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Stability Theory of Time-Marching Methods with Extensions to the Theory of Ill-Conditioning of Steady-State Methods

- by -

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STABILITY THEORY OF TIME-MARCHING METHODS WITH EXTENSIONS TO THE THEORY OF ILL-CONDITIONING OF STEADY-STATE METHODS

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S.M. Richardson

SUMMARY

A relatively straightforward method for obtaining the local stability properties of multi-variable time-marching schemes for predicting compressible flows is developed. Applications of the method to several one-, two- and three-dimensional examples are presented. An extension of the techniques used in the stability analysis to the prediction of ill-conditioning of steady-state methods, in particular near sonic conditions, is also developed, and examples given which show how to overcome the ill-conditioning problems.

*Replaces A.R.C.36 910.

I - STABILITY THEORY

The equations of motion of an unsteady, inviscid, compressible flow may always be expressed in the form:-

$$\frac{\partial \underline{V}}{\partial t} = \underline{F}(\underline{V}) \qquad (1)$$

where t denotes time, $\underline{V}$ denotes the vector of m (say) flow variables, and $\underline{F}$ denotes a (generally, non-linear) spatial differential operator.

[In one-dimensional isentropic flow, for example, the flow equations are:-

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} &= 0 \\ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} &= 0 \end{aligned}\right\} \qquad (2)$$

where $c^2 = \gamma RT$. In the form (1), equations (2) become:-

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \end{pmatrix} = -\begin{pmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ u \end{pmatrix} \qquad (3)$$

so that:-

$$\underline{V} = \begin{pmatrix} \rho \\ u \end{pmatrix} \qquad (4)$$

and

$$\underline{F} = -\begin{pmatrix} u & \rho \\ \frac{c^2}{\rho} & u \end{pmatrix} \frac{\partial}{\partial x} \qquad (5)$$
]

Numerical schemes for solving the partial differential equations (1) generally involve (possibly, non-uniform) discretisation in time and space, and lead to a set of algebraic equations to be solved for each flow variable at each point in the (discretised) flow field. (Boundary conditions, of course, may be treated somewhat differently.) The set of algebraic equations, which we may regard as the basic equations defining a time-marching method, will always be of the form:-

$$\underline{V}^{n+1}_{ijk} = \underline{\underline{A}}\,\underline{V}^{n}_{ijk} + \underline{\underline{B}}\,\underline{V}^{n}_{i-1\,jk} + \underline{\underline{C}}\,\underline{V}^{n}_{i+1\,jk} + \cdots \qquad (6)$$

where $t = t_n$, $x = x_i$, $y = y_j$, $z = z_k$, and the matrices $\underline{\underline{A}}$, $\underline{\underline{B}}$, $\underline{\underline{C}}$ etc may vary with position and time.

[Thus, if forward-time, upwind-space differencing is used for (2), we have, if $u \geqslant 0$:-

$$\left.\begin{aligned} \left(\frac{u_i^{n+1} - u_i^n}{\Delta t}\right) + u_i^n\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right) + \frac{c^2}{\rho_i^n}\left(\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}\right) &= 0 \\ \left(\frac{\rho_i^{n+1} - \rho_i^n}{\Delta t}\right) + u_i^n\left(\frac{\rho_i^n - \rho_{i-1}^n}{\Delta x}\right) + \rho_i^n\left(\frac{u_i^n - u_{i-1}^n}{\Delta x}\right) &= 0 \end{aligned}\right\} \quad (7)$$

which may be rewritten:-

$$\left.\begin{aligned} u_i^{n+1} &= u_i^n - \frac{u_i^n}{c}\frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) - \frac{c}{\rho_i^n}\frac{c\Delta t}{\Delta x}(\rho_i^n - \rho_{i-1}^n) \\ \rho_i^{n+1} &= \rho_i^n - \frac{u_i^n}{c}\frac{c\Delta t}{\Delta x}(\rho_i^n - \rho_{i-1}^n) - \frac{\rho_i^n}{c}\frac{c\Delta t}{\Delta x}(u_i^n - u_{i-1}^n) \end{aligned}\right\} \quad (8)$$

or:-

$$\underline{v}_i^{n+1} = \begin{pmatrix} 1 - MQ & -Q\rho/c \\ -Qc/\rho & 1 - MQ \end{pmatrix}\underline{v}_i^n + \begin{pmatrix} MQ & Q\rho/c \\ Qc/\rho & MQ \end{pmatrix}\underline{v}_{i-1}^n \quad (9)$$

where:- $$\underline{v}_i^n = \begin{pmatrix} \rho_i^n \\ u_i^n \end{pmatrix} \quad (10)$$

$$Q = c\,\Delta t/\Delta x \qquad \text{(Courant number)} \quad (11)$$

and $$M = u/c \qquad \text{(Mach number)} \quad (12)$$

i.e.

$$\underline{v}_i^{n+1} = \underline{\underline{A}}\,\underline{v}_i^n + \underline{\underline{B}}\,\underline{v}_{i-1}^n \quad (13)$$

which is of the form (6).

