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Galerkin's Method in Mechanics and Differential Equations

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§1. *Introduction and summary.*—The method to be described here is attributed to the Russian investigator V. G. Galerkin, whose original papers are inaccessible to the present writer. His knowledge of the method is derived from a description given in a paper by E. P. Grossman.[1] Grossman states that the method was given by Galerkin in his treatise " Rods and Plates " (Vestnik Ingeneroff, 1915, p. 897), and that applications to oscillation problems were first made by V. P. Lyskov. It is pointed out by Grossman that Galerkin's process in applications to mechanics leads to the same results as Lagrange's principle of virtual work, but employs a special co-ordinate system.

The method of Galerkin belongs to the same general class as those of Rayleigh and Ritz, for it seeks to obtain an approximate solution of a differential equation with given boundary conditions by taking a function which satisfies these conditions exactly, and proceeds to specialise the function in such a manner as to secure approximate satisfaction of the differential equation. The selected function is a linear combination of n independent functions, and the coefficients are determined by a process of integration.

The Galerkin process can be considered from two points of view, (a) simply as a means for the approximate solution of differential equations, and (b) as a method specially adapted for the treatment of problems concerning the statics and dynamics of elastic and other deformable bodies. These two aspects are treated separately in Parts I and II of the paper respectively, and will now be briefly discussed.

(a) Let the result of substituting the given function, which satisfies the boundary conditions, in the differential equation be ε. Since the result should be zero, ε is the error in the differential equation. Then the Galerkin process consists in choosing the n coefficients in the function in such a manner that n distinct weighted means of the error, taken throughout a certain range of representation, shall all be zero.

(b) In mechanical applications ε can be interpreted as a generalised force, and the multipliers used to weight the errors are the virtual displacements corresponding to increments of each of the generalised co-ordinates in turn. Thus the vanishing of the weighted mean is here interpreted as the vanishing of the virtual work in the appropriate displacement.

The degree of accuracy attained can be increased indefinitely by increasing the number of independent functions employed, but this entails a great increase of labour. However, when the functions are well chosen, an excellent approximation can be obtained by the use of a very small number, as is sufficiently shown by the examples included in this paper. The result of the Galerkin process proper

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is a set of n linear equations*, possibly involving a characteristic number, and the further treatment of these may follow any of the established methods. For instance, if n were large, it might be of advantage to apply matrices.

The illustrative examples have all been deliberately chosen of a simple nature, and all have known solutions, so that the accuracy of the approximations obtained can be tested. The examples in Part I are both of one-point boundary problems, while the examples of mechanical applications given in Part II are as follows :—

(1) Flexural oscillation of a uniform cantilever. (See §10.)

(2) Torsional oscillation of a uniform cantilever. (See §11.)

(3) Torsional oscillation of a uniform cantilever carrying a flywheel. (See §12.)

(4) Flexural oscillation of a uniform cantilever carrying a massive particle. (See §13.)

(5) Determination of the critical loads of struts. (See §14.)

(6) Solution of the St. Venant torsion problem for certain boundaries. (See §15.)

The last is an instance of an application to a partial differential equation in two dimensions.

There is probably scarcely a mechanical problem concerning elastic or other continuously deformable bodies to which the Galerkin method cannot be applied with success. Here are a few obvious applications which are not included in the list of examples :—

(1) Forced motions.

(2) Motions of bodies subject to the action of damping forces, or aerodynamical forces, e.g., wings or blades placed in an airstream.

(3) Oscillations of diaphragms.

(4) Oscillations of rotating blades.

(5) Deflexions of struts with eccentric and lateral loads.

(6) The St. Venant flexure problem and others of the Dirichlet type.

Acknowledgments.—The writer is greatly indebted to Miss H. M. Lyon, M.A., who worked the examples on flexural and torsional oscillations given in §§10 to 13.

* If the differential equation is non-linear, then these equations will also be non-linear.

PART I.

The Galerkin Method for the Approximate Solution of Linear Ordinary Differential Equations

§2. *Statement of the method.*—Suppose that it is desired to find the solution of the linear ordinary differential equation

$$P_0(x)\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + P_{n-1}(x)\frac{dy}{dx} + P_n(x)\,y + Q(x) = 0 \qquad (1)$$

for the range $a \lessgtr x \lessgtr b$ of the independent variable, given certain "boundary conditions" which render the solution unique*. A typical condition will be

$$A_0\frac{d^n y}{dx^n} + A_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + A_n y + B = 0 \qquad (2)$$

to be satisfied when $x = x_0$, where $A_0, \ldots\ldots A_n$ and B are given constants. Let Y be a function which satisfies the complete set of conditions, and let $Y_1, Y_2, \ldots\ldots Y_m$ be a sequence of linearly independent functions which all satisfy the set of homogeneous conditions obtained from (2) by replacing B by zero. Then clearly the function

$$y = Y + \sum_{r=1}^{r=m} c_r Y_r\,, \qquad (3)$$

where the coefficients c are independent of x, also satisfies all the boundary conditions. In the important case where the boundary conditions are of the homogeneous type the function Y will be omitted. It remains to determine the coefficients so that y shall be a good approximation to the exact solution of (1) for the prescribed range of x.

The typical Galerkin process for the determination of the coefficients is as follows. Substitute the expression (3) in the differential equation (1), multiply by Y_s, integrate the result from a to b, and equate to zero. When s is made 1, 2, . . . m in succession, m linear equations are obtained which determine the coefficients c. It is shown in Part II that in certain mechanical applications of the method it may be preferable to employ a differential coefficient of Y_s in place of Y_s as multiplier, and plainly the method can be extended by taking as multipliers any convenient set of linearly independent functions of x. The discussion given in the following section shows that any of these variants must yield a good approximation when m is sufficiently large.

* The "boundary conditions" need not be restricted to conditions to be satisfied for the extreme values a and b of x.

§3. *The method of least mean squared error.*—The justification of the Galerkin process will be approached by the consideration of an alternative method whose correctness appears to be self-evident.

Let the result of substituting the expression (3) for y on the left-hand side of (1) be ε. Then ε is the error in the differential equation corresponding to the approximation (3), which, it is to be remembered, satisfies the boundary conditions exactly. Hence, if ε were zero for all values of x within the range a to b the solution would be exact, since it is by hypothesis unique. Failing this, the criterion will be adopted that the approximation is best when the mean squared error in the differential equation is a minimum*. Thus the coefficients c are to be such that

$$J = \int_a^b \varepsilon^2 dx \qquad \qquad (4)$$

is a minimum†. Hence

$$\frac{\partial J}{\partial c_s} = 0 \ (s = 1, 2, \ldots . m) . \qquad \qquad (5)$$

J is a quadratic function of the coefficients, so that the m equations (5) are linear and serve to determine the coefficients uniquely. Now

$$\frac{\partial J}{\partial c_s} = 2 \int_a^b \varepsilon \frac{\partial \varepsilon}{\partial c_s} dx ,$$

and
$$\begin{aligned}\frac{\partial \varepsilon}{\partial c_s} &= P_0(x) \frac{d^n Y_s}{dx^n} + P_1(x) \frac{d^{n-1} Y_s}{dx^{n-1}} + \ldots \ldots + P_{n-1}(x) \frac{dY_s}{dx} + P_n(x) Y_s \\ &= Z_s(x), \text{ say.} \qquad \qquad (6)\end{aligned}$$

With this notation

$$\varepsilon = Z(x) + Q(x) + \sum_{r=1}^{r=m} c_r Z_r(x) , \qquad \qquad (7)$$

where $Z(x)$ results from the substitution of Y for Y_s in (6). Hence equation (5) becomes

$$\int_a^b \varepsilon Z_s(x)\, dx = 0 , \qquad \qquad (8)$$

or
$$\sum_{r=1}^{r=m} c_r \int_a^b Z_r(x) Z_s(x)\, dx + \int_a^b [Z(x) + Q(x)] Z_s(x)\, dx = 0 . \qquad \qquad (9)$$

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* This, of course, does not imply that the mean squared error in y is a minimum.

† It is to be understood that (1) is purely real.

