r^3}. \quad \ldots \quad (78)$$

Equations (77) and (78) are strictly true only if $e_1 - e_2$ is of the form $\frac{\lambda}{r^2} + \frac{\mu}{r}$ where λ and μ are constants and if $q_0 r^2$ is constant (cf. equation (64)); but in practice it appears reasonable to use them more generally. This extension may be justified by analogy with the case of the parallel beam.

A.3. *The Simple Theory of Bending for a Parallel Beam as a Special Case.*—For a parallel beam, when r_0 is very large in comparison with $r_0 - r$, writing $r_0 - r = x$, equations (77) and (78) become

$$\xi = -\int_{0}^{x}\frac{e_1 - e_2}{d}\,dx + \frac{q_0 x}{\mathrm{G}r_0} \quad \ldots \quad (77a)$$

and

$$\mathrm{V}_0 = -\int_{0}^{x}\int_{0}^{x}\frac{e_1 - e_2}{d}\,dxdx - \frac{q_0 x}{\mathrm{G}}. \quad \ldots \quad (78a)$$

The second term in equation (77a) is negligible: the other terms correspond to the ordinary simple theory of bending, including the deflection due to shear.

A.4. *Nature of the Simple Solution for the Tapered Beam.*—The nature of the simple solution expressed in equations (64) to (66) is best understood by reference to the values of the bending moment M and total shear force F at the plane section through r' perpendicular to $\theta = 0$. The direct stress p and the shear stress q on this plane are given by

$$p = \overline{rr} \cos^2 \theta - \overline{r\theta} \sin 2\theta \quad \ldots \quad (79)$$

$$q = \overline{rr} \sin \theta \cos \theta + \overline{r\theta} \cos 2\theta. \quad \ldots \quad (80)$$

Then using (64), but ignoring the C and M terms which contribute nothing to the shear or bending moment, and writing $r = r' \sec \theta$, $y = r' \tan \theta$ $dy = r' \sec^2\theta \, d\theta$ where y is the perpendicular distance from the line $\theta = 0$,

$$p = -\frac{A}{r^2} \sin 2\theta - \frac{B}{r^2} (\sin 2\theta + \sin 4\theta) - \frac{L}{r} \sin \theta \cos^2\theta \quad \ldots \quad (81)$$

$$q = \frac{A}{r^2} \cos 2\theta + \frac{B}{r^2} \cos 4\theta - \frac{L}{r} \sin^2 \theta \cos \theta \quad \ldots \quad (82)$$

and

$$\begin{aligned} M &= \int_{-\alpha}^{\alpha} pr' \tan \theta \; r' \sec^2 \theta \, d\theta = \int_{-\alpha}^{\alpha} pr^2 \tan \theta \, d\theta \\ &= -\{A(2\alpha - \sin 2\alpha) + B(\sin 2\alpha - \tfrac{1}{2} \sin 4\alpha) + Lr'(\alpha - \tfrac{1}{2} \sin 2\alpha)\} \end{aligned} \quad (83)$$

$$F = \int_{-\alpha}^{\alpha} qr' \sec^2 \theta \, d\theta = \frac{(A + B \cos 2\alpha) \sin 2\alpha}{r'} - L(\alpha - \tfrac{1}{2} \sin 2\alpha) \quad \ldots \quad (84)$$

Finally, suppressing the primes which are no longer necessary,

$$F = \frac{(A + B \cos 2\alpha) \sin 2\alpha}{r} - L(\alpha - \tfrac{1}{2} \sin 2\alpha) \quad \ldots \quad (85)$$

and

$$Fr - M = (A + B \cos 2\alpha) \tan 2\alpha + A(2\alpha - \tan 2\alpha) \quad \ldots \quad (86)$$

Then, writing $A + B \cos 2\alpha = q_\alpha r^2$, so that q_α is the shear stress $(\overline{r\,\theta})$ at $\pm\alpha$ and since $A + B = q_0 r^2$, $B(1 - \cos 2\alpha) = (q_0 - q_\alpha) r^2$ and $A(1 - \cos 2\alpha) = (q_\alpha - q_0 \cos 2\alpha) r^2$.

