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A TWO-DIMENSIONAL AEROFOIL OSCILLATING AT LOW FREQUENCIES IN HIGH SUBSONIC FLOW

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SUMMARY

The integral equation formulation developed in Ref.(1) for the analysis of the high subsonic shock free flow past a steady two-dimensional aerofoil is extended to the problem of an aerofoil oscillating in simple harmonic motion. The essential non-linearities in the transonic potential equations are retained in the analysis. Results are obtained for the in and out of phase force and moment derivatives at the fundamental frequency for a NACA 0012 aerofoil oscillating in pitch at a low reduced frequency in subcritical conditions. It is found that both the aerodynamic stiffness and damping derivatives are significantly affected by the transonic non-linearities. Unfortunately, because of uncertainties of wind tunnel interference effects at the present time, it does not seem possible to compare these theoretical results with experimental results. Neither are there any other theoretical results for comparison.

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*Replaces A.R.C.34 259

≠Communicated by Professor G. J. Hancock

CONTENTS

NOTATION

$a_o(\bar{\xi})$	=	Scaling functions defined in Eq.(101)
$b_1(\bar{\xi})$	=	Scaling functions defined in Eq.(107)
C	=	Boundary of the domain S and shown in Fig.2
c_n	=	Coefficients defined in Eq.(67)
$C(k)$	=	$\dfrac{H^{(2)}(k)}{H_1^{(2)}(k) + i\,H_o^{(2)}(k)}$
c	=	Aerofoil chord
$c_p(x,z,t)$	=	Pressure coefficient
$c_{p_n}(\bar{x},\bar{z})$	=	Reduced pressure coefficient defined in Eq.(18)
$E_{c_1}(x)$	=	Function defined by Eq.(108)
$E_{c_2}(x)$	=	Function defined by Eq.(108)
$E_{c_3}(x)$	=	Function defined by Eq.(108)
$g_n(\bar{x},\bar{z})$	=	Function defined by Eq.(13)
$h_1^{(R)}(\dfrac{\bar{\zeta}}{a_o(\bar{\xi})})$	=	Approximating function defined by Eq.(101)
$h_1^{(I)}(\dfrac{\bar{\zeta}}{a_1(\bar{\xi})}$	=	Approximating function defined by Eq.(103)
$I(\bar{x})$	=	Integral defined by Eq.(59)
$I_{\tau n}(\bar{x})$	=	Integral defined by Eq.(83)
$I_{c_n}(\bar{x})$	=	Integral defined by Eq.(85)
K	=	$M_\infty \Omega$
k	=	$(\gamma+1)\, M_\infty^2$
$\bar{l}_1(\bar{x})$	=	Loading defined by Eq.(63)
$\left.\begin{matrix} l_\alpha \\ l_{\dot{\alpha}} \end{matrix}\right\}$	=	Derivatives defined by Eq.(74)

$M(x,z,t)$	= Local Mach number
M_∞	= Freestream Mach number
m_α, $m_{\dot\alpha}$	= Derivatives defined by Eq.(75)
$N(\bar{x},\bar{z})$	= Inward drawn normal to the curve C
n	= Number of harmonic
R	= The radius of the outer boundary of S
r	$= \left[(\bar{x}-\bar{\xi})^2 + (\bar{z}-\bar{\zeta})^2\right]^{\frac{1}{2}}$
S	= The domain in which Green's theorem is valid and shown in Fig. 2.
t	= Time variable
$u(x,z,t)$	= Perturbation velocity in the freestream direction
$\bar{u}_n(x,z)$	$= \frac{\partial}{\partial \bar{x}} \bar{\phi}_n(\bar{x},\bar{z})$
U_∞	= Freestream velocity
$w(x,z,t)$	= Perturbation velocity normal to the freestream
$\bar{w}_n(\bar{x},\bar{z})$	$= \frac{\partial \bar{\phi}_n(\bar{x},\bar{z})}{\partial \bar{z}}$
x, z	= Cartesian co-ordinate system
x_o	= Location of pitching axis
$\bar{x}$	$= x$
$\bar{z}$	$= \beta z$
$z_{ou}(x)$	= Mean ordinate of the upper surface of the aerofoil
$z_{oL}(x)$	= Mean ordinate of the lower surface of the aerofoil
$z_c(x)$, $z_T(x)$	= Components of $z_{ou}(x)$, $z_{oL}(x)$
$\bar{Z}_T(\bar{x})$	$= \frac{k}{\beta^3} z_T(x)$
$\bar{Z}_c(\bar{x})$	$= \frac{k}{\beta^3} z_c(x)$

$z_1(x)$ = Time dependent ordinate of the aerofoil surfaces

$\bar{Z}_1(\bar{x})$ $= \dfrac{k}{\beta^3} z_1(x)$

α = Mean angle of incidence

$\bar{A}$ $= \dfrac{k}{\beta^3} \alpha$

α_o = Amplitude of pitching oscillation

$\bar{A}_o$ $= \dfrac{k}{\beta^3} \alpha_o$

β $= (1-M_\infty^2)^{\frac{1}{2}}$

$\Gamma_n(\theta)$ = Functions defined in Eq.(69) and Eq.(110)

γ = Ratio of specific heats

δ $= \ln(\dfrac{M_\infty}{2}) + (1-M_\infty^2)^{\frac{1}{2}} \ln\left[\dfrac{1 + (1-M_\infty^2)^{\frac{1}{2}}}{M_\infty}\right]$

$\left.\begin{matrix}\bar{\xi}\\ \bar{\zeta}\end{matrix}\right\}$ = Cartesian co-ordinate system corresponding to $(\bar{x},\bar{z})$

θ $= \cos^{-1}(1-2\bar{x})$

θ^1 $= \cos^{-1}(1-2\bar{\xi})$

$\Lambda(\bar{x},\bar{\xi})$ $= \dfrac{1}{2} \ln\left[\dfrac{1-\bar{x}\bar{\xi} + (1-\bar{\xi}^2)^{\frac{1}{2}}(1-\bar{x}^2)^{\frac{1}{2}}}{1-\bar{x}\bar{\xi} - (1-\bar{\xi}^2)^{\frac{1}{2}}(1-\bar{x}^2)^{\frac{1}{2}}}\right]$

ν $= \dfrac{\omega c}{U_\infty}$: the frequency parameter

$\phi(x,z,t)$ = Perturbation velocity potential

$\phi_n(x,z)$ = Function defined by Eq.(7)

$\bar{\phi}_n(\bar{x},\bar{z})$ $= \dfrac{k}{\beta^2} \phi_n(x,z)\, e^{-iM_\infty^2 n\Omega x}$

$\overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z})$ $= \bar{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) + in\, M_\infty^2 \Omega \bar{\phi}_n(\bar{x},\bar{z})$

$\chi(K|\bar{x}-\bar{\xi}|)$ = Function defined in Eq.(58)

Ω $= \dfrac{\nu}{1-M_\infty^2}$

ω = Frequency of oscillation of the aerofoil

1. INTRODUCTION

At the present time many problems within the overall field of inviscid wing theory rely for their solution on linearisation of the fundamental equations. Although two-dimensional steady aerofoil theory is a major area where significant progress has been made beyond linearised theory, two-dimensional oscillatory aerofoil theory still remains primarily a linearised theory throughout the Mach number range, from subsonic through transonic to supersonic speeds. The range and combination of parameters for which linearisation is valid have been classified by Landahl(2) and by Miles(3); linearised oscillatory wing theory is described in the AGARD 'Manual of Aeroelasticity'(4) and surveyed more recently by Landahl and Stark(5).

The aim of this paper is to improve the standard linearised solution for the two-dimensional oscillating aerofoil in the high subsonic flow regime, where linearised theory is suspect, by extending the ideas developed in Ref.(1) for the solution of the steady two-dimensional aerofoil problem.

At the present time linearised theory is the standard approach for the determination of oscillating aerofoil characteristics at high subsonic speed, so a few brief introductory remarks on linearised theory are appropriate.

The linearised differential equation representing the flow characteristics about an aerofoil oscillating in a high subsonic stream can be formulated either in terms of a perturbation velocity potential(6) or in terms of a reduced acceleration potential(7). Application of Green's theorem satisfying the appropriate boundary conditions reduces both of these formulations to singular integral form; the problem then is to invert the integral equation in order to evaluate (numerically) the pressure distribution, the overall lift and moment. Formulation of the problem using the velocity potential is fairly straightforward but difficulties arise because both the wing and wake have to be considered. When the reduced acceleration potential is used the wake does not appear explicitly but then the application of the boundary condition on the wing becomes more complicated. Jones[6] solved the problem of the oscillating aerofoil at subsonic speeds for low values of the frequency parameter using the velocity potential; an iterative procedure was employed to solve the integral equation starting from the incompressible flow solution. Dietze[7], on the other hand, developed a similar iterative approach, also starting with the incompressible solution, but using the reduced acceleration potential. Direct collocation methods for the solution of the integral equation mainly using the reduced acceleration potential have been developed; Possio[8] and Frazer[9] were the forerunners of this type of approach; a more recent development is by Zwaan[10]. Analytic solutions can be found in terms of an infinite series of Mathieu functions[11] but the convergence of the series is slow for high freestream Mach numbers.

Non-linearities become important at transonic speeds and although the potential equations can be linearised for high values of the frequency parameter at Mach numbers close to unity there is a practical range of conditions involving Mach number and frequency parameter where the non-linear potential equation must be used in order to obtain meaningful results.

As mentioned earlier, significant advances have been made on the solution of the non-linear equations for the flow about steady two-dimensional aerofoils at transonic speeds. Virtually all of the current methods and techniques being investigated for the steady problem are reviewed in Ref.(1) where they are assessed from the point of view of their capability for extension not only to oscillatory aerofoil problems but also to finite wings. The conclusion of that evaluation is that although several accurate numerical techniques[12] [13] [14] exist, these techniques are restricted to the steady problem either by their analysis or by their computational complexity; the most promising steady theory which offers hope for extension to oscillatory motions is the integral equation formulation[15]. One main advantage is that this approach essentially extends standard linearised theory and so the considerable expertise built up over the years in solving the linearised equation can be utilised.

As already stated this paper is an extension of the earlier work[1] to an oscillatory motion of an aerofoil at high subcritical Mach numbers. The aerofoil is assumed to be oscillating in simple harmonic motion about a mean incidence at a particular fundamental frequency. As a consequence of the non-linear nature of the potential differential equation the solution for the perturbation velocity and pressure must take the form of a series in multiples of the fundamental frequency. An infinite sequence of potential differential equations is formed; each differential equation consists of the harmonic under consideration and a non-linear combination of all preceding harmonics. Steady flow about the aerofoil at its mean incidence appears as the first term in the series. It is shown that the magnitude of the response in successive harmonics decreases.

Application of Green's theorem, satisfying the appropriate boundary conditions, reduces each of the differential equations to two simultaneous integral equations in which the symmetric and anti-symmetric effects are coupled through double (surface) integrals involving non-linear quadratic terms. By approximating the variation of the flow field normal to the aerofoil surface the double integrals are reduced to single integrals over the wing chord and downstream wake; the coupling between the symmetric 'thickness' and the antisymmetric 'lifting' effect is still retained. A particular result for the case of a symmetric wing oscillating about a zero mean incidence is that even and odd numbered harmonics of the velocity potential are antisymmetric and symmetric respectively.

For simplicity at this stage the problem is limited to that of a symmetric wing oscillating about a zero mean incidence. First the steady flow past the aerofoil at zero incidence is obtained by the application of the method of Ref.(1). Then the technique used by Jones[6] is applied to obtain the characteristics at the fundamental frequency. Since the latter method is restricted to low values of the frequency parameter the non-linear theory presented here is similarly restricted.

Following the technique used by Jones[6] the integral equation is solved by assuming the conventional Fourier series to represent the wing loading. Although the series is slowly convergent in respect of the pressure loading at any one point on the aerofoil surface, a combination of the first four terms in the series gives the overall lift and moment. Only overall lift and moment are derived in this paper; the calculation of the pressure loading distribution is not attempted. To estimate the pressure loading collocation techniques are required; this alternative approach will be investigated in a later report.

Results for the stiffness derivatives l_α, m_α and the damping derivatives $l_{\dot\alpha}$, $m_{\dot\alpha}$ are presented. With the pitching axis at the leading edge the non-linear derivatives differ by a significant amount from the corresponding linear values. Variation of $l_{\dot\alpha}$, m_α, $m_{\dot\alpha}$ with changes in the pitching axis is also obtained.

Harmonics depend solely on the non-linearities and difficulties arise in the approximation of the variations in those terms throughout the flow field which are needed if useful results are to be derived.

No solutions for the harmonic terms are given in this paper.

Unfortunately no reliable experimental results seem to be available for two-dimensional aerofoils oscillating in an infinite uniform stream; although wind tunnel measurements have been made the uncertainties due to the large wind tunnel wall interference effects are sufficient to cause considerable doubt on the interpretation of the measurements. There is furthermore an absence of any comparable theoretical results. Thus no comparisons are possible at the present time with either other theories or with experimental results.

2. MATHEMATICAL FORMULATION

2.1 Basic Equations

To study the problem of a two-dimensional aerofoil oscillating in a uniform stream, a cartesian co-ordinate system is set up as shown in Fig. 1; the origin is taken at the mean position of the wing leading edge; the x-axis is in the freestream direction with the z-axis normal to the freestream; the co-ordinates x and z are non-dimensionalised with respect to the wing chord c.

A non-dimensional perturbation velocity potential $\phi(x,z,t)$ may be defined by

$$\frac{\partial\phi(x,z,t)}{\partial x} = u(x,z,t) \ , \ \frac{\partial\phi(x,z,t)}{\partial z} = w(x,z,t), \qquad (1)$$

where $u(x,z,t)$ and $w(x,z,t)$ are the non-dimensional perturbation velocities in the x and z directions respectively normalised with respect to the freestream velocity U_∞.

The unsteady transonic potential equation for inviscid non-conducting, isentropic, irrotational flow around a two-dimensional aerofoil as derived by several authors (e.g. Guderly[16]) is

$$(1-M_\infty^2 - (\gamma+1)M_\infty^2\,\phi_x)\,\phi_{xx} + \phi_{zz} - \frac{2\,M_\infty^2\,c}{U_\infty}\,\phi_{xt} - \frac{M_\infty^2\,c^2}{U_\infty^2}\,\phi_{tt} = 0 \qquad (2)$$

In the derivation of Eq.(2) it is assumed that ϕ and its derivatives are small compared to unity; the second order term $(\gamma+1)\,M_\infty^2\,\phi_x\,\phi_{xx}$ is retained at transonic speeds because it is of comparable order of magnitude to the term $(1-M_\infty^2)\,\phi_{xx}$.

The boundary conditions are:

(i) the flow at the aerofoil surface remains tangential to the moving aerofoil surface;

(ii) the pressure is continuous off the wing, particularly across the wake;

(iii) the perturbation velocity potential $\phi(x,z,t)$ vanishes at large distances upstream of the aerofoil;

(iv) the Kutta trailing edge condition must be satisfied, namely the velocities at the trailing edge must remain finite.

For small amplitude oscillations the upper and lower surfaces of the wing at any time, t, may be denoted by the non-dimensional functions $z_u(x,t)$ and $z_L(x,t)$ respectively. It is now assumed that the wing oscillates in simple harmonic motion with frequency ω and that the upper and lower surfaces of the wing at any time can be expressed as

$$\begin{aligned} z_u(x,t) &= z_{ou}(x) + z_1(x)e^{i\omega t}, \\ z_L(x,t) &= z_{oL}(x) + z_1(x)e^{i\omega t}, \end{aligned} \tag{3}$$

where it is assumed that $z_{ou}(x)$ and $z_{oL}(x)$ represent the mean profile and $z_1(x)$ is the mode shape of the oscillation. The mean surface ordinates $z_{ou}(x)$ and $z_{oL}(x)$ can be expressed as

$$\begin{aligned} z_{ou}(x) &= -\alpha x + z_c(x) + z_T(x) , \\ z_{oL}(x) &= -\alpha x + z_c(x) - z_T(x) ; \end{aligned}$$

where α is the mean angle of incidence, $z_c(x)$ and $z_T(x)$ are the mean camber and thickness distributions.

