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The Influence of Fibre Distribution on the Moduli of Unidirectional Fibre Reinforced Composites

by

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1976

PRICE *£1-80* NET

UDC 621-419.9 : 539.382 : 539.42 : 539.415.2

*CP No.1347
December 1974

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SUMMARY

Theoretical analyses are made of the influence of the following features on the longitudinal shear and transverse tensile moduli of unidirectional fibre reinforced composites: (i) random variations in positioning of individual fibres, (ii) overall variations leading to fibre bunching, (iii) overall variations leading to interspersed matrix layers, (iv) localised variations resulting in matrix pockets. The degree to which these features occur in CFRP is considered, and observed discrepancies between measured values of the moduli and theoretical predictions assuming a regular hexagonal array are explained.

* Replaces RAE Technical Report 74182 - ARC 36140.

CONTENTS

1 INTRODUCTION

It is well known that certain fibres (notably those of glass, carbon or boron) have exceptional longitudinal strength and/or stiffness properties. When such fibres are embedded in a matrix to form a 'fibre reinforced composite' they can be used in a structural context. Unidirectional fibre reinforced composites are used as struts, ties or stringers in which configurations the longitudinal properties of the fibres show up to maximum advantage. A proper understanding of the transfer of load into such members requires a knowledge of the longitudinal shear modulus G_1^c of the composite and, to a lesser extent, the transverse modulus E_2^c. A rigorous theoretical determination of G_1^c and E_2^c for unidirectional composites based on carbon or glass fibre is not possible because the smallness of individual fibres makes their precise dimensions and positioning beyond control. However, theoretical predictions of G_1^c have been made for idealised composites in which, for example, the fibres are in regular hexagonal or square arrays[1-6]. For small values of the fibre volume fraction, roughly $v_f < 0.4$, the choice of array makes little difference to the predicted values of G_1^c and both are in fair agreement with experimentally determined values. However, for the practically important range in which $v_f > 0.4$, the predicted values of G_1^c for the square array exceed those for the hexagonal array; for example, at $v_f = 0.7$, which is near the practical upper limit for v_f, the difference is some 15% for CFRP. Furthermore, the experimentally determined values of G_1^c are now often significantly higher[7] than the predictions for the square array, but insofar as this array yields better agreement than the hexagonal array, its use for theoretical prediction is favoured by some authors. But this is no more than empiricism for the natural array is hexagonal - like the stable array of the red balls in a snooker frame - and, indeed, a close examination of the distribution of fibres over typical cross-sections supports this view, although there is also an all-pervading randomisation and other distributional features associated with the manufacturing process. Another objection to the adoption of the square array lies in the fact that the derived values of the transverse moduli are not independent of the orientation of the composite cross-section, as they should be in an actual composite with a basically random distribution of fibres.

The present paper considers theoretically the influence on the moduli of the following features: (i) random variations in positioning of individual fibres, (ii) overall variations leading to fibre bunching, (iii) overall variations leading to interspersed matrix layers, (iv) localised variations resulting in matrix pockets. It is shown that these features, particularly numbers (iii) and (iv), can adequately account for the erstwhile discrepancies between theory and experiment. In all cases it is assumed that the fibres are parallel, a previous paper[8] having shown that longitudinal fibre waviness and misalignment have a negligible influence on the composite moduli.

2 PRINCIPAL NOTATION

d	diameter of fibres
E	Young's (or tensile) modulus
$\bar{E}$	plane strain modulus $(\varepsilon_1 = 0)$
G	shear modulus
h	height of repeating rectangle
k	proportion of pure matrix in annular bands, layers or pockets
p	pitch of fibres in regular hexagonal array
r	radius of fibre/p
v	fibre volume fraction
v_f	overall fibre volume fraction
x, y, z	Cartesian axes parallel to 1, 2, 3 directions. Fibres in 1-direction. See Figs.1, 5
Y_n, Z_n	coordinates of fibre centres
Γ	introduced in equation (6) and elsewhere
Δ	'random' displacement of fibre
ε	direct strain
η	G^m/G_1^f
ν	Poisson's ratio.

The following indices and suffices are used:

a	away from resin-rich pockets
b	bunches of relatively higher fibre volume fraction
c	composite
f	fibre
ℓ	layers of relatively higher fibre volume fraction
m	matrix
p	pockets of pure matrix
s	matrix bands surrounding bunches
1, 2, 3	indicate Cartesian axes (see Figs.1, 5)

3 EFFECT OF LOCAL RANDOM VARIATIONS IN FIBRE SPACING

A rigorous analysis to determine the elastic moduli G_1^c and E_2^c of CFRP, or other fibre composite, is difficult, even with the simplifying assumption of parallel fibres of equal circular cross-section, and published numerical results[1-6] are restricted to such fibres in regular diamond or rectangular arrays. These results include as a special case the hexagonal array which is particularly important because it is the only one which yields values of G_1^c, E_2^c which are independent of the orientation of the composite cross-section. This property of isotropy must also be assumed to hold for a random distribution of fibres, provided the number of fibres is sufficiently large as in CFRP.

What is needed is the analysis of arrays which are isotropic like the hexagon but, nevertheless, exhibit 'random' deviations. To be amenable to analysis they must also exhibit a repeating pattern. Fig.1 indicates how such an array can be formed:-

Starting from a regular hexagonal array with pitch p between adjacent fibre centres, we fix the position of those fibres (one in four) which lie on a regular hexagonal array with pitch $2p$. The six fibres adjacent to each of these fixed fibres are now displaced radially by amounts $+\Delta$ at $\theta = 0^o, \pm 120^o$ and $-\Delta$ at $\theta = \pm 60^o, 180^o$. As may be seen from Fig.1 the resulting pattern of fibres repeats itself every 120^o and the derived values of G_1^c and E_2^c are therefore independent of the orientation of the composite cross-section. The essential property of isotropy has thus been maintained despite the introduction of an element of randomness into the idealised hexagonal array. [The reader may note that other such schemes are available, but this is the simplest.] The magnitude of the displacement Δ may be derived by considerations of randomness. Thus, although Δ has been arbitrarily defined as positive and its direction specified, this direction should be regarded as specifying a vanishingly narrow sector in which the centre of the displaced fibre must lie, e.g. $\theta = 0^o$ specifies a vanishingly narrow sector bounded by lines at $\theta = \pm\delta$ as $\delta \to 0$, as indicated by the broken lines in Fig.1. The radius of this sector, and hence the maximum possible value of Δ, is $(p - d)$ at which value the displaced fibres touch the fixed ones. The chosen value of Δ is the *average* of all possible displacements in this sector, i.e.

$$\Delta = \frac{2}{3}(p - d) \quad . \tag{1}$$

Figs.2,3 show sections of the derived composite for $v_f = 0.6$ and 0.75.

It is hoped to derive a rigorous analysis of such randomized hexagonal arrays at a later date. For the present we rely on an approximate analysis, referred to as the Slicing technique[9], which yields lower bounds to the composite moduli. For the regular hexagonal array and for values of $v_f \geqslant 0.4$, this analysis underestimates the correct values of G_1^c by about 20%. To draw any meaningful conclusions we must therefore confine comparisons to these lower bounds, making the tacit assumption that the correction factors for given values of v_f, G^m/G_1^f are the same for the randomized and regular hexagonal arrays. Further, although a rigorous solution would yield identical values of G_1^c for shearing in the 12 or 13 planes, the Slicing technique yields slightly different values. Also, when the influence of fibre randomisation is considered the Slicing technique predicts an increase in the shear modulus in the 12 plane and a decrease in the 13 plane. The resulting uncertainties in interpretation are minimised by adopting an average value G_1^{c*}, say, for the shear moduli appropriate to the 12 and 13 planes. Similar remarks apply for the transverse tensile modulus.

According to the Slicing technique lower bounds for the shear moduli are given by

$$\left.\begin{aligned} \frac{G_{12}^c}{G^m} &= \frac{1}{h_z}\int_0^{h_z} \left\{1 - (1-\eta)v_f(z)\right\}^{-1} dz \,, \\ \frac{G_{13}^c}{G^m} &= \frac{1}{h_y}\int_0^{h_y} \left\{1 - (1-\eta)v_f(y)\right\}^{-1} dy \,, \\ \text{where}\qquad \eta &= \frac{G^m}{G_1^f} \,, \end{aligned}\right\} \qquad (2)$$

and h_z, h_y are the heights of rectangles such as OABC, ODEF which repeat themselves in the y and z directions, respectively, as shown in Fig.1. The terms $v_f(z)$ and $v_f(y)$ are the local values of the fibre volume fractions in vanishingly thin slices at z and y. Thus, adopting for simplicity a unit value for the basic hexagonal fibre pitch p, and introducing Y_n, Z_n for the coordinates of the fibre centres, it can be shown that

$$v_f(z) = \frac{1}{\sqrt{3}} \sum_{n=1}^{6} \left[r^2 - (z - Z_n)^2 \right]^{\frac{1}{2}}$$

where (3)

$$Z_1 = 0 , \quad Z_2 = 1 , \quad Z_3 = \Delta ,$$

$$Z_4 = 1 + \Delta , \quad Z_5 = Z_6 = \tfrac{1}{2}(1 - \Delta) ,$$

and

$$v_f(y) = \sum_{n=1}^{4} \left[r^2 - (y - Y_n)^2 \right]^{\frac{1}{2}}$$

where (4)

$$Y_1 = Y_2 = 0 ,$$

$$Y_3 = \frac{\sqrt{3}}{2}(1 - \Delta) , \quad Y_4 = \frac{\sqrt{3}}{2}(1 + \Delta) ,$$

and

$$r = \left(\frac{v_f \sqrt{3}}{2\pi} \right)^{\frac{1}{2}} ,$$

$$\Delta = \frac{2}{3}(1 - 2r) \quad \text{or zero} , \qquad (5)$$

depending on whether the randomised or regular hexagonal array is being considered.

The results are shown in Table 1 which tabulates the ratio $G^{c*}_{1,random}/G^{c*}_{1,regular}$ for various values of v_f and G^f_1/G^m . It will be seen that while randomisation always causes an increase in the shear modulus, the magnitude of the increase is in practice negligible. Finally we note that similar conclusions can be drawn for the transverse tensile moduli E^c_2, E^c_3 for, according to the Slicing technique, these moduli are determined by equations similar in character to equation (2).