In order to determine the complete discretised flow field at time t_{n+1} from the field at time t_n, it is necessary to solve the complete set of equations (6) at all points in the flow field. The question of importance here is:- <u>will the numerical scheme be stable?</u> More precisely, will any error or disturbance in the flow field (due to round-off error or actual physical disturbances) be unrealistically and disastrously amplified? To formalise this concept of stability, we use ideas developed by Douglas (1961).

Consider the one-variable linear differential equation:-

$$\frac{\partial w}{\partial t} = A\,w + b \quad (14)$$

where A involves spatial derivatives, and may vary with time. Assume that:-

$$\left.\begin{aligned} w(\underline{x}, 0) &= w_0(\underline{x}) && \text{at time } t = 0 \\ w(\underline{x}, t) &= W(\underline{x}, t) && \text{on the boundary of the region of interest.} \end{aligned}\right\} \quad (15)$$

Replace the differential system (14)/(15) by the difference system:-

$$\left.\begin{aligned} w^{*0} &= w_0 \\ w^{*n+1} &= C^n w^{*n} + q^n \end{aligned}\right\} \quad (16)$$

where w^{*n} is the discrete approximation to w at time t_n. Assume that (16) is consistent i.e.

if $$w^{n+1} = C^n w^n + q^n + e^n \quad (17)$$

then $$(1/\Delta t)\, e^n \longrightarrow 0 \quad (18)$$

for sufficiently differentiable solutions of (14) as the increments of the independent variables tend to zero.

Put:- $$z^n = w^n - w^{*n} \quad (19)$$

and let $\underline{z}$ be the vector of the variables z at all points in the region of interest. Then:-

$$\left.\begin{aligned} \underline{z}^{n+1} &= \underline{\underline{C}}^n \underline{z}^n + \underline{e}^n \\ \text{and} \quad \underline{z}^0 &= \underline{0} \end{aligned}\right\} \quad (20)$$

Thus the analysis of (16) is replaced by that of (20). The natural norm to use for (20) may vary from time-step to time-step, as the simplest choice may depend on $\underline{\underline{C}}^n$, which may depend on n since A may depend on t. We introduce the sequence of (vector) norms $\|\underline{z}^n\|_n$; n = 0, 1, 2, ..., where each norm satisfies the usual norm axioms.

[The norm $\|x\|$ of a quantity x describes its magnitude. Any definition of a norm will suffice provided it satisfies the so-called "norm axioms" (see Liusternik & Sobolev (1961)):-

(i) $\|x\| > 0$ unless $x \equiv 0$, when $\|x\| = 0$ i.e. the norm is non-negative;

(ii) $\|x\| + \|y\| \geqslant \|x + y\|$ i.e. a triangular inequality;

(iii) $\|\lambda x\| = |\lambda|\,\|x\|$ for any scalar λ; $|\lambda|$ denotes the magnitude of λ.]

We introduce also the induced (matrix operator) norms:-

$$\|\underline{\underline{C}}\|_n = \max_{\underline{z} \neq \underline{0}} \cdot \frac{\|\underline{\underline{C}}\,\underline{z}\|_n}{\|\underline{z}\|_n} \qquad (21)$$

Then (20) implies that:-

$$\|\underline{z}^{n+1}\|_n \leqslant \|\underline{\underline{C}}^n\|_n \|\underline{z}^n\|_n + \|\underline{e}^n\|_n \qquad (22)$$

by the triangular inequality. We need to be able to compare successive norms to be able to use the recursion relation (22). Assume that:-

$$\|\underline{z}\|_{n+1} \leqslant \{1 + a\Delta t\}\|\underline{z}\|_n \qquad n = 0, 1, 2, \ldots \qquad (23)$$

for all $\underline{z}$. (We assume (23) holds because we need it for our proof, and because common schemes obey it.) Then:-

$$\|\underline{z}^{n+1}\|_{n+1} \leqslant \{1 + a\Delta t\}\{ \|\underline{\underline{C}}^n\|_n \|\underline{z}^n\|_n + \|\underline{e}^n\|_n\} \qquad (24)$$

whence, using (22), we obtain:-

$$\|\underline{z}^n\|_n \leqslant \sum_{k=0}^{n-1} \{1 + a\Delta t\}^{n-k} \|\underline{e}^k\|_k \prod_{j=k+1}^{n-1} \|\underline{\underline{C}}^j\|_j \qquad (25)$$

where:-

$$\prod_{j=n}^{n-1} \|\underline{\underline{C}}^j\|_j \equiv 1$$

Note that the index n-k is a <u>power</u>, not a superscript.