The tangency boundary conditions are

$$\begin{aligned} \frac{w(x,z_u,t)}{1+u(x,z_u,t)} &= \frac{\partial z_u(x,t)}{\partial x} + \frac{\partial z_u(x,t)}{\partial t} \cdot \frac{c}{U_\infty(1+u(x,z_u,t))} \\ &= \frac{\partial z_{ou}(x)}{\partial x} + \left[\frac{\partial z_1(x)}{\partial x} + \frac{i\nu z_1(x)}{(1+u(x,z_u,t))}\right]e^{i\omega t} , \\ \frac{w(x,z_L,t)}{1+u(x,z_L,t)} &= \frac{\partial z_L(x,t)}{\partial x} + \frac{\partial z_L(x,t)}{\partial x} \cdot \frac{c}{U_\infty(1+u(x,z_L,t))} \\ &= \frac{\partial z_{oL}(x)}{\partial x} + \left[\frac{\partial z_1(x)}{\partial x} + \frac{i\nu z_1(x)}{(1+u(x,z\ ,t))}\right]e^{i\omega t} , \end{aligned} \tag{4}$$

where ν is the frequency parameter ($\omega c/U_\infty$).

By means of a Taylors series expansion Eq.(4) can be expanded as

$$w(x,+0,t) = \left[\frac{\partial z_{ou}(x)}{\partial x} + \frac{\partial z_1(x)}{\partial x} e^{i\omega t}\right]\left[1 + u(x,+0,t)\right] + i\nu z_1(x)e^{i\omega t} - \left[\frac{\partial w(x,z,t)}{\partial z}\right]_{z=+0}\left[z_{ou}(x) + z_1(x)e^{i\omega t}\right] + \cdots$$

$$w(x,-0,t) = \left[\frac{\partial z_o(x)}{\partial x} + \frac{\partial z_1(x)}{\partial x} e^{i\omega t}\right]\left[1 + u(x,-0,t)\right] + i\nu z_1(x)e^{i\omega t} - \left[\frac{\partial w(x,z,t)}{\partial z}\right]_{z=-0}\left[z_o(x) + z_1(x)e^{i\omega t}\right] + \cdots \tag{5}$$

As discussed later, to ensure that second order accuracy is introduced around the nose it is necessary to include some second order terms in the boundary conditions, Eqs.(5). However, at this stage to the order of approximation of the transonic potential equation, Eq.(2), the tangency boundary condition in Eq.(5) reduces to the linearised form on the plane $z = 0$, thus

$$w(x,+0,t) = \left[\frac{\partial \phi(x,z,t)}{\partial z}\right]_{z=+0} = \frac{\partial z_{ou}(x)}{\partial x} + \left[\frac{\partial z_1(x)}{\partial x} + i\nu z_1(x)\right]e^{i\omega t}$$

$$w(x,-0,t) = \left[\frac{\partial \phi(x,z,t)}{\partial z}\right]_{z=-0} = \frac{\partial z_{oL}(x)}{\partial x} + \left[\frac{\partial z_1(x)}{\partial x} + i\nu z_1(x)\right]e^{i\omega t} \tag{6}$$

Because of the non-linear terms in the basic differential equation it is necessary to express the perturbation velocity potential, $\phi(x,z,t)$ in terms of a series in the fundamental frequency, thus

$$\phi(x,z,t) = \sum_{n=0}^{\infty} \phi_n(x,z)e^{i\omega nt} \tag{7}$$

On substitution of the series expressed in Eq.(7) into Eq.(2), then equating coefficients of $e^{i\omega nt}$, the following set of equations is obtained:

$$(1-M_\infty^2)\,\phi_{nxx} + \phi_{nzz} - 2i\,M_\infty^2 n\nu\,\phi_{nx} + M_\infty^2 n^2\nu^2\,\phi_n$$

$$= (\gamma+1)\,M_\infty^2 \frac{\partial}{\partial x}\left\{\frac{1}{2}\sum_{r=0}^{n}(\phi_{(n-r)_x}\,\phi_{r_x})\right\} \tag{8}$$

On substitution of Eq.(7) the boundary conditions expressed in Eq.(6) become

$$\left(\frac{\partial\,\phi_0(x,z)}{\partial z}\right)_{z=\pm 0} = -\alpha + z_c'(x) \pm z'(x),$$

$$\left(\frac{\partial\,\phi_1(x,z)}{\partial z}\right)_{z=\pm 0} = z_1'(x) + i\nu z_1(x), \tag{9}$$

$$\left(\frac{\partial\,\phi_n(x,z)}{\partial z}\right)_{z=\pm 0} = 0 \quad \text{for} \quad n \geqslant 2,$$

where the dash denotes differentiation with respect to x.

Eq.(8) can be transformed by introducing the parameters

$$\beta^2 = 1-M_\infty^2,\ k = (\gamma+1)M_\infty^2,\ \Omega = \frac{\nu}{(1-M_\infty^2)},\ K = M_\infty\Omega$$

and the variables

$$\bar{x} = x,\ \bar{z} = \beta z,\ \bar{\phi}_n(\bar{x},\bar{z}) = \frac{k}{\beta^2}\,\phi_n(x,z)e^{-iM_\infty^2\Omega n\bar{x}}$$

$$\bar{u}_n(\bar{x},\bar{z}) = \frac{\partial\,\bar{\phi}_n(\bar{x},\bar{z})}{\partial\bar{x}},\ \bar{w}_n(\bar{x},\bar{z}) = \frac{\partial\,\bar{\phi}_n(\bar{x},\bar{z})}{\partial\bar{z}} \tag{10}$$

Substitution of the variables defined in Eq.(10) into Eq.(8) leads to

$$\bar{\phi}_{n\bar{x}\bar{x}} + \bar{\phi}_{n\bar{z}\bar{z}} + n^2K^2\,\bar{\phi}_n = g_{n\bar{x}}, \tag{11}$$

where

$$g_{n\bar{x}}(\bar{x},\bar{z}) = e^{-iM_\infty^2 n\Omega\bar{x}}\,\frac{\partial}{\partial\bar{x}}\left\{\frac{1}{2}\sum_{r=0}^{n}(\bar{\phi}_{n-r}(\bar{x},\bar{z})e^{iM_\infty^2(n-r)\Omega\bar{x}})_{\bar{x}}\,(\bar{\phi}_r(\bar{x},\bar{z})e^{iM_\infty^2 r\Omega\bar{x}})_{\bar{x}}\right\}. \tag{12}$$

In the subsequent analysis it is $g_n(\bar{x},\bar{z})$ which is required; integration of Eq.(12) gives

$$g_n(\bar{x},\bar{z}) = \frac{1}{2}\sum_{r=0}^{n} \overset{*}{\phi}_{n-r_{\bar{x}}}(\bar{x},\bar{z})\ \overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) + \frac{iM_\infty^2 n\Omega}{2}\int_{-\infty}^{\bar{x}} \sum_{r=0}^{n} (\overset{*}{\phi}_{n-r_{\bar{\xi}}}(\bar{\xi},\bar{z})\ \overset{*}{\phi}_{r_{\bar{\xi}}}(\bar{\xi},\bar{z}))\, d\bar{\xi}\ , \tag{13}$$

where

$$\overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) = \bar{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) + iM_\infty^2 n\Omega\ \bar{\phi}_n(\bar{x},\bar{z}) \tag{14}$$

On application of the transformed variables of Eq.(10) the tangency boundary conditions expressed in Eq.(9) become

$$\left(\frac{\partial\bar{\phi}_0(\bar{x},\bar{z})}{\partial\bar{z}}\right)_{\bar{z}=\pm 0} = -\bar{A} + \bar{Z}_c'(\bar{x}) \pm \bar{Z}_\tau'(\bar{x})\ , \tag{15a}$$

$$\left(\frac{\partial\bar{\phi}_1(\bar{x},\bar{z})}{\partial\bar{z}}\right)_{\bar{z}=\pm 0} = \bar{w}_1(\bar{x},\pm 0) = \left[Z_1'(\bar{x}) + i\nu Z_1(\bar{x})\right] e^{-iM_\infty^2\Omega\bar{x}}\ , \tag{15b}$$

$$\left(\frac{\partial\bar{\phi}_n(\bar{x},\bar{z})}{\partial\bar{z}}\right)_{\bar{z}=\pm 0} = \bar{w}_n(\bar{x},\pm 0) = 0 \text{ for } n \geqslant 2\ , \tag{15c}$$

where

$$\bar{A} = \frac{k}{\beta^3}\alpha,\ \bar{Z}_c(\bar{x}) = \frac{k}{\beta^3} z_c(x)\ ,\ \bar{Z}_\tau(\bar{x}) = \frac{k}{\beta^3} z_\tau(x),\ \bar{Z}_1(\bar{x}) = \frac{k}{\beta^3} z_1(x)\ .$$

The first equation from Eq.(11), when n = 0, is

$$\bar{\phi}_{0\bar{x}\bar{x}} + \bar{\phi}_{0\bar{z}\bar{z}} = \frac{\partial}{\partial\bar{x}}\left[\frac{1}{2}\bar{\phi}_{0\bar{x}}^2\right] \tag{16a}$$

which, together with boundary conditions Eq.(15a), represents the steady flow.

The second equation from Eq.(11), when n = 1 is

$$\bar{\phi}_{1\bar{x}\bar{x}} + \bar{\phi}_{1\bar{z}\bar{z}} + K^2\bar{\phi}_1 = \frac{\partial}{\partial\bar{x}}(\overset{*}{\phi}_{1\bar{x}}\ \bar{\phi}_{0\bar{x}}) + iM_\infty^2\Omega\overset{*}{\phi}_{1\bar{x}}\ \bar{\phi}_{0\bar{x}} \tag{16b}$$

which, together with boundary condition Eq.(15b), represents the fundamental response. It is noted that Eq.(16b) is linear in $\bar{\phi}_1$.

On putting n = 2 in Eq.(11)

$$\bar{\phi}_{2\bar{x}\bar{x}} + \bar{\phi}_{2\bar{z}\bar{z}} + 4\,K^2\,\bar{\phi}_2 = \frac{\partial}{\partial\bar{x}}\left(\phi^{\star}_{2\bar{x}}\,\bar{\phi}_{o\bar{x}}\right) + 2iM_\infty^2\Omega\phi^{\star}_{2\bar{x}}\,\bar{\phi}_{o\bar{x}}$$

$$+ \frac{\partial}{\partial\bar{x}}\left[\frac{1}{2}\left(\phi^{\star}_{1\bar{x}}\right)^2\right] + iM_\infty^2\Omega\left(\phi^{\star}_{1\bar{x}}\right)^2 \tag{16c}$$

which, together with boundary condition Eq.(15c), represents the first harmonic.

The pressure coefficient, $c_p(x,z,t)$ can be found from the unsteady Bernoulli equation; to the order of approximation of Eq.(2)

$$c_p(x,z,t) = -2\left[\frac{\partial\phi(x,z,t)}{\partial x} + \frac{c}{U_\infty}\,\frac{\partial\phi(x,z,t)}{\partial t}\right] \tag{17}$$

Expanding $c_p(x,z,t)$ in a series in the frequency parameter in a similar manner as the velocity potential.

$$c_p(x,z,t) = \sum_{n=0}^{\infty} c_{p_n}(x,z)\;e^{i\omega nt}\;. \tag{18}$$

And from Eqs. (7), (12), (17), (18)

$$c_{p_n}(x,z) = -\,2\,\frac{\beta^2}{k}\left[\bar{\phi}_{n\bar{x}}(\bar{x},\bar{z}) + in\Omega\;\bar{\phi}_n(\bar{x},\bar{z})\right]e^{iM_\infty^2 n\Omega\bar{x}} \tag{19}$$

The boundary condition of continuous pressure off the wing across the wing wake can, by reference to Eq.(19), be expressed as

$$\Delta\,\bar{\phi}_{n\bar{x}}(\bar{x}) + in\Omega\;\Delta\,\bar{\phi}_n(\bar{x}) = 0\;,\quad \bar{x} \geqslant 1 \tag{20}$$

where

$$\Delta\,\bar{\phi}_{n\bar{x}}(\bar{x}) = \left[\bar{\phi}_{n\bar{x}}(\bar{x},+0) - \bar{\phi}_{n\bar{x}}(\bar{x},-0)\right],$$

$$\Delta\,\bar{\phi}_n\,(\bar{x}) = \left[\bar{\phi}_n\,(\bar{x},+0) - \bar{\phi}_n\,(\bar{x},-0)\right].$$

The ordinary differential equation, Eq.(20), can be solved to give

$$\Delta\,\bar{\phi}_n(\bar{x}) = \Delta\,\bar{\phi}_n(1)\;e^{in\Omega(1-\bar{x})},\;\bar{x} \geqslant 1\;. \tag{21}$$

Eqs. (20, 21) automatically satisfy the Kutta trailing edge condition.

2.2 Application of Green's Theorem

A particular form of Green's theorem is used in the following analysis; further details are given by Lanczos[17].

If $\mathcal{L}^{(n)}(f)$ is the linear differential operator acting on the function f and defined by

$$\mathcal{L}^{(n)}(f) = \frac{\partial^2 f}{\partial \bar{x}^2} + \frac{\partial^2 f}{\partial \bar{z}^2} + n^2 K^2 f \ , \tag{22}$$

then Green's theorem states

that

$$\iint_S \left[\psi_n \mathcal{L}^{(n)}(\bar{\phi}_n) - \bar{\phi}_n \bar{\mathcal{L}}^{(n)}(\psi_n)\right] dS = -\oint_C \left[\psi_n \frac{\partial \bar{\phi}_n}{\partial N} - \bar{\phi}_n \frac{\partial \psi_n}{\partial N}\right] dC \ , \tag{23}$$

where C is a closed circuit, S is the domain enclosed by C, N is the inward drawn normal to the curve C and $\bar{\mathcal{L}}^{(n)}(f)$ is the adjoint operator related to $\mathcal{L}^{(n)}(f)$; in this case the operator is self-adjoint, thus

$$\bar{\mathcal{L}}^{(n)}(f) = \mathcal{L}^{(n)}(f) \ .$$

It is necessary that $\psi_n(\bar{x},\bar{z})$ and $\bar{\phi}_n(\bar{x},\bar{z})$ are two functions that are continuous and have continuous first and second derivatives in the domain S.

In Eq.(23) the function $\bar{\phi}_n(\bar{x},\bar{z})$ is taken to be the transformed perturbation velocity potential as defined in Eq.(10) and ψ_n is chosen to be an elementary solution of

$$\mathcal{L}^{(n)}(\psi_n) = \delta(r) \ , \tag{24}$$

where $\delta(r)$ is the delta function at $r = 0$ and

$$r = \left[(\bar{x}-\bar{\xi})^2 + (\bar{z} - \bar{\zeta})^2\right]^{\frac{1}{2}} \tag{25}$$

Since $\bar{\phi}_n(\bar{x},\bar{z})$ satisfies the equation

$$\mathcal{L}^{(n)}(\bar{\phi}_n) = g_{n\bar{x}}$$

and since ψ_n satisfies Eq.(24), then Eq.(23) becomes

$$\iint_S \psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\ g_{n_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})\ dS$$

$$= -\oint_C \left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial \bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N} - \bar{\phi}_n(\bar{\xi},\bar{\ }) \frac{\partial \psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N} \right] dC\ . \qquad (26)$$

For n = 0 the fundamental solution for Eq.(24) is

$$\psi_o(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{2}{\pi} \ln(r)\ . \qquad (27a)$$

For $n \geqslant 1$ the fundamental solution to Eq.(24) is

$$\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = i\ H_o^{(2)}(nKr) \qquad n \geqslant 1\ ; \qquad (27b)$$

$H_o^{(2)}(nKr)$ is the Hankel function of the second kind and of zero order.

The asymptotic behaviour of $H_\mu^{(2)}(nKr)$, where μ is the order of the Hankel function, which in this analysis is always an integer, for large and small values of the argument is[18]

$$\left.\begin{aligned} H_\mu^{(2)}(nKr) &\sim \left(\frac{2}{\pi nKr}\right)^{\frac{1}{2}} \exp\left[- i\left(nKr - \frac{\pi}{2}\mu - \frac{\pi}{4}\right)\right] + \underline{0}\ \left(\frac{1}{r}{}^{3/2}\right) \\ & \qquad\qquad \text{as } nKr \to \infty \\ H_o^{(2)}(nKr) &\sim \frac{2i}{\pi} \ln\left(\frac{2}{nKr}\right) \qquad \text{as } nKr \to 0 \\ H_\mu^{(2)}(nKr) &\sim \frac{i}{\pi}\left(\frac{2}{nKr}\right)^\mu \mu! \qquad \text{as } nKr \to 0\ . \end{aligned}\right\} \qquad (28)$$

Differentiation of the Hankel function of order μ is given[18] by

$$\frac{d}{dx}\left[x^{-\mu} H_\mu^{(2)}(x)\right] = - x^{-\mu}\ H_{\mu+1}^{(2)}(x) \qquad (29)$$

By noting the asymptotic behaviour of the Hankel functions as defined by Eq.(28), $\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ $(n \geqslant 1)$ is logarithmically singular at the point $(\bar{x},\bar{z})$ and tends to zero as $\{e^{-inKr}/(nKr)^{\frac{1}{2}}\}$ at infinity. From Eq.(27a) it can be seen that $\psi_o(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ is also logarithmically singular at the point $(\bar{x},\bar{z})$.