Table 1

Values of $G^{c*}_{1,random}/G^{c*}_{1,regular}$

v_f	$G^f_1/G^m = 20$	50	∞
0.4	1.003	1.002	1.003
0.5	1.003	1.003	1.005
0.6	1.004	1.005	1.008
0.7	1.007	1.012	1.018
0.8	1.008	1.014	1.024

4 OVERALL VARIATIONS LEADING TO FIBRE BUNCHING

An examination of the overall distribution of fibres, particularly in CFRP, frequently shows a bunching effect with regions of relatively higher fibre volume fraction surrounded by bands of pure, or nearly pure, matrix. These bunches may well have their origin in the tows used in manufacture. The influence of such bunching on the longitudinal shear modulus of the composite as a whole can be determined adequately by an adaptation of the technique and formula due to Hashin[10,11].

Hashin has shown that a composite circular cylinder of diameter d_c consisting of an inner circular cylinder of diameter d_i and a surrounding concentric annular cylinder (a tube) with shear moduli G^i and G^a, respectively, is elastically identical (under longitudinal shear) to a homogeneous cylinder with longitudinal shear modulus G^h_1, where

$$\frac{G^h_1}{G^a} = \frac{1 + \Gamma_i}{1 - \Gamma_i} ,$$

where

$$\Gamma_i = v_i\left(\frac{G^i - G^a}{G^i + G^a}\right) ,$$

and

$$v_i = (d_i/d_c)^2 ,$$

$$= \text{volume fraction of inner material.} \qquad (6)$$

Hashin represents a fibre-matrix composite by an amalgam of such composite cylinders of varying sizes (rather like a two-dimensional version of the graded aggregate used in concrete) and hence derives the following expression for the longitudinal shear modulus of the composite:

$$\left.\begin{aligned} \frac{G_1^c}{G^m} &= \frac{1+\Gamma_f}{1-\Gamma_f} \quad , \\ \text{where} \qquad & \\ \Gamma_f &= v_f\left(\frac{G_1^f - G^m}{G_1^f + G^m}\right) \quad . \end{aligned}\right\} \qquad (7)$$

This type of model can be used to investigate the effect of fibre bunching by assuming that the bunched fibres are themselves grouped in circular cylinders with a surrounding matrix band of constant thickness. If the overall fibre volume fraction of such a 'composite composite cylinder' is v_f and the surrounding matrix band comprises a proportion k_s of the total volume, it follows that the fibre volume fraction in the bunched regions, v_b, is given by

$$v_b = \frac{v_f}{1-k_s} \quad . \qquad (8)$$

The longitudinal shear modulus of the bunched regions G_1^b is given by

$$\left.\begin{aligned} \frac{G_1^b}{G^m} &= \frac{1+\Gamma_b}{1-\Gamma_b} \quad , \\ \text{where} \qquad & \\ \Gamma_b &= v_b\left(\frac{G_1^f - G^m}{G_1^f + G^m}\right) \quad , \end{aligned}\right\} \qquad (9)$$

while the longitudinal shear modulus of the 'composite composite cylinder' G_1^c is given by

$$\left.\begin{aligned} \frac{G_1^c}{G^m} &= \frac{1+\Gamma_k}{1-\Gamma_k} , \\ \text{where} \qquad & \\ \Gamma_k &= (1-k_s)\left(\frac{G_1^b - G^m}{G_1^b + G^m}\right) . \end{aligned}\right\} \qquad (10)$$

Substitution of equations (8), (9) into (10) and comparison with (7) shows that

$$G_1^c = \left[G_1^c\right]_{k_s=0} . \qquad (11)$$

Thus the bunching effect has no influence on the longitudinal shear modulus of the composite as a whole; the effect of the increased modulus in the bunched regions is exactly balanced by the reduced modulus in the surrounding matrix bands. What is more, equation (11) shows that the model adopted for describing fibre bunching is unnecessarily restrictive because the analysis is also valid for arbitrary mixtures of bunched fibres with surrounding matrix bands (with, perhaps, differing values of k_s) and homogeneous (non-bunched) arrays provided only that equation (8) is satisfied. Similar conclusions can be drawn for the transverse moduli.

Of course, the underlying concept in Hashin's model - that cylinders of fibre-cum-matrix with ever-decreasing size are available to fill up the gaps between larger cylinders - must be regarded primarily as a mathematical convenience. It is clearly only an approximation in the context of CFRP, say, where the fibres are all about the same size. However, despite this shortcoming equation (7) gives excellent agreement, up to $v_f = 0.6$, with Symm's[5] accurate numerical values, reproduced in Table 2, for a composite with equal fibres in an hexagonal array. Beyond this value of v_f equation (7) gives values of G_1^c/G^m which increasingly underestimate Symm's results; thus $v_f = 0.7$ it is in error by about 2%, depending on the ratio G_1^f/G^m. We regard these differences as too small to detract from the general validity of equation (11); indeed, if we take $G_1^f/G^m = 20$ and $v_f = 0.7$, $k_s = 0.067$ so that $v_b = 0.75$ and adopt Symm's value for G_1^b in equation (10), it is found that G_1^c exceeds $\left[G_1^c\right]_{k_s=0}$ by only 0.5%.

Table 2

Values of G_1^c/G^m for hexagonal array

v_f	$G_1^f/G^m = 6$	12	20	120	∞
0.40	1.80	2.02	2.14	2.30	2.33
0.45	1.95	2.23	2.37	2.59	2.64
0.50	2.11	2.47	2.65	2.94	3.00
0.55	2.30	2.75	2.99	3.37	3.46
0.60	2.50	3.08	3.39	3.91	4.03
0.65	2.74	3.47	3.89	4.61	4.78
0.70	3.02	3.95	4.53	5.55	5.81
0.75	3.34	4.57	5.38	6.93	7.35
0.80	3.73	5.39	6.60	9.18	9.96
0.85	4.22	6.58	8.54	13.70	15.70

It may likewise be shown that the bunching effect has little influence on the transverse tensile modulus. Numerical values of $E_2^c \left(= E_3^c\right)$ for a composite with equal fibres in an hexagonal array are not known to the same accuracy as for G_1^c but Table 3 presents estimates of E_2^c given in Ref.9.

Table 3

Values of E_2^c/E^m for hexagonal array (ν = 0.25)

v_f	$\bar{E}_2^f/\bar{E}^m = 6$	12	20	120	∞
0.40	1.84	2.10	2.25	2.45	2.49
0.45	2.00	2.32	2.49	2.76	2.82
0.50	2.16	2.57	2.78	3.12	3.20
0.55	2.35	2.86	3.13	3.58	3.69
0.60	2.56	3.20	3.55	4.15	4.30
0.65	2.80	3.60	4.07	4.90	5.10
0.70	3.09	4.08	4.73	5.88	6.20
0.75	3.40	4.71	5.62	7.36	7.84
0.80	3.80	5.56	6.85	9.71	10.60
0.85	4.27	6.73	8.77	14.50	16.80

5 OVERALL VARIATIONS LEADING TO INTERSPERSED MATRIX LAYERS

The curing process in the manufacture of CFRP sometimes results in a banded structure, as exemplified in Fig.4, in which the composite is interspersed with roughly parallel layers of pure, or nearly pure, matrix. It is easy to determine the influence of such bands on the longitudinal shear moduli of the composite,

G^c_{12} and G^c_{13} , because the stiffnesses of the individual layers combine either in series or parallel and the shear moduli depend only on the total thickness of the matrix layers, rather than on individual thicknesses and dispositions. Indeed, the influence of more than two types of layer (specifying the type by its fibre volume fraction) can readily be determined. However, such theoretical refinement will seldom be justified and in the ensuing analysis attention is confined to two types, namely purely matrix layers (i.e. zero fibre volume fraction) and composite layers with fibre volume fraction v_ℓ , (see Fig.5). The *overall* fibre volume fraction is again denoted by v_f and hence, if the total thickness of the purely matrix layers comprises a proportion k_ℓ of the total thickness, it follows that

$$v_\ell = \frac{v_f}{1 - k_\ell} \quad . \tag{12}$$

Although it is not orientation-dependent, the longitudinal shear modulus of the purely composite layers will be denoted by G^ℓ_{12} or G^ℓ_{13} depending on whether G^c_{12} or G^c_{13} is under consideration.

5.1 The longitudinal shear modulus G^c_{12}

From Fig.5 it is seen that the *stiffnesses* of individual layers are additive and accordingly

$$G^c_{12} = (1 - k_\ell)G^\ell_{12} + k_\ell G^m \quad . \tag{13}$$

The influence of matrix layers on the shear modulus G^c_{12} is most conveniently expressed by the ratio $G^c_{12}\Big/\left[G^c_{12}\right]_{k_\ell=0}$, for which a simple closed form expression may be derived if it is assumed that G^ℓ_{12} is given adequately by equation (7) with, of course, v_f replaced by v_ℓ . It may now be shown that

$$\frac{G^c_{12}}{\left[G^c_{12}\right]_{k_\ell=0}} = \frac{1 - k_\ell\left(\dfrac{1 + 2\Gamma_f}{1 + \Gamma_f}\right)}{1 - \dfrac{k_\ell}{1 - \Gamma_f}} \quad , \tag{14}$$

where Γ_f is defined in equation (7).

It follows that in unidirectional fibre reinforced composites with a *given overall fibre volume fraction* the presence of resin-rich layers in the 12 plane always *increases* the longitudinal shear modulus G^c_{12} . The variation of $G^c_{12}\Big/\left[G^c_{12}\right]_{k_\ell=0}$ with k_ℓ is shown in Fig.6 for various values of Γ_f .