Hence

$$\begin{aligned} Fr - M &= q_\alpha r^2 \tan 2\alpha + \frac{2\alpha - \tan 2\alpha}{1 - \cos 2\alpha} (q_\alpha - q_0 \cos 2\alpha) r^2 \\ &= \frac{2\alpha - \sin 2\alpha}{1 - \cos 2\alpha} q_\alpha r^2 + \frac{\sin 2\alpha - 2\alpha \cos 2\alpha}{1 - \cos 2\alpha} q_0 r^2 \\ &= \frac{2\alpha}{3} (q_\alpha r^2 + 2q_0 r^2) + \frac{4\alpha^3}{45} (q_\alpha - q_0) r^2 + \ldots\ldots \end{aligned} \quad (87)$$

If α is small, all terms after the first are negligible and writing $2r\alpha = d$, where d is the depth of the beam at section r, equation (87) may be put in the form

$$q_0 = \frac{3}{2 + (q_\alpha/q_0)} \cdot \frac{F - (M/r)}{d} \quad \ldots \quad (87a)$$

If r be large, this reduces to

$$q_0 = \frac{3}{2 + (q_\alpha/q_0)} \frac{F}{d}$$

the usual form for a parallel beam when q varies over the depth.

In any case whatever the values of r and α, equation (87) shows that the shear stress $\overline{r\theta}$ is proportional to the moment $Fr - M$ about the apex of the beam ; this conclusion is otherwise obvious because a load at the apex of the beam produces only direct (radial) stresses. This second conclusion is confirmed by equations (64) and (85) which shows that the part of the stress $\overline{rr}$, which is proportional to $1/r$, is also proportional to the shear load F.

If $q_\alpha = q_0$ when $B = 0$, the tapered beam loaded only at its tip exhibits the peculiar inversion that shear load causes only direct (radial) stresses, whilst bending moment causes only shear stresses ; of course in the latter case the necessary radial loads are carried by the heavy booms implied by the assumption $q_\alpha = q_0$.

From equation (83) it may be shown that

$$-M = (4B\cos\alpha + Lr)(\alpha - \tfrac{1}{2}\sin 2\alpha)$$
$$+ \frac{r^2}{1 - \cos 2\alpha}\{(2\alpha - 2\sin 2\alpha + \tfrac{1}{2}\sin 4\alpha + 4\alpha\cos\alpha - 4\sin\alpha\cos^2\alpha)q_\alpha$$
$$+ (\sin 2\alpha - 2\alpha\cos 2\alpha - 4\alpha\cos\alpha + 4\sin\alpha\cos^2\alpha)\, q_0\}$$

or approximately

$$-\frac{12M}{d^3} = -\frac{3M}{2r^3\alpha^3} = \left(\frac{4B\cos\alpha}{r^3} + \frac{L}{r^2}\right)\left(1 - \frac{\alpha^2}{5}\right) + \frac{3}{5r}\left(q_0 + \frac{7q_\alpha}{3}\right)\Big/\left(1 - \frac{\alpha^2}{3}\right). \qquad (88)$$

Or neglecting α^2 in comparison with unity* and substituting $\dfrac{4B\cos\alpha}{r^3} + \dfrac{L}{r^2} = -E\dfrac{e_1 - e_2}{d}$

from equation (70) $$M = \frac{Ed^2}{12}(e_1 - e_2) - \frac{1}{20}\left(q_0 + \frac{7q_\alpha}{3}\right)\frac{d^3}{r}. \quad \ldots \quad (88a)$$

The first term is the usual bending formula, just as for a parallel beam ; the second term represents a small correction, which is usually negligible. Using equation (87a), this term may be written in the alternative form

$$\frac{3}{20}\,\frac{1 + (7q_\alpha/3q_0)}{2 + (q_\alpha/q_0)}\left(F - \frac{M}{r}\right)\frac{d^2}{r} = \left(F - \frac{M}{r}\right)\frac{d^2}{6r} \text{ if } q_\alpha = q_0.$$

Even in the extreme case $d = r$ ($\alpha = \tan^{-1}\tfrac{1}{2} = 27°$) this term represents a correction to the bending moment equal to the shear load carried by the web acting at a radius of only one sixth the depth of the web.

A.5. *Summary of Results.*—It is concluded that the simple theory of bending for tapered beams is described in the following rules :—

(*a*) The distribution of radial stress may be computed from the value of the bending moment M and the modulus of section of the beam at each section just as for a parallel beam.

(*b*) The distribution of shear stress $\overline{r\theta}$ (apart from the effect of buckling) may be computed from the shape and dimensions of the section as for a parallel beam, except that in place of the total shear load F the effective shear load $F - \dfrac{M}{r}$, where r is the distance of the section from the apex of the beam, must be used.

(*Note.*—In the body of the paper this reduction of the total shear load is made *ab initio* by the use of the conception of " equivalent booms " ; the term M/r is then represented by the components of the boom loads resolved parallel to the section.)

*The correction factor $1 - \dfrac{\alpha^2}{5}$ represents the appropriate mean between 1 and $\cos\alpha$; if actual booms are present their contribution to the bending moment is of course reduced in the ratio $\cos\alpha$.

(*c*) The bending deflection may be computed from the *radial* strains exactly as for a parallel beam, the inclination of the booms or edges being disregarded.