To the present order of approximation $\bar{\phi}_n(\bar{x},\bar{z})$ and its derivatives are discontinuous across the wing planform and the wake, that is across the slit $(\bar{\xi} > 0\ ;\ \bar{\zeta} = \pm 0)$. Since $\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$, $\bar{\phi}_n(\bar{x},\bar{z})$ and their derivatives must be continuous in the domain S the point $(\bar{x},\bar{z})$ and the slit $(\bar{\xi} > 0\ ;\ \bar{\zeta} = \pm 0)$ must be excluded from S. The curve C and the enclosed domain S are shown in Fig. 2.

Eq.(26) can be written as

$$\iint_S \psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) g_{n_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})\, dS = -\oint_{C_1+C_W+C_\infty} \left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N} - \bar{\phi}_n(\bar{\xi},\bar{\zeta}) \frac{\partial\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N}\right] dC \tag{30}$$

where C_1 is that part of the boundary C surrounding the point $(\bar{x},\bar{z})$ and is taken to be a small circle of radius ε, C_W is that part of the boundary surrounding the positive $\bar{\xi}$-axis and C_∞ completes the boundary C, thus denoting the outer limiting boundary of S, taken to be a large circle, centre at the origin and of radius R. These boundaries, C_1, C_W, C_∞, together with the sense of integration, are also shown in Fig. 2.

The surface integral on the left-hand side of Eq.(30) is defined for $\bar{z} > 0$ by

$$\iint_S F\, dS = \lim_{\varepsilon\to 0} \Big\{ \int_{-\infty}^{\bar{x}-\varepsilon} \Big(\int_{+0}^{\infty} F\, d\bar{\zeta}\Big) d\bar{\xi} + \int_{\bar{x}+\varepsilon}^{\infty} \Big(\int_{+0}^{\infty} F\, d\bar{\zeta}\Big) d\bar{\xi} + \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{-0} F\, d\bar{\zeta}\Big) d\bar{\xi}$$
$$+ \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \Big[\int^{\bar{z}-\left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}} F\, d\bar{\zeta} + \int_{\bar{z}+\left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}}^{\infty} F\, d\bar{\zeta} \Big] d\bar{\xi} \Big\} \tag{31a}$$

while for $\bar{z} < 0$

$$\iint_S F\, dS = \lim_{\varepsilon\to 0} \Big\{ \int_{-\infty}^{\bar{x}-\varepsilon} \Big(\int_{-\infty}^{-0} F\, d\bar{\zeta}\Big) d\bar{\xi} + \int_{\bar{x}+\varepsilon}^{+\infty} \Big(\int_{-\infty}^{-0} F\, d\bar{\zeta}\Big) d\bar{\xi} + \int_{-\infty}^{\infty} \Big(\int_{+0}^{\infty} F\, d\bar{\zeta}\Big) d\bar{\xi}$$
$$+ \int_{\bar{x}-\varepsilon}^{\bar{x}+\varepsilon} \Big[\int_{-\infty}^{\bar{z}-\left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}} F\, d\bar{\zeta} + \int_{\bar{z}+\left[\varepsilon^2-(\bar{x}-\bar{\xi})^2\right]^{\frac{1}{2}}}^{-0} F\, d\bar{\zeta} \Big] d\bar{\xi} \Big\} \tag{31b}$$

It has been found that for the present purposes it is simpler to work in cartesian rather than in polar co-ordinates. As $\bar{z} \to \pm 0$ the limiting process is found using the appropriate formula of Eq.(31a) or Eq.(31b).

The integrals around C_∞ and C_1 in the limit as $R \to \infty$ and $\varepsilon \to 0$ are determined next.

For steady flow the velocity potential $\bar{\phi}_0(\bar{\xi},\bar{\zeta})$ on C_∞ when R is large can be assumed to be of the form $\{a\theta + b/R\}$ where a and b are constants and R,θ are the polar co-ordinates on C_∞ (a is zero for the non-lifting case). And since on C_∞

$$\psi_0(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \underline{O}(\ln(R))$$

the line integral on the right hand side of Eq.(30) around C_∞ is then a constant which is zero for the non-lifting problem; the constant is immaterial since it disappears when the whole equation is later differentiated.

In unsteady flows the disturbances are small on C_∞ so the linearised form of the differential equations is valid and all solutions $\bar{\phi}_n(\bar{\xi},\bar{\zeta})$ $(n > 1)$ behave like

$$\bar{\phi}_n \sim \frac{\text{const.}}{R^{\frac{1}{2}}} \exp\left[-i(nKR - \frac{\pi}{4})\right] n \geqslant 1 \quad , \tag{32}$$

on the argument that at large distances from the aerofoil $\bar{\phi}_n$ has the behaviour of an elementary solution of Eq.(11) as given by Eqs.(26,27). Physically Eq.(32) implies that waves are propagated outward as pure radiation waves towards infinity. Since the behaviour of ψ_n and $\bar{\phi}_n$ at infinity are similar it can be seen from Eq.(28, 32) that as $R \to \infty$

$$\left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N}\right]_{C_\infty} - \left[\bar{\phi}_n(\bar{\xi},\bar{\zeta}) \frac{\partial\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N}\right]_{C_\infty} \sim \underline{O}\left(\frac{1}{R^{3/2}}\right) ,$$

and since

$$\delta C_\infty = R\delta\theta$$

then

$$\lim_{R\to\infty} \oint_{C_\infty} \left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N} - \bar{\phi}_n(\bar{\xi},\bar{\zeta}) \frac{\partial\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N}\right] dC = 0 \quad . \tag{33}$$

For the line integral around C_1 in Eq.(26) the radius of the circle surrounding $(\bar{x},\bar{z})$ is ε and so

$$\delta C_1 = \varepsilon\delta\theta$$

taking θ to increase in a clockwise sense. For small ε

$$\psi_n \sim \frac{2}{\pi}\ln(\frac{nK\varepsilon}{2}) \quad ,$$

hence it follows that as $\varepsilon \to 0$, $\bar{\phi}_n(\bar{\xi},\bar{\zeta})$ and its derivatives take their value at $(\bar{x},\bar{z})$ and all the terms in the integral over C_1 vanish except those involving radial derivatives of ψ_n; thus, remembering that $\frac{\partial}{\partial N} \equiv \frac{\partial}{\partial \varepsilon}$,

$$\lim_{\varepsilon\to 0} \oint_{C_1} \left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N} - \bar{\phi}_n(\bar{\xi},\bar{\zeta}) \frac{\partial\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N}\right] dC = 4\,\bar{\phi}_n(\bar{x},\bar{z}) \ . \tag{34}$$

Finally for the line integral around C_w in Eq.(30); the curve lies around the positive $\bar{\xi}$-axis, as shown in Fig. 2, thus

$$\oint_{C_w} \left[\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \frac{\partial\bar{\phi}_n(\bar{\xi},\bar{\zeta})}{\partial N} - \bar{\phi}_n(\bar{\xi},\bar{\zeta}) \frac{\partial\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})}{\partial N}\right] dC$$
$$= \int_0^\infty \left[\psi_n(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) - \psi_{n_{\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},0)\,\Delta\bar{\phi}_n(\bar{\xi})\right] d\bar{\xi} \ , \tag{35}$$

where

$$\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) = \left[\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi},+0) - \bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi},-0)\right]$$

$$\Delta\bar{\phi}_n(\bar{\xi}) = \left[\bar{\phi}_n(\bar{\xi},+0) - \bar{\phi}_n(\bar{\xi},-0)\right]$$

On substitution of Eq.(33, 34, 35), Eq.(30) becomes

$$4\,\bar{\phi}_n(\bar{x},\bar{z}) = \int_0^\infty \left[\psi_n(\bar{x},\bar{\xi};\bar{z},0)\,\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) - \psi_{n_{\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},0)\,\Delta\phi_n(\bar{\xi})\right] d\bar{\xi}$$
$$+ \iint_S \psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\, g_{n_{\bar{\zeta}}}(\bar{\xi},\bar{\zeta})\, dS \tag{36}$$

The double integral in Eq.(36) can be integrated by parts with respect to ξ, using Eq.(31); thus

$$4\,\bar{\phi}_n(\bar{x},\bar{z}) = \int\left[\psi_n(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) - \psi_{n_{\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_n(\bar{\xi})\right]d\bar{\xi}$$

$$- \iint_S \psi_{n_{\bar{\xi}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\;g_n(\bar{\xi},\bar{\zeta})\;dS \qquad (37)$$

where $g_n(\bar{x},\bar{z})$ is defined in Eqs.(13,14).

The standard procedure is now followed: Eq.(37) is first differentiated with respect to $\bar{x}$ and the limiting conditions found as $\bar{z} \to \pm 0$; secondly Eq.(37) is differentiated with respect to $\bar{z}$ and again the limiting condition as $\bar{z} \to \pm 0$ is found. This leads to the two fundamental integral equations for the coupled symmetric and asymmetric problems.

On differentiation with respect to $\bar{x}$ Eq.(37) gives

$$4\,\bar{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) = \int_0^\infty\left[\psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) - \psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_n(\bar{\xi})\right]d\bar{\xi}$$

$$+ 2\,g_n(\bar{x},\bar{z}) - \iint_S \psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\;g_n(\bar{\xi},\bar{\zeta})\;dS. \qquad (38)$$

The term $2\,g_n(\bar{x},\bar{z})$ in Eq.(38) arises from the differentiation of the limit of integration around the singular point $(\bar{x},\bar{z})$ in the surface integral.

Differentiation of Eq.(37) with respect to $\bar{z}$ gives

$$4\,\bar{\phi}_{n_{\bar{z}}}(\bar{x},\bar{z}) = \int_0^\infty\left[\psi_{n_{\bar{z}}}(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi}) - \psi_{n_{\bar{\zeta}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},0)\;\Delta\bar{\phi}_n(\bar{\xi})\right]d\bar{\xi}$$

$$- \iint_S \psi_{n_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\;g_n(\bar{\xi},\bar{\zeta})\;dS\;. \qquad (39)$$

The various derivatives of ψ_n in Eqs.(38,39) can be listed by reference to Eqs.(27,29):

n = 0

$$\begin{aligned}
\psi_o(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= \ln(r) \\
\psi_{o_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\psi_{o_{\bar{\xi}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{(\bar{x}-\bar{\xi})}{r} \\
\psi_{o_{\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\psi_{o_{\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{(\bar{z}-\bar{\zeta})}{r} \\
\psi_{o_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\psi_{o_{\bar{\zeta}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{\left[(\bar{x}-\bar{\xi})^2-(\bar{z}-\bar{\zeta})^2\right]}{r^4} \\
\psi_{o_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= \psi_{o_{\bar{\zeta}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{2(\bar{x}-\bar{\xi})(\bar{z}-\bar{\zeta})}{r^4}
\end{aligned} \tag{40a}$$

$n \geqslant 1$

$$\psi_n(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = i\, H_0^{(2)}(nKr)$$

$$\begin{aligned}
\psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\psi_{n_{\bar{\xi}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = -\frac{inK(\bar{x}-\bar{\xi})}{r}\, H_1^{(2)}(nKr) \\
\psi_{n_{\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\psi_{n_{\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = -\frac{inK(\bar{z}-\bar{\zeta})}{r}\, H_1^{(2)}(nKr) \\
\psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\frac{in^2K^2(\bar{x}-\bar{\xi})^2}{r^2}\, H_2^{(2)}(nKr) + nKH_1^{(2)}(nKr) \\
\psi_{n_{\bar{\zeta}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= -\frac{in^2K^2(\bar{z}-\bar{\zeta})^2}{r^2}\, H_2^{(2)}(nKr) + nKH_1^{(2)}(nKr) \\
\psi_{n_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) &= \psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = -\frac{in^2K^2(\bar{z}-\bar{\zeta})(\bar{x}-\bar{\xi})}{r^2}\, H_2^{(2)}(nKr)
\end{aligned} \tag{40b}$$

The limit as $\bar{z} \to \pm\, 0$ is now taken for both Eqs. (38, 39).

For Eq.(38) as $\bar{z} \to +\, 0$ then, by reference to Eq.(40)

$$4\,\bar{\phi}_{n_{\bar{x}}}(\bar{x},+0) = \int_0^\infty \psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};+0,0)\, \Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi})d\bar{\xi} + \lim_{\bar{z}\to+0} \int_0^\infty \psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\, 0)\, \Delta\bar{\phi}_n(\bar{\xi})\, d\bar{\xi}$$

$$+ 2\, g_n(\bar{x},+0) - \lim_{\bar{z}\to+0} \iint_S \psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\, g_n(\bar{\xi},\bar{\zeta})\, dS \quad , \tag{41}$$

while in the limit as $\bar{z} \to -0$

$$4\,\bar{\phi}_{n_{\bar{x}}}(\bar{x},-0) = \int_0^\infty \psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};-0,0)\,\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi})\,d\bar{\xi} + \lim_{\bar{z}\to -0}\int_0^\infty \psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},0)\,\Delta\bar{\phi}_n(\bar{\xi})\,d\bar{\xi}$$

$$+\,2\,g_n(\bar{x},-0) - \lim_{\bar{z}\to -0}\iint_S \psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\,g_n(\bar{\xi},\bar{\zeta})\,dS \quad . \tag{42}$$

It can be seen from the form of $\psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ in Eq.(40) that

$$\lim_{\bar{z}\to +0}\psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{0}) = -\lim_{\bar{z}\to -0}\psi_{n_{\bar{\zeta}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{0}) \quad . \tag{43}$$

Because $\psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ involves only $(\bar{z}-\bar{\zeta})^2$ and noting the definition of the surface integral given by Eq.(31), then it follows that

$$\lim_{\bar{z}\to +0}\iint_S \psi_{n_{\xi x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\,g_n(\bar{\xi},\bar{\zeta})\,dS = \lim_{\bar{z}\to -0}\iint_S \psi_{n_{\bar{\xi}\bar{x}}}(x,\xi;z,\zeta)\,g_n(\bar{\xi},\bar{\zeta})\,dS \quad . \tag{44}$$

Thus on addition of Eqs.(41,42), substituting Eqs.(43,44)

$$2\left[\bar{\phi}_{n_{\bar{x}}}(\bar{x},+0) + \bar{\phi}_{n_{\bar{x}}}(\bar{x},-0)\right] = \int_0^\infty \psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};0,0)\,\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi})\,d\bar{\xi}$$

$$+\left[g_n(\bar{x},+0) + g_n(\bar{x},-0)\right]$$

$$-\lim_{z\to +0}\frac{1}{2}\iint_S \psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\left[g_n(\bar{\xi},\bar{\zeta}) + g_n(\bar{\xi},-\bar{\zeta})\right]dS \tag{45}$$

Following similar arguments Eq.(39) becomes

$$2\left[\bar{\phi}_{n_{\bar{z}}}(\bar{x},+0) + \bar{\phi}_{n_{\bar{z}}}(\bar{x},-0)\right] = -\lim_{\bar{z}\to +0}\int_0^\infty \psi_{n_{\bar{\zeta}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},0)\,\Delta\bar{\phi}_n(\bar{\xi})\,d\bar{\xi}$$

$$-\lim_{\bar{z}\to +0}\frac{1}{2}\iint_S \psi_{n_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\left[g_n(\bar{\xi},\bar{\zeta}) - g_n(\bar{\xi},-\bar{\zeta})\right]dS \tag{46}$$

Eqs.(45,46) express the two fundamental relationships for each harmonic contribution in a two-dimensional aerofoil oscillating in a high subsonic flow.

3. PROPERTIES OF THE NON-LINEAR EQUATIONS

Before discussing their solution it is helpful to establish some of the characteristics of the non-linear equations.

3.1 Magnitude of the Higher Harmonics

The fundamental assumption in all perturbation methods is that the terms retained in an equation and its boundary conditions are of comparable order of magnitude. The order of magnitude of the higher harmonic terms may be estimated by reference to the fundamental equation.