5.2 The longitudinal shear modulus G^c_{13}

From Fig.5 it is seen that the *flexibilities* of individual layers are additive and accordingly

$$G^c_{13} = \left(\frac{1 - k_\ell}{G^\ell_{13}} + \frac{k_\ell}{G^m}\right)^{-1} \quad . \tag{15}$$

The ratio $G^c_{13}\Big/\left[G^c_{13}\right]_{k_\ell=0}$ may be expressed in a form analogous to equation (14):

$$\frac{G^c_{13}}{\left[G^c_{13}\right]_{k_\ell=0}} = \frac{1 - \dfrac{k_\ell}{1 + \Gamma_f}}{1 - k_\ell\left(\dfrac{1 - 2\Gamma_f}{1 - \Gamma_f}\right)} \quad . \tag{16}$$

It follows that in unidirectional fibre reinforced composites with a *given overall fibre volume fraction* the presence of resin-rich layers in the 12 plane always *decreases* the longitudinal shear modulus G^c_{13} . The variation of $G^c_{13}\Big/\left[G^c_{13}\right]_{k_\ell=0}$ with k_ℓ is shown in Fig.7 for various values of Γ_f .

Example

The difference between G^c_{12} and G^c_{13} can be quite marked, as indicated by the following example in which

$$G^f_{12} = G^f_{13} = 20G^m \quad ,$$

$$v_f = 0.65 \quad ,$$

$$k_\ell = 0.133 \quad .$$

From equation (12) it follows that

$$v_\ell = 0.75$$

and hence, from Table 2,

$$G^\ell_{12} = G^\ell_{13} = 5.38G^m \quad .$$

Equations (13), (15) now yield

$$G^c_{12} = 4.80G^m$$

and

$$G^c_{13} = 3.40G^m \quad .$$

If these values are compared with the modulus appropriate to a uniform distribution of fibres, with $v_f = 0.65$, it is seen that the presence of matrix layers increases the modulus G^c_{12} by 23% and reduces the modulus G^c_{13} by 13%. It may be confirmed that the simplified equations (14), (16) agree closely with these results.

5.3 The transverse modulus E^c_2

The determination of the transverse moduli of a composite with interspersed matrix layers is a little more complicated because of an interaction between the differing layers due to Poisson's ratio effects. Furthermore, expressions are first required for the longitudinal tensile modulus E^c_1 and the relevant Poisson's ratios. Some simplification, with negligible loss of accuracy, is achieved by assuming that

$$\nu^m = \nu^f_{12} = \nu^f_{13} = \nu \ , \quad \text{say.} \tag{17}$$

It now follows that

$$\nu^\ell_{12} = \nu^\ell_{13} = \nu^c_{12} = \nu^c_{13} = \nu \quad , \tag{18}$$

and

$$E^\ell_1 = v_\ell E^f_1 + (1 - v_\ell)E^m \quad , \tag{19}$$

so that

$$E_1^c = (1 - k_\ell)E_1^\ell + k_\ell E^m$$

$$= v_f E_1^f + v_m E^m \quad , \tag{20}$$

in virtue of equation (12).

Thus the presence of matrix layers does not influence the longitudinal tensile modulus E_1^c . Also, in virtue of equation (18) and the Reciprocal Theorem it follows that

$$\nu_{21}^\ell / \nu = E_2^\ell / E_1^\ell \ , \quad \nu_{31}^\ell / \nu = E_3^\ell / E_1^\ell \quad , \tag{21}$$

and

$$\nu_{21}^c / \nu = E_2^c / E_1^c \ , \quad \nu_{31}^c / \nu = E_3^c / E_1^c \quad . \tag{22}$$

The simplest way to determine E_2^c is *via* the plane strain moduli $\bar{E}_2^c$, $\bar{E}_2^\ell$, $\bar{E}_2^m$ where the bar indicates conditions of zero ε_1 . Use is made of the relation

$$E_2^c = \left(1 - \nu_{12}^c \nu_{21}^c\right)\bar{E}_2^c \quad , \tag{23}$$

where

$$\left.\begin{aligned} \bar{E}_2^c &= (1 - k_\ell)\bar{E}_2^\ell + k_\ell \bar{E}_2^m \quad , \\ \bar{E}_2^\ell &= E_2^\ell \Big/ \left(1 - \nu_{12}^\ell \nu_{21}^\ell\right) \quad , \\ \bar{E}_2^m &= E^m/(1 - \nu^2) \quad . \end{aligned}\right\} \tag{24}$$

Substitution of equation (22) in equation (23) and re-arranging yields

$$E_2^c = \bar{E}_2^c \Big/ \left\{1 + \nu^2\left(\bar{E}_2^c / E_1^c\right)\right\} \quad . \tag{25}$$

This relationship can be simplified by neglecting terms of order $\nu^2 E_2^c$ in comparison with E_1^c , whence

$$E_2^c \approx (1 - k_\ell)E_2^\ell + k_\ell E^m \quad , \tag{26}$$

which is identical in form to equation (13).

5.4 The transverse modulus E_3^c

The previous analysis for determining E_2^c was facilitated by the fact that under an applied transverse stress σ_2^c conditions of plane stress exist throughout, with σ_3^m and σ_3^ℓ zero. However, under an applied transverse stress σ_3^c the stress components σ_1^m, σ_2^m, σ_1^ℓ, σ_2^ℓ in the matrix and composite layers are all non-zero because of Poisson's ratio effects and differing stiffness properties in the differing layers. A rigorous expression for E_3^c is accordingly more difficult to derive and to interpret. In practice, however, because of their relatively low stiffness the matrix layers tend to have imposed on them the naturally occurring strains $-\nu_{31}^\ell\varepsilon_3^\ell$, $-\nu_{32}^\ell\varepsilon_3^\ell$ in the composite layers due to the Poisson's ratio contractions. Furthermore, because of the much greater stiffness in the fibre direction $\nu_{31}^\ell\varepsilon_3^\ell$ is negligible in comparison with the naturally occurring strain if the matrix layers were unconstrained. This, under an applied stress σ_3^c the matrix layers effectively deform so that $\varepsilon_1^m = 0$, $\varepsilon_2^m = -\nu_{32}^\ell\varepsilon_3^\ell$, and the layers therefore have an effective tensile modulus given by

$$E_3^{m*} = (1 - \nu)\left\{\frac{(1 + \nu)(1 - 2\nu)}{E^m} + \frac{\nu\nu_{32}^\ell}{E_3^\ell}\right\}^{-1} \quad . \tag{27}$$

By the same argument the composite layers are virtually unaffected by the matrix layers and they therefore have an effective tensile modulus which is equal to E_3^ℓ . The transverse modulus of the composite as a whole is accordingly given adequately by

$$E_3^c = \left(\frac{1 - k_\ell}{E_3^\ell} + \frac{k_\ell}{E_3^{m*}}\right)^{-1} \quad , \tag{28}$$

which is identical in form to equation (15).

6 LOCALISED VARIATIONS RESULTING IN ISOLATED POCKETS OF MATRIX

A closer examination of the distribution of fibres over typical cross-sections, as exemplified in Fig.8 which shows a CFRP section in which $v_f \approx 0.6$, frequently shows pockets of matrix whose sizes are comparable to the cross-section of individual fibres. Such pockets are to be expected when the fibre volume fraction is high ($v_f \geq 0.6$, say), because the surrounding fibres are so closely packed as to offer considerable resistance in the cure cycle to the multiplicity of fibre movements required to replace such a line of matrix with a fibre. Indeed, at higher values of v_f a typical cross-section has many of the characteristics of the crystalline structure in metals as exemplified in Fig.9 which shows a CFRP section in which $v_f = 0.7$. This is because the hexagonal array is now much in evidence, but where such arrays occur (the 'crystal grains') their orientation is random. From purely kinematic reasons it follows that between such arrays (see Fig.9) there are likely to be 'loose' fibres and pockets of matrix. Figs.10,11 are versions of the theoretically derived random distribution of Fig.3 modified by the introduction of pockets of matrix to reduce the overall fibre volume fraction to 0.65 and 0.7 respectively.

The influence of such pockets of matrix on the longitudinal shear rigidity can be investigated in a manner similar to that considered in section 4. We denote the overall fibre volume fraction by v_f, the (local) fibre volume fraction away from matrix pockets by v_a and the volume fraction of such pockets by k_p, so that

$$v_a = \frac{v_f}{1 - k_p} . \tag{29}$$

[Note that if 1 in 20 fibres, say, is 'replaced' by matrix we have $k_p = 0.05$, etc.]

The (local) longitudinal shear modulus of the composite G_1^a away from matrix pockets is given approximately by a modified version of equation (7):

$$\left. \begin{aligned} \frac{G_1^a}{G^m} &= \frac{1 + \Gamma_a}{1 - \Gamma_a} , \\ \text{where} \qquad & \\ \Gamma_a &= v_a \left(\frac{G_1^f - G^m}{G_1^f + G^m} \right) . \end{aligned} \right\} \tag{30}$$

The influence of the matrix pockets on the longitudinal shear modulus G_1^c can now be determined by regarding them as flexible fibres in a stiff matrix. Thus a reinterpretation of equation (7) gives

$$\left.\begin{aligned} \frac{G_1^c}{G_1^a} &= \frac{1 - \Gamma_p}{1 + \Gamma_p} , \\ \text{where} \\ \Gamma_p &= k_p \left(\frac{G_1^a - G^m}{G_1^a + G^m} \right) . \end{aligned}\right\} \qquad (31)$$

In assessing the significance of such pockets it is expedient to determine the ratio $G_1^c \Big/ \left[G_1^c\right]_{k_p=0}$. After some elementary manipulation equations (29), (30) and (31) yield

$$\frac{G_1^c}{\left[G_1^c\right]_{k_p=0}} = \frac{\{1 - k_p(1 + \Gamma_f)\}\{1 - k_p/(1 + \Gamma_f)\}}{\{1 - k_p(1 - \Gamma_f)\}\{1 - k_p/(1 - \Gamma_f)\}} , \qquad (32)$$

where Γ_f is defined in equation (7).

It follows that in unidirectional fibre reinforced composites with a *given overall fibre volume fraction* the presence of pockets of matrix always *increases* the longitudinal shear modulus. The variation of $G_1^c \Big/ \left[G_1^c\right]_{k_p=0}$ with k_p is shown in Fig.12 for various values of Γ_f.