(*d*) The shear deflection may be computed from the shear strains by dividing these strains by the distance from the apex of the beam, integrating and multiplying the integrals by the distance from the apex of the beam. (*Note.*—This procedure is necessary because pure shear of a tapered beam consists in relative rotation round the apex of the beam.)

These rules are approximate in that the square of half the angle between the edges of the beam has been assumed negligible in comparison with unity. Provided that the angle subtended at the apex of the beam is less than 20°, the error thus introduced is unlikely to exceed 5 per cent. and will normally be very much less.

A.6. *Comments on the Extension of the Simple Theory to Cases Strictly Beyond its Scope.*—The rules (*a*) to (*d*) stated in §A.5 represent a considerable extension of the actual results of §A.2–4 in that the restrictions on the type of loading have been entirely ignored. This extension corresponds exactly to the similar extension of the simple theory of bending in application to parallel beams, which is common practice and justified by general experience. Appeal by analogy to justify the similar extension for tapered beams is perhaps sufficient ; but in view of certain unfamiliar characteristics of the tapered beam, some further discussion, based on the simplest illustrative case may be desirable.

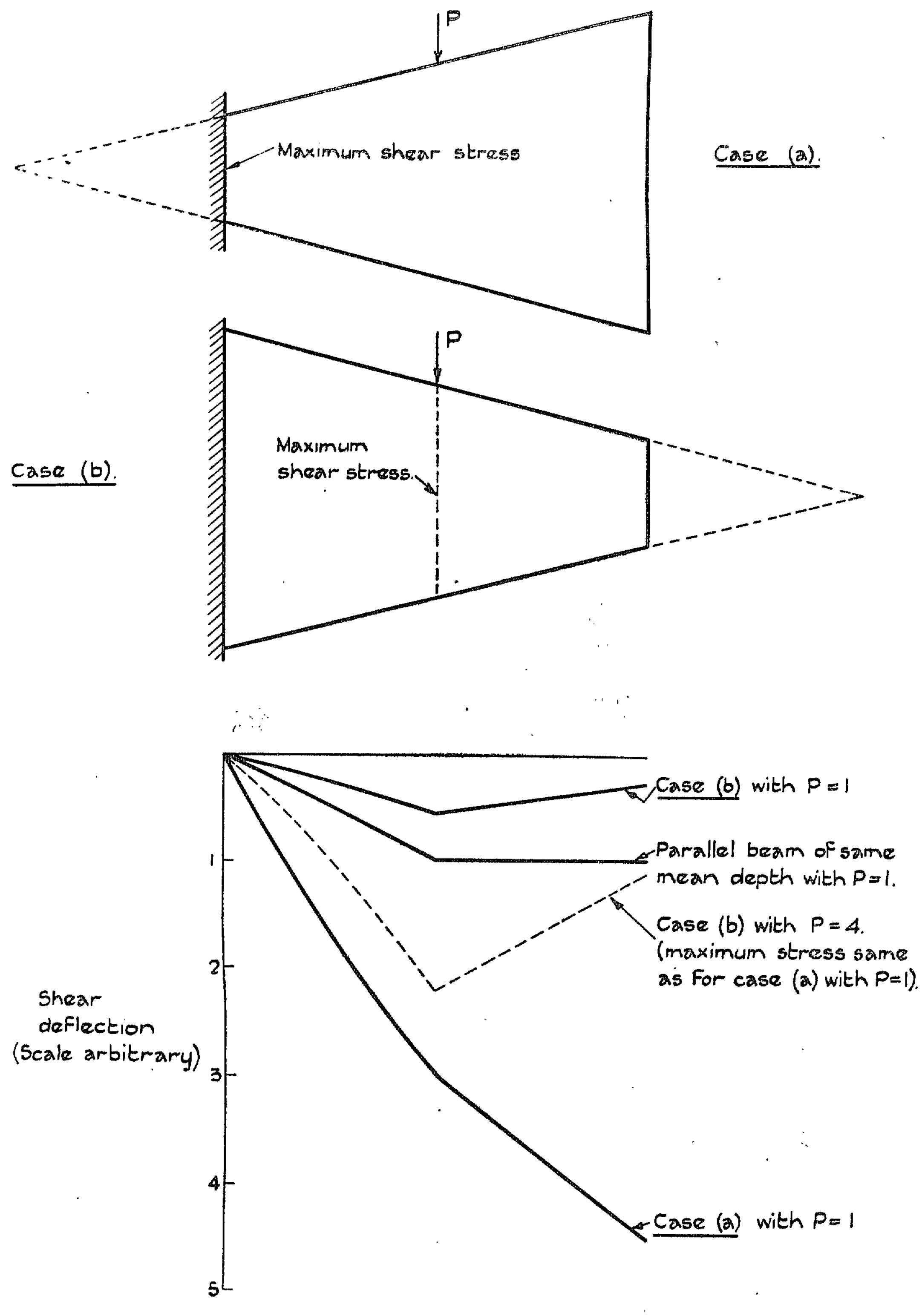
If a tapered beam held as a cantilever at section r_0 be subjected to a shear force F at section r_1, $M = F\ (r - r_1)$ and $F - \frac{M}{r} = \frac{Fr_1}{r}$. If the beam is of uniform thickness, the shear stress is proportional to $\frac{Fr_1}{r^2}$, which conforms to equation (64c). The radial direct stresses are proportional to M/r^2, that is to $\frac{F}{r} - \frac{Fr_1}{r^2}$, which conforms to equation (64a). The bending stresses and strains are then calculable by the rules stated and the bending deflection follows.