From Eq.(8) it can be implied that

$$\phi_{n_x}(x,z) \sim \underline{0}\left\{\frac{(\gamma+1)M_\infty^2}{(1-M_\infty^2)} \cdot \frac{1}{2} \cdot \sum_{r=0}^{n} (\phi_{n-r_x}(x,z) \cdot \phi_{r_x}(x,z))\right\}$$

or in terms of $\overset{*}{\phi}(\bar{x},\bar{z})$, using Eq.(14),

$$\overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) \sim \underline{0}\left\{\frac{1}{2}\sum_{r=0}^{n} (\overset{*}{\phi}_{n-r,\bar{x}}(\bar{x},\bar{z}) \cdot \overset{*}{\phi}_{r_{\bar{x}}}(\bar{x},\bar{z}))\right\} \tag{47}$$

Eq.(47) may be rewritten as

$$\overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) \sim \underline{0}\left\{\bar{u}_0(\bar{x},\bar{z})\ \overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z}) + \frac{1}{2}\sum_{r=1}^{n} (\overset{*}{\phi}_{n-r,\bar{x}}(\bar{x},\bar{z})\ \overset{*}{\phi}_{r_{\bar{x}}}(\bar{x},\bar{z}))\right\}. \tag{48}$$

Now

$$\left[1-M_\infty^2 - (\gamma+1)\ M_\infty^2\ \phi_x(x,z,t)\right] = 1 - M^2(x,z,t)$$

where $M(x,z,t)$ is the local Mach number given in the transformed variables of Eqs.(7, 10, 14) by

$$1-M^2(\bar{x},\bar{z},t) = (1-M_\infty^2)\left[1 - \sum_{n=0}^{\infty} (\overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z})\ e^{i(M_\infty^2 n\Omega\bar{x}+\omega t)})\right].$$

For a flow which remains subsonic throughout then

$$1-M^2(\bar{x},\bar{z},t) > 0$$

for all $(\bar{x},\bar{z},t)$, which implies that

$$\sum_{n=0}^{\infty} |\ \overset{*}{\phi}_{n_{\bar{x}}}(\bar{x},\bar{z})| < 1\ . \tag{49}$$

Thus

$$|\overset{*}{\phi}_{0_{\bar{x}}}(\bar{x},\bar{z})| = |\bar{u}_0(\bar{x},\bar{z})| < 1$$

and (50)

$$|\overset{*}{\phi}_{1_{\bar{x}}}(\bar{x},\bar{z})| < 1 - |\bar{u}_0(\bar{x},\bar{z})|$$

The convergence of the higher harmonics is established by Eq.(49); the order of magnitude of the first few harmonics can be found on using Eq.(48,50).

From Eq.(48)

$$|\overset{*}{\phi}_{2_{\bar{x}}}(\bar{x},\bar{z})| \leq \underline{0}\ \left\{\frac{|\overset{*}{\phi}_{1_{\bar{x}}}(\bar{x},\bar{z})|^2}{2\left[1-|\bar{u}_0(\bar{x},\bar{z})|\right]}\right\} \tag{51}$$

Thus it would be expected that from Eq.(50)

$$|\overset{*}{\phi}_{2_{\bar{x}}}(\bar{x},\bar{z})| < |\overset{*}{\phi}_{1_{\bar{x}}}(\bar{x},\bar{z})|$$

Similarly it can be argued that

$$|\overset{*}{\phi}_{3_{\bar{x}}}(\bar{x},\bar{z})| < \left|\{\overset{*}{\phi}_{2_{\bar{x}}}(\bar{x},\bar{z})\right|$$

Thus it is seen that the first two harmonics decrease in magnitude as the number of the harmonic increases. Since the order of magnitude of $\overset{*}{\phi}_{1_{\bar{x}}}(\bar{x},\bar{z})$ depends on amplitude of the oscillation it can be deduced from Eqs. (48, 50) that the smaller the amplitude the faster the magnitude of the harmonics approaches zero. Provided the amplitude of oscillation is small, then it is expected that the flow is adequately described by the first few terms in the series expansion for the velocity potential. If the amplitude is sufficiently small it is expected that the first term alone will suffice.

3.2 Oscillation of a Rigid Symmetric Aerofoil about Zero Incidence

For a symmetric aerofoil at zero incidence the antisymmetric boundary condition for $\bar{\phi}_{0_{\bar{z}}}(\bar{x},\pm 0)$, given by Eq.(15a), implies that $\bar{\phi}_0(\bar{x},\bar{z})$ is symmetric with respect to $\bar{z}$. On consideration of the symmetric boundary condition for $\bar{\phi}_{1_{\bar{z}}}(\bar{x},\pm 0)$, given by Eq.(15b), together with the differential equation for $\bar{\phi}_1(\bar{x},\bar{z})$, it follows that since $\bar{\phi}_0(\bar{x},\bar{z})$ is symmetric with respect to $\bar{z}$, $\bar{\phi}_1(\bar{x},\bar{z})$ is antisymmetric with respect to $\bar{z}$. Furthermore the symmetry or otherwise of the higher harmonics can be deduced directly from Eqs.(11, 12), together with the boundary condition

$$\bar{\phi}_{n_{\bar{z}}}(\bar{x},\pm 0) = 0 \qquad n \geq 2.$$

Since $\bar{\phi}_0(\bar{x},\bar{z})$ is symmetric and $\bar{\phi}_1(\bar{x},\bar{z})$ is antisymmetric it follows from Eq.(11, 12) that $\bar{\phi}_2(\bar{x},\bar{z})$ is symmetric with respect to $\bar{z}$.

Similar reasoning indicates that $\bar{\phi}_3(\bar{x},\bar{z})$ is antisymmetric.

For the symmetric wing oscillating about zero incidence $\bar{\phi}_n(\bar{x},\bar{z})$ is symmetric with respect to $\bar{z}$ for n even but antisymmetric for n odd.

For the general problem of a wing, oscillating about a mean non-zero incidence, all $\bar{\phi}_n(\bar{x},\bar{z})$ have symmetric and antisymmetric components.

3.3 Limiting Case of Zero Frequency

If the problem of a symmetric aerofoil oscillating about a mean non-zero incidence is considered then as the frequency, ω, tends to zero the problem tends to the steady problem of the symmetric aerofoil at the higher steady incidence equal to the initial mean incidence plus the oscillation amplitude incidence. It is instructive to appreciate what happens in the series solution presented here as the frequency, ω, tends to zero; the steady solution for an aerofoil at any steady incidence is given in Ref.(1) and is therefore available for comparison.

In the limit as ω tends to zero, Eq.(11) becomes

$$\bar{\phi}_{n_{\bar{x}\bar{x}}} + \bar{\phi}_{n_{\bar{z}\bar{z}}} = \frac{\partial}{\partial \bar{x}} \{\frac{1}{2} \sum_{r=0}^{n} \bar{\phi}_{n-r_{\bar{x}}} \bar{\phi}_{r_{\bar{x}}}\} \tag{52}$$

and the tangency boundary condition given by Eq.(15) becomes

$$\begin{aligned} \bar{\phi}_{0_{\bar{z}}}(\bar{x},\pm 0) &= -\bar{A} + \bar{Z}_c(\bar{x}) \pm \bar{Z}'_T(\bar{x}) \\ \bar{\phi}_{1_{\bar{z}}}(\bar{x},\pm 0) &= \bar{Z}'_1(\bar{x}) \\ \bar{\phi}_{n_{\bar{z}}}(\bar{x},\pm 0) &= 0 \qquad n \geq 2 \end{aligned} \tag{53}$$

On summation over all n, using Eqs. (7, 10), the set of equations, Eq.(52) gives

$$\bar{\phi}_{\bar{x}\bar{x}} + \bar{\phi}_{\bar{z}\bar{z}} = \frac{\partial}{\partial \bar{x}} \frac{1}{2} (\bar{\phi}_{\bar{x}})^2 \tag{54}$$

where

$$\bar{\phi}(\bar{x},\bar{z}) = \frac{k}{\beta^2} \phi(x,z) \quad .$$

After a similar summation the tangency boundary condition of Eq.(53) becomes

$$\bar{\phi}_{\bar{z}}(\bar{x},\pm 0) = -\bar{A} + \bar{Z}'_c(\bar{x}) \pm \bar{Z}'_T(\bar{x}) + \bar{Z}'_1(\bar{x}) \quad .$$

The Kutta condition of finite velocity at the trailing edge is still valid as is the condition of zero load off the wing; the latter condition is now given by

$$\Delta \bar{\phi}_{n_{\bar{x}}}(\bar{x}) = 0 \quad , \qquad \bar{x} \geq 1 \quad .$$

In the limit as ω tends to zero the equivalent steady problem of the aerofoil at an initial mean incidence plus the oscillation amplitude incidence is obtained as the sum of an infinite series.

It is shown in Section 3.1 that for small amplitudes of pitch the magnitude of the higher harmonics is small. A good test of the accuracy of any solution procedure used in the unsteady analysis therefore can be obtained by comparing the limiting solution for zero frequency at small amplitudes found by summing only the first few terms in the series with the more accurate result derived directly from the method outlined in Ref.(1).

4. THE LINEAR PROBLEM

The set of equations Eqs.(45, 46) which, together with their boundary conditions Eq.(15) formulate the non-linear problem, are extensions to the much simpler set of equations that formulate the linear problem. The standard linear set of equations are obtained by neglecting all second order terms, i.e. the g_n terms in Eqs.(45, 46). Substituting the boundary conditions given by Eq.(15) the linear equations can be reduced to

$$\left.\begin{aligned} \frac{1}{2}\left[\bar{\phi}_{o_{\bar{x}}}(\bar{x},+0) + \bar{\phi}_{o_{\bar{x}}}(\bar{x},-0)\right] &= \frac{1}{\pi}\int_0^1 \frac{\bar{Z}'(\bar{\xi})}{(\bar{x}-\bar{\xi})}\,d\bar{\xi} \\ -\bar{A} + \bar{Z}_c'(\bar{x}) &= \frac{-1}{2\pi}\int_0^1 \frac{\Delta\bar{\phi}_{o_{\bar{\xi}}}(\bar{\xi})}{(\bar{x}-\bar{\xi})}\,d\bar{\xi} \\ \bar{\phi}_{1_{\bar{x}}}(\bar{x},\bar{z}) &= -\bar{\phi}_{1_{\bar{x}}}(\bar{x},-\bar{z}) \end{aligned}\right\} \qquad (55a)$$

$$\bar{w}_1(\bar{x},\pm 0) = \left[\bar{Z}_1'(\bar{x}) + i\nu\bar{Z}_1(\bar{x})\right]e^{-iM_\infty^2\Omega\bar{x}} = -\frac{1}{4}\lim_{\bar{z}\to+0}\int_0^\infty \psi_{1_{\bar{z}_\zeta}}(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}_1(\bar{\xi})\,d\bar{\xi} \qquad (55b)$$

$$\bar{\phi}_n(\bar{x},\bar{z}) = 0 \qquad n > 1 \qquad (55c)$$

The first pair of equations, Eq.(55a) is recognised as the linear integral equations for the steady thickness and camber problems in subsonic flow; the second pair, Eq.(55b) are linear equations for the additional terms induced by the aerofoil oscillation which are independent of the steady field $\bar{\phi}_o$; the third equation Eq.(55c) states that the higher harmonic terms are a consequence only of the non-linearities.

In order to discuss the solution of the non-linear Eqs.(45, 46) with some degree of confidence, it is helpful to start with the solution procedure of the linear problems as defined by Eqs. (55).

The pair of equations, Eq.(55a), govern the steady flow about an aerofoil whose ordinates are given by the mean profile of the aerofoil, that is $(\bar{A}, \bar{Z}_T(\bar{x}), \bar{Z}_C(\bar{x}))$: these equations may be solved by one of the several methods currently available, the method of Weber[19] has been used here.

The solution of the equation for the fundamental oscillatory potential $\Delta\bar{\phi}_1(\bar{x},\bar{z})$, Eq.(55b), is more difficult.

As described in the Introduction there are various methods of solving Eq.(55b) for $\Delta\bar{\phi}_1(\bar{x},\bar{z})$. Rewriting Eq.(55b) in a form similar to that for the incompressible unsteady problem, a solution can be obtained for low frequencies[6]. The integral equation may also be solved directly by collocation. Because of its relative simplicity the low frequency solution is derived in this paper. It is hoped to investigate the application of a more direct collocation solution in a later paper.

The kernel function $\psi_{1_{\bar{z}\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ which appears in Eq.(55b) can by reference to Eq.(40) be expressed in the form

$$\psi_{1_{\bar{z}\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) = \frac{-iK^2(\bar{z}-\bar{\zeta})^2}{r^2} H_2^{(2)}(Kr) + K H_1^{(2)}(Kr)$$

$$= \psi_{1_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) + \frac{iK^2\left[(\bar{x}-\bar{\xi})^2-(z-\bar{\zeta})^2\right]}{r^2} H_2^{(2)}(Kr) \tag{56}$$

Thus Eq.(55b) becomes in the limit as $\bar{z}\to+0$ after integration by parts of the term involving $\psi_{1_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$

$$2\pi\bar{w}_1(\bar{x},0) = \int_0^\infty \frac{1}{(\bar{x}-\bar{\xi})} \frac{\partial}{\partial\bar{\xi}} \left[\Delta\bar{\phi}_1(\bar{\xi}) \frac{\pi}{2} K|\bar{x}-\bar{\xi}|\; iH_1^{(2)}(K|\bar{x}-\bar{\xi}|)\right] d\bar{\xi} \tag{57}$$

A function $\chi(K|\bar{x}-\bar{\xi}|)$ is introduced, where

$$\chi(K|\bar{x}-\bar{\xi}|) = 1 + \frac{\pi}{2} K|\bar{x}-\bar{\xi}| \; iH_1^{(2)}(K|\bar{x}-\bar{\xi}|) \tag{58}$$

hence Eq.(57) can be written as

$$2\pi(\bar{w}_1(\bar{x},0) + I(\bar{x})) = - \int_0^\infty \frac{\Delta\bar{\phi}_{1_{\bar{\xi}}}(\bar{\xi})}{(\bar{x}-\bar{\xi})} \, d\bar{\xi} \tag{59}$$

where

$$2\pi I(\bar{x}) = - \int_0^\infty \frac{1}{(\bar{x}-\bar{\xi})} \frac{\partial}{\partial\bar{\xi}}\left[\Delta\bar{\phi}_1(\bar{\xi}) \; \chi(K|\bar{x}-\bar{\xi}|)\right] d\bar{\xi}$$

Using Eq.(21), Eq.(59) gives

$$2\pi(\bar{w}_1(\bar{x},0) + I(\bar{x})) = - \int_0^1 \frac{\Delta\bar{\phi}_{1_{\bar{\xi}}}(\bar{\xi})}{(\bar{x}-\bar{\xi})} \, d\bar{\xi} + i\Omega\Delta\bar{\phi}_1(1)\int_1^\infty \frac{e^{-i\Omega(\bar{\xi}-1)}}{(\bar{x}-\bar{\xi})} d\bar{\xi} \tag{61}$$

If $I(\bar{x})$ is assumed known on the left hand wide of Eq.(61) then the problem can be regarded as the incompressible unsteady problem for which a formal solution is known[20,21], namely

$$\bar{l}_1(\bar{x}) = \frac{-4}{\pi} \left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \{(1 - C(\tfrac{\Omega}{2})) \int_0^1 \left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} (\bar{w}_1(\bar{\xi},0) + I(\bar{\xi}))d\bar{\xi}\}$$

$$- \frac{4}{\pi} \int_0^1 \{\left(\frac{\bar{\xi}}{1-\bar{\xi}}\right)^{\frac{1}{2}} \left(\frac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} \frac{1}{(\bar{x}-\bar{\xi})} - \frac{i\Omega}{2} \Lambda(\bar{x},\bar{\xi})\}(\bar{w}_1(\bar{\xi},0) + I(\bar{\xi})) \, d\bar{\xi} \tag{62}$$

where $\bar{l}(\bar{x})$ is the load distribution defined by

$$\left.\begin{aligned} \bar{l}_1(\bar{x}) &= \frac{-k}{\beta^2} \left[c_p \; (\bar{x},+0) - c_p \; (\bar{x},-0)\right] e^{-iM_\infty^2\Omega\bar{x}} \\ C(\tfrac{\Omega}{2}) &= \text{Theodorsen's Function} = \frac{H_1^{(2)}(\frac{\Omega}{2})}{H_1^{(2)}(\frac{\Omega}{2}) + iH_0^{(2)}(\frac{\Omega}{2})} \\ \Lambda(\bar{x},\bar{\xi}) &= \frac{1}{2} \ln\left[\frac{1-\bar{x}\bar{\xi} + (1-\bar{\xi}^2)^{\frac{1}{2}}(1-\bar{x}^2)^{\frac{1}{2}}}{1-\bar{x}\bar{\xi} - (1-\bar{\xi}^2)^{\frac{1}{2}}(1-\bar{x}^2)^{\frac{1}{2}}}\right] \end{aligned}\right\} \tag{63}$$

In terms of transformed variables θ, θ' defined by

$$\bar{x} = \frac{1}{2}(1 - \cos\theta)$$

$$\bar{\xi} = \frac{1}{2}(1 - \cos\theta') \tag{64}$$

Eq.(62) becomes

$$\bar{l}_1(\theta) = \frac{-4}{\pi}\left\{\int_0^\pi (\bar{w}(\theta') + I(\theta'))\left\{\left[C(\tfrac{\Omega}{2})(1-\cos\theta') + \cos\theta'\right]\cot(\tfrac{\theta}{2})\right.\right.$$

$$\left.\left. + \frac{i\Omega}{2}\Lambda(\theta,\theta')\sin\theta' + \frac{\sin\theta}{\cos\theta-\cos\theta'}\right\} d\theta'\right\} \tag{65}$$

where

$$\Lambda(\theta,\theta') = \frac{1}{2}\ln\left[\frac{1-\cos(\theta+\theta')}{1-\cos(\theta-\theta')}\right]$$

It is shown in the Appendix that to a first order in frequency

$$2\pi I(\bar{x}) = i\,\Omega\,\delta\,\Delta\,\bar{\phi}_1(1) \tag{66a}$$

where

$$\delta = \ln(\frac{M_\infty}{2}) + (1-M_\infty^2)^{\frac{1}{2}}\ln\left[\frac{1+(1-M_\infty^2)^{\frac{1}{2}}}{M_\infty}\right] \tag{66b}$$

Thus to a first order in frequency $I(x)$ is independent of $\bar{x}$ although it depends on $\Delta\bar{\phi}_1(1)$ which is unknown at this stage.