Finally we note that, according to the Slicing technique, the ratio $E_2^c \Big/ \left[E_2^c\right]_{k_p=0}$ is given approximately by the rhs of equation (32) if Γ_f is redefined as

$$\Gamma_f = v_f \left(\frac{E_2^f - E^m}{E_2^f + E^m} \right) .$$

6.1 Application of results to CFRP

The point has already been made that the underlying concept in Hashin's model becomes increasingly invalid at high values of v_f. Some improvement in equation (32) can therefore be expected if instead of equations (7) and (30) we adopt Symm's accurate values for hexagonal fibre arrays. However, if this is done the elegant simplicity of equation (32) is lost for instead of a relation of the form

$$G_1^c \Big/ \left[G_1^c\right]_{k_p=0} = F(k_p, \Gamma_f)$$

we require

$$G_1^c \Big/ \left[G_1^c\right]_{k_p=0} = F\left(k_p, v_f, G_1^f \big/ G^m\right)$$

which does not lend itself so readily to graphical presentation.

As for the value of k_p itself, it is to be expected that for a given matrix/fibre combination and manufacturing process, k_p will vary with the overall volume fraction v_f. Clearly it does *not* depend on the ratio $G_1^f \big/ G^m$ *per se*, but a relation of the form

$$k_p = k_p(v_f) \tag{33}$$

may be valid for a wide range of composites.

For the particular ratio of $G_1^f \big/ G^m = 20$, typical of CFRP, Fig.13 shows the variation of $G_1^c \Big/ \left[G_1^c\right]_{k_p=0}$ with v_f, k_p using Symm's accurate values[5] (augmented by Mansfield[8] and reproduced in Table 2) for $\left[G_1^c\right]_{k_p=0}$ and G_1^a. The superimposed broken line shows the increase in the longitudinal shear modulus for composites with square rather than hexagonal arrays. For example, at $v_f = 0.7$, the longitudinal shear modulus appropriate to the square array is 1.155 times that for the hexagonal array. The same increase in this modulus would occur if an (otherwise) regular hexagonal array were interspersed with matrix pockets specified by $k_p = 0.105$.

7 COMBINED VARIATIONS

The discussion so far has been deliberately simplified by treating in isolation the effects of the different variations in fibre distribution. In practice the different variations seldom occur in isolation and this can modify some of the earlier conclusions. The most important coupling of effects stems from the presence of pockets of matrix in a composite with a banded structure (see sections 6, 5). It will be seen, however, that the analysis can be readily adapted to cater for this more complex situation.

7.1 Isolated pockets of matrix in a banded composite

The geometry of the composite section is specified by the parameters k_p, k_ℓ and the overall fibre volume fraction v_f. In terms of these parameters the fibre volume fraction in the composite layers is given by

$$v_\ell = \frac{v_f}{1 - k_\ell} \quad , \qquad \text{(12 bis)}$$

while a reinterpretation of equation (29) shows that

$$\left.\begin{aligned} v_a &= \frac{v_\ell}{(1 - k_p)} \\ &= \frac{v_f}{(1 - k_\ell)(1 - k_p)} \quad . \end{aligned}\right\} \qquad (34)$$

The longitudinal shear modulus G_1^a away from matrix pockets is now given by Table 2, or approximately by equation (30), while a reinterpretation of equation (31) shows that the longitudinal shear modulus of the purely composite layers $\left(G_{12}^\ell \text{ or } G_{13}^\ell\right)$ is given by

$$\left.\begin{aligned} \frac{G_1^\ell}{G_1^a} &= \frac{1 - \Gamma_p}{1 + \Gamma_p} \\ \text{where} \qquad \Gamma_p &= k_p\left(\frac{G_1^a - G^m}{G_1^a + G^m}\right) \quad . \end{aligned}\right\} \qquad (35)$$

Turning now to section 5 it will be seen that G^c_{12}, G^c_{13} are given by equations (13), (15).

Example

Consider again the example on p.13, modified by the presence of pockets of matrix specified by

$$k_p = 0.05 , \quad \text{say} \quad .$$

From equation (34):

$$v_a = 0.79 \quad ,$$

and hence, by interpolation in Table 2,

$$G^a_1 = 6.35G^m \quad .$$

Equation (35) now yields

$$G^\ell_1 = 5.91G^m \quad ,$$

and hence equations (13), (15) yield

$$G^c_{12} = 5.26G^m \quad ,$$

$$G^c_{13} = 3.57G^m \quad .$$

If these values are compared with those for which k_p is zero it is seen that the presence of matrix pockets increases G^c_{12} by about 9.5% and G^c_{13} by about 5%.

7.2 Isolated pockets of matrix in a composite with fibre bunching

It is shown here that the presence of fibre bunching (see section 4) in a composite with isolated pockets of matrix causes an *increase* in the longitudinal shear modulus. For example, let us compare two composites specified by

$$G_1^f / G^m = 20 \quad ,$$

$$v_f = 0.6 \quad ,$$

$$k_p = 0.1 \quad ,$$

$$k_s = 0 \quad \text{or} \quad 0.05 \quad .$$

In general we have

$$v_b = \frac{v_f}{1 - k_s} \quad , \tag{8 bis}$$

$$= 0.60 \quad \text{or} \quad 0.667$$

and hence

$$v_a = \frac{v_f}{(1 - k_s)(1 - k_p)} \quad , \tag{36}$$

$$= 0.632 \quad \text{or} \quad 0.703 \ .$$

Interpolation in Table 2 now yields

$$G_1^a / G^m = 3.70 \quad \text{or} \quad 4.58 \ ,$$

while a reinterpretation of equation (31) yields

$$\left. \begin{aligned} G_1^b / G_1^a &= \frac{1 - \Gamma_p}{1 + \Gamma_p} \quad , \\ \text{where} \qquad \Gamma_p &= k_p \left(\frac{G_1^a - G^m}{G_1^a + G^m} \right) \quad . \end{aligned} \right\} \tag{37}$$

Thus

$$G_1^b / G^m = 4.28$$

and, finally, equation (31) yields

$$G_1^c / G^m = 3.49 \quad \text{if} \quad k_s = 0 \quad ,$$

and equation (10) yields

$$G_1^c / G^m = 3.89 \quad \text{if} \quad k_s = 0.05 \quad .$$

Thus the presence of fibre bunching causes an increase in the longitudinal shear modulus of about 11%. The reason for this lies in the fact that the presence of matrix pockets causes a greater increase in G_1^b than that predicted by equation (9) or Table 2. Similar proportional increases can be expected for the transverse tensile modulus.

8 CONCLUSIONS

This paper considers theoretically the influence of fibre distribution on the longitudinal shear moduli G_{12}^c, G_{13}^c and the transverse tensile moduli E_2^c, E_3^c of unidirectional fibre reinforced composites. The distributions which occur in practice, particularly for CFRP, have been noted and, insofar as they deviate from a uniform regular hexagonal array, they are categorized as follows:

(i) random variations in positioning of individual fibres,

(ii) overall variations leading to fibre bunching,

(iii) overall variations leading to resin-rich layers in the 12 plane,

(iv) localised variations resulting in isolated pockets of matrix.

Simple geometrical parameters are introduced for specifying the magnitude of these deviations, and formulae are derived for the moduli of the composite in terms of these parameters and the stiffness characteristics of the fibres and matrix. The deviations in fibre distribution are first considered in isolation when it is shown that for composites with a given overall fibre volume fraction:

(i) and (ii) result in a negligible increase in the moduli of the composite,

(iii) results in increases in the moduli G_{12}^c, E_2^c (typically up to 20%), and decreases in the moduli G_{13}^c, E_3^c (typically up to 15%),

(iv) results in an increase in all the moduli (typically up to 20%).

The paper concludes with a treatment of combined deviations, focussing attention on the combinations (ii), (iv) and (iii), (iv). The results are capable of explaining the observed discrepancies between measured values of the composite moduli[7] and theoretical predictions assuming a regular hexagonal array.

Acknowledgment

The author is indebted to Miss Doreen Best for the computation.

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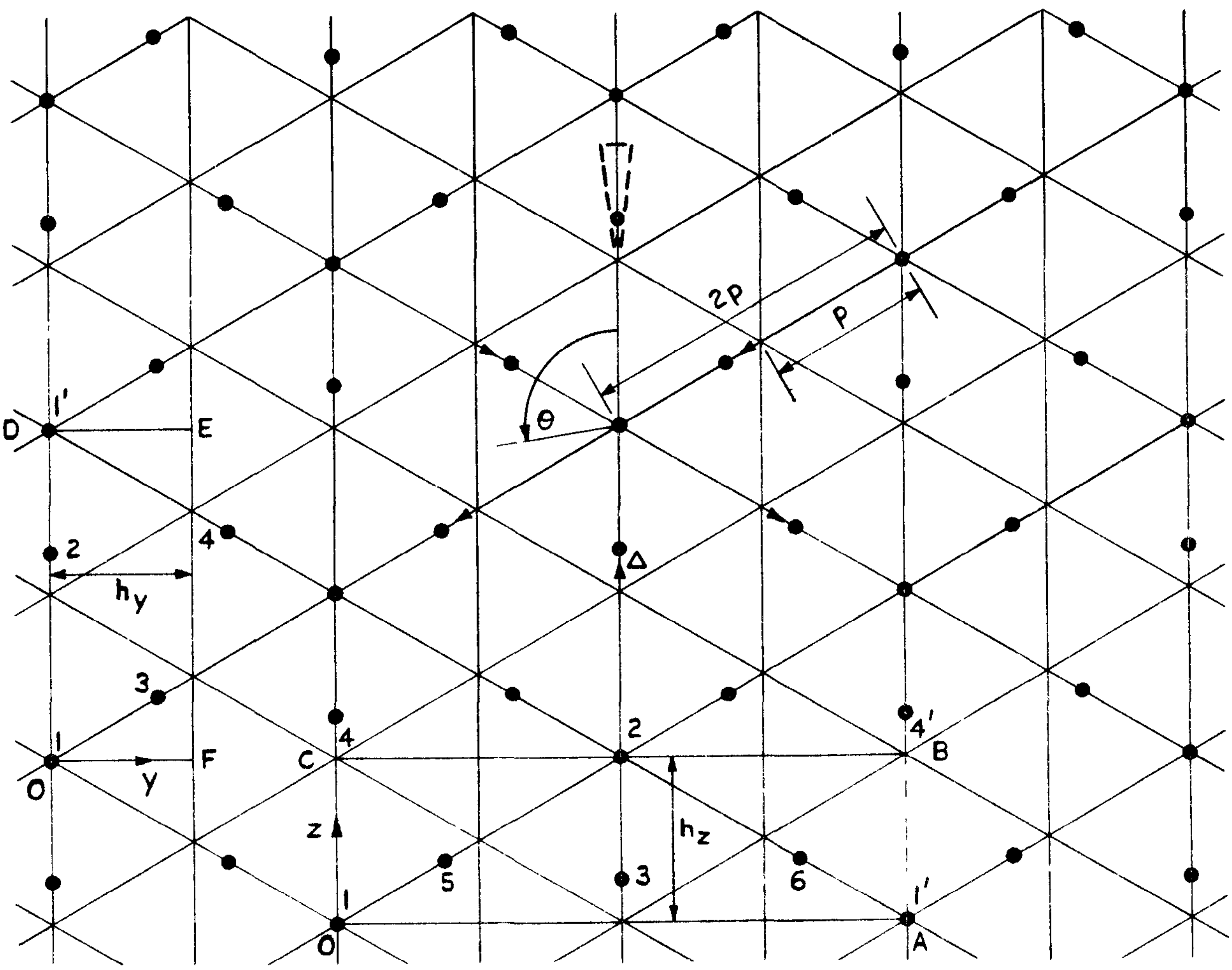