The Galerkin process leads to m equations typified by

$$\int_a^b \varepsilon \, Y_s \, dx = 0 ,$$

or
$$\sum_{r=1}^{r=m} c_r \int_a^b Z_r Y_s \, dx + \int_a^b [Z(x) + Q(x)] \, Y_s \, dx = 0 . \quad \ldots \quad \ldots \quad (10)$$

Consequently

$$\int_a^b \varepsilon \, (\Sigma \, g_r Y_r) \, dx = 0 , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

where the constants g are arbitrary. Now Z_s can be expanded approximately as a series in Y_1, Y_2, etc., so that

$$Z_s = \Sigma g_{sr} Y_r + \eta_s , \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where η_s is the error in the series. Hence the equation (8) can be written

$$\int_a^b \varepsilon \, (\Sigma \, g_{sr} Y_r + \eta_s) = 0 , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

and the Galerkin equations (11) only differ by the omission of the small quantities η. When m is large, η_s will be small, and it is therefore obvious that the two methods will then lead to almost identical results. A similar argument can be applied when arbitrary multipliers $W_s(x)$ are used in place of the multipliers Y_s of the Galerkin process.

It may be remarked that the introduction of an additional function Y_{m+1} in the least mean squared error method must lead to an improved approximation in the sense that the mean squared error will be reduced. For, when the coefficient c_{m+1} is zero and the other coefficients are the same as before the value of J is the same as before. But there will in general be a set of values of $c_1 \ldots \ldots c_{m+1}$ which will give J a minimum value which is still smaller.

§4. *Primary and secondary boundary conditions, and the choice of the functions Y.*—Consider the differential equation

$$\frac{dy}{dx} - y = 0 ,$$

and suppose that the boundary condition is $y = 1$ when $x = 0$. Then, in order that the differential equation shall be satisfied at $x = 0$ it is necessary that $\frac{dy}{dx} = 1$ at this point. This, then, is the secondary boundary condition which is a consequence of the primary boundary condition $y = 1$ and of the differential equation itself. In general, the secondary boundary conditions will be defined as those which are the consequence of the satisfaction of the differential equation and of the primary boundary conditions jointly.

Now it is only necessary that the function (3) shall satisfy the primary boundary conditions, but in general a much better approximation is secured with a given number of disposable coefficients c when the function also satisfies the secondary boundary conditions exactly. This amounts to a combination of the Galerkin method with the method of collocation which is discussed in a companion paper[2].

§5. *Illustrative examples.*—The following examples are intended merely to illustrate the method, and very simple equations with known solutions have been selected. Since most of the examples in Part II are two point boundary problems, those given here are both one point boundary problems for the sake of variety.

Example 1.—Take the differential equation

$$\frac{dy}{dx} - y = 0$$

with the condition $y = 1$ when $x = 0$. Then the exact solution is e^x.

Let the range of representation be 0 to 1, and first take a function (3) which satisfies the primary but not the secondary boundary condition. Such a function is

$$y = 1 + c_1 x + c_2 x^2 + c_3 x^3 \quad \ldots \quad (14)$$

Then

$$\varepsilon = (c_1 - 1) + (2c_2 - c_1)x + (3c_3 - c_2)x^2 - c_3 x^3 ,$$

and the Galerkin equations are

$$\int_0^1 \varepsilon x \, dx = \int_0^1 \varepsilon x^2 \, dx = \int_0^1 \varepsilon x^3 \, dx = 0 .$$

When cleared of fractions these become

$$\left. \begin{aligned} 10c_1 + 25c_2 + 33c_3 &= 30 , \\ 5c_1 + 18c_2 + 26c_3 &= 20 , \\ 21c_1 + 98c_2 + 150c_3 &= 105 , \end{aligned} \right\} \quad \ldots \quad (15)$$

and the corresponding expression for y is

$$116y = 116 + 120x + 45x^2 + 35x^3 . \quad \ldots \quad (16)$$

If the approximation $y = 1 + c_1x$ had been employed, then the Galerkin equation would have been the first of equations (15) with the terms in c_2 and c_3 omitted, while if $y = 1 + c_1x + c_2x^2$ had been used, the two Galerkin equations would be obtained from the first pair of equations (15) by omission of the terms in c_3. In this way the following earlier approximations are obtained :—

$$y = 1 + 3x , \qquad (17)$$

and

$$11y = 11 + 8x + 10x^2 . \qquad (18)$$

The three approximations are compared with the true solution in the following table :—

Value of x.	Approximation.			e^x.
	Linear.	Quadratic.	Cubic.	
0	1·0	1·0	1·0	1·0
0·2	1·6	1·1818	1·2248	1·2214
0·4	2·2	1·4364	1·4952	1·4918
0·6	2·8	1·7636	1·8255	1·8221
0·8	3·4	2·1636	2·2303	2·2255
1·0	4·0	2·6364	2·7241	2·7183

The same problem will now be treated using a function which satisfies the primary and secondary boundary conditions. Clearly such a function is

$$y = 1 + x + c_2x^2 + c_3x^3 + c_4x^4 . \qquad (19)$$

Then

$$\varepsilon = (2c_2 - 1)x + (3c_3 - c_2)x^2 + (4c_4 - c_3)x^3 - c_4x^4 ,$$

and the Galerkin equations are

$$\int_0^1 \varepsilon x^2\, dx = \int_0^1 \varepsilon x^3\, dx = \int_0^1 \varepsilon x^4\, dx = 0 .$$

These yield

$$126c_2 + 182c_3 + 220c_4 = 105 ,$$
$$196c_2 + 300c_3 + 375c_4 = 168 ,$$
$$96c_2 + 153c_3 + 196c_4 = 84 ,$$

and the corresponding approximate solution is

$$1198y = 1198 + 1198x + 609x^2 + 168x^3 + 84x^4 . \qquad (20)$$

Approximations of lower degree can be obtained at once in the same manner as before. They are

$$6y = 6 + 6x + 5x^2 , \quad \ldots \quad \ldots \quad \ldots \quad (21)$$

and

$$76y = 76 + 76x + 33x^2 + 21x^3 . \quad \ldots \quad \ldots \quad (22)$$

The approximations are compared with the true solution in the following table :—

Value of x.	Approximation.			e^x.
	Quadratic.	Cubic.	Quartic.	
0	1·0	1·0	1·0	1·0.
0·2	1·2333	1·2196	1·2216	1·2214
0·4	1·5333	1·4872	1·4921	1·4918
0·6	1·9000	1·8160	1·8224	1·8221
0·8	2·3333	2·2194	2·2259	2·2255
1·0	2·8333	2·7105	2·7187	2·7183

It will be seen that the approximation when a given number of coefficients c is employed is much better than before.

Example 2.—The differential equation is

$$\frac{d^2y}{dx^2} + xy = 0 ,$$

and the boundary conditions $y = 1$, $\frac{dy}{dx} = 0$, when $x = 0$. It can be shown that the solution is

$$y = \frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{3}} \sqrt{x}\, J_{-\frac{1}{3}}\left(\frac{2}{3} \sqrt{x^3}\right).$$

It is obvious that the function

$$Y = 1 - \frac{x^3}{6}$$

satisfies the primary and secondary boundary conditions, and it is further clear that the series for y must proceed in powers of x^3. Hence it will be advantageous to take the approximation

$$y = 1 - \frac{x^3}{6} + c_6x^6 + c_9x^9 . \quad \ldots \quad \ldots \quad \ldots \quad (23)$$

Let the range of representation be 0 to b. Then the Galerkin equations are

$$\int_0^b \varepsilon x^6 \, dx = \int_0^b \varepsilon x^9 \, dx = 0 ,$$

where

$$\varepsilon = -\frac{x^4}{6} + 30c_6x^4 + (c_6 + 72c_9)\, x^7 + c_9x^{10} .$$

When simplified the equations become

$$c_6\left(\frac{30}{11}+\frac{b^3}{14}\right)+c_9b^3\left(\frac{36}{7}+\frac{b^3}{17}\right)=\frac{1}{66},$$

and

$$c_6\left(\frac{15}{7}+\frac{b^3}{17}\right)+c_9b^3\left(\frac{72}{17}+\frac{b^3}{20}\right)=\frac{1}{84},$$

and it follows that

$$c_6=\frac{1+\frac{7}{360}b^3}{180+\frac{7}{2}b^3+\frac{77}{2040}b^6}, \quad \ldots \quad (24)$$

and

$$c_9=\frac{-1}{12{,}960+252b^3+\frac{231}{85}b^6}. \quad \ldots \quad (25)$$

If $b = 2$ these formulae yield

$$c_6 = 5\cdot491774 \times 10^{-3}$$

and

$$c_9 = -6\cdot60066 \times 10^{-5}.$$

The approximate and true solutions are compared in the following table :—

Value of x.	Value of y.	
	Approximate.	True.
0	1·0	1·0
0·2	0·99867	0·99867
0·4	0·98936	0·98936
0·6	0·96426	0·96426
0·8	0·91610	0·91611
1·0	0·83876	0·83881
1·2	0·72806	0·72819
1·4	0·58265	0·58294
1·6	0·40493	0·40540
1·8	0·20169	0·20230
2·0	−0·01566	−0·01498

It may be remarked that when $b \to 0$ equations (24) and (25) yield

$$c_6 = \frac{1}{180},$$

and

$$c_9 = \frac{-1}{12{,}960}.$$

These are the true values of the coefficients of x^6 and x^9 in the infinite series for y.