If the known downwash distribution function $\bar{w}(\bar{x},0)$ can be expanded as a cosine series

$$\bar{w}_1(\bar{x},0) = c_0 + c_1(\frac{1}{2} + \cos\theta) + \sum_{n=2}^{\infty} c_n \cos n\theta \tag{67}$$

it follows from Eqs.(64, 66) that to a first order in frequency

$$\bar{l}_1(\theta) = -2\left[c_0 + \frac{i\Omega\delta\Delta\bar{\phi}_1(1)}{2\pi}\right]\Gamma_0(\theta) - 2\sum_{n=1}^{\infty} c_n\,\Gamma_n(\theta) \tag{68}$$

where

$$\Gamma_0(\theta) = 2\left[C(\tfrac{\Omega}{2})\cot\frac{\theta}{2} + \frac{i\Omega}{2}\sin\theta\right]$$

$$\Gamma_1(\theta) = -2\sin\theta + \cot\frac{\theta}{2} + \frac{i\Omega}{2}(\sin\theta + \frac{\sin 2\theta}{2})$$

$$\Gamma_n(\theta) = -2\sin(n\theta) + \frac{i\Omega}{2}(\frac{\sin(n+1)\theta}{(n+1)}) \tag{69}$$

Eq.(68) expresses $\bar{l}_1(\theta)$ in terms of $\Delta\bar{\phi}_1(1)$ so a linear algebraic equation for $\Delta\bar{\phi}_1(1)$ can be found by application of the condition

$$\Delta\bar{\phi}_1(\bar{x}) = \frac{1}{2} e^{-iM_\infty^2\Omega\bar{x}} \int_0^{\bar{x}} \bar{l}_1(\bar{\xi})\, e^{iM_\infty^2\Omega\bar{\xi}}\, d\bar{\xi} \;, \; \bar{x} \le 1 \tag{70}$$

which gives when $\bar{x} = 1$

$$\Delta\bar{\phi}_1(1) = -\pi\, c_0 + \underline{O}(\Omega)$$

The final result is for low frequencies

$$\bar{l}_1(\theta) = -\,2\, c_0(1+\frac{i\Omega\delta}{2})\, \Gamma_0(\theta) - 2\sum_{n=1}^{\infty} c_n\Gamma_n(\theta) \tag{71}$$

where the coefficients c_0, c_n are obtained from Eq.(67), δ from Eq.(66b) and $\Gamma_n(\theta)$ from Eq.(69).

The total lift, L, and the pitching moment M, about the mid-chord point are defined by

$$L = \frac{1}{4} \cdot \frac{\beta^2}{k} \rho_\infty U_\infty^2 \{\int_0^\pi \bar{l}_1(\theta) \sin\theta\, e^{i\frac{M_\infty^2}{2}\Omega(1-\cos\theta)} d\theta \} e^{i\omega t} \tag{72a}$$

$$M = -\,\frac{1}{8} \cdot \frac{\beta^2}{k} \cdot \rho_\infty U_\infty^2 \{\int_0^\pi \bar{l}_1(\theta) \cos\theta \sin\theta\, e^{i\frac{M_\infty^2}{2}\Omega(1-\cos\theta)} d\theta \} e^{i\omega t} \tag{72b}$$

nose up pitching moments are positive.

If Eqs.(69,71) are used then the integrals in Eq.(72) may be evaluated. To a first order in frequency the total lift and moment are given by

$$L = \frac{-\beta^2}{k} \rho_\infty U_\infty^2\pi \{ (1-\frac{\pi}{4}\Omega)\, c_0 + i\frac{\Omega}{2} \Big[(\ln\frac{\Omega}{4} + 0.577 + 1 + \delta + \frac{M_\infty^2}{2})\, c_0 + \frac{(2-M_\infty^2)}{4} c_1 - \frac{M_\infty^2}{8} c_2\Big]\} e^{i\omega t} \tag{73a}$$

$$M = \frac{-\beta^2}{k} \rho_\infty U_\infty^2 \frac{\pi}{4} \{(1-\frac{\pi\Omega}{4})\, c_0 + \frac{c_1}{2} - \frac{c_2}{2} + i\frac{\Omega}{2} \Big[(\ln\frac{\Omega}{4} + 0.577 + \delta)\, c_0 + \frac{(1+M_\infty^2)}{4} c_1 - \frac{M_\infty^2}{2} c_2 + \frac{M_\infty^2}{4} c_3\Big]\} e^{i\omega t} \tag{73b}$$

where Euler's constant 0.577 arises in the low frequency expansion of $c(\frac{\Omega}{4})$

It can be seen that the total force and moment depend only on the first four coefficients of the cosine series, Eq.(67).

Although Eq.(73b) gives the moment about the mid-chord point, the moment about any point can be found by a linear combination of Eqs.(73a, 73b).

The stiffness and damping derivatives for an aerofoil oscillating in pitch about the mid-chord point are defined as

$$\frac{L}{\rho_\infty U_\infty^2} = \alpha_0(l_\alpha + i\nu l_{\dot{\alpha}})\, e^{i\omega t} \tag{74}$$

$$\frac{M}{\rho_\infty U_\infty^2} = \alpha_0(m_\alpha + i\nu m_{\dot{\alpha}})\, e^{i\omega t} \tag{75}$$

where α_0 is the amplitude of the pitching motion.

The flutter derivatives can then be found to first order in frequency by using Eq.(74, 75).

4.1 Linearised Theory for an Aerofoil Pitching at Low Frequency.

If an aerofoil is oscillating in pitch about a given pitching axis, X_0, with low frequency ω, then, with reference to Eq.(3),

$$z_1(x) = -\alpha_0(x-x_0) \tag{76}$$

where α_0 is the amplitude of the oscillation.

Hence, using Eq.(15)

$$\bar{w}_1(\bar{x},0) = -\bar{A}_0(1 + i\nu(\bar{x}-\bar{x}_0))\, e^{-iM_\infty^2\Omega\bar{x}}$$

where

$$\bar{A}_0 = \frac{k}{\beta^3}\alpha_0$$

After transformation to the variables defined in Eq.(64) $\bar{w}_1(\bar{x},0)$ can then be expanded in the cosine series Eq.(67) with

$$c_0 = \bar{A}_0\{1 + i\left[(1-M_\infty^2)\cos\theta_0 - 2M_\infty^2 + 0.5\right]\tfrac{\Omega}{2}\} \qquad (78)$$

$$c_1 = \bar{A}_0 i\{2M_\infty^2 - 1\}\tfrac{\Omega}{2}$$

$$c_n = 0 \qquad n \geq 2$$

where

$$\bar{x}_0 = \tfrac{1}{2}(1 - \cos\theta_0)$$

To first order in frequency, $\bar{l}_1(\theta)$ can be found from Eqs.(68, 69, 71, 78) as

$$\bar{l}_1(\theta) = 2\bar{A}_0\{2(1-\tfrac{\pi}{4}\Omega)\cot\tfrac{\theta}{2} + i\Omega\{\left[(1-M_\infty^2)\cos\theta_0 + 0.577 - M_\infty^2 + \delta + \ln\tfrac{\Omega}{4}\right]\cot\tfrac{\theta}{2} + 2(1-M_\infty^2)\sin\theta\}\} \qquad (79)$$

The flutter derivatives can be obtained by substitution of the coefficients c_0, c_1, c_n from Eqs.(78) into Eqs.(73, 74, 75). These values are shown as functions of frequency parameter as the linearised curves in Figs. 3, 4, 5, 6.

A function which is required later in the subsequent non-linear analysis is $\phi^*_{1\bar{x}_\ell}(\bar{x},\pm 0)$ which is given by

$$\phi^*_{1\bar{x}_\ell}(\bar{x},0) = \frac{\bar{l}_1(\bar{x})}{4} - \frac{i\,(1-M_\infty^2)\Omega}{2}\Delta\bar{\phi}_1(\bar{x}) \qquad (80)$$

Using Eqs.(70, 79) and Eq.(63) $\phi^*_{1\bar{x}}(\bar{x},0)$ is found to first order in frequency for the case of an aerofoil oscillating in pitch about its mid-point, as

$$\phi^*_{1\bar{x}_\ell}(\bar{x},0) = \bar{A}_0\Big\{(1-\tfrac{\pi\Omega}{4})\left(\tfrac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} + i\tfrac{\Omega}{2}\Big\{\Big[(1-M_\infty^2)(1-2\bar{x}_0) + 0.577 - M_\infty^2 + \delta + \ln\tfrac{\Omega}{4}\Big]\left(\tfrac{1-\bar{x}}{\bar{x}}\right)^{\frac{1}{2}} - (1-M_\infty^2)\arccos(1-2\bar{x})\Big\}\Big\} \qquad (81)$$

5. THE NON-LINEAR PROBLEM

The linear problem discussed in the previous section is valid if the local Mach number does not differ much from the freestream Mach number, an assumption that becomes less accurate as the latter approaches unity. At high subsonic speeds linearisation is inadequate and the linear potential equation must be replaced by the corresponding non-linear equation. As shown in Eqs.(45, 46) a double integral involving the non-linear terms over the entire flow field is now introduced. If the freestream Mach number is not too high, the flow remains subcritical so the additional non-linear terms may be regarded as correction terms in the linear equations. For example, if linear theory is 25% in error as regards the pressure, say, and if the non-linear terms can be approximated to a similar order of accuracy, then the overall error in the pressure will become of the order of 5% which is a considerable improvement over linear theory.

The technique described in Section 4 for low frequencies is now extended to the non-linear problem.

The problem considered here is concerned with a rigid symmetric aerofoil oscillating about zero mean incidence. For this problem it has already been shown in Section 3.2 that the solutions for successive harmonics are alternately symmetric and antisymmetric with respect to z. Also it has been shown in Section 3.1 that if the amplitude of the oscillation is sufficiently small then the magnitude of successive harmonic terms in the series expansion for $\phi(x,z,t)$ decreases rapidly. For small amplitude oscillations it is thought that only the first term $\bar{\phi}_1(\bar{x},\bar{z})$ need be determined.

Eq.(45) may be written as

$$2\left[\bar{\phi}_{n_{\bar{x}}}(\bar{x},+0) + \bar{\phi}_{n_{\bar{x}}}(\bar{x},-0)\right] = \int_0^\infty \psi_{n_{\bar{x}}}(\bar{x},\bar{\xi};0,0)\Delta\bar{\phi}_{n_{\bar{\zeta}}}(\bar{\xi})d\bar{\xi} + \left[g_n(\bar{x},+0)+g_n(\bar{x},-0)\right] - I_{\tau_n}(\bar{x}) \tag{82}$$

where

$$I_{\tau_n}(\bar{x}) = \lim_{\bar{z}\to+0} \frac{1}{2} \iint_S \psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta}) \left[g_n(\bar{\xi},\bar{\zeta}) + g_n(\bar{\xi},-\bar{\zeta})\right] dS \tag{83}$$

and $g_n(\bar{\xi},\bar{\zeta})$ is defined in Eqs.(13, 14).

Eq.(46) may be written as

$$2\left[\bar{\phi}_{n_{\bar{z}}}(\bar{x},+0) + \bar{\phi}_{n_{\bar{z}}}(\bar{x},-0)\right] = -\lim_{\bar{z}\to+0} \int_0^\infty \psi_{n_{\bar{z}\bar{\zeta}}}(\bar{x},\bar{\xi};\bar{z},0)\Delta\bar{\phi}_n(\bar{\xi})\, d\bar{\xi} - I_{c_n}(\bar{x}) \tag{84}$$

where

$$I_{c_n}(\bar{x}) = \lim_{\bar{z}\to+0} \frac{1}{2} \iint_S \psi_{n_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\left[g_n(\bar{\xi},\bar{\zeta}) - g_n(\bar{\xi},-\bar{\zeta})\right] dS \tag{85}$$

5.1 Case when n = 0

The case when n = 0 in Eq.(82, 84), together with the appropriate boundary conditions of Eq.(15) represents the flow about the steady aerofoil at the mean incidence: this problem is solved approximately in Ref.(1). For a symmetric aerofoil at zero mean incidence Eq.(84) is identically zero while Eq.(82) becomes

$$\bar{u}_{0\tau}(\bar{x}) - \frac{\bar{u}^2_{0\tau}(\bar{x})}{4} = \bar{u}_{\tau\ell}(\bar{x}) + I_{\tau_0}(\bar{x}) \tag{86}$$

since

$$\begin{aligned} \bar{u}_{0\tau}(\bar{x}) &= \text{symmetric velocity on upper surface} \\ &= \bar{u}_0(\bar{x},+0) = \bar{u}(\bar{x},-0) \\ &= \frac{1}{2}\left[\bar{u}_0(\bar{x},+0) + \bar{u}_0(\bar{x},-0)\right] \end{aligned} \tag{87}$$

$\bar{u}_{\tau L}(\bar{x})$ = linearised solution modified at leading edge

$$= \{\frac{1}{\pi}\int_0^1 \frac{\bar{Z}'_\tau(\bar{\xi})}{(\bar{x}-\bar{\xi})}\, d\bar{\xi} + \frac{k}{\beta^2}\}\ \{1 + (\frac{\beta^2\bar{Z}'_\tau(\bar{x})}{k})^2\}^{-\frac{1}{2}} - \frac{k}{\beta^2} \tag{88}$$

$$\bar{u}^2_{0\tau}(\bar{x}) = g_0(\bar{x},+0) + g_0(\bar{x},-0)$$

$$I_{\tau_0}(\bar{x}) = \lim_{z\to+0} \iint \psi_{0\bar{\xi}\bar{x}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})\, \frac{\bar{u}^2_0(\bar{\xi},\bar{\zeta})}{2}\, dS$$

The formula for $\bar{u}_{\tau L}(\bar{x})$ given by Eq.(88) contains appropriate second order terms in order to remove the leading edge singularities that are a feature of linearised solutions: the nature of these singularities is discussed further in Section 6.