Fig. 1 Fibre centres in randomized hexagonal array

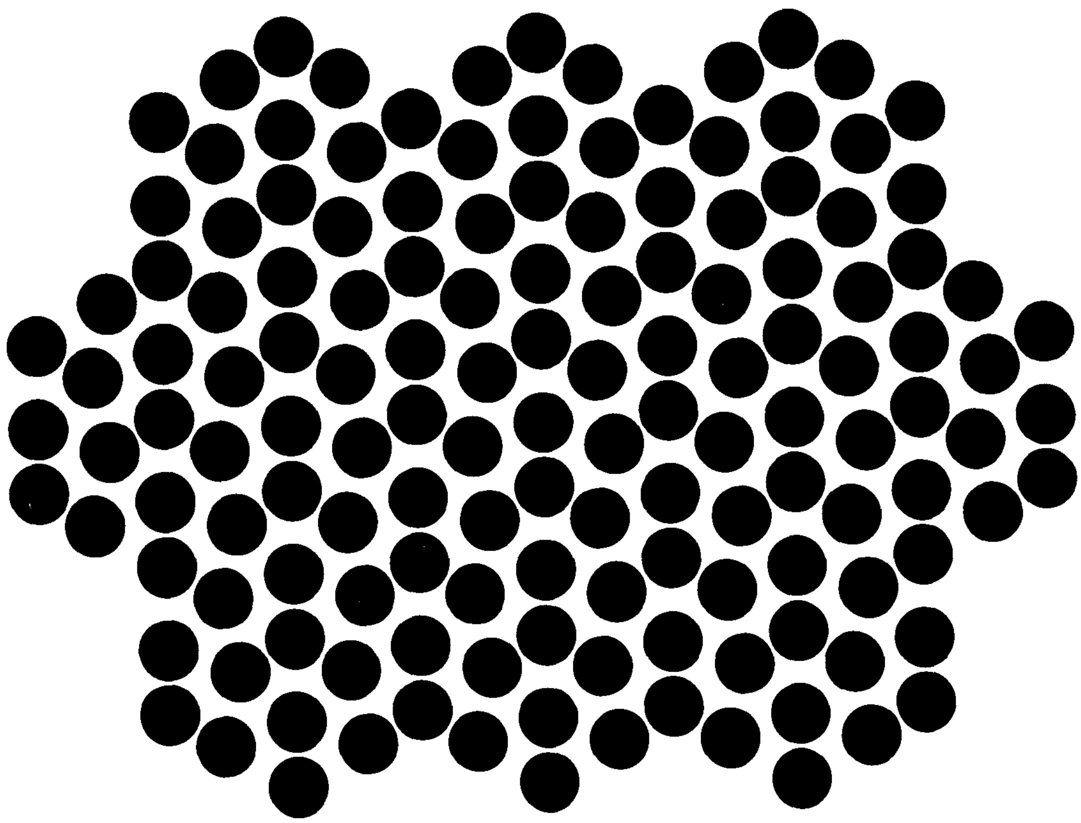

Fig.2 Randomized hexagonal array (v_f = 0.6)

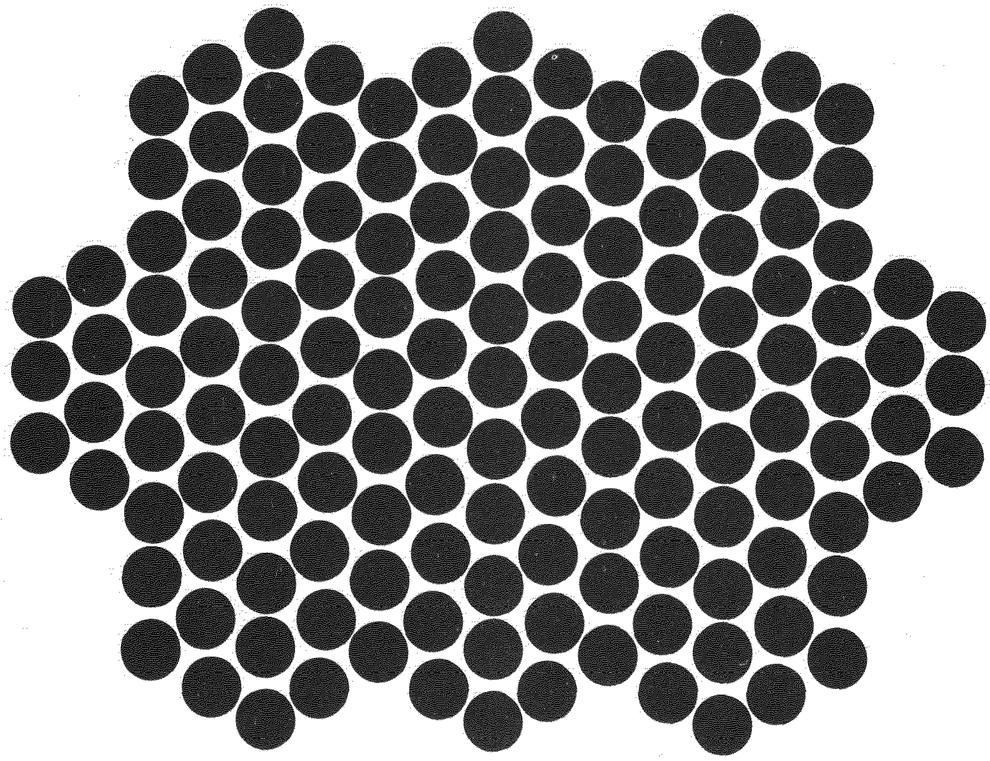

Fig.3 Randomized hexagonal array (v_f = 0.75)

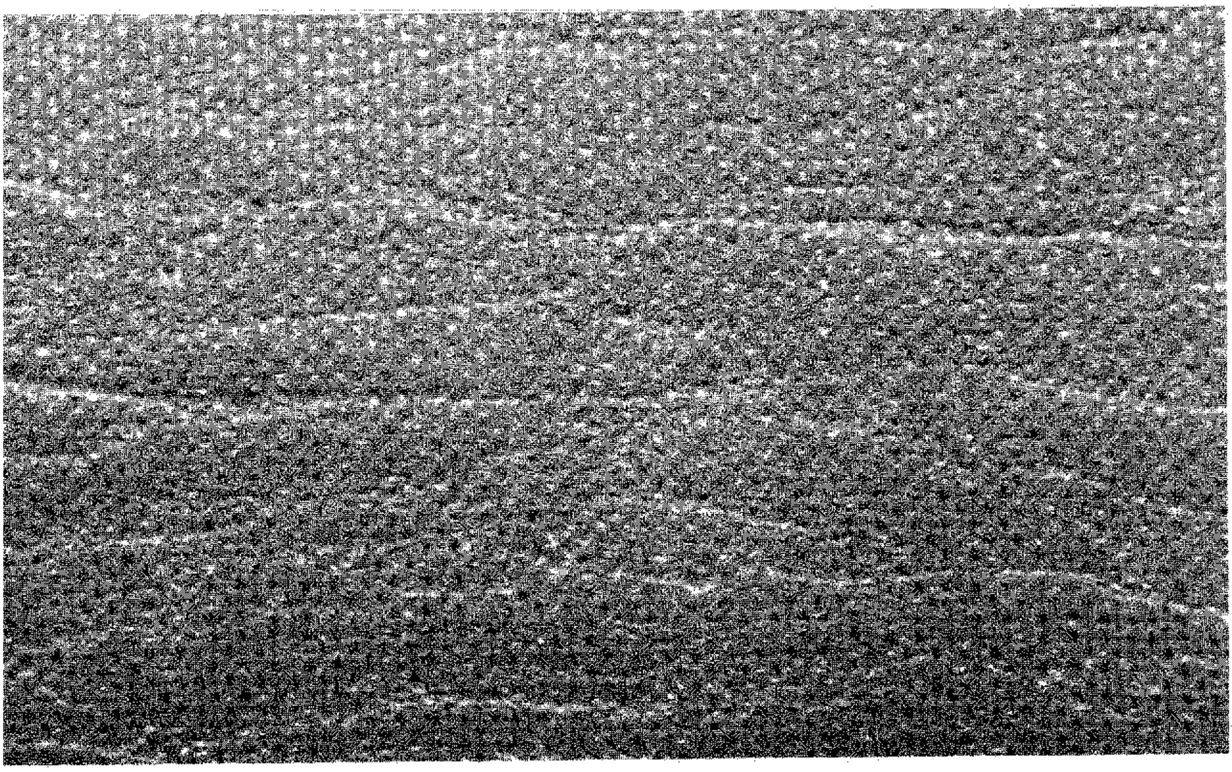

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Fig.4 Section of CFRP showing banded structure