Over the portion of the beam r_0 to r_1 the shear deflection is proportional to $\frac{r}{{r_0}^2} - \frac{1}{r}$ (equation (78)). So far as the bending deflection is concerned, the " slope " of the bent beam defined by ξ is identical with $-\frac{\delta V_0}{\delta r}$, the rate of change of bending deflection ; but the shear strain also causes a change of slope, which is proportional to the shear deflection divided by the radius r. If $r_0 > r$ this change of slope is exactly *opposite* to the rate of change of shear deflection ; in fact the shear deflection always represents rotation through angles proportional to $1 - \frac{{r_0}^2}{r^2}$ about the apex of the beam. In computing the deflection at sections beyond the applied load F, the bending deflection is found in the normal way as for a parallel beam, but the shear deflection instead of being constant, now varies in proportion to r.

The effect of the local distortion of the section r_1 under the load F is disregarded just as in the parallel beam case, and the principle of superposition enables this simple case to be extended to cover all *loading* cases. In short, deep beams the effect of local distortion under the loads may not be negligible, but this source of error is no more serious for tapered than for parallel beams.

If the section of the beam varies otherwise than by pure taper, the complete beam may be regarded as a composite one built up from a number of beams of uniform thickness loaded in parallel and sharing the load so that their deflections are everywhere the same. The problem is thus reduced to a number in each of which the anomaly concerns only the loading conditions and this is covered by the principle of superposition.

The variation of shear deflection (alone) along the length of a tapered cantilever is illustrated in Fig. 12, where the contrast between the two cases $r_0 > r_1$ and $r_0 < r_1$ is shown.



The Simple Theory of Bending and Shear for Tapered Beams

Shear Deflection.

FIG. 12.

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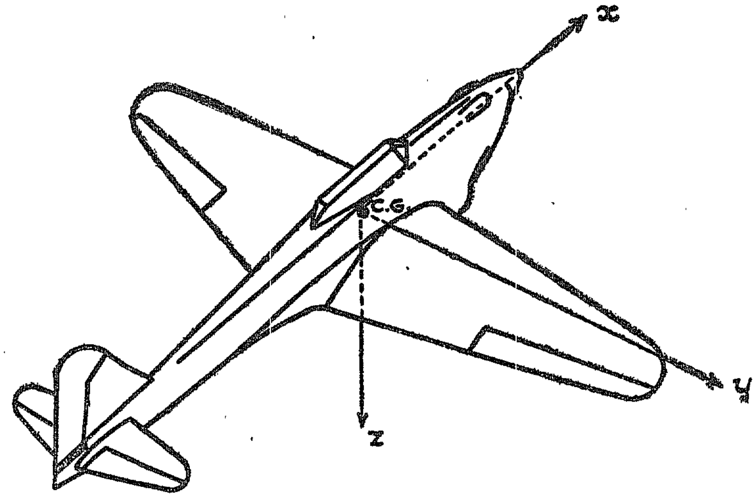
SYSTEM OF AXES



Axes	Symbol Designation Positive direction	x longitudinal forward	y lateral starboard	z normal downward
Force	Symbol	X	Y	Z
Moment	Symbol Designation	L rolling	M pitching	N yawing
Angle of Rotation	Symbol	ϕ	θ	ψ
Velocity	Linear Angular	u p	v q	w r
Moment of Inertia		A	B	C

Components of linear velocity and force are positive in the positive direction of the corresponding axis.

Components of angular velocity and moment are positive in the cyclic order y to z about the axis of x, z to x about the axis of y, and x to y about the axis of z.

The angular movement of a control surface (elevator or rudder) is governed by the same convention, the elevator angle being positive downwards and the rudder angle positive to port. The aileron angle is positive when the starboard aileron is down and the port aileron is up. A positive control angle normally gives rise to a negative moment about the corresponding axis.

The symbols for the control angles are :—

- ξ aileron angle
- η elevator angle
- η_T tail setting angle
- ζ rudder angle

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