As mentioned previously, the non-linearities may be regarded as correction terms to the linear problem, at least for subcritical flows, and it has been suggested that adequate results can be obtained even if the non-linearities in Eq.(86), expressed by the double integral $I_{\tau 0}(\bar{x},\bar{u}_{0\tau}(\bar{x}))$, are approximated to only a fair degree of accuracy. In Ref.(1) the double integral is evaluated by approximating the variation of the perturbation velocity $\bar{u}_0(\bar{\xi},\bar{\zeta})$ in the $\bar{\zeta}$ direction by a suitable interpolation function involving only the ordinate, $\bar{\zeta}$, the value of $\bar{u}_0(\bar{\xi},\bar{\zeta})$ on the aerofoil surface and the aerofoil geometry. The integration with respect to $\bar{\zeta}$ is then performed and the double integral is reduced to a single integral over the chord.

It is shown in Ref.(1) that

$$|I_{\tau 0}(\bar{x},\bar{u}_{0\tau}(\bar{x}))| << \frac{\bar{u}^2_{0\tau}(\bar{x})}{4} \tag{89}$$

for values of $\bar{u}_{0\tau}(\bar{x})$ close to unity; this result is less accurate as $\bar{u}_{0\tau}(\bar{x})$ decreases but then the importance of the non-linear terms decreases and linearised theory becomes more valid. A useful approximate solution to the steady problem is therefore

$$\bar{u}_{0\tau}(\bar{x}) - \frac{\bar{u}^2_{0\tau}(\bar{x})}{4} = \bar{u}_{\tau L}(\bar{x}) \tag{90}$$

On the basis of Eq.(90) a first approximation for the non-linear $\bar{u}_{0\tau}(\bar{x})$ could be quickly evaluated from the linearised solution. An improved solution for $\bar{u}_{0\tau}(\bar{x})$ is obtained by substituting the first approximation into $I_{\tau 0}(\bar{x},\bar{u}_{0\tau}(\bar{x}))$ and then solving the quadratic Eq.(86) for $\bar{u}_{0\tau}(\bar{x})$; thus an iterative procedure is established.

5.2 Case when n = 1

The case when n = 1 in Eqs.(82 - 85), together with the boundary condition, Eq.(15b), represents the fundamental response.

Since the aerofoil is symmetric and is oscillating about a zero mean incidence, Eq.(82) is identically zero by virtue of the argument presented in Section 3.2 that $\bar{\phi}_1(\bar{x},\bar{z})$ is an antisymmetric function of $\bar{z}$.

Using Eq.(15), Eq.(84) becomes for n = 1, noting the argument leading to Eq.(56)

$$2\pi\bar{w}_1(\bar{x},\pm 0) = \int_0^\infty \frac{1}{(\bar{x}-\bar{\xi})} \frac{\partial}{\partial\xi}\left[\Delta\bar{\phi}_1(\bar{\xi})\,\frac{\pi}{2}\,K|\bar{x}-\bar{\xi}|\;iH_1^{(2)}(K|\bar{x}-\bar{\xi}|)\right] d\bar{\xi}$$

$$- \frac{\pi}{2}\, I_{c_1}(\bar{x}) \tag{91}$$

where

$$I_{c_1}(\bar{x}) = \lim_{\bar{z}\to +0} \frac{1}{2} \iint_S \frac{-K^2(\bar{z}-\bar{\zeta})(\bar{x}-\bar{\xi})}{\left[(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2\right]} H_2^{(2)}(Kr)\quad g_1(\bar{\xi},\bar{\zeta})\;\; dS \tag{92}$$

$$g_1(\bar{\xi},\bar{\zeta}) = \bar{u}_0(\bar{\xi},\bar{\zeta})\;\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) + i\,M_\infty^2\Omega \int_{-\infty}^{\bar{\xi}} \bar{u}_0(\bar{\xi}',\bar{\zeta})\;\phi^*_{1_{\bar{\xi}}}(\bar{\xi}',\bar{\zeta})\; d\bar{\xi}'$$

$$\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) = \bar{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) + i\,M_\infty^2\,\Omega\bar{\phi}_1(\bar{\xi},\bar{\zeta}) \tag{93}$$

To a first order in frequency Eq.(92) is

$$I_{c1}(\bar{x}) = \lim_{\bar{z}\to +0} \iint_S \frac{4}{\pi}\,\frac{(\bar{x}-\bar{\xi})(\bar{z}-\bar{\zeta})}{\left[(\bar{x}-\bar{\xi})^2+(\bar{z}-\bar{\zeta})^2\right]^2}\, g_1(\bar{\xi},\bar{\zeta})\; dS \tag{94}$$

The double integral for $I_{C1}(\bar{x})$ given by Eq.(94) can be evaluated assuming suitable interpolation functions to approximate the variation of $\bar{u}_0(\bar{\xi},\bar{\zeta})\ \overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ normal to the aerofoil surface in a similar way to that used for the steady problem[1]. In this manner the double integral $I_{c_1}(\bar{x})$ is reduced to a single integral involving the values of $\bar{u}_0(\bar{\xi},\pm 0)$, $\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0)$, the aerofoil surface and the aerofoil geometry.

5.3 Approximating Functions

In general the kernels in the double integrals $I_{T_n}(\bar{x})$, $I_{C_n}(\bar{x})$ represented by $\psi_{n_{\bar{\xi}\bar{x}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ and $\psi_{n_{\bar{\xi}\bar{z}}}(\bar{x},\bar{\xi};\bar{z},\bar{\zeta})$ respectively, decrease rapidly away from the surface (like $|\bar{\zeta}|^{-2}$ and $|\bar{\zeta}|^{-3}$ respectively). Except in the case of the steady flow and the low frequency expansion, Eq.(94), this initially rapid decrease slows down for larger values of $|\bar{\zeta}|$ because of the asymptotic nature of the Hankel functions. It is suggested that if the variation of $\bar{u}_0(\bar{\xi},\bar{\zeta})$ and $g_n(\bar{\xi},\bar{\zeta})$ with $\bar{\zeta}$ is approximated by a function that is exact for small $\bar{\zeta}$ then the double integrals may be reasonably accurate since any error in the approximating functions away from the surface is offset by the rapidly diminishing kernel function. Only the steady flow (n = 0) and the fundamental response are considered in detail here.

The potential $\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ is complex and needs to be divided into its real and imaginary parts in this analysis: thus

$$\phi^*_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) = \overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) + i\ \overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) \tag{95}$$

where $\overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ and $\overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ are both real functions.

Approximating functions are chosen to be of the form for $\bar{\zeta} > 0$

$$\begin{aligned}
\bar{u}_0(\bar{\xi},\bar{\zeta}) &= \bar{u}_0(\bar{\xi},+0)\; h_0\left(\frac{\bar{\zeta}}{a_0(\bar{\xi})}\right) \\
\overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) &= \overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},+0)\; h_1^{(R)}\left(\frac{\bar{\zeta}}{a_1(\bar{\xi})}\right) \qquad (96) \\
\overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) &= \overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},+0)\; h_1^{(I)}\left(\frac{\bar{\zeta}}{b_1(\bar{\xi})}\right)
\end{aligned}$$

where $a_0(\bar{\xi})$, $a_1(\bar{\xi})$ and $b_1(\bar{\xi})$ are real functions of $\bar{\xi}$ chosen so that Eq.(99) are exact for small values of $\bar{\zeta}$. It should be remembered that in the present case $\bar{u}_0(\bar{\xi},\bar{\zeta})$ is a symmetric function of $\bar{\zeta}$ whereas $\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ is an antisymmetric function of $\bar{\zeta}$; thus the relations corresponding to Eq.(96) for $\bar{\zeta} < 0$ are straightforward.

The value of $\bar{u}_0(\bar{\xi},\bar{\zeta})$ and $\overset{*}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ close to the surface can be found in terms of the surface conditions by means of McLaurin's series expansion; the second term in the McLaurin's series can be expressed in terms of surface values and the aerofoil geometry by using the boundary conditions in Eq.(15b) and the condition of irrotationality. A necessary condition for any approximation is then that for $|\bar{\zeta}|$ close to zero $h_0\left(\frac{\bar{\zeta}}{a_0(\bar{\xi})}\right)$, $h_1^{(R)}\left(\frac{\bar{\zeta}}{a_1(\bar{\xi})}\right)$, $h_1^{(I)}\left(\frac{\bar{\zeta}}{b_1(\bar{\xi})}\right)$ should have identical series expansions in $\bar{\zeta}$ to the McLaurin series expansion for $\bar{u}_0(\bar{\xi},\bar{\zeta})$, $\overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$, $\overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ to first order in $\bar{\zeta}$. It is also desirable that the approximating functions should represent at least the qualitative behaviour of $\bar{u}_0(\bar{\xi},\bar{\zeta})$, $\overset{*(R)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ and $\overset{*(I)}{\phi}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ as $|\bar{\zeta}| \to \infty$ in that these functions then tend to zero. It is shown in Ref.(1) however that for steady flows the total integral is not critically dependent on the

behaviour of the approximating functions at large values of $|\bar{\zeta}|$ and although it is helpful to represent the far field behaviour correctly it is suggested that in the light of experience in the steady problem[1] this is a secondary consideration compared to that of accurately representing the behaviour close to the aerofoil surface.

The derivation of the function $h_0(\frac{\bar{\zeta}}{a_0(\bar{\xi})})$ is discussed fully in Ref.(1); it is sufficient to note that for a symmetric aerofoil

$$h_0(\frac{\bar{\zeta}}{a_0(\bar{\xi})}) = \frac{1}{\left[1 + \frac{\bar{\zeta}}{a_0(\bar{\xi})}\right]^2} \tag{97}$$

where

$$a_0(\bar{\xi}) = \frac{-2\ \bar{u}_0(\bar{\xi},\pm 0)}{\bar{Z}_T''(\bar{\xi})} \tag{98}$$

The McLaurin series expansion for $\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ in the neighbourhood of the aerofoil surface is

$$\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) = \phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0) + \bar{\zeta}\ \phi^{*(R)}_{1_{\bar{\xi}\bar{\zeta}}}(\bar{\xi},\pm 0) + \ldots . \tag{99}$$

From Eqs.(13, 14, 15) Eq.(99) becomes for $(0 \leqslant \bar{\xi} \leqslant 1)$

$$\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) = \phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0) + \underline{0}(\Omega^2) \tag{100}$$

since $\bar{Z}_1''(\bar{x}) = 0$ for an oscillating rigid aerofoil. Thus to first order in frequency the variation of $\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ with $\bar{\zeta}$ for small $\bar{\zeta}$ is zero. Thus it is assumed that

$$h_1^{(R)}\ (\frac{\bar{\zeta}}{a_1(\bar{\xi})}) = 1 \tag{101}$$

The McLaurin series expansion for $\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ can be expressed for $(0 \leqslant \bar{\xi} \leqslant 1)$, using Eqs.(13, 14, 15), as

$$\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta}) = \phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0) + \bar{\zeta}\nu\bar{Z}_1'(\bar{\xi}) \tag{102}$$

It is therefore assumed that

$$h_1^{(I)}\left(\frac{\bar{\zeta}}{b_1(\bar{\xi})}\right) = \frac{1}{\left(1+\frac{\bar{\zeta}}{b_1(\bar{\xi})}\right)} \tag{103}$$

where

$$b_1(\bar{\xi}) = \frac{-\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0)}{\nu\,\bar{Z}_1'(\bar{\xi})} \tag{104}$$

Since it is to be expected that $\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},\pm 0)$ will be of the same order in frequency as the linear solution, it can be seen on using Eq.(15b, 78) that

$$b_1(\bar{\xi}) \sim \underline{0}\left(\frac{1}{\ln(\Omega)}\right) \tag{105}$$

The approximations for $\phi^{*}_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ given above in Eqs.(101, 104) have only been deduced over the wing chord $(0 \leqslant \bar{\xi} \leqslant 1)$. It is argued that the contribution from the wake to the double integral is negligible since it is implied in the wake boundary condition, Eq.(21) that the velocity in the wake is equal to the freestream velocity U_∞ and hence

$$\bar{u}_0(\bar{\xi}) \approx 0 \quad , \quad \bar{\xi} \geqslant 1 \tag{106}$$

Substituting Eqs.(96, 101, 104) into the integral $I_{c_1}(\bar{x})$ given by Eq.(94) the integration with respect to $\bar{\zeta}$ can be performed, leading to

$$\begin{aligned} I_{c_1}(\bar{x}) &= I_{c_1}(\bar{x},\bar{u}_{0T}(\bar{x}),\ \phi^{*}_{1_{\bar{x}}}(\bar{x},\pm 0)) = \frac{-2}{\pi}\Bigg\{\int_0^1 \Bigg\{\frac{2\bar{u}_{0T}(\bar{\xi})\left[\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},0)+i\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},0)\right]}{(\bar{x}-\bar{\xi})} \\ &+ \bar{Z}_T''(\bar{\xi})\left[\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},0) + i\,\phi^{*(I)}_{1_{\bar{\xi}}}(\bar{\xi},0)\right] E_{c_1}\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_T''(\bar{\xi})}{2\,\bar{u}_{0T}(\bar{\xi})}\right] \\ &-(1-M_\infty^2)\Omega\bar{Z}'(\bar{\xi})\bar{u}_{0T}(\bar{\xi})\left[E_{c_2}\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_T''(\bar{\xi})}{2\,\bar{u}_{0T}(\bar{\xi})}\right] - 2\,E_{c_1}\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_T(\bar{\xi})}{2\,\bar{u}_{0T}(\bar{\xi})}\right]\right]d\bar{\xi} \\ &- iM_\infty^2\Omega\int_0^1 2\,\bar{u}_{0T}(\bar{\xi})\,\phi^{*(R)}_{1_{\bar{\xi}}}(\bar{\xi},0)\,E_{c_3}\left[\frac{-(\bar{x}-\bar{\xi})\bar{Z}_T''(\bar{\xi})}{2\,\bar{u}_{0T}(\bar{\xi})}\right]d\bar{\xi}\Bigg\} \end{aligned} \tag{107}$$

where

$$\left.\begin{aligned}
E_{c_1}(x) &= \frac{1}{(1+x^2)^3}\{2x(3-x^2)\ln|x| + x(1+x^2)(5+x^2) \\
&\qquad\qquad +(1-3x^2)\pi\,\mathrm{sgn}(a_0x)\} \\
E_{c_2}(x) &= \frac{1}{(1+x^2)^2}\{2x\ln|x| + x(1+x^2)+\frac{(1+x^2)}{2}\pi\,\mathrm{sgn}(a_0x)\} \\
E_{c_3}(x) &= \frac{1}{(1+x^2)^2}\{|x|\pi\,\mathrm{sgn}(a_0) - (1+x^2) - (1-x^2)\ln|x|\}
\end{aligned}\right\} \qquad (108)$$

In Eq.(107), by reference to Eq.(93), it is seen the first integral arises from the term $\bar{u}_0(\bar{\xi},\bar{\zeta})\ \phi^*_{1_{\bar{\xi}}}(\bar{\xi},\bar{\zeta})$ whereas the second integral arises from the term involving

$$\int_{-\infty}^{\bar{\xi}} \bar{u}_0(\bar{\xi}',\bar{\zeta})\ \phi^*_{1_{\bar{\xi}}}(\bar{\xi}',\bar{\zeta})\ d\bar{\xi}' \quad ;$$

it is necessary to change the order of integration in order to obtain the second integral in Eq.(107).

To evaluate Eq.(107) it is necessary to modify the behaviour of $\phi^*_{1_{\bar{\xi}}}(\bar{\xi},0)$ in the neighbourhood of the wing leading edge. A modification has already been implied since it is necessary for the determination of $\bar{u}_{0\tau}(\bar{\xi})$ as discussed in Ref.(1). These leading edge modifications are described in the following section.

No attempt is made here to determine the higher harmonics since there are difficulties associated with the formulation of suitable approximating functions. It is shown in Section 3.2 that in the present example the pressure distribution for the first harmonic is symmetric, in which case the flutter derivatives are unaffected by the first harmonic and since the higher harmonics are small in comparison to the fundamental, it is expected that their effects will be negligible.

6. SOME SECOND ORDER CORRECTIONS

One consequence of linearised theory is the appearance of leading edge singularities. The assumption of small perturbations breaks down near the leading edge of an aerofoil, both at a stagnation point and in regions of local high velocity. Singular terms appear at the leading edge in both 'lifting' and 'thickness' linearised solutions. For example the integral in Eq.(62) for the loading distribution shows a leading edge singularity of the order of $(\bar{x}^{-\frac{1}{2}})$. In the present analysis, since account is being taken of second order terms it is necessary to modify accordingly the linearised singularities in the nose region. As already stated the integrals for $I_{c_1}(\bar{x})$ involve second order terms which can only be evaluated if the linearised singularities are removed.