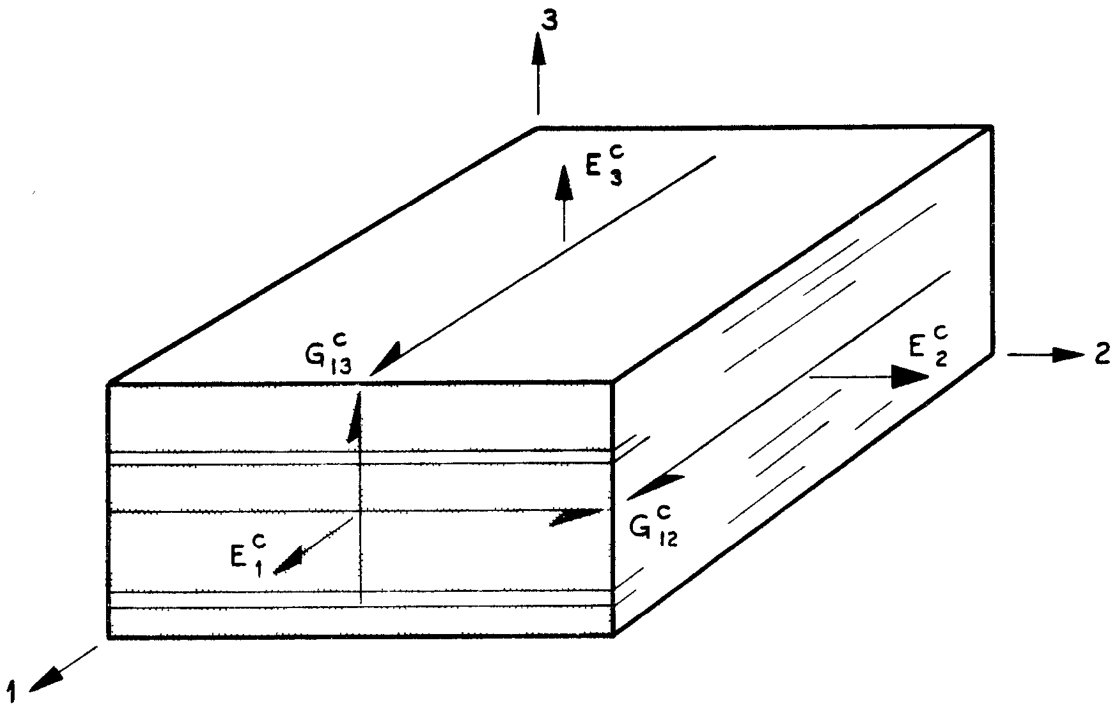


Fig. 5 Composite with matrix layers in 12 plane

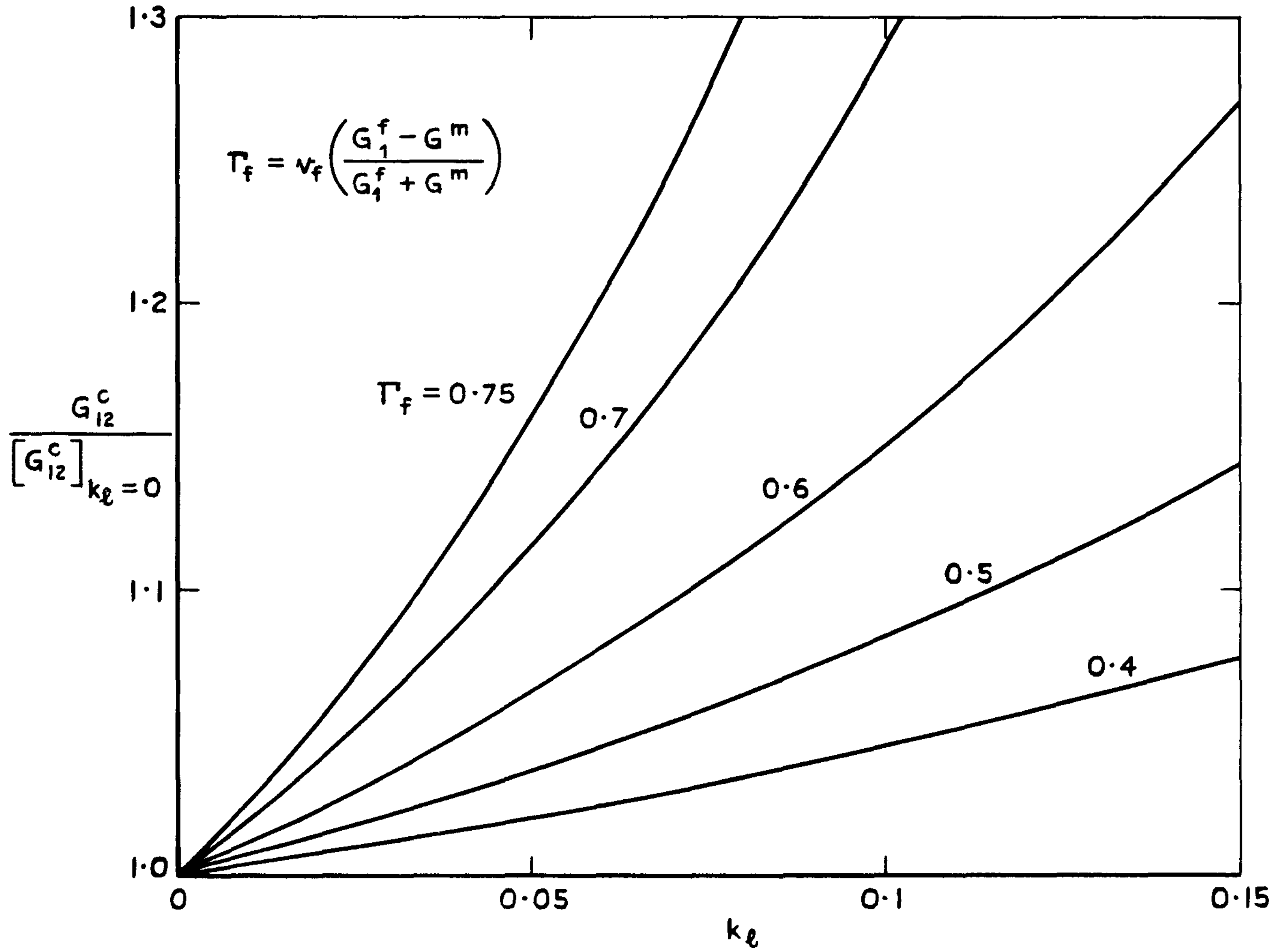


Fig.6 Influence of matrix layers on longitudinal shear modulus G_{12}^c: Simplified general relationship