The scheme (16) is <u>defined</u> to be <u>stable</u> with respect to the given sequence of norms if:-

$$\|\underline{\underline{C}}^k\|_k \leqslant \{1 + b\Delta t\} \qquad k = 0, 1, 2, \ldots \qquad (26)$$

as $\Delta t \longrightarrow 0$. Thus the numerical scheme (16) is <u>stable</u> if the norm of the matrix operator $\underline{\underline{C}}$, which is the discrete analogue of the differential operator A in (14), is, at every time-step, less than one plus some (problem-dependent) constant multiplied by the magnitude of the time-step length Δt. If (26) holds, then (25) becomes:-

$$\|\underline{z}^n\|_n \leqslant K \sum_{k=0}^{n-1} \|\underline{e}^k\|_k \qquad (27)$$

where:-

$$K = c + d\,\Delta t + O(\Delta t^2) \qquad (28)$$

How do we apply this definition of stability to the numerical scheme (6)? First, we note that, instead of <u>one</u> unknown, w, we have m unknowns, $\underline{V}$, at each point in the discretised region of interest. Secondly, the stability definition (26) applies globally (i.e. at all points in the discretised region simultaneously); ensuring that the norm of the differential operator analogue satisfies an equivalent of (26) globally generally involves an inordinate amount of work, unless particularly simple equations and boundary conditions are involved. Finally, the stability definition (26) applies to linear differential operators (recall that A in (14) is linear). So we wish:-

(i) to generalise (26) to $\underline{m}$ variables, instead of one;

(ii) to obtain a local, rather than a global, stability criterion;

(iii) to generalise (26) to non-linear differential operators.

Suppose we expand $\underline{V}^n_{ijk}$ from (6) in (spatial) Fourier series. We assume Δx, Δy and Δz are all constant, though this condition can be relaxed. Then we have:-

$$\underline{V}^n_{ijk} = \sum_p \sum_q \sum_r \underline{S}^n_{pqr}\, e^{iIp\Delta x}\, e^{jIq\Delta y}\, e^{kIr\Delta z} \qquad (29)$$

where:-

$$I^2 = -1$$

Put:-

$$\underline{V}^n_{ijk}\Big|_{pqr} = \underline{S}^n_{pqr}\, e^{iIp\Delta x}\, e^{jIq\Delta y}\, e^{kIr\Delta z} \qquad (30)$$

so that:-

$$\underline{V}^n_{ijk} = \sum_p \sum_q \sum_r \underline{V}^n_{ijk}\Big|_{pqr} \qquad (31)$$

Then (6) becomes:-

$$\underline{V}^{n+1}_{ijk}\Big|_{pqr} = \underline{\underline{A}}\,\underline{V}^n_{ijk}\Big|_{pqr} + \underline{\underline{B}}\,\underline{V}^n_{ijk}\Big|_{pqr} e^{-Ip\Delta x} + \underline{\underline{C}}\,\underline{V}^n_{ijk}\Big|_{pqr} e^{+Ip\Delta x} + \ldots \qquad (32)$$

whence:-

$$\underline{\underline{A}}^*\underline{V}^n_{ijk}\Big|_{pqr} = \underline{V}^{n+1}_{ijk}\Big|_{pqr} \qquad (33)$$

where:-

$$\underline{\underline{A}}^* = \underline{\underline{A}} + e^{-Ip\Delta x}\,\underline{\underline{B}} + e^{+Ip\Delta x}\,\underline{\underline{C}} + \ldots. \qquad (34)$$

Equation (33) shows how the (pqr)-th Fourier component of the m variables $\underline{V}^n_{ijk}$ is transformed locally into the (pqr)-th Fourier component of $\underline{V}^{n+1}_{ijk}$.

So, by analogy with (26), we may postulate that scheme (6) is locally stable if:-

$$\| \underline{\underline{A}}^* \| \leqslant \{1 + b\Delta t\} \tag{35}$$

at all time-steps, as $\Delta t \longrightarrow 0$. A suitable norm for the (matrix) operator $\underline{\underline{A}}^*$ is the spectral radius (i.e. the magnitude of the largest eigenvalue):-

$$\| \underline{\underline{A}}^* \| = \rho(\underline{\underline{A}}^*) = |\lambda_{max}(\underline{\underline{A}}^*)| \tag{36}$$

Thus we postulate that the scheme (6) is locally stable if:-

$$\rho(\underline{\underline{A}}^*) \leqslant \{1 + b\Delta t\} \tag{37}$$

which is assured if the spectral radius of $\underline{\underline{A}}^*$ is not greater than one, i.e. if:-

$$\rho(\underline{\underline{A}}^*) \leqslant 1 \tag{38}$$

We can regard this stability analysis as a generalisation of the von Neumann analysis for one-variable equations (see, for example, Roache (1972)). Clearly, since (37) is a local, rather than a global, criterion, the stability analysis possesses neither the necessity nor the sufficiency properties of a global analysis. Nevertheless, there is evidence (see again Roache (1972)) that the von Neumann method is dependable in practice, at least for one-variable problems. Its main drawback indeed arises not so much from whether it works or not, but from obtaining the spectral radius $\rho(\underline{\underline{A}}^*)$; for complex problems, a numerical, rather than an analytical, method may be necessary, as we shall see.