The nature of the leading edge singularities in steady flow is fully discussed in Ref.(1) where suitable correction terms are derived. It is shown that correction terms involving only the thickness distribution near the nose, even for lifting aerofoils, are sufficient. An example of the leading edge correction for steady flow past a symmetric aerofoil is given in Eq.(88) where if

$$\bar{u}_{T\ell}(\bar{x}) = \frac{1}{\pi}\int_0^1 \frac{\bar{Z}_T'(\bar{\xi})}{(\bar{x}-\bar{\xi})}\, d\bar{\xi}$$

is the standard linearised velocity then a non-singular approximation to the linearised solution is given by $\bar{u}_{TL}(\bar{x})$ where

$$\bar{u}_{TL}(\bar{x}) = \frac{\left[\bar{u}_{T\ell}(\bar{x}) + \frac{k}{\beta^2}\right]}{\left[1 + \left(\frac{\beta^2\bar{Z}_T'(\bar{x})}{k}\right)^2\right]^{\frac{1}{2}}} - \frac{k}{\beta^2} \qquad (109)$$

In the present unsteady analysis similar correction terms involving only the steady thickness distribution are introduced; no undue complications arise since the thickness distribution, and hence the corrections, are independent of time.

In the series solution in the transformed variable θ, given by Eq.(64), outlined in Section 4, the singularities in $\bar{l}_1(\theta)$ appear only in the functions $\Gamma_n(\theta)$, given by Eq.(69), as $\cot\frac{\theta}{2}$. On further examination of Eq.(69) it is seen that $\cot\frac{\theta}{2}$, the leading edge singularity, appears only in the $\Gamma_0(\theta)$ and $\Gamma_1(\theta)$ terms. The chordwise loading distribution $\bar{l}_1(\theta)$ consists in general of an infinite series in $\Gamma_n(\theta)$ but it is argued that no significant error will result if the leading edge correction is applied only to $\Gamma_0(\theta)$ and $\Gamma_1(\theta)$, the first two terms of the series. Proceeding in a similar manner to the steady problem outlined in Ref.(1), a non-singular approximation to $\bar{l}_1(\theta)$ for the linear problem is given by Eq.(68) with

$$\Gamma_0(\theta) = 2\{c(\tfrac{\Omega}{2}) \frac{\cot\frac{\Omega}{2}}{\left[1 + \left(\frac{\beta^2 \bar{Z}_T'(\bar{x}(\theta))}{k}\right)^2\right]^{\frac{1}{2}}} + i\frac{\Omega}{2}\sin\theta\}$$

$$\Gamma_1(\theta) = -2\sin\theta + \frac{\cot\frac{\theta}{2}}{\left[1 + \left(\frac{\beta^2 \bar{Z}_T'(\bar{x}(\theta))}{k}\right)^2\right]^{\frac{1}{2}}} + i\frac{\Omega}{2}\left(\sin\theta + \frac{\sin 2\theta}{2}\right)$$

$$\Gamma_n(\theta) = -2\sin(n\theta) + i\frac{\Omega}{2}\frac{\sin(n+1)\theta}{(n+1)} \tag{110}$$

Associated with this modified form of the loading is a modified form of the function $\phi^*_{,\bar{x}_{\ell}}(\bar{x},0)$, given by Eq.(81), which is denoted by $\phi^*_{,\bar{x}_L}(\bar{x},0)$ and can be found by using Eqs.(68, 70, 80, 110).

Another separate issue concerns the form of the linearised boundary conditions in the neighbourhood of the leading edge.

The extended boundary conditions of Eq.(5) include second order terms like

$$\{\frac{\partial z_{ou}(x)}{\partial x} + e^{i\omega t}\frac{\partial z_1(x)}{\partial x}\}u(x,+0,t) - \frac{\partial w(x,+0,t)}{\partial z}\{z_{ou}(x) + e^{i\omega t}z_1(x)\}$$

In linearised theory it has been noted that the loading distribution, and hence the perturbation velocity u(x,+0,t) has a leading edge singularity and it is possible that at least some of the non-linear terms in Eq.(5) become significant in the region of the leading edge. In order to evolve a more accurate solution the magnitude of the second order terms in the boundary conditions should be investigated. Considering only the case of a symmetric aerofoil at zero incidence the boundary condition for the fundamental response can be separated from Eq.(5) and expressed in the transformed variables of Eq.(10); thus

$$\bar{w}_1(\bar{x},\pm 0) = \bar{\phi}_{1\bar{z}}(\bar{x},\pm 0) = \left[\bar{Z}_1'(\bar{x}) + i\nu\bar{Z}_1(\bar{x})\right] e^{-iM_\infty^2\Omega\bar{x}} + \frac{\beta^2}{k}\left(1-\frac{1}{\beta^2}\right)\left[\bar{Z}_T(\bar{x})\bar{\phi}_{1\bar{x}\bar{x}}(\bar{x},\pm 0)\right.$$

$$\left. + \bar{Z}_1(\bar{x})\bar{u}_{0\bar{x}}(\bar{x},\pm 0)\right] + \frac{\beta^2}{k}\left\{\frac{\partial}{\partial\bar{x}}\left[\bar{Z}_T(\bar{x})\bar{\phi}_{1\bar{x}}(\bar{x},\pm 0)\right] + \frac{\partial}{\partial\bar{x}}\left[\bar{Z}_1(\bar{x})\bar{u}_0(\bar{x},\pm 0)\right]\right\}$$

$$+ \frac{\beta^2}{k}\bar{Z}_T(\bar{x})\; K^2\bar{\phi}_1(\bar{x}) \qquad (111)$$

using the conditions that in linearised flow

$$\frac{\partial^2\bar{\phi}_0}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}_0}{\partial\bar{z}^2} = 0$$

$$\frac{\partial^2\bar{\phi}_1}{\partial\bar{x}^2} + \frac{\partial^2\bar{\phi}_1}{\partial\bar{z}^2} + K^2\bar{\phi}_1 = 0 \qquad (112)$$

The corresponding second order boundary condition for steady flow can be found by taking the zero frequency limit of Eq.(111): thus

$$\lim_{\omega\to 0}\bar{w}_1(\bar{x},0) = \bar{Z}_1'(\bar{x}) + \frac{\beta^2}{k}\lim_{\omega\to 0}\left\{\frac{\partial}{\partial\bar{x}}\left[\bar{Z}_T(\bar{x})\;\bar{\phi}_{1\bar{x}}(\bar{x},\pm 0)\right] + \frac{\partial}{\partial\bar{x}}\left[\bar{Z}_1(\bar{x})\bar{u}_0(\bar{x},0)\right]\right.$$

$$\left. + \left(1-\frac{1}{\beta^2}\right)\left[\bar{Z}_T(\bar{x})\bar{\phi}_{1\bar{x}\bar{x}}(\bar{x},0) + \bar{Z}_1(\bar{x})\bar{u}_{0\bar{x}}(\bar{x},0)\right]\right\} \qquad (113)$$

Eq.(113) corresponds to the second order antisymmetric or 'lifting' boundary condition in steady flow.

For the case of a symmetric aerofoil at incidence it was found in Ref.(1) that if in Eq.(113) the term

$$\frac{\beta^2}{k} \lim_{\omega\to 0} \{\frac{\partial}{\partial\bar{x}} \left[\bar{\bar{Z}}_T(\bar{x})\bar{\phi}_{1_{\bar{x}}}(\bar{x},0)\right]\}$$

is retained, the accuracy of the overall solution is greatly improved: thus

$$\lim_{\omega\to 0} \bar{w}_1(\bar{x},0) = \bar{\bar{Z}}_t'(\bar{x}) + \frac{\beta^2}{k} \lim_{\omega\to 0} \{\frac{\partial}{\partial\bar{x}} \left[\bar{\bar{Z}}_T(\bar{x})\ \bar{\phi}_{1_{\bar{x}}}(\bar{x},0)\right]\} \tag{114}$$

In unsteady theory, as has been stated earlier, there is a leading edge singularity in the loading distribution of the order $(\bar{x}^{-\frac{1}{2}})$. In linear theory $\bar{\phi}_{1_{\bar{x}}}(\bar{x},0)$ can be found for a pitching aerofoil on using Eqs.(68, 69): thus

$$\bar{\phi}_{1_{\bar{x}}}(\bar{x},0) = \bar{A}_0\{ (1 - \frac{\pi\Omega}{4})(\frac{1-\bar{x}}{\bar{x}})^{\frac{1}{2}} + i\frac{\Omega}{2} \{\left[(1-M_\infty^2)(1-2\bar{x}_0)\right.$$

$$\left. + 0.577 - M_\infty^2 + \delta + \ln(\frac{\Omega}{4}) \right] (\frac{1-\bar{x}}{\bar{x}})^{\frac{1}{2}} + 2(1-M_\infty^2)\left[\bar{x}(1-\bar{x})\right]^{\frac{1}{2}}\}\} \tag{115}$$

The non-singular terms in Eq.(115) can be neglected for the present purpose and since Eq.(114) yields good results in the steady case it is argued that a more accurate boundary condition than Eq.(15b) is

$$\bar{w}_1(\bar{x},0) = \bar{\phi}_{1_{\bar{z}}}(\bar{x},0) = \left[\bar{Z}_t'(\bar{x}) + i\nu\bar{Z}_t(\bar{x})\right] e^{-iM_\infty^2\Omega\bar{x}}$$

$$+ \bar{A}_0 \frac{\beta^2}{k} \{ (1 - \frac{\pi\Omega}{4}) + i\frac{\Omega}{2} \left[(1-M_\infty^2)(1-2\bar{x}_0)\right.$$

$$\left. + 0.577 - M_\infty^2 + \delta + \ln(\frac{\Omega}{4})\right] \frac{d}{d\bar{x}} \left[\bar{\bar{Z}}_T(\bar{x})(\frac{1-\bar{x}}{\bar{x}})^{\frac{1}{2}}\right] \} \tag{116}$$

7. APPLICATIONS

Calculations have been performed for a NACA aerofoil oscillating in pitch about a point x_0 on the aerofoil chord at a freestream Mach number, $M_\infty = 0.63$; only the response at the fundamental mode is considered.

The method of calculation is to rewrite Eq.(91) in the form

$$2\pi\,\bar{w}_1(\bar{x},0) + \frac{I_{c_1}(\bar{x},\bar{u}_{0T}(\bar{x}),\phi^*_{1_{\bar{x}}}(\bar{x},0))}{4} = \int_0^\infty \frac{1}{(\bar{x}-\bar{\xi})}\frac{\partial}{\partial\bar{\xi}}\{\Delta\bar{\phi}_1(\bar{\xi})\tfrac{\pi}{2}\,K|\bar{x}-\bar{\xi}|\,iH_1^{(2)}(K|\bar{x}-\bar{\xi}|)\}d\bar{\xi} \tag{117}$$

where the left hand side is taken for small ω. The function $I_{c_1}(\bar{x},\bar{u}_{0T}(\bar{x}),\ \phi^*_{1_{\bar{x}}}(\bar{x},0)$ expressed by Eq.(107) is evaluated with the non-linear function $\bar{u}_{0T}(\bar{x})$ already determined from the steady case, and assuming

$$\phi^*_{1_{\bar{x}}}(\bar{x},0) = \phi^*_{1_{\bar{x}L}}(\bar{x},0)$$

that is by assuming that $\phi^*_{1_{\bar{x}}}(\bar{x},0)$ is given by the linearised value in Eq.(81) modified for the leading edge singularity as described in Section (6).

Thus $I_{c_1}(\bar{x},\bar{u}_{0T}(\bar{x},),\ \phi^*_{1_{\bar{x}L}}(\bar{x},0)$ is regarded as a known function which modifies the downwash function $\bar{w}_1(\bar{x},0)$; the overall problem then reduces to the linearised form discussed in Section (4). The solution in Section (4) is obtained by expressing the left-hand side of Eq.(117) as a cosine series; once the coefficients of the series are established the derivatives l_α, $l_{\dot{\alpha}}$, m_α, $m_{\dot{\alpha}}$ can be obtained from Eqs.(73, 74, 75).

The variation of the flutter derivatives with reduced frequency for the pitching axis at the leading-edge is shown in Figs. 3, 4, 5, 6. The effect of the non-linearity is to increase considerably the magnitude of all the derivatives over the corresponding linear values. The variation of $l_{\dot{\alpha}}$, m_α, $m_{\dot{\alpha}}$ with change in pitching axis is shown in Figs. 7, 8, 9.

The conclusion reached in Section 3.3 states that in the limit of zero frequency the unsteady flow approaches the equivalent steady flow at an increased angle of incidence equal to the pitching amplitude α_0, and by the argument in Section 3.1 it is expected that no great error will result if only the steady solution and the fundamental response are used to estimate the exact flow. Therefore a test of the accuracy of the present method is to calculate, in the limit of zero frequency, only the fundamental solution and compare the values obtained to the more exact steady flow calculation of Ref.(1) for the NACA 0012 aerofoil at $M_\infty = 0.7$ and $\alpha_0 = 0.01$ rads. The respective values are shown in Table 1: the moments are about the leading edge.

Derivative	m_α (about leading edge)	l_α	A/c aft of leading edge
Non-linear steady values (Ref.1) $\alpha_0 = 0.01$ rads.	- 1.29	5.13	0.252
Zero frequency limit (present paper)	- 1.29	4.94	0.262
Standard linear values	- 1.095	4.4	0.25

Table 1

The agreement between the results of Ref.(1) and the zero frequency limit of the present method is excellent for m_α and fairly good for l_α. This good agreement between the two values of m_α is partly due to taking the moments about the leading edge and hence minimising the effect of errors in the pressure distribution near the nose. If moments are taken about the mid-chord, say, there is a discrepancy between the result of Ref.(1) and the zero frequency limit of the present method. A probable source of error is in computing the coefficients of the cosine series Eq.(96). These require considerable accuracy near the extremities of the aerofoil since in this region

$$\cos(n\theta) \sim \underline{O}(1) \quad :$$

errors in the calculation of $I_{c_1}(\bar{x},\bar{u}_{0_T}(\bar{x})$, $\phi^{*}_{1_{\bar{x}}}(\bar{x},0)$ are likely to be large in this region because of the somewhat arbitrary nature of the nose correction terms, Eq.(115).

It is possible that a more accurate solution may be obtained using a collocation technique but this procedure is not discussed further in this paper; it is intended to direct further effort into this type of solution.

8. CONCLUDING REMARKS

An integral equation method previously developed for steady high subsonic flows[1] has been extended to include oscillatory flows. The non-linearity of the problem gives rise to higher harmonic terms and the solution is formulated by two infinite sets of equations, each pair giving the symmetric and antisymmetric components of one harmonic. Each of this pair of equations involves not only the harmonic in question but a non-linear combination of all preceding harmonics. It is indicated that the higher harmonics are small for small amplitudes of oscillation. A solution is obtained for low values of the reduced frequency by means of a cosine expansion and although this procedure is probably less accurate than a collocation technique the results obtained give the non-linear effects to a reasonable degree of accuracy.

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APPENDIX

The integral $I(\bar{x})$ defined in Eq.(60) can be expressed in the form

$$2\pi I(\bar{x}) = \int_0^1 \frac{1}{(\bar{x}-\bar{\xi})} \cdot \frac{\partial}{\partial\bar{\xi}}\Big[\Delta\bar{\phi}_1(\bar{\xi})\ \chi(K|\bar{x}-\bar{\xi}|)\Big]d\bar{\xi}$$

$$+ \int_1^\infty \frac{\Delta\bar{\phi}_1(1)}{(\bar{x}-\bar{\xi})}\ \frac{\partial}{\partial\bar{\xi}}\Big[-\chi(K|\bar{x}-\bar{\xi}|)\ e^{-i\Omega(\bar{\xi}-1)}\Big]d\bar{\xi} \qquad 1.1$$

since $\Delta\bar{\phi}_1(\bar{\xi}) = \Delta\bar{\phi}_1(1)\ e^{-i\Omega(\bar{\xi}-1)}$ in the wake.