PART II

Applications of the Galerkin Method to Mechanics

§6. *Introductory.*—The writer's interest in the Galerkin method chiefly concerns its applications to mechanical problems, and he believes the method to be of great value in the treatment of the statics and dynamics of elastic and other continuously deformable bodies. As already stated, the method may be regarded as an easy way of deriving the Lagrangian dynamical equations with a special choice of the generalised co-ordinates, but the specific problem requires examination in order that the proper multipliers shall be used in the Galerkin process. (See, for instance, §14.)

The coefficients obtained in the Galerkin equations are always definite integrals. In the illustrative examples which follow these integrals are all readily obtained exactly, but in more complicated instances, or in cases where the mechanical properties of the bodies concerned are specified by graphs or tables, approximate methods of integration, such as Simpson's rule, must be used. For the sake of ease in integration it is advisable to employ rational integral functions in (3) wherever possible.

Many of the examples are problems concerning the determination of characteristic numbers and the corresponding modes of displacement. Now the Galerkin process yields as many modes as there are independent functions employed, and the labour involved rises with great rapidity as the number of these increases. Hence, if it is desired to investigate one of the higher modes, it will be most advantageous to employ a small number of functions which are known to resemble the required mode. It may be added that the choice of the functions is of great importance, and provides opportunity for the display of skill; the choice should be guided by the greatest possible knowledge of analogous problems. The advantage of choosing functions which satisfy the secondary boundary conditions (see §4 and §14, Ex. 1) should not be overlooked.

The following are some brief miscellaneous notes on the method :—

(*a*) When an elastic body is subject to a concentrated load of any kind it may be advantageous to employ one discontinuous function. (See discussion in §§12 and 13.)

(*b*) If it is desired to obtain the value of a stress from the approximate solution, then an integral expression rather than a differential coefficient should be used wherever possible. A consideration of the case of an oscillating cantilever will make this clear.

(*c*) In dealing with a problem in n dimensions it would theoretically be necessary in general to use an n-ply infinite set of functions in order to obtain convergence to the exact solution. (See, for example, the discussion of the torsion of prisms in §15.)

(*d*) If a linear combination of the functions in use happens to be an exact solution, then the Galerkin method will yield that exact solution. (See, for example, §14, Ex. 2.)

§7. *Flexural motion of a cantilever beam.*—The general discussion of the connection between Galerkin's method and the dynamical equations of Lagrange will be approached by a discussion of the flexural motion of a cantilever beam as the argument is specially simple in this case.

The differential equation governing the deflexion of a thin beam not subject to end load parallel to itself is

$$\frac{d^2}{dx^2}\left(\mathrm{EI}\,\frac{d^2y}{dx^2}\right) = w\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (26)$$

where y = normal deflexion,

x = distance from the root,

EI = flexural rigidity,

and w = load per unit span.

In general EI will be a function of x, and w will be a function of x and of the time t. When the beam is in motion, w must be taken to include the reversed effective force (inertia force) per unit span, given by $-m\left(\frac{d^2y}{dt^2}\right)$, where m is the mass per unit span. Let w_i be the inertia load, and w_a the resultant external load, both per unit span. Then (26) becomes

$$\frac{d^2}{dx^2}\left(\mathrm{EI}\,\frac{d^2y}{dx^2}\right) = w_a + w_i\,. \qquad \ldots \quad \ldots \quad \ldots \qquad (27)$$

Now this equation simply expresses the equality of the elastic reaction and applied load. Let the elastic force per unit span be

$$w_e = -\frac{d^2}{dx^2}\left(\mathrm{EI}\,\frac{d^2y}{dx^2}\right). \qquad \ldots \quad \ldots \qquad (28)$$

Then equation (27) becomes simply

$$w_a + w_i + w_e = 0\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (29)$$

The point to be emphasised is that (26) or (29) is an expression of the balance of normal forces per unit span.

The deflexion y must satisfy the following boundary conditions for the case of a cantilever without tip load :—

At the root $(x = 0)$ $\qquad y = \frac{dy}{dx} = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (30)$

and at the tip $(x = l)$

$$\mathrm{EI}\,\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\mathrm{EI}\,\frac{d^2y}{dx^2}\right) = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad (31)$$

The equations (31) express the conditions that the bending moment and shearing force vanish at the tip. Provided that EI does not vanish at the tip they are equivalent to

$$\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad (32)$$

Suppose that Y_1, Y_2, Y_3, etc., are a sequence of linearly independent functions of x only which all satisfy (30) and (31). Then if

$$y = \Sigma\, q_r\, Y_r\,, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (33)$$

the quantities q may be regarded as the Lagrangian dynamical co-ordinates of the beam, for, when they are assigned, the deflexion at all points becomes definite. Since the beam is an elastic body it would strictly be necessary to employ an infinite number of dynamical co-ordinates, but this does not affect the argument.

The Lagrangian dynamical equation corresponding to the co-ordinate q_r is the expression of the fact that the total work done by all the forces applied to the beam (including the inertia forces) in a virtual displacement corresponding to the increment δq_r of q_r (with all the other co-ordinates constant) is zero. Hence the equation is

$$\int_0^l (w_a + w_i + w_e)\,\frac{\partial y}{\partial q_r}\,dx = 0\,,$$

or

$$\int_0^l (w_a + w_i + w_e)\,Y_r\,dx = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad (34)$$

This is precisely the equation given by Galerkin's method, and it is therefore equivalent to the employment of Lagrange's equations with the special co-ordinate system expressed by (33). The identity of (34) with the usual Lagrangian equation

$$\frac{d}{dt}\left(\frac{\partial \mathrm{T}}{\partial \dot{q}_r}\right) - \frac{\partial \mathrm{T}}{\partial q_r} + \frac{\partial \mathrm{V}}{\partial q_r} = \mathrm{Q}_r\,, \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

follows from

$$\mathrm{T} = \frac{1}{2}\int_0^l m\left(\frac{dy}{dt}\right)^2 dx\,, \quad \ldots \quad \ldots \quad \ldots \quad (36)$$

and

$$\mathrm{V} = \frac{1}{2}\int_0^l \mathrm{EI}\left(\frac{d^2y}{dx^2}\right)^2 dx\,. \quad \ldots \quad \ldots \quad \ldots \quad (37)$$

It can be shown that

$$\frac{\partial V}{\partial q_r} = \int_0^l Y_r \frac{d^2}{dx^2}\left(EI \frac{d^2y}{dx^2}\right) dx$$

by integration by parts and use of the boundary conditions (30) and (31).

It is evident that the results of applying Galerkin's process would not agree with Lagrange's equations if the equation (26) had been multiplied throughout by an arbitrary factor, as might appear permissible. In fact *it is always necessary to know the physical meaning of the differential equation in order that the proper multipliers may be employed.* If the wrong multipliers are used a fair approximation may still be obtained, but the process will not be equivalent to the employment of Lagrange's equations.

§8. *Flexural-torsional motion of a blade or wing.*—It is clear that a torsional motion of a beam or blade can be treated in the same general manner as the flexural motion. The fundamental equation of motion corresponding to (29) is

$$t_a + t_i + t_e = 0 , \qquad \ldots \qquad (38)$$

where each t is a twisting moment per unit span. Let θ be the angle of twist at any point, and let Θ_1, Θ_2, etc., be a sequence of linearly independent functions of x which all satisfy the same boundary conditions as θ. Then if

$$\theta = \Sigma q'_r \Theta_r , \qquad \ldots \qquad (39)$$

the quantities q'_r may be regarded as the dynamical co-ordinates. The Lagrangian equation corresponding to q'_r is obviously

$$\int_0^l (t_a + t_i + t_e) \Theta_r \, dx = 0 , \qquad \ldots \qquad (40)$$

which is also the Galerkin equation.

If the motion is actually both flexural and torsional both sets of equations (34) and (40) must be employed. The combined set can be solved by ordinary methods. For instance, in the case of free motion it would be assumed that each dynamical co-ordinate was proportional to $e^{\lambda t}$, and this would lead to a determinantal equation for λ.