II - APPLICATIONS

1 - Isentropic flow

(i) one-dimensional flow:- consider the flow equations (2), and suppose we use the forward-time, upwind-space differencing scheme (9) to solve them. We put:-

$$\underline{V}_i^n = \sum_p \underline{S}_p^n \, e^{iIp\Delta x} = \sum_p \underline{V}_i^n\Big|_p \tag{39}$$

Then:-

$$\underline{V}_{i-1}^n\Big|_p = e^{-Ip\Delta x}\,\underline{V}_i^n\Big|_p \tag{40}$$

so that (9) becomes:-

$$\underline{V}_i^{n+1}\Big|_p = \begin{pmatrix} 1 - MQ(1 - e^{-Ip\Delta x}) & -\frac{Q\varrho}{c}(1 - e^{-Ip\Delta x}) \\ -\frac{Qc}{\varrho}(1 - e^{-Ip\Delta x}) & 1 - MQ(1 - e^{-Ip\Delta x}) \end{pmatrix} \underline{V}_i^n\Big|_p \tag{41}$$

We put $\theta = -p\Delta x$. Then, by hypothesis, the scheme (9) is stable if the spectral radius of $\underline{\underline{A}}^*$ satisfies (37), where $\underline{\underline{A}}^*$ is given by:-

$$\underline{\underline{A}}^* = \begin{pmatrix} 1 - MQ(1 - e^{I\theta}) & -\frac{Q\varrho}{c}(1 - e^{I\theta}) \\ -\frac{Qc}{\varrho}(1 - e^{I\theta}) & 1 - MQ(1 - e^{I\theta}) \end{pmatrix} \tag{42}$$

The eigenvalues of $\underline{\underline{A}}^*$ are given by:-

$$\lambda = 1 - Q(M\pm 1)(1 - e^{I\theta}) \tag{43}$$

Now $e^{I\theta} = \cos\theta + I\sin\theta$, whence:-

$$|\lambda| = \sqrt{(1 - Q(M\pm 1)(1 - \cos\theta))^2 + (Q(M\pm 1)\sin\theta)^2} \tag{44}$$

The extrema of $|\lambda|$ with respect to θ occur when $d|\lambda|/d\theta$ vanishes. Now:-

$$\frac{d|\lambda|}{d\theta} = \frac{1}{2|\lambda|}\left[-2Q(M\pm 1)(1 - Q(M\pm 1))\sin\theta\right] \tag{45}$$

So $d|\lambda|/d\theta$ vanishes non-trivially (i.e. excluding $Q \equiv 0$) when:-

$$\theta = n\pi \;;\; n \text{ integer} \tag{46}$$

(a) if n is even, (44) gives:-

$$|\lambda| \equiv 1 \tag{47}$$

(b) if n is odd, (44) gives:-

$$|\lambda| = |1 - 2Q(M\pm 1)| \tag{48}$$

For stability, we require:-

$$|\lambda_{max}| \leqslant 1 + b\Delta t \tag{49}$$

or, to assure stability (insofar as a local criterion can assure stability), we require:-

$$|\lambda_{max}| \leqslant 1 \tag{50}$$

Clearly, case (a) is always stable. Case (b) gives:-

$$-1 \leqslant 1 - 2Q(M\pm1) \leqslant +1$$

i.e.

$$+1 \geqslant Q(M\pm1) \geqslant 0 \tag{51}$$

The left-hand inequality in (51) gives:-

$$\frac{c\Delta t}{\Delta x}\left(\frac{u}{c} \pm 1\right) \leqslant 1$$

i.e.

$$(u \pm c)\Delta t/\Delta x \leqslant 1 \tag{52}$$

which is the Courant-Friedrichs-Lewy (C-F-L) criterion. The right-hand inequality in (51) gives, since $Q \geqslant 0$:-

$$M \pm 1 \geqslant 0$$

or, since M is assumed positive (to give upwind-differencing):-

$$M \geqslant 1 \tag{53}$$

Thus forward-time, upwind-space differencing is stable if the flow is supersonic (by (53)) and the C-F-L criterion (52) on the magnitude of the time-step length Δt is satisfied.