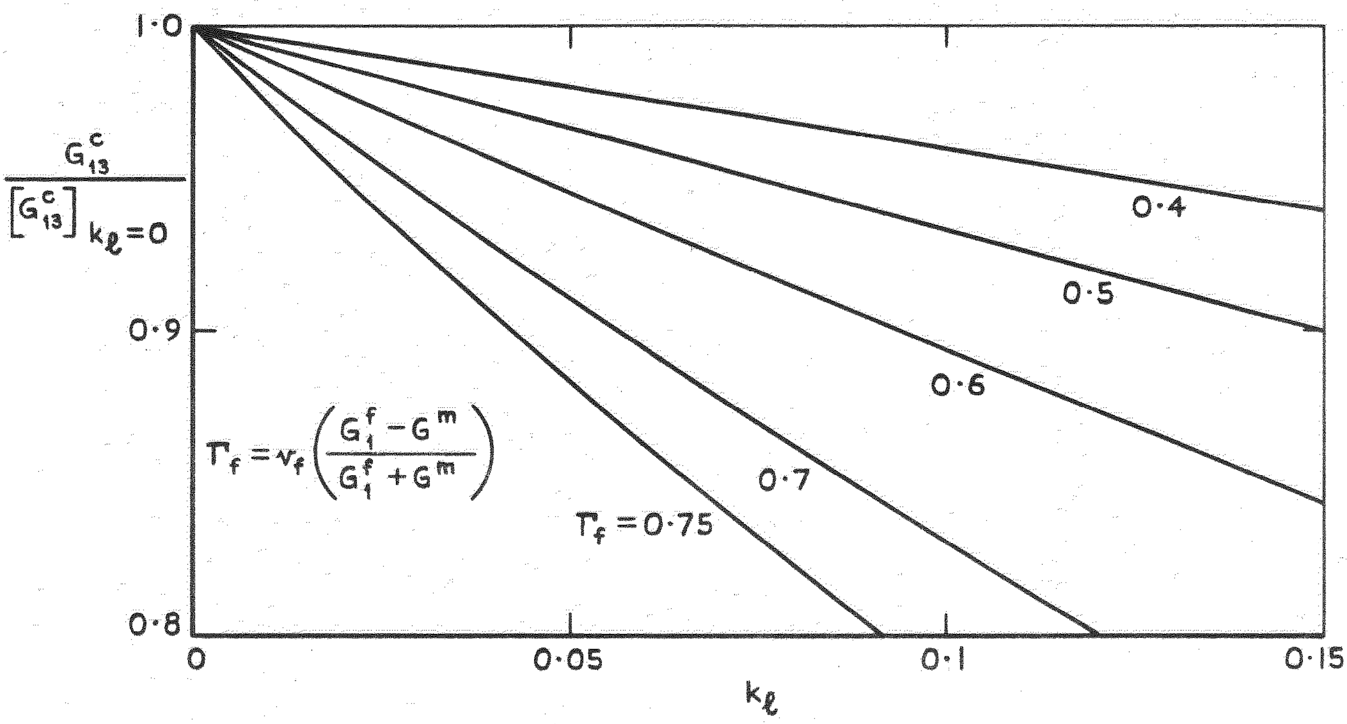
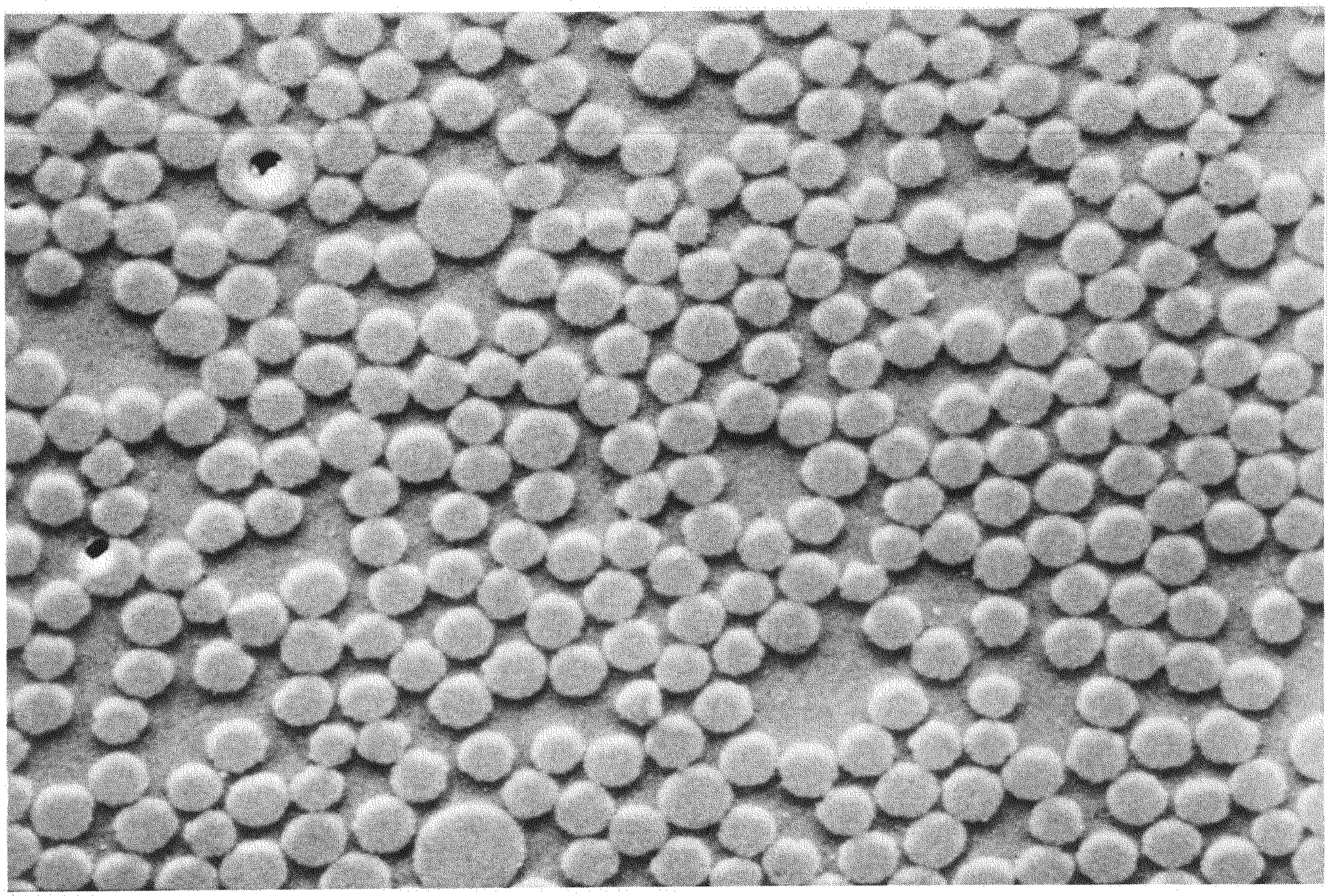
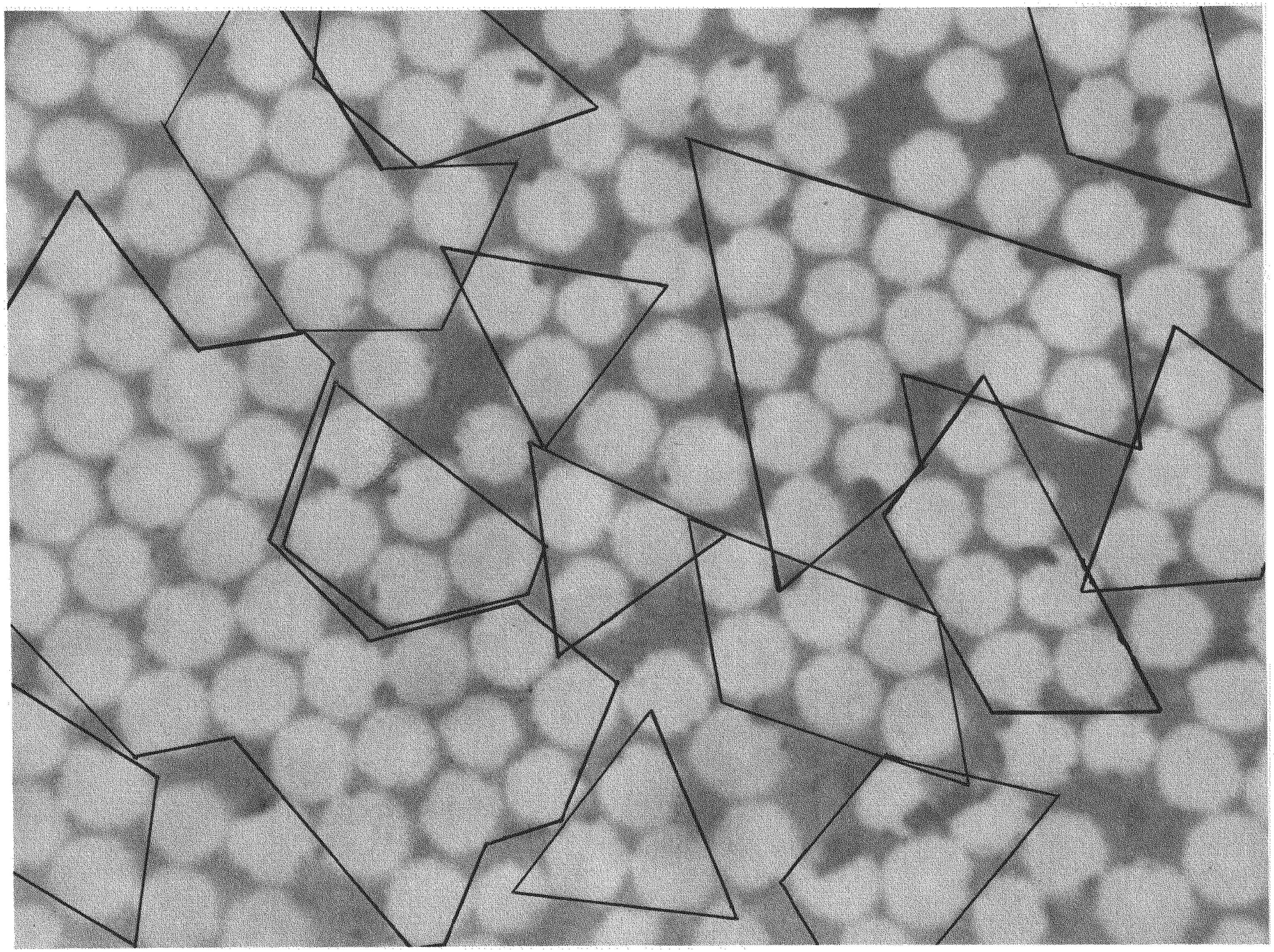


Fig.7 Influence of matrix layers on longitudinal shear modulus G_{13}^c: Simplified general relationship



x1000

Fig.8 Section of CFRP showing pockets of matrix ($v_f = 0.6$)



x2000

Fig.9 Composite showing crystalline structure ($v_f = 0.7$)

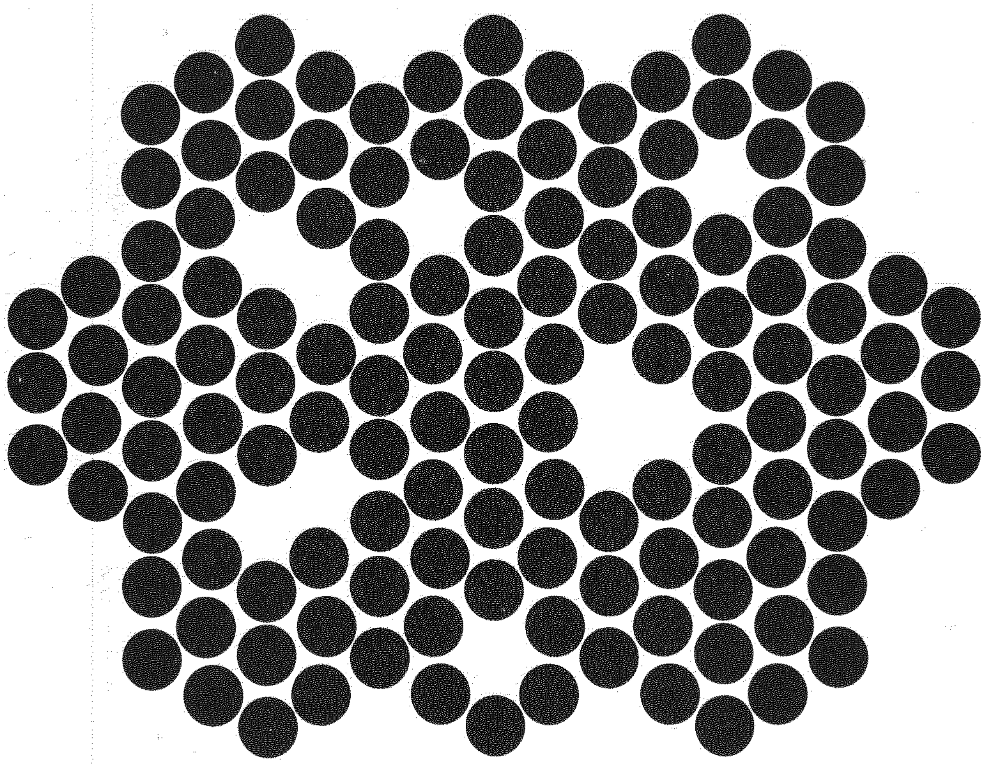


Fig.10 Randomized hexagonal array with matrix pockets (v_f = 0.65, k_p = 0.14)