§9. *Application of Galerkin's method to elastic bodies in general.*—It is evident that a procedure similar to that explained above can be applied to problems on the dynamics or statics of an elastic body when the condition of support is zero displacement on a certain surface. Here the three components, u, v, w of the displacement would be written

$$\left.\begin{aligned} u &= \Sigma q_r U_r (x, y, z) , \\ v &= \Sigma q_r V_r (x, y, z) , \\ w &= \Sigma q_r W_r (x, y, z) , \end{aligned}\right\} \qquad \ldots \qquad (41)$$

and the functions U, V, W would be chosen to satisfy the boundary conditions for displacement. The fundamental equations of motion are, in the case of an isotropic body,

$$(\lambda + \mu)\frac{\partial \Delta}{\partial x} + \mu \nabla^2 u - \varrho \frac{\partial^2 u}{\partial t^2} + \varrho X = 0\,, \quad \ldots \quad \ldots \qquad (42)$$

together with two similar equations. Now this equation expresses the balance of the components of force in the direction OX, and may be written concisely

$$x_e + x_i + x_a = 0\,, \quad \ldots \quad \ldots \quad \ldots \qquad (43)$$

where x_e, x_i and x_a are the components of elastic, inertia and external force respectively, all per unit volume. Hence the Lagrange or Galerkin equation corresponding to the co-ordinate q_r is

$$\iiint \begin{bmatrix} U_r\,(x_e + x_i + x_a) + V_r(y_e + y_i + y_a) \\ + W_r(z_e + z_i + z_a) \end{bmatrix} dx\,dy\,dz = 0\,. \quad \ldots \quad \ldots \qquad (44)$$

Surface tractions, if any, at points where the displacements are not zero must be introduced through surface integrals whose form is sufficiently obvious.

§10. *Flexural oscillation of a uniform cantilever.*—The case of a uniform cantilever will be considered since the object in view is to illustrate the method, and to test the approximation obtained by comparison with a known solution. The oscillations of a cantilever of variable section can be treated in exactly the same way.

The first step is to obtain a convenient set of functions satisfying the conditions:—

$$y = \frac{dy}{dx} = 0 \text{ when } x = 0\,,$$

and

$$\frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0 \text{ when } x = l\,.$$

Let

$$\xi = \frac{x}{l}\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (45)$$

Then one suitable function is

$$y_r = 1 - \cos r\pi\xi + \tfrac{1}{2}(-1)^{r+1}\, r^2\pi^2\xi^2\,, \quad \ldots \quad \ldots \qquad (46)$$

where r is a positive integer. However, it is usually convenient to employ a rational integral function if possible, and it will be found that the trinomial

$$Y_r = \frac{1}{6}(r+2)(r+3)\,\xi^{r+1} - \frac{1}{3}r(r+3)\,\xi^{r+2} + \frac{1}{6}r(r+1)\,\xi^{r+3} \quad \ldots \qquad (47)$$

satisfies all the conditions when r is positive*. It may be noted in passing that these functions have the properties :—

$$Y_r = 1 \text{ when } \xi = 1 \,. \qquad \ldots \quad \ldots \quad \ldots \qquad (48)$$

$$\int_0^1 Y_r \, d\xi = \frac{2}{r+4} \,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (49)$$

$$\left.\begin{aligned} \int_0^1 \frac{d^4 Y_r}{d\xi^4} \, d\xi &= 8 \qquad (\text{when } r = 1) \\ &= -20 \ (\text{when } r = 2) \\ &= 0 \qquad (\text{when } r > 2) \,. \end{aligned}\right\} \quad \ldots \quad \ldots \qquad (50)$$

$$\int_0^1 Y_r \frac{d^4 Y_s}{d\xi^4} \, d\xi = \int_0^1 Y_s \frac{d^4 Y_r}{d\xi^4} \, d\xi \,. \qquad \ldots \quad \ldots \quad \ldots \qquad (51)$$

The last equation follows from the reciprocal theorem, since Y_r and Y_s are possible displacements of a uniform cantilever, and the corresponding loads per unit span are proportional to the fourth differential coefficients of the displacements.

The general equation of motion is

$$EI \frac{\partial^4 y}{\partial x^4} + m \frac{\partial^2 y}{\partial t^2} = 0 \,,$$

and if the beam is oscillating purely in one mode with frequency $p/2\pi$, this becomes

$$\frac{d^4 y}{dx^4} - \frac{m p^2 y}{EI} = 0 \,.$$

It is convenient to change the independent variable to ξ and the equation then becomes

$$\frac{d^4 y}{d\xi^4} - a y = 0 \,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (52)$$

where

$$a = \frac{m p^2 l^4}{EI} \,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (53)$$

Now substitute

$$y = q_1 Y_1 + q_2 Y_2 \,, \qquad \ldots \quad \ldots \quad \ldots \qquad (54)$$

* Analogous functions for other types of support are easily obtained.

and form the Galerkin equations :—

$$q_1 \int_0^1 Y_1 \left(\frac{d^4 Y_1}{d\xi^4} - \alpha Y_1\right) d\xi + q_2 \int_0^1 Y_1 \left(\frac{d^4 Y_2}{d\xi^4} - \alpha Y_2\right) d\xi = 0 .$$

$$q_1 \int_0^1 Y_2 \left(\frac{d^4 Y_1}{d\xi^4} - \alpha Y_1\right) d\xi + q_2 \int_0^1 Y_2 \left(\frac{d^4 Y_2}{d\xi^4} - \alpha Y_2\right) d\xi = 0 .$$

These yield

$$\left.\begin{aligned} \frac{q_1}{q_2}\left(\frac{16}{5} - \frac{104}{405}\alpha\right) + \left(\frac{8}{3} - \frac{1304}{5670}\alpha\right) &= 0 , \\ \text{and} \quad \frac{q_1}{q_2}\left(\frac{8}{3} - \frac{1304}{5670}\alpha\right) + \left(\frac{80}{21} - \frac{1304}{6237}\alpha\right) &= 0 . \end{aligned}\right\} \quad \ldots \qquad (55)$$

The condition of compatibility of these equations is

$$\left(\frac{104}{405}\alpha - \frac{16}{5}\right)\left(\frac{1304}{6237}\alpha - \frac{80}{21}\right) - \left(\frac{1304}{5670}\alpha - \frac{8}{3}\right)^2 = 0 , \quad \ldots \qquad (56)$$

and the roots of this quadratic are found to be 12·3625 and 515·86. The correct value for the lower root is the fourth power of the lowest root of

$$1 + \cosh m \cos m = 0 , \quad \ldots \quad \ldots \quad \ldots \qquad (57)$$

and is 12·3623, so that the approximation is excellent. The correct value of the second root is very nearly $(3\pi/2)^4 = 493 \cdot 13$. Thus, as would be expected, the second root given by the quadratic (56) is considerably in error. When α has been found, q_1/q_2 can be obtained from (55) and the mode of displacement calculated from (54).

Further calculations have been made using the three functions Y_1, Y_2 and Y_3. The results of all the calculations are summarised in the following table :—

Fundamental Flexural Mode and Frequency Coefficient for a Uniform Cantilever

		Approximation.			Correct Result.
		1st. ($q_2 = q_3 = 0$)	2nd. ($q_3 = 0$)	3rd.	
$\alpha = \dfrac{mp^2 l^4}{EI}$		12·46	12·3625	12·3624	12·3623*
$\dfrac{y}{y_l}$	$\xi = 0 \cdot 25$	0·1055	0·0972	0·0973	0·0973
	0·5	0·3542	0·3397	0·3395	0·3395
	0·75	0·6680	0·6580	0·6577	$0 \cdot 6577_6$
	1·0	1·0	1·0	1·0	1·0

* The last digit is doubtful.

The mode is specified as the ratio of y to y_1, the value of y at the tip. It will be seen that both the second and third approximations give excellent approximations to the mode as well as to the frequency.

§11. *Torsional oscillation of a uniform cantilever.*—The differential equation governing the torsional motion of a cantilever is

$$\frac{\partial}{\partial x}\left(C\,\frac{\partial \theta}{\partial x}\right) = J\,\frac{\partial^2 \theta}{\partial t^2}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (58)$$

where θ = angle of twist,

C = torsional stiffness of unit length,

and J = moment of inertia of unit length.