If, instead of upwind-space differencing, we use centred-space differencing, with constant mesh size Δx, then (41) would become, in the same notation:-

$$\underline{V}_i^{n+1}\Big|_p = \begin{pmatrix} 1 - \frac{1}{2}MQ(e^{+I\theta} - e^{-I\theta}) & -\frac{Q\rho}{2c}(e^{+I\theta} - e^{-I\theta}) \\ -\frac{Qc}{2\rho}(e^{+I\theta} - e^{-I\theta}) & 1 - \frac{1}{2}MQ(e^{+I\theta} - e^{-I\theta}) \end{pmatrix} \underline{V}_i^n\Big|_p \tag{54}$$

whence:-

$$|\lambda| = \sqrt{1 + Q^2 \sin^2\theta\,(M\pm1)^2} \tag{55}$$

Clearly, for real Q and M, we have:-

$$Q^2 \sin^2\theta\,(M+1)^2 \geqslant 0 \tag{56}$$

whence, from (55):- $$|\lambda| \geqslant 1 \tag{57}$$

so forward-time, centred-space differencing is unstable except in the trivial case of zero Courant number Q.

(ii) <u>two</u>-dimensional flow:- the flow equations are:-

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{c^2}{\rho}\frac{\partial \rho}{\partial y} &= 0 \\
\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} &= 0
\end{aligned}\right\} \qquad (58)$$

Forward-time, upwind-space differencing of (58) gives, if u and $v \geqslant 0$:-

$$\left.\begin{aligned}
u^{n+1}_{ij} &= u^n_{ij} - MQ(u^n_{ij} - u^n_{i-1j}) - NR(u^n_{ij} - u^n_{ij-1}) - \frac{Qc}{\rho}(\rho^n_{ij} - \rho^n_{i-1j}) \\
v^{n+1}_{ij} &= v^n_{ij} - MQ(v^n_{ij} - v^n_{i-1j}) - NR(v^n_{ij} - v^n_{ij-1}) - \frac{Rc}{\rho}(\rho^n_{ij} - \rho^n_{ij-1}) \\
\rho^{n+1}_{ij} &= \rho^n_{ij} - MQ(\rho^n_{ij} - \rho^n_{i-1j}) - NR(\rho^n_{ij} - \rho^n_{ij-1}) - \frac{Q\rho}{c}(u^n_{ij} - u^n_{i-1j}) \\
&\qquad - \frac{R\rho}{c}(v^n_{ij} - v^n_{ij-1})
\end{aligned}\right\} \qquad (59)$$

Expansion in a (double) Fourier series gives:-

$$\underline{V}^{n+1}_{ij}\Big|_{pq} = \begin{pmatrix} 1 - MQa - NRb & -\frac{Q\rho a}{c} & -\frac{N\rho b}{c} \\ -\frac{Qca}{\rho} & 1 - MQa - NRb & 0 \\ -\frac{Ncb}{\rho} & 0 & 1 - MQa - NRb \end{pmatrix} \underline{V}^n_{ij}\Big|_{pq} \qquad (60)$$

where:- $a = 1 - e^{I\theta}$, $b = 1 - e^{I\phi}$, $\theta = -p\Delta x$, $\phi = -q\Delta y$,

$Q = c\Delta t/\Delta x$, $R = c\Delta t/\Delta y$, $M = u/c$, and $N = v/c$.

The eigenvalues of the matrix in (60) are given by:-

$$\lambda = 1 - MQ(1 - e^{I\theta}) - NR(1 - e^{I\phi}) \begin{cases} + \sqrt{Q^2(1 - e^{I\theta})^2 + R^2(1 - e^{I\phi})^2} \\ - \text{------ '' ------} \\ 0 \end{cases} \qquad (61)$$

Extrema of $|\lambda|$ with respect to θ and ϕ occur when $\partial|\lambda|/\partial\theta$ and $\partial|\lambda|/\partial\phi$ <u>both</u> vanish; this occurs when:-

$$\left.\begin{aligned} \theta &= n_1\pi \\ \phi &= n_2\pi \end{aligned}\right\} n_1, n_2 \text{ integers} \qquad (62)$$

(a) if n_1 and n_2 are both even, (61) gives:-

$$|\lambda| = 1 \qquad (63)$$

(b) if n_1 is odd and n_2 is even, (61) gives:-

$$|\lambda| = \left| 1 - 2MQ \begin{cases} + 2Q \\ - 2Q \\ 0 \end{cases} \right| \qquad (64)$$

(c) if n_1 is even and n_2 is odd, (61) gives:-

$$|\lambda| = \left| 1 - 2NR \begin{cases} +2R \\ -2R \\ 0 \end{cases} \right| \qquad (65)$$

(d) if n_1 and n_2 are both odd, (61) gives:-

$$|\lambda| = \left| 1 - 2MQ - 2NR \begin{cases} +2\sqrt{Q^2 + R^2} \\ -2 \quad \text{—}\ " \text{—} \\ 0 \end{cases} \right| \qquad (66)$$

The stability condition (38), which assures that (37) holds, is clearly satisfied in case (a). In cases (b) and (c), the first two conditions in each just reproduce the one-dimensional results (52) and (53) i.e.