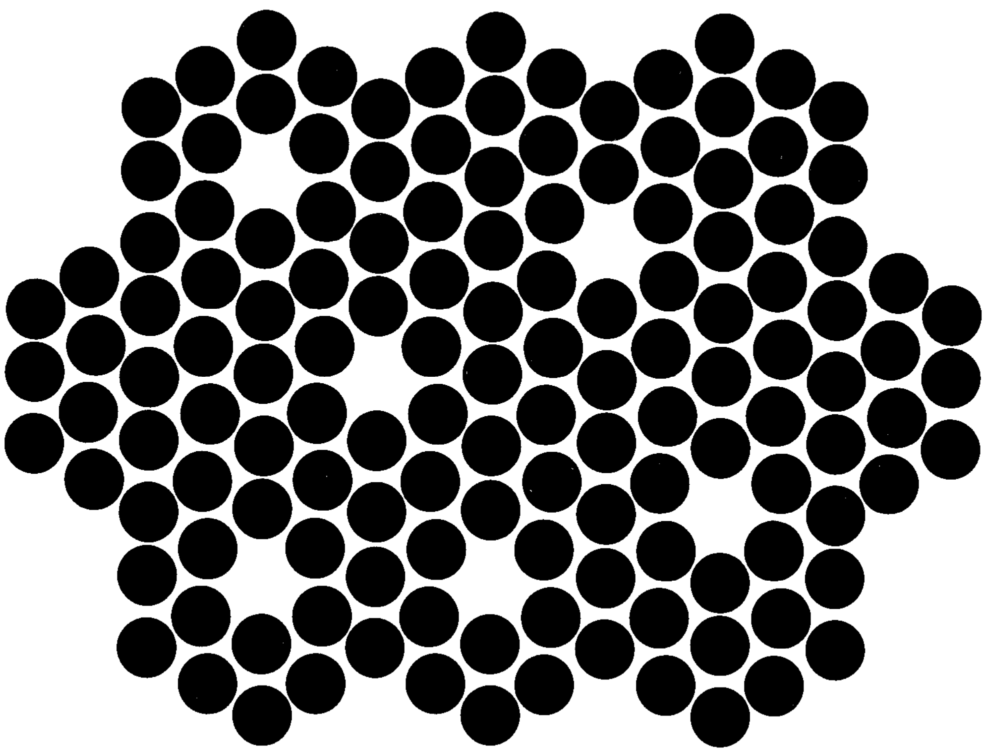


Fig.11 Randomized hexagonal array with matrix pockets (v_f = 0.7, k_p = 0.07)

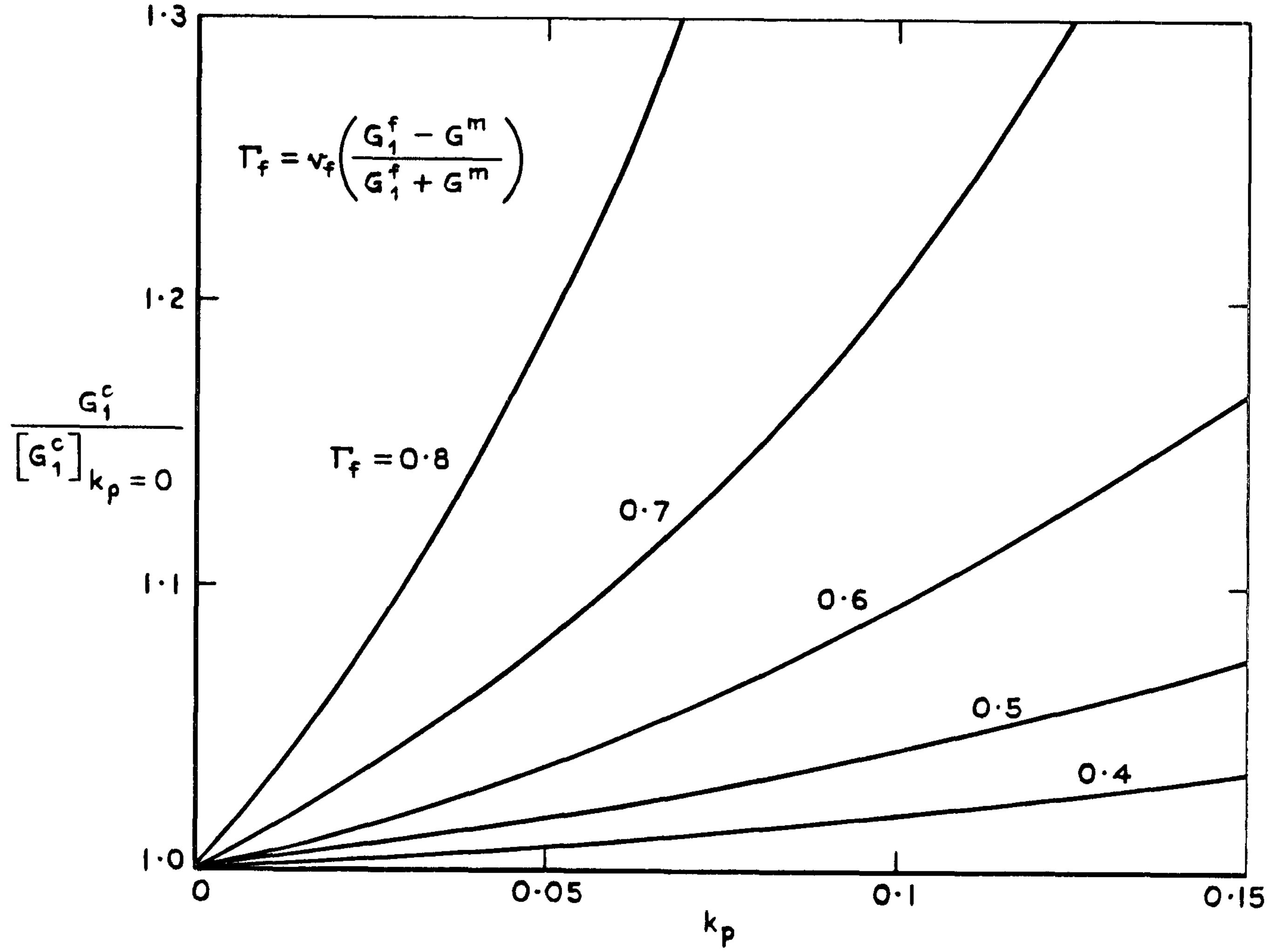


Fig.12 Influence of matrix pockets on longitudinal shear modulus: Simplified general relationship

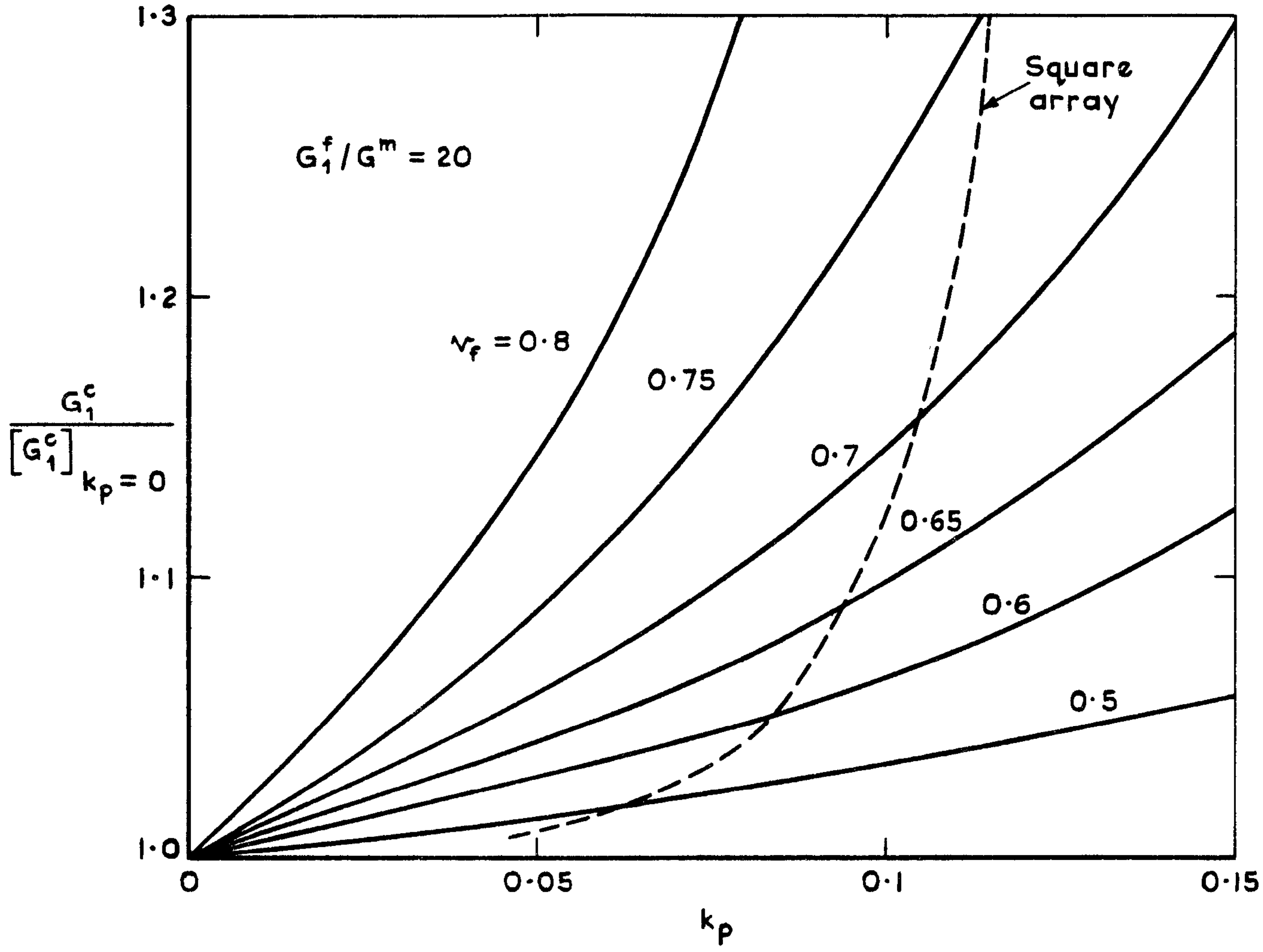


Fig.13 Influence of matrix pockets on longitudinal shear modulus: Accurate particular relationship

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