When C and J are constant, and when the cantilever is oscillating in a single mode with frequency $p/2\pi$, this becomes

$$C\,\frac{d^2\theta}{dx^2} + Jp^2\theta = 0\,.$$

Let l be the length of the cantilever, and define ξ as in (45). Then the last equation can be written

$$\frac{d^2\theta}{d\xi^2} + k\theta = 0\,, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (59)$$

where

$$k = \frac{Jp^2l^2}{C}\,. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (60)$$

The boundary conditions are

$$\theta = 0 \text{ when } \xi = 0\,,$$

and

$$\frac{d\theta}{d\xi} = 0 \text{ when } \xi = 1\,.$$

Clearly the function

$$\theta_r = \sin \tfrac{1}{2}(2r - 1)\,\pi\xi\,, \qquad \ldots \quad \ldots \qquad (61)$$

where r is a positive integer, satisfies the boundary conditions, and is in fact a possible mode for a uniform cantilever. But it will be convenient to select rational integral functions when C and J are variable, and the following binomial satisfies the boundary conditions :—

$$\Theta_r = (r + 1)\,\xi^r - r\xi^{r+1}\,. \qquad \ldots \quad \ldots \qquad (62)$$

The torsion binomials have the properties :—

$$\Theta_r = 1 \text{ when } \xi = 1 \quad \ldots \quad \ldots \quad \ldots \quad (63)$$

$$\int_0^1 \Theta_r \, d\xi = \frac{2}{r+2} \quad \ldots \quad \ldots \quad \ldots \quad (64)$$

$$\left.\begin{aligned} \int_0^1 \frac{d^2\Theta_r}{d\xi^2}\, d\xi &= -2 \; (r = 1) \\ &= 0 \; (r > 1) \, . \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (65)$$

$$\int_0^1 \Theta_r \frac{d^2\Theta_s}{d\xi^2}\, d\xi = \int_0^1 \Theta_s \frac{d^2\Theta_r}{d\xi^2}\, d\xi \, . \quad \ldots \quad \ldots \quad \ldots \quad (66)$$

The Galerkin method will now be applied to the uniform cantilever, and the approximation

$$\theta = q_1\Theta_1 + q_2\Theta_2 \quad \ldots \quad \ldots \quad \ldots \quad (67)$$

will be adopted. The Galerkin equations obtained from (59) are

$$q_1 \int_0^1 \Theta_1 \left(\frac{d^2\Theta_1}{d\xi^2} + k\Theta_1\right) d\xi + q_2 \int_0^1 \Theta_1 \left(\frac{d^2\Theta_2}{d\xi^2} + k\Theta_2\right) d\xi = 0 \, ,$$

and

$$q_1 \int_0^1 \Theta_2 \left(\frac{d^2\Theta_1}{d\xi^2} + k\Theta_1\right) d\xi + q_2 \int_0^1 \Theta_2 \left(\frac{d^2\Theta_2}{d\xi^2} + k\Theta_2\right) d\xi = 0$$

These reduce to

$$\left.\begin{aligned} &\frac{q_1}{q_2}\left(-\frac{4}{3} + \frac{8}{15}k\right) + \left(-1 + \frac{13}{30}k\right) = 0 \, , \\ \text{and} \quad &\frac{q_1}{q_2}\left(-1 + \frac{13}{30}k\right) + \left(-\frac{6}{5} + \frac{13}{35}k\right) = 0 \, . \end{aligned}\right\} \quad (68)$$

The condition of compatibility of these equations yields a quadratic for k whose roots are

$$\frac{12}{130}\left(141 \pm \sqrt{13{,}056}\right) ,$$

i.e., $2 \cdot 4680$ and $23 \cdot 5625$. The true values are $\frac{\pi^2}{4} = 2 \cdot 4674$ and $\frac{9\pi^2}{4} = 22 \cdot 207$. It will be seen that the fundamental root is very closely correct.

Further calculations have been made using the functions Θ_1, Θ_2, and Θ_3. The results of all the calculations are given in the following table, which includes a comparison of the true and approximate modes.

Fundamental Torsional Mode and Frequency Coefficient for a Uniform Cantilever

		Approximation.			Correct result.
		1st. ($q_2 = q_3 = 0$)	2nd. ($q_3 = 0$)	3rd.	
$k = \frac{Jp^2l^2}{C}$		2·5	2·4680	2·4674	2·4674
$\frac{\theta}{\theta_1}$	$\xi = 0{\cdot}25$	0·4375	0·3821	0·3827	0·3827
	0·5	0·7500	0·7008	0·7071	0·7071
	0·75	0·9375	0·9190	0·9236	0·9239
	1·0	1·0	1·0	1·0	1·0

§12. *Torsional oscillation of a cantilever carrying a flywheel.*—The procedure explained in the last two sections is adequate for the treatment of the oscillations of cantilevers of constant or variable section, but a new problem arises when the beam carries an isolated mass or masses. In this case the true solution will exhibit a discontinuity at each carried mass, and different functional expressions for the displacement will hold for the several intervals between the masses. It is possible to obtain a good approximation even here by the Galerkin process employing only continuous functions, just as it is possible to approximate to a discontinuous function by a Fourier series. But as a rule a much more accurate result will be obtained by using *one* suitable discontinuous function together with a set of continuous functions of the ordinary kind. It appears that the advantage obtained in this way is greatest when there is a discontinuity in the *first* differential coefficient of the displacement (as in torsion), and that the advantage becomes less and less as the order of the first discontinuous differential coefficient increases. For flexural oscillations the earliest differential coefficient to exhibit discontinuity at a carried mass is the third, and here the advantage gained from the use of a discontinuous function is very slight. (See §13.)

In order to derive the Galerkin equations correctly it is always best to imagine the added mass to be distributed over a short distance, and then suppose this distance to tend to zero. When this is not done terms are apt to be omitted from the equations.

The choice of the discontinuous function must be considered. Now it is clear that this function must remain suitable even when the density of the cantilever itself tends to zero. Clearly, therefore, the most suitable function is the modal function for a massless beam carrying an isolated mass. In the case of torsion, for instance, the function will be obviously

$$\left.\begin{aligned}\Theta_0 &= \xi \ (0 \leqslant \xi \leqslant h) \, , \\ &= h \ (h \leqslant \xi \leqslant 1) \, ,\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \qquad (69)$$

when the added mass lies at the distance hl from the root.

When there are several carried masses the advantage obtained from the use of discontinuous functions is reduced (at any rate so far as the fundamental mode is concerned), since the individual discontinuities at the masses will be less important.

The following problem will now be discussed :—Find the fundamental frequency and mode for the torsional motion of a uniform cantilever carrying a flywheel whose moment of inertia is twice that of the cantilever itself at a distance of one third of the span from the root. This will be treated by the Galerkin method employing one discontinuous and one continuous function. Accordingly

$$\theta = q_0\Theta_0 + q_1\Theta_1 \, , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (70)$$

where Θ_0 is as defined by (69) with $h = \frac{1}{3}$, and Θ_1 is given by (62). The differential equation governing the motion is

$$\mathrm{C}\frac{d^2\theta}{dx^2} + \mathrm{J}p^2\theta = 0 \, .$$

Consider the equation which results when this is multiplied by Θ_r and integrated over the span. Take first the contribution of the term $\mathrm{C}\dfrac{d^2\theta}{dx^2}$, which is the sum of the continuous part $\mathrm{C}q_1\dfrac{d^2\Theta_1}{dx^2}$ and the discontinuous part $\mathrm{C}q_0\dfrac{d^2\Theta_0}{dx^2}$. The first of these only calls for the comment that the range of integration must be split into two parts if the multiplier Θ_r is discontinuous. On the other hand $d^2\Theta_0/dx^2$ vanishes except at the discontinuity, and it is easy to see that the value of $\int \Theta_r \, \mathrm{C}q_0 \dfrac{d^2\Theta_0}{dx^2} \, dx$ taken over the discontinuity is

$$\mathrm{C}q_0\Theta_r(\tfrac{1}{3})\left\{\left(\frac{d\Theta_0}{dx}\right)_{\frac{1}{3}+\epsilon} - \left(\frac{d\Theta_0}{dx}\right)_{\frac{1}{3}-\epsilon}\right\} = -\frac{\mathrm{C}q_0\Theta_r(\frac{1}{3})}{l} \, ,$$

since $$\left(\frac{d\Theta_0}{dx}\right)_{\frac{l}{3}+\epsilon} = 0\,,$$

and $$\left(\frac{d\Theta_0}{dx}\right)_{\frac{l}{3}-\epsilon} = \frac{1}{l}\,.$$

Next take the contribution of the term $Jp^2\theta$ to the integral. The part of this due to the beam itself requires no comment, but there is obviously the additional term $Ip^2\theta(\frac{l}{3})\Theta_r(\frac{l}{3})$ due to the flywheel of moment of inertia $I = 2lJ$ in the present example.