$$(u \pm c)\Delta t/\Delta x \leqslant 1; \quad (v \pm c)\Delta t/\Delta x \leqslant 1 \qquad (67)$$

$$M \geqslant 1 \quad ; \quad N \geqslant 1 \qquad (68)$$

while the third gives, assuming M and N both to be positive (which upwind-differencing implies):-

$$u\Delta t/\Delta x \leqslant 1 \quad ; \quad v\Delta t/\Delta x \leqslant 1 \qquad (69)$$

Conditions (67) correspond to the motion of pressure waves, and conditions (69) to the motion of vorticity waves. Case (d) gives:-

$$u\frac{\Delta t}{\Delta x} + v\frac{\Delta t}{\Delta x} \pm c\Delta t\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}} \leqslant 1 \qquad (70)$$

as the pressure wave condition (or, generalising (52), the C-F-L condition),

$$\left.\begin{aligned} M + \alpha N &\geqslant \sqrt{1 + \alpha^2} \\ \text{where:-} \quad \alpha &= \Delta x/\Delta y \end{aligned}\right\} \qquad (71)$$

and:-

$$u\frac{\Delta t}{\Delta x} + v\frac{\Delta t}{\Delta y} \leqslant 1 \qquad (72)$$

as the vorticity wave condition.

Condition (71) may be rewritten in a more convenient form. If M* denotes the actual Mach number, then:-

$$M = M^* \cos\beta \quad \text{and} \quad N = M^* \sin\beta \qquad (73)$$

where β is the flow angle relative to the x-axis. Thus (71) becomes:-

$$M^* \cos\beta + \alpha M^* \sin\beta \geqslant \sqrt{1 + \alpha^2}$$

Let:- $$\sin\gamma = 1/\sqrt{1 + \alpha^2} \quad \text{and} \quad \cos\gamma = \alpha/\sqrt{1 + \alpha^2} \qquad (74)$$

Then:- $$M^* \cos\beta \sin\gamma + M^* \sin\beta \cos\gamma \geqslant 1$$

or:- $$M^* \sin(\beta + \gamma) \geqslant 1 \tag{75}$$

which gives a condition on the actual Mach number in terms of β defined by (73) and γ defined by (74).

(iii) <u>three</u>-dimensional flow:- forward-time, upwind-space differencing of the three-dimensional flow equations gives a result similar to (8) and (59), assuming all velocities to be positive. Expansion in a (triple) Fourier series then gives a result similar to (41) and (60). The eigenvalues of the matrix involved are given by:-

$$\lambda = 1 - MQ(1 - e^{I\theta}) - NR(1 - e^{I\phi}) - PS(1 - e^{I\psi}) \begin{cases} +\zeta \\ -\zeta \\ 0 \end{cases} \tag{76}$$

where:- $\theta = -p\Delta x$, $\phi = -q\Delta y$, $\psi = -r\Delta z$, $Q = c\Delta t/\Delta x$, $R = c\Delta t/\Delta y$, $S = c\Delta t/\Delta z$, $M = u/c$, $N = v/c$, $P = w/c$, and

$$\zeta = \sqrt{Q^2(1 - e^{I\theta})^2 + R^2(1 - e^{I\phi})^2 + S^2(1 - e^{I\psi})^2}$$

Thus the principal stability conditions are, by (38):-

$$u\frac{\Delta t}{\Delta x} + v\frac{\Delta t}{\Delta y} + w\frac{\Delta t}{\Delta z} \pm c\Delta t\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}} \leqslant 1 \tag{77}$$

which is the (C-F-L) condition on pressure waves,

$$u\frac{\Delta t}{\Delta x} + v\frac{\Delta t}{\Delta y} + w\frac{\Delta t}{\Delta z} \leqslant 1 \tag{78}$$

which is the condition on vorticity waves, and:-

$$\left.\begin{array}{l} M + \alpha N + \beta P \geqslant \sqrt{1 + \alpha^2 + \beta^2} \\ \text{where} \quad \alpha = \Delta x/\Delta y \quad \text{and} \quad \beta = \Delta x/\Delta z \end{array}\right\} \tag{79}$$

Just as in the two-dimensional case (which included one-dimensional stability conditions in each flow direction), the three-dimensional example here includes one-dimensional and two-dimensional stability conditions in each flow direction and pair of flow directions, respectively. They are easy to derive, and consequently not given here.

2 - <u>Constant stagnation enthalpy flow</u>

Consider the one-dimensional flow equations:-

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \dfrac{1}{\rho}\dfrac{\partial p}{\partial x} = 0 \\[2ex] \dfrac{\partial \rho}{\partial t} + u\dfrac{\partial \rho}{\partial x} + \rho\dfrac{\partial u}{\partial x} = 0 \end{array}\right\} \tag{80}$$

At steady-state (and it is the steady-state result we are usually interested in), the flow occurs at constant stagnation enthalpy h_o. We assume, following Denton (1975) that there is the correct heat transfer at any instant of time to ensure that h_o is constant. Then it is easy to show that, for a perfect gas flow:-

$$p = \rho \phi (h_o - \tfrac{1}{2}u^2) \qquad (81)$$

where $\phi = (\gamma - 1)/\gamma$, and $\gamma = C_p/C_v$ is the specific heat ratio, assumed constant. Thus:-

$$\frac{\partial p}{\partial x} = \frac{\partial \rho}{\partial x} \phi (h_o - \tfrac{1}{2}u^2) - \rho \phi u \frac{\partial u}{\partial x} \qquad (82)$$

If we use forward-time, upwind-space differencing for velocity and density derivatives, and centred-space differencing for pressure derivatives (again in the manner of Denton (1975)), we get, in the usual notation:-

$$\left.\begin{aligned} u_i^{n+1} &= u_i^n - MQ(u_i^n - u_{i-1}^n) - \frac{\phi c Q}{2\rho}\frac{(h_o - \tfrac{1}{2}u^2)}{c^2}(\rho_{i+1}^n - \rho_{i-1}^n) \\ &\qquad + \frac{\phi M Q}{2}(u_{i+1}^n - u_{i-1}^n) \\ \rho_i^{n+1} &= \rho_i^n - MQ(\rho_i^n - \rho_{i-1}^n) - \frac{Q\rho}{c}(u_i^n - u_{i-1}^n) \end{aligned}\right\} \qquad (83)$$

The usual Fourier series substitution gives:-

$$\left.\underline{V}_i^{n+1}\right|_p = \begin{pmatrix} 1 - MQ(1 - e^{-I\theta}) & -\frac{Q\rho}{c}(1 - e^{-I\theta}) \\ -\frac{Qc\phi\alpha}{2\rho}(e^{+I\theta} - e^{-I\theta}) & 1 - MQ\left[(1 - e^{-I\theta}) - \frac{\phi}{2}(e^{+I\theta} - e^{-I\theta})\right] \end{pmatrix} \left.\underline{V}_i^n\right|_p \qquad (84)$$

$$\left.\begin{aligned} \text{where} \quad \alpha &= \left(\frac{h_o}{c^2} - \tfrac{1}{2}M^2\right) \\ \text{and} \quad \theta &= + p\Delta x \end{aligned}\right\} \qquad (85)$$

The eigenvalues of the matrix in (84) are given by:-

$$\lambda = 1 - MQ(1 - \cos\theta + I\sin\theta\,(1 - \tfrac{1}{2}\phi)) \pm \left[-\tfrac{1}{4}M^2Q^2\phi^2\sin^2\theta - Q^2 I\sin\theta\,\alpha\phi(\cos\theta - 1 - I\sin\theta)\right]^{1/2} \qquad (86)$$

Extrema of $|\lambda|$ occur when $d|\lambda|/d\theta$ vanishes. This occurs when $\theta = n\pi$; n integer. If n is even, then:-

$$|\lambda| = +1 \qquad (87)$$

which, by the stability condition (38), is <u>always</u> stable. If n is odd, then:-

$$|\lambda| = |1 - 2MQ| \qquad (88)$$

and, for stability, (88) gives:-

$$-1 \leqslant 1 - 2MQ \leqslant +1$$

i.e.

$$0 \leqslant 2MQ \leqslant +2 \qquad (89)$$

If M is positive (as <u>upwind</u>-differencing implies), the left-hand inequality in (89) is <u>automatic</u>, and the right-hand inequality gives:-

$$MQ \leqslant 1$$

i.e.

$$u \, \Delta t/\Delta x \leqslant 1 \qquad (90)$$

which is a <u>vorticity</u> wave condition.

We note that $d|\lambda|/d\theta$ also vanishes at some other values of θ which cannot be obtained explicitly; instead, a numerical method must be used to find either the extrema of $|\lambda|$ or to solve the equation $d|\lambda|/d\theta = 0$, using local values of the parameters M etc. Whatever the values of θ are, we can easily see from (86) that the result obtained will <u>not</u> be the C-F-L condition (52), which suggests that the stability analysis presented by Denton (1976), in which he obtains the C-F-L condition for his scheme, may be incorrect.

III - EXTENSION TO ILL-CONDITIONING THEORY

The equations of motion of a steady-state inviscid compressible flow may always be expressed in the form:-

$$\underline{F}(\underline{V}) = \underline{0} \qquad (91)$$

by analogy with equation (1). Suitable spatial discretisation of the flow field leads to a set of algebraic equations of the form:-

$$\underline{\underline{A}}\,\underline{V}_{ijk} + \underline{\underline{B}}\,\underline{V}_{i-1\,jk} + \underline{\underline{C}}\,\underline{V}_{i+1\,jk} + \dots = \underline{0} \qquad (92)$$

in some sense equivalent to (91), by analogy with (6), which, by expansion in spatial Fourier series may be written:-

$$\underline{\underline{A}}^{*}\,\underline{V}_{ijk}\Big|_{pqr} = \underline{0} \qquad (93)$$

by analogy with (33).