It follows from the foregoing that the Galerkin equation corresponding to Θ_0 is

$$Cq_1\int_0^{\frac{l}{3}}\Theta_0\frac{d^2\Theta_1}{dx^2}\,dx + Cq_1\int_{\frac{l}{3}}^{l}\Theta_0\frac{d^2\Theta_1}{dx^2}\,dx - \frac{Cq_0\Theta_0(\frac{l}{3})}{l}$$
$$+ Jp^2\int_0^{\frac{l}{3}}\Theta_0\,(q_0\Theta_0 + q_1\Theta_1)\,dx + Jp^2\int_{\frac{l}{3}}^{l}\Theta_0\,(q_0\Theta_0 + q_1\Theta_1)\,dx$$
$$+ 2Jp^2l\Theta_0(\tfrac{l}{3})\{q_0\Theta_0(\tfrac{l}{3}) + q_1\Theta_1(\tfrac{l}{3})\} = 0\,, \quad \ldots \quad \ldots \quad \ldots \quad (71)$$

and the equation corresponding to Θ_1 is

$$Cq_1\int_0^{l}\Theta_1\frac{d^2\Theta_1}{dx^2}\,dx - \frac{Cq_0\Theta_1(\frac{l}{3})}{l}$$
$$+ Jp^2\int_0^{\frac{l}{3}}\Theta_1\,(q_0\Theta_0 + q_1\Theta_1)\,dx + Jp^2\int_{\frac{l}{3}}^{l}\Theta_1\,(q_0\Theta_0 + q_1\Theta_1)\,dx$$
$$+ 2Jp^2l\Theta_1(\tfrac{l}{3})\{q_0\Theta_0(\tfrac{l}{3}) + q_1\Theta_1(\tfrac{l}{3})\} = 0\,. \quad \ldots \quad \ldots \quad \ldots \quad (72)$$

Change the variable from x to ξ, multiply by l/C, and reduce. The equations become

$$q_0\left(-\frac{1}{3} + \frac{25}{81}k\right) + q_1\left(-\frac{5}{9} + \frac{565}{972}k\right) = 0\,,$$

and $$q_0\left(-\frac{5}{9} + \frac{565}{972}k\right) + q_1\left(-\frac{4}{3} + \frac{466}{405}k\right) = 0\,. \quad \ldots \quad \ldots \quad (73)$$

These yield a quadratic for k whose lower root is 1·0337. Now it can be shown that the exact value of k is λ^2, where λ is a root of

$$\cos\lambda = \left(\frac{I}{lJ}\right)\lambda\sin h\lambda\cos(1-h)\lambda\,. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (74)$$

Here h is the distance from the root to the flywheel as a fraction of the overhang. In the present instance the equation becomes

$$\cos \lambda = 2\lambda \sin \frac{\lambda}{3} \cos \frac{2\lambda}{3} ,$$

and it is found that the smallest root is 1·0164468 radians. Hence $k = \lambda^2 = 1 \cdot 033164$, and it will be seen that the approximation obtained by the use of one discontinuous and one continuous function is excellent. The following table gives a comparison of the true and approximate modes, and also shows the results of some calculations in which only continuous functions were used. It is evident that it would be necessary to employ a large number of such functions in order to obtain a good approximation.

Fundamental Torsional Mode and Frequency Coefficient for a Uniform Cantilever carrying a Flywheel

Flywheel situated at one third of the overhang from the root

Moment of inertia of the flywheel twice that of the cantilever

		Functions used in Approximations*.			Correct result.
		Θ_1 & Θ_2	Θ_1, Θ_2 & Θ_3	Θ_0 & Θ_1	
$k = \frac{Jp^2l^2}{C}$		1·1019	1·0983	1·0337	1·0332
$\frac{\theta}{\theta_1}$	ξ = 0·25	0·5693	0·5754	0·5982	0·5892
	0·5	0·8672	0·8968	0·8785	0·8736
	0·75	0·9814	1·0015	0·9696	0·9679
	1·0	1·0	1·0	1·0	1·0

* Θ_0 is discontinuous.
Θ_1, Θ_2, and Θ_3 are continuous.

§13. *Flexural oscillation of a uniform cantilever carrying an isolated mass.*—The most convenient discontinuous function for use here is the modal function appropriate to the oscillation of a massless cantilever carrying a massive particle. This is obviously identical with the static deflexion function for an isolated load. Hence the discontinuous function will be taken as

$$\left.\begin{aligned} Y_0 &= 3h\xi^2 - \xi^3 \quad (0 \leqslant \xi \leqslant h) , \\ &= 3h^2\xi - h^3 \quad (h \leqslant \xi \leqslant 1) . \end{aligned}\right\} \qquad \ldots \quad \ldots \qquad (75)$$

The differential equation is (see §10)

$$\frac{d^4y}{dx^4} - \frac{mp^2y}{EI} = 0 ,$$

and the only point requiring special attention in forming the Galerkin equations is the influence of the discontinuity in Y_0 on the value of $\int_0^l Y_r \frac{d^4y}{dx^4} dx$. It is easy to see that the contribution of the discontinuity at $\xi = h$ to this integral is

$$q_0 \, Y_r(h) \left\{ \left(\frac{d^3Y_0}{dx^3}\right)_{h+\epsilon} - \left(\frac{d^3Y_0}{dx^3}\right)_{h-\epsilon} \right\} = \frac{6q_0 \, Y_r(h)}{l^3} . \quad .. \quad .. \quad .. \quad (76)$$

The following specific problem has been treated by the Galerkin method, employing the discontinuous function Y_0 in conjunction with the continuous functions Y_r, and using the continuous functions only :—

Find the fundamental mode and frequency for a uniform cantilever carrying a particle of mass equal to 1·5 times the mass of the cantilever at a distance of one third of the overhang from the root.

In view of the detailed discussion of the corresponding torsional problem given in the preceding section it will not be necessary to discuss further the treatment by the Galerkin method, but a few words must be said about the orthodox solution of the problem. It can be shown that the frequency parameter a defined by equation (53) is given by

$$a = \mu^4 , \quad .. \quad .. \quad .. \quad .. \quad (77)$$

where μ is a root of the equation

$$1 + \cosh \mu \cos \mu = \left(\frac{M}{2ml}\right) \mu F(\mu, h) . \quad .. \quad .. \quad (78)$$

Here M is the mass of the carried particle, and

$$\begin{aligned} F(\mu, h) = {} & \cosh \mu h \sin \mu h - \cos \mu h \sinh \mu h \\ & + \cos \mu (1 - h) \sinh \mu(1 - h) - \cosh \mu(1 - h) \sin \mu(1 - h) \\ & + \cosh \mu h \sin \mu \cosh \mu(1 - h) - \cos \mu h \sinh \mu \cos \mu(1 - h) . \end{aligned} \qquad (79)$$

Exact formulae for the displacements of the two segments of the beam can also be obtained, but these will not be quoted.

The results obtained by the various methods are compared in the following table. It will be seen that the advantage gained by the use of a discontinuous function is here negligible, and the reason for this has been given at the beginning of §12.

Fundamental Flexural Mode and Frequency Coefficient for a Uniform Cantilever carrying a Massive Particle

Particle situated at one third of the overhang from the root
Mass of particle 1·5 times mass of cantilever

		Functions used in Approximations*.					Correct result.
		Y_1	Y_1 & Y_2	Y_1 & Y_3	Y_1, Y_2 & Y_3	Y_0 & Y_1	
	$\alpha = \frac{mp^2l^4}{EI}$	10·5346	10·5346	10·5345	10·5328	10·5340	10·5284
$\frac{y}{y_1}$	ξ = 0·25	0·1055	0·1056	$0·1051_5$	0·1058	0·1062	0·1059
	0·5	0·3542	0·3543	0·3535	0·3535	0·3554	0·3534
	0·75	0·6680	0·6681	0·6674	0·6669	0·6687	0·6665
	1·0	1·0	1·0	1·0	1·0	1·0	1·0

* Y_0 is discontinuous.
Y_1, Y_2, Y_3 are continuous.

§14. *Determination of the critical loads of struts.*—Consider a straight strut of length l, pin-jointed at the ends, and subject to an axial compressive load P. Then the differential equation governing the deflections of the strut is

$$EI \frac{d^2y}{dx^2} + Py = 0, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (80)$$

and, if the origin is at midspan, the primary boundary conditions are $y = 0$ when $x = \pm l/2$. On account of (80) there is the secondary boundary condition $EI \frac{d^2y}{dx^2} = 0$ when $x = \pm l/2$, and, provided that EI does not vanish at the ends, this implies that

$$\frac{d^2y}{dx^2} = 0 \text{ when } x = \pm \frac{l}{2}.$$

Now suppose that

$$y = \Sigma q_r y_r ; \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad 81)$$

where the functions y_r satisfy at least the primary boundary conditions, and regard the quantities q_r as dynamical co-ordinates. In order to obtain the Lagrangian equations of equilibrium it will first be necessary to obtain an expression for u, the shortening of the distance between the ends of the strut due to the lateral bowing. Let ds be an element of arc of the bowed centre line. Then

$$\frac{dx}{ds} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{-\frac{1}{2}} = 1 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

since $\frac{dy}{dx}$ is very small. Hence clearly

$$u = \frac{1}{2}\int_{-\frac{l}{2}}^{+\frac{l}{2}} \left(\frac{dy}{dx}\right)^2 dx\,, \qquad \ldots \qquad (82)$$

and

$$\frac{\partial u}{\partial q_r} = \frac{1}{2}\int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{\partial}{\partial q_r}\left(\frac{dy}{dx}\right)^2 dx$$

$$= \int_{-\frac{l}{2}}^{+\frac{l}{2}} \frac{dy_r}{dx}\,\frac{dy}{dx}\,dx$$

$$= \left[y\,\frac{dy_r}{dx}\right]_{-\frac{l}{2}}^{+\frac{l}{2}} - \int_{-\frac{l}{2}}^{+\frac{l}{2}} y\,\frac{d^2y_r}{dx^2}\,dx$$

$$= -\int_{-\frac{l}{2}}^{+\frac{l}{2}} y\,\frac{d^2y_r}{dx^2}\,dx\,.$$

Therefore the work done by P in a virtual displacement corresponding to the increment δq_r is

$$-\,P\,\delta q_r \int_{-\frac{l}{2}}^{+\frac{l}{2}} y\,\frac{d^2y_r}{dx^2}\,dx\,.$$

Also the elastic potential energy is

$$V = \frac{1}{2}\int_{-\frac{l}{2}}^{+\frac{l}{2}} \mathrm{EI}\left(\frac{d^2y}{dx^2}\right)^2 dx\,,$$

and

$$\frac{\partial V}{\partial q_r} = \int_{-\frac{l}{2}}^{+\frac{l}{2}} \mathrm{EI}\,\frac{d^2y}{dx^2}\,\frac{d^2y_r}{dx^2}\,dx\,.$$

Now

$$\frac{\partial V}{\partial q_r}\,\delta q_r = \text{virtual work of P.} \quad \text{Hence}$$

$$\int_{-\frac{l}{2}}^{+\frac{l}{2}} \mathrm{EI}\,\frac{d^2y}{dx^2}\,\frac{d^2y_r}{dx^2}\,dx + \mathrm{P}\int_{-\frac{l}{2}}^{+\frac{l}{2}} y\,\frac{d^2y_r}{dx^2}\,dx = 0\,. \qquad (83)$$

This is precisely the Galerkin equation obtained from the differential equation (80) by use of the multiplier d^2y_r/dx^2. The reason why the multiplier is this and not merely y_r is that (80) expresses a balance of moments, not of normal forces, and the proper multiplier must therefore be a rate of rotation.

Applications will now be made to two concrete examples.

Example 1.—*Uniform pin-jointed strut.*—The first step is the choice of a suitable set of functions. In the present case, or in any other in which the strut is symmetrical about its mid-point, only even functions of x should be employed, since interest is confined to the fundamental mode of deflexion. The set

$$y_r = \left(\frac{2x}{l}\right)^{2r} - 1 \qquad (84)$$

satisfy the primary, but not the secondary, boundary conditions, while the set

$$\mathrm{Y}_r = (4r + 1) - (r + 1)(2r + 1)\left(\frac{2x}{l}\right)^{2r} + r(2r - 1)\left(\frac{2x}{l}\right)^{2r+2} \qquad (85)$$

satisfy both the primary and secondary conditions. The problem will be worked with both sets of functions, as this will make apparent the advantage of adopting functions satisfying both the primary and secondary boundary conditions.

First use the functions y_1 and y_2. The corresponding multipliers are $8/l^2$ and $192x^2/l^4$, so that the Galerkin equations are

$$\frac{8q_1}{l^2}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left(\mathrm{EI}\,\frac{d^2y_1}{dx^2}+\mathrm{P}y_1\right)dx+\frac{8q_2}{l^2}\int_{-\frac{l}{2}}^{+\frac{l}{2}}\left(\mathrm{EI}\,\frac{d^2y_2}{dx^2}+\mathrm{P}y_2\right)dx = 0$$

and $$\frac{192q_1}{l^4}\int_{-\frac{l}{2}}^{+\frac{l}{2}}x^2\left(\mathrm{EI}\,\frac{d^2y_1}{dx^2}+\mathrm{P}y_1\right)dx+\frac{192q_2}{l^4}\int_{-\frac{l}{2}}^{+\frac{l}{2}}x^2\left(\mathrm{EI}\,\frac{d^2y_2}{dx^2}+\mathrm{P}y_2\right)dx = 0\,.$$

These become, when multiplied by $105l^3/16\mathrm{EI}$,

$$\left.\begin{aligned} q_1\,(420-35\beta)+q_2\,(840-42\beta) &= 0\,,\\ \text{and}\quad q_1\,(840-42\beta)+q_2\,(3024-60\beta) &= 0\,,\end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad (86)$$

where $$\beta=\frac{\mathrm{P}l^2}{\mathrm{EI}}\;. \quad \ldots \quad \ldots \quad \ldots \quad (87)$$

The eliminant of the equations (86) is

$$\beta^2-180\beta+1680=0\,, \quad \ldots \quad \ldots \quad \ldots \quad (88)$$

of which the roots are 9·8751 and 170·125. Now the true fundamental value of β is $\pi^2 = 9{\cdot}8696$, so that the approximation is fair.

The problem has also been worked out using the functions Y_1 and Y_2, but it will not be necessary to enter into details. The final quadratic for β is

$$13\beta^2-1332\beta+11{,}880=0\,, \quad \ldots \quad \ldots \quad \ldots \quad (89)$$

of which the roots are 9·86961 and 92·5919. The first of these is an extremely close approximation to π^2, and the second is not far from $9\pi^2 = 88{\cdot}826$*. The results of all the calculations are given in the following table.

Critical Load Coefficient for a Uniform Pin-Jointed Strut

	Functions used in Approximations†.				Correct result.
	y_1	y_1 & y_2	Y_1	Y_1 & Y_2	
$\beta=\frac{\mathrm{P}l^2}{\mathrm{EI}}$	12·0	9·8751	9·8824	9·86961	$\pi^2 = 9{\cdot}869604$

†y_1 y_2 satisfy primary boundary condition only.
Y_1 Y_2 satisfy both primary and secondary boundary conditions.

*The root $4\pi^2$ is absent since this corresponds to a displacement which is not symmetrical about the mid-point.

Example 2.—Pin-jointed tapered strut.—The strut considered has a flexural rigidity given by

$$\mathrm{EI} = \frac{\mathrm{B}}{7}\left\{7 - 3\left(\frac{x}{l}\right)^2\right\}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (90)$$

where B is the maximum value of EI, occurring at the end $x = 0$, and l is the length. It can easily be verified that the differential equation

$$\frac{\mathrm{B}}{7}\left\{7 - 3\left(\frac{x}{l}\right)^2\right\}\frac{d^2y}{dx^2} + \mathrm{P}y = 0, \qquad \ldots \quad \ldots \quad \ldots \qquad (91)$$

has the exact solution

$$y = 7\left(\frac{x}{l}\right) - 10\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^5 \qquad \ldots \quad \ldots \qquad (92)$$

with $$\mathrm{P} = \frac{60\mathrm{B}}{7l^2}. \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (93)$$

Since the strut is not symmetrical about its mid-point, the functions must not be restricted to the symmetrical type. The conditions (primary and secondary) to be satisfied are :—

$$y = \frac{d^2y}{dx^2} = 0$$

when $x = 0$ and when $x = l$. The following functions are suitable :—

$$y_0 = \left(\frac{x}{l}\right) - 2\left(\frac{x}{l}\right)^3 + \left(\frac{x}{l}\right)^4, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (94)$$

$$y_r = (r + 3)\left(\frac{x}{l}\right)^{r+2} - (2r + 5)\left(\frac{x}{l}\right)^{r+3} + (r + 2)\left(\frac{x}{l}\right)^{r+4}, \qquad \ldots \qquad (95)$$

where in (95) r is not to be less than 1.