Solution of the system (91) for $\underline{V}$ will involve (local) inversion of the matrix $\underline{\underline{A}}^{*}$ in (93). The question of importance is:- under what conditions will it be difficult - or impossible - to perform this matrix inversion? Clearly, if $\underline{\underline{A}}^{*}$ is <u>singular</u> (i.e. det $\underline{\underline{A}}^{*} \equiv 0$), inversion is impossible. More generally, if we define the <u>condition number</u> q by:-

$$q = \left| \lambda_{max} / \lambda_{min} \right| \qquad (94)$$

where λ_{max} and λ_{min} denote the maximum and minimum eigenvalues of $\underline{\underline{A}}^{*}$ in magnitude, then if $q \gg 1$, we say that $\underline{\underline{A}}^{*}$ is <u>ill-conditioned</u>. Clearly, if $\lambda_{min} \equiv 0$, det $\underline{\underline{A}}^{*}$ vanishes, and $q \to \infty$. So infinite condition number implies singularity of the matrix. If $q \gg 1$, then inversion of $\underline{\underline{A}}^{*}$ will be difficult, so we can get an idea of the ease of solution of the discrete analogue (92) of the equations (91) by examining q at all points in the (discretised) flow field. We consider some illustrative isentropic one-dimensional flow examples.

Suppose, first, that upwind-spatial differencing is used (as in (9) with no time dependence). Then the usual Fourier series substitution gives:-

$$\begin{pmatrix} M(1 - e^{-I\theta}) & \frac{\rho}{c}(1 - e^{-I\theta}) \\ \frac{c}{\rho}(1 - e^{-I\theta}) & M(1 - e^{-I\theta}) \end{pmatrix} \underline{V}_i\Big|_p = \underline{0} \qquad (95)$$

where $\theta = p\Delta x$, $M = u/c$, and $I^2 = -1$. The eigenvalues of the matrix in (95) are given by:-

$$\lambda = (1 - e^{-I\theta})(M \pm 1) \qquad (96)$$

so that the condition number is given by:-

$$q = \left| \frac{M + 1}{M - 1} \right| \qquad (97)$$

assuming $M \geqslant 0$ (as upwind-differencing implies). When $M \doteqdot 1$, $q \gg 1$, so that the scheme (95) becomes ill-conditioned near sonic conditions.

Suppose now that centred-space, and not upwind-space, differencing is used, so that (95) becomes:-

$$\begin{pmatrix} \frac{1}{2}M(e^{+I\theta} - e^{-I\theta}) & \frac{\rho}{2c}(e^{+I\theta} - e^{-I\theta}) \\ \frac{c}{2\rho}(e^{+I\theta} - e^{-I\theta}) & \frac{1}{2}M(e^{+I\theta} - e^{-I\theta}) \end{pmatrix} \underline{V}_i\Big|_p = \underline{0} \qquad (98)$$

Then, as before, we have:-

$$q = \left| \frac{M + 1}{M - 1} \right| \qquad (99)$$

so that scheme (98) becomes ill-conditioned at sonic conditions also.

If upwind-space differencing is used for velocity and density derivatives, and centred-space differencing for pressure derivatives, we get:-

$$\begin{pmatrix} M(1 - e^{-I\theta}) & \frac{\rho}{c}(1 - e^{-I\theta}) \\ \frac{c}{2\rho}(e^{+I\theta} - e^{-I\theta}) & M(1 - e^{-I\theta}) \end{pmatrix} \underline{V}_i\Big|_p \qquad (100)$$

The eigenvalues of the matrix in (100) are given by:-

$$\lambda = M(1 - e^{-I\theta}) \pm \sqrt{\tfrac{1}{2}(1 - e^{-I\theta})(e^{+I\theta} - e^{-I\theta})} \qquad (101)$$

and the matrix becomes singular if:-

$$\det \underline{\underline{A}}^* = M^2(1 - e^{-I\theta})^2 - \tfrac{1}{2}(1 - e^{-I\theta})(e^{+I\theta} - e^{-I\theta}) \equiv 0 \qquad (102)$$

so that equation (100) becomes singular only for certain Fourier components at $M = \pm 1$ (specifically, those components with $\theta = 2n\pi$; n integer) and ill-conditioned near $M = \pm 1$ again only for certain Fourier components.

We see, therefore, that by the use of different spatial differencing schemes for the coupling derivatives (i.e. the velocity derivatives in the continuity equation, and the pressure derivatives in the momentum equations) from the other derivatives (i.e. the density derivatives in the continuity equation, and the velocity derivatives in the momentum equations) we can eliminate the problem of ill-conditioning. Thus, by a quite simple technique, we can overcome at least one problem associated with solving the steady-state flow equations.

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