The functions y_0 and y_1 will be used in the present case, and the Galerkin equations become when reduced

$$\left.\begin{aligned} q_0(2064 - 34\gamma) + q_1(852 - 17\gamma) &= 0, \\ \text{and} \quad q_0(852 - 17\gamma) + q_1(2136 - 16\gamma) &= 0, \end{aligned}\right\} \qquad \ldots \quad \ldots \qquad (96)$$

where $$\gamma = \frac{7\mathrm{P}l^2}{\mathrm{B}}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (97)$$

so that the exact (fundamental) value of γ is 60 by equation (93).

The eliminant of equations (96) is

$$17\,\gamma^2 - 5112\,\gamma + 245{,}520 = 0\,,$$

of which the roots are 60 (exact) and 240·705. The substitution of $\gamma = 60$ in the first of equations (96) yields

$$24q_0 - 168q_1 = 0\,,$$

or

$$q_0 = 7q_1\,.$$

Put $q_1 = 1$. Then the mode of deflexion is given by

$$y = 7y_0 + y_1 = 7\left(\frac{x}{l}\right) - 10\left(\frac{x}{l}\right)^3 + 3\left(\frac{x}{l}\right)^5,$$

which agrees with (92). This result exemplifies the fact that if any linear combination of the functions employed happens to be an exact solution of the differential equation, then the Galerkin method will produce that exact solution.

It only remains to add that the approximation to γ obtained by the use of y_0 only is 2064/34 = 60·706 (see the coefficient of q_0 in the first of equations (96)).

§15. *Applications to the St. Venant torsion problem.*—The St. Venant problem of the torsion of a solid cylinder or prism can be reduced to the following form[3] :—Find a function Ψ of x and y which vanishes on the boundary of the section of the cylinder and satisfies

$$\nabla^2\Psi + 2 = 0 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (98)$$

everywhere within the section. Then the components of shearing stress are given by

$$\left.\begin{aligned} X_z &= \mu\,\tau\,\frac{\partial \Psi}{\partial y}\,, \\ Y_z &= -\,\mu\,\tau\,\frac{\partial \Psi}{\partial x}\,, \end{aligned}\right\} \qquad \ldots \quad \ldots \quad \ldots \qquad (99)$$

where τ = twist (radians) per unit length,

and μ = modulus of rigidity of the isotropic material.

Also the torsional stiffness of unit length is

$$C = 2\mu \iint \Psi\, dx\, dy\,. \qquad \ldots \quad \ldots \quad \ldots \qquad (100)$$

In the present discussion attention will be confined to cylinders or prisms whose cross-sections possess an axis of symmetry OX*. The boundary will therefore be specified by

$$y = \pm t \,, \qquad \ldots \qquad (101)$$

where t is a known function of x. It is clear that the functions

$$T_r = t^{2r} - y^{2r}$$

all vanish on the boundary, and it might be supposed that $\Sigma c_r T_r$ would be capable of representing the solution of the problem. But on inspection it will be found that this is the difference of a function of x only and of a function of y only, and manifestly this is too restricted in form to represent the true solution in all cases. Again $(t^2 - y^2)(k_0 + k_1 x + k_2 x^2 + \ldots)$ is insufficiently general since it is parabolic in y. The conclusion is that the general expression for the torsion stress function is

$$\Psi = \Sigma (t^{2r} - y^{2r})(k_{0r} + k_{1r}x + k_{2r}x^2 + \ldots) \quad \ldots \quad (102)$$

The question now arises as to the proper multipliers to be used in the Galerkin process. This can be answered by reference to the well known membrane analogue of the torsion problem. In that analogue the equation (98) expresses a balance of forces normal to the membrane, while the deflexion is proportional to Ψ. Hence the equations of virtual work are obtained by multiplication by $\delta\Psi$, and integration over the section.

Example 1.—*Elliptic cylinder.*—Here

$$t^2 = \frac{b^2}{a^2}(a^2 - x^2) \,.$$

Assume

$$\Psi = k_0 (t^2 - y^2)$$

$$= \frac{k_0}{a^2}(a^2b^2 - b^2x^2 - a^2y^2) \,.$$

Then

$$\varepsilon = \nabla^2\Psi + 2 = -\frac{2k_0}{a^2}(a^2 + b^2) + 2 \,.$$

This vanishes if

$$k_0 = \frac{a^2}{a^2 + b^2} \,,$$

and the exact solution is accordingly

$$\Psi = \frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2 + b^2}$$

* The method can easily be extended to cylinders with unsymmetrical sections.

Example 2.—Cylinder whose section is a cubic oval (symmetrical aerofoil).—No "classical" solution has been found in this case, but the problem has been treated in detail by the thickness-parameter method[4,5].

The cross-section of the cylinder is here specified by

$$t^2 = \theta^2\left(cx - 2x^2 + \frac{x^3}{c}\right), \qquad (103)$$

where c is the chord and θ is the thickness parameter. The problem will be worked using first the approximation

$$\Psi = k_0\,(t^2 - y^2)\,, \qquad (104)$$

and then with

$$\Psi = k_0\,(t^2 - y^2) + k_1 x\,(t^2 - y^2)\,. \qquad (105)$$

Let the result of substituting the approximation to Ψ in (98) be ε. Then the Galerkin equation corresponding to (104) is

$$\iint \varepsilon\,(t^2 - y^2)\,dx\,dy = 0\,,$$

where in the present instance ε is a function of x only. Accordingly the last equation becomes

$$\frac{4}{3}\int_0^c t^3\,\varepsilon\,dx = 0\,,$$

which can be reduced to

$$13\,(1 - k_0 - 2k_0\theta^2) + 15k_0\theta^2 = 0\,.$$

Hence
$$k_0 = \frac{13}{13 + 11\theta^2}\,,$$

and the corresponding expression for the torsional stiffness yielded by (100) is

$$C = \frac{256\,\mu c^4\theta^3}{3465}\left(\frac{13}{13 + 11\theta^2}\right). \qquad (106)$$

The thickness parameter method gives the following approximation which is correct up to the 9th power of θ :—

$$C = \frac{256\,\mu c^4\theta^3}{3465}\left(1 - \frac{11}{13}\,\theta^2 + \theta^4 - \frac{379}{221}\,\theta^6\right). \qquad (107)$$

Equation (106) agrees with this as far as the fifth power of θ.

When the expression (105) is used the Galerkin equations are

$$\iint \varepsilon \, (l^2 - y^2) \, dx \, dy = 0 \, ,$$

and

$$\iint \varepsilon \, x \, (l^2 - y^2) \, dx \, dy = 0 \, ,$$

where

$$\varepsilon = 2 - (k_0 + k_1 x)\left\{2 + \theta^2\left(4 - 6\frac{x}{c}\right)\right\} + 2k_1\theta^2\left(c - 4x + \frac{3x^2}{c}\right).$$

The Galerkin equations reduce to

$$k_0\,(13 + 11\theta^2) + k_1 c\,(5 + 3\theta^2) = 13 \, ,$$

and

$$k_0\,(255 + 153\theta^2) + k_1 c\,(119 + 81\theta^2) = 255 \, .$$

Hence

$$k_0 = \frac{17 + 18\theta^2}{17 + 52\theta^2 + 27\theta^4} \, ,$$

and

$$ck_1 = \frac{51\theta^2}{17 + 52\theta^2 + 27\theta^4} \, .$$

The approximate stress function is therefore

$$\Psi = \frac{(l^2 - y^2)\left\{(17 + 18\theta^2) + 51\theta^2\left(\frac{x}{c}\right)\right\}}{17 + 52\theta^2 + 27\theta^4} \, , \qquad \ldots \qquad (108) \, .$$

and the corresponding expression for the stiffness is

$$C = \frac{256\,\mu c^4 \theta^3}{3465}\left(\frac{221 + 489\theta^2}{221 + 676\theta^2 + 351\theta^4}\right). \qquad \ldots \qquad (109)$$

A comparison of the results given by (106), (109) and (107) for the case where $\theta = 1/5$ is given below where the quantity tabulated is the multiplier of $256\mu c^4\theta^3/3465$:—

			Error.
Galerkin (1 function)	..	0·967262	−0·000382
Galerkin (2 functions)	..	0·967652	+0·000008
Thickness parameter ..	..	0·967644	—

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