When $K \to 0$

$$\chi(K|\bar{x}-\bar{\xi}|) = \frac{K^2}{2}\ (\bar{x}-\bar{\xi})^2\ \ln\left(\frac{K}{2}\right) + \underline{O}\ (K^2) \qquad 1.2$$

and so the first integral on the right hand side of Eq.(1.1) is $\underline{O}(K^2 \ln K)$ and can thus be neglected in the present analysis. The second integral, $I_2(\bar{x})$ say, can be written as

$$I_2(\bar{x}) = -\ \Delta\bar{\phi}_1(1)\ e^{-i\Omega(\bar{x}-1)}Q - \int_{\bar{x}}^1 \frac{\Delta\bar{\phi}_1(1)}{(\bar{x}-\bar{\xi})} \cdot \frac{\partial}{\partial\bar{\xi}}\Big[\chi(K|\bar{x}-\bar{\xi}|)\ e^{-i\Omega(\bar{\xi}-1)}\Big]d\bar{\xi} \qquad 1.3$$

where

$$Q = -\int_{\bar{x}}^\infty \frac{1}{(\bar{x}-\bar{\xi})}\ \frac{\partial}{\partial\bar{\xi}}\Big[\chi(K|\bar{x}-\bar{\xi}|)\ e^{-i\Omega(\bar{\xi}-\bar{x})}\Big]d\bar{\xi} \qquad 1.4$$

or

$$Q = \int_0^\infty \frac{\Omega}{y}\frac{\partial}{\partial y}\ \chi(M_\infty y)\ e^{-iy}\ dy \qquad 1.5$$

and

$$y = \frac{(\bar{x}-\bar{\xi})}{\Omega}$$

By virtue of Eq.(1.2) the second term in Eq.(1.3) is $\underline{O}(K^2 \ln K)$ and can be neglected in the present approximation.

Consider now the limiting form of $Q(\varepsilon)$ where

$$Q(\varepsilon) = \int_\varepsilon^\infty \frac{\Omega}{y} \cdot \frac{\partial}{\partial y}\Big[\chi(M_\infty y)\ e^{-iy}\Big]dy \qquad 1.6$$

The function χ is defined by Eq.(57) and may be expressed as

$$\chi(M_\infty y) = 1 + \frac{\pi}{2} M_\infty y \left[Y_1(M_\infty y) + i\, J_1(M_\infty y) \right] \qquad 1.7$$

On using Eq.(1.7) it may be shown that

$$\frac{Q(\varepsilon)}{\Omega} = \int_\varepsilon^\infty \{ \frac{\pi}{2} i \frac{\partial}{\partial y} \left[e^{-iy} (Y_0(M_\infty y) + i\, J_0(M_\infty y)) \right]$$
$$- \frac{\pi}{2} (1-M_\infty^2)^{\frac{1}{2}} (Y_0(M_\infty y) + i\, J_0(M_\infty y))\, e^{-iy} - i \frac{e^{-iy}}{y} \}\, dy \qquad 1.8$$

Now

$$\int_\varepsilon^\infty \frac{e^{-iy}}{y}\, dy = - 0.577 - \ln\varepsilon - i \frac{\pi}{2} + \underline{O}(\varepsilon) \qquad 1.9$$

where 0.577 is Eulers constant, and

$$\frac{\pi}{2} i \left\{ \left[Y_0(M_\infty y) + i\, J_0(M_\infty y) \right] e^{-iy} \right\}_\varepsilon^\infty$$

$$= - \frac{\pi i}{2} e^{-i\varepsilon} \left[Y_0(M_\infty \varepsilon) + i\, J_0(M_\infty \varepsilon) \right]$$

$$= - i \left[0.577 + \ln \frac{M_\infty \varepsilon}{2} + \frac{i\pi}{2} \right] + \underline{O}(\varepsilon) \qquad 1.10$$

It is proved in Ref.(22) that

$$\int_0^\infty \left[Y_0(M_\infty y) + i\, J_0(M_\infty y) \right] e^{-iy}\, dy = \frac{-2i}{\pi (1-M_\infty^2)^{\frac{1}{2}}} \ln \left| \frac{1-(1-M_\infty^2)^{\frac{1}{2}}}{M_\infty} \right| \qquad 1.11$$

Hence as $\varepsilon \to 0$

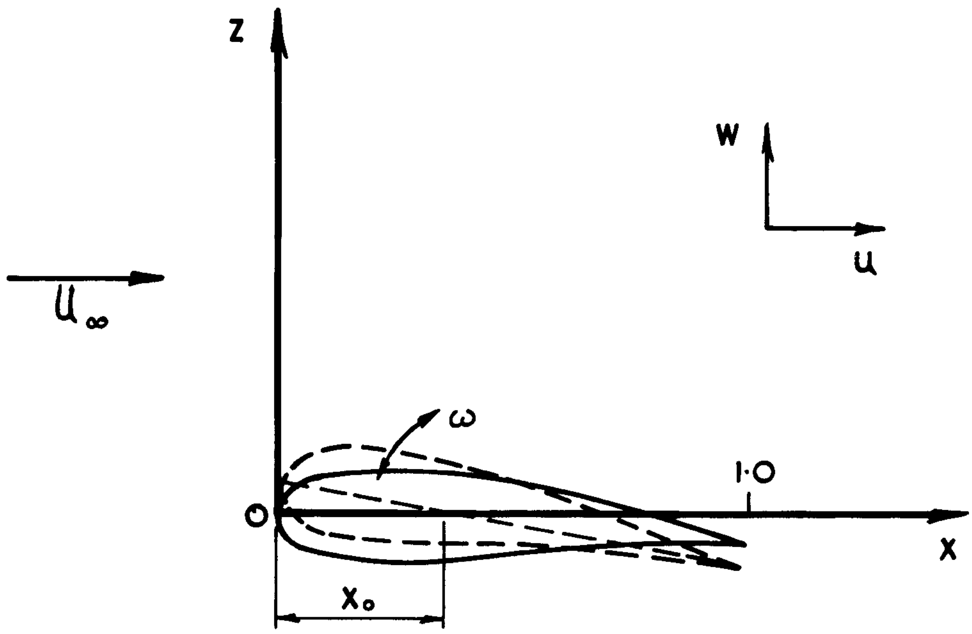
$$\frac{Q(\varepsilon)}{\Omega} \to i \left[(1-M_\infty^2)^{\frac{1}{2}} \ln \left| \frac{1-(1-M_\infty^2)^{\frac{1}{2}}}{M_\infty} \right| - \ln\left(\frac{M_\infty}{2}\right) \right]$$

and it follows from Eqs.(1.1, 1.3) that

$$2\pi I(\bar{x}) = i\Omega\delta\Delta\bar{\phi}_1(1) + \underline{O}(K^2 \ln K) \qquad 1.12$$

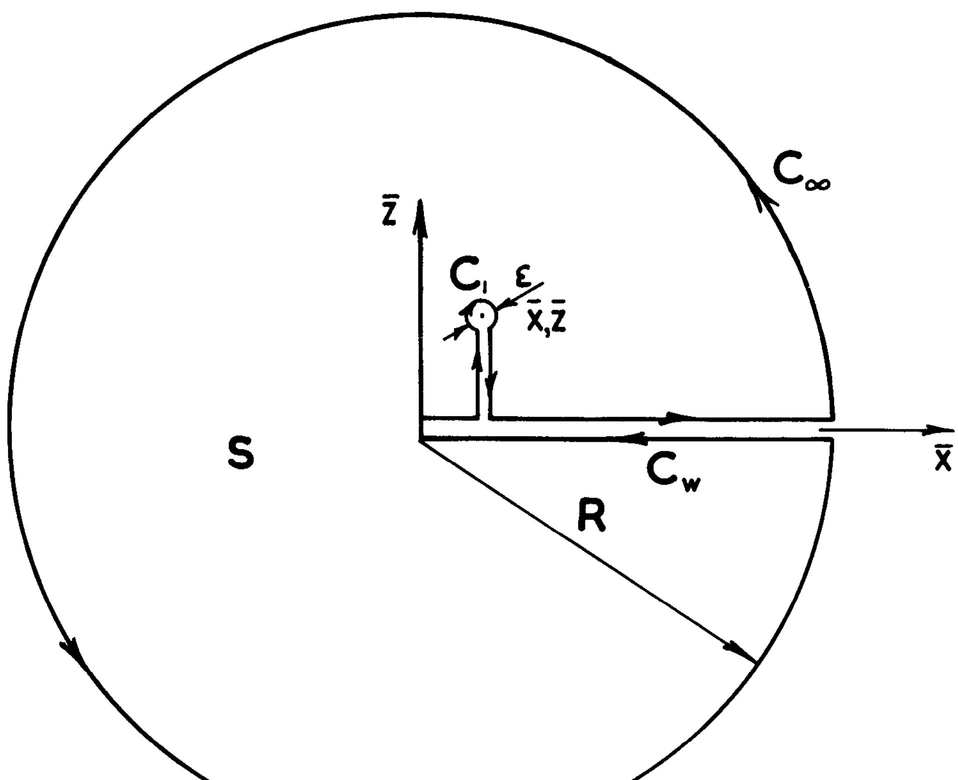
where

$$\delta = \ln\left(\frac{M_\infty}{2}\right) + (1-M_\infty^2)^{\frac{1}{2}} \ln \left| \frac{1-(1-M_\infty^2)^{\frac{1}{2}}}{M_\infty} \right|$$



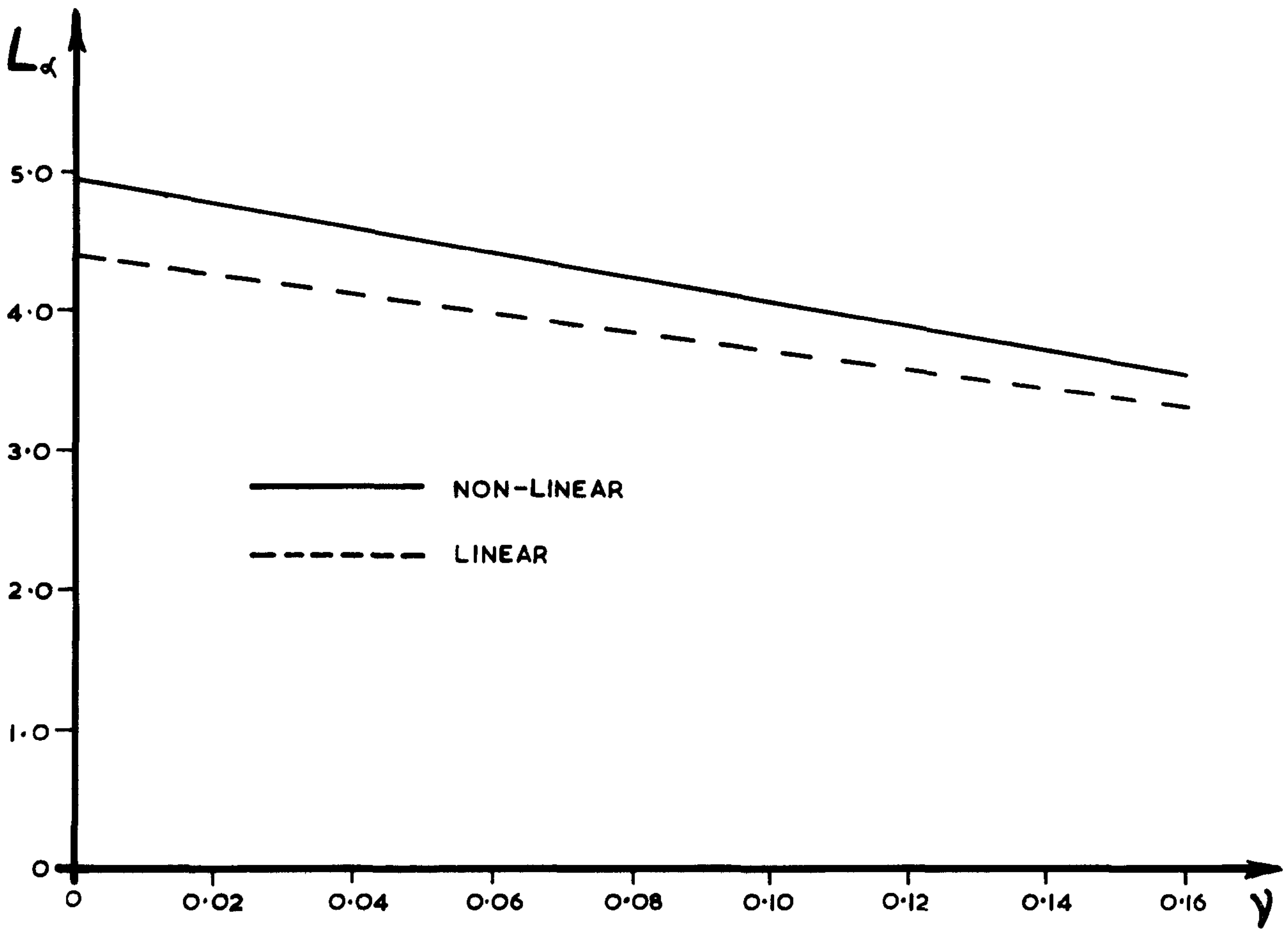
Co-ordinate system

Fig. 1



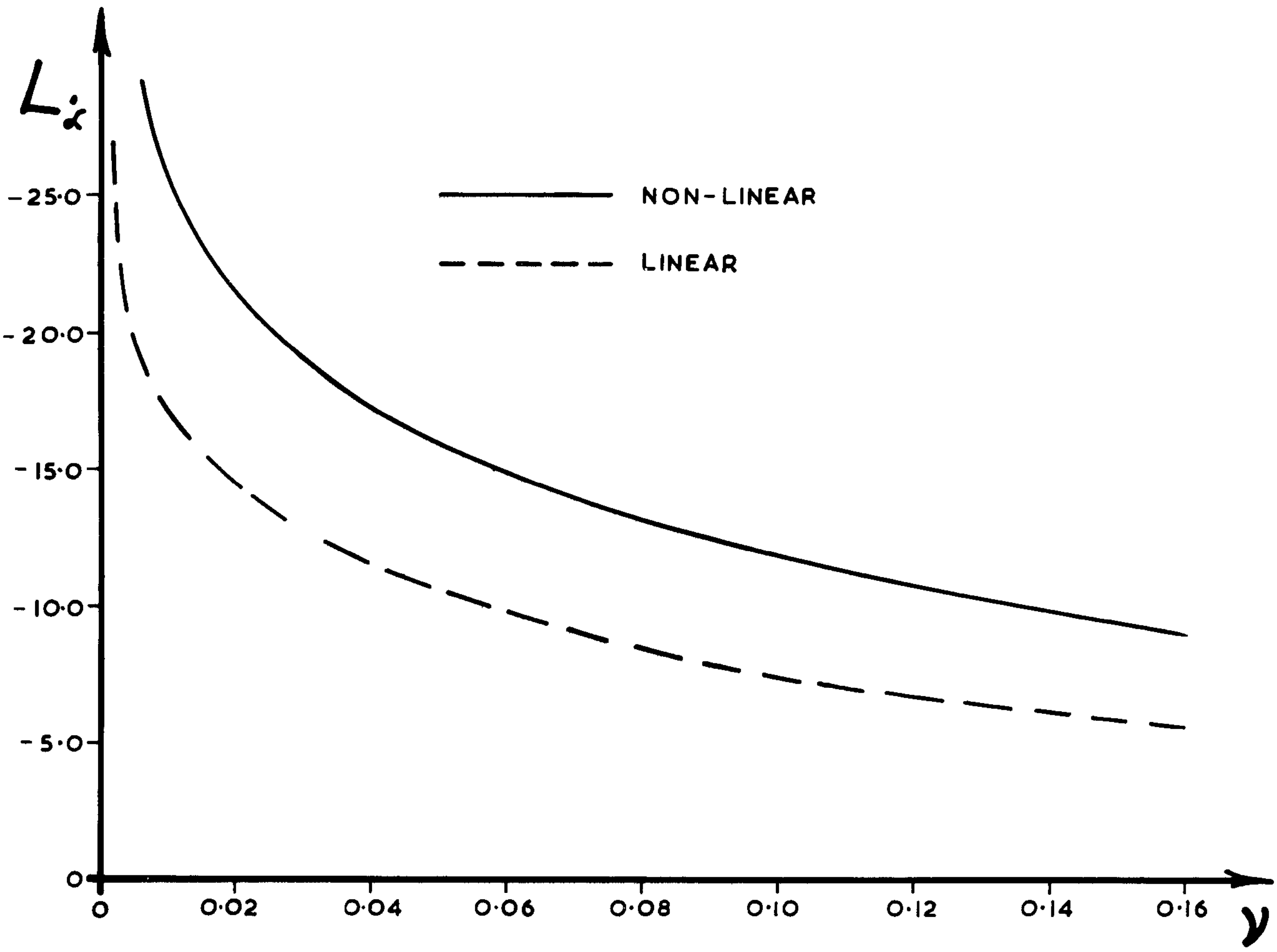
Domain of integration for Green's theorem

Fig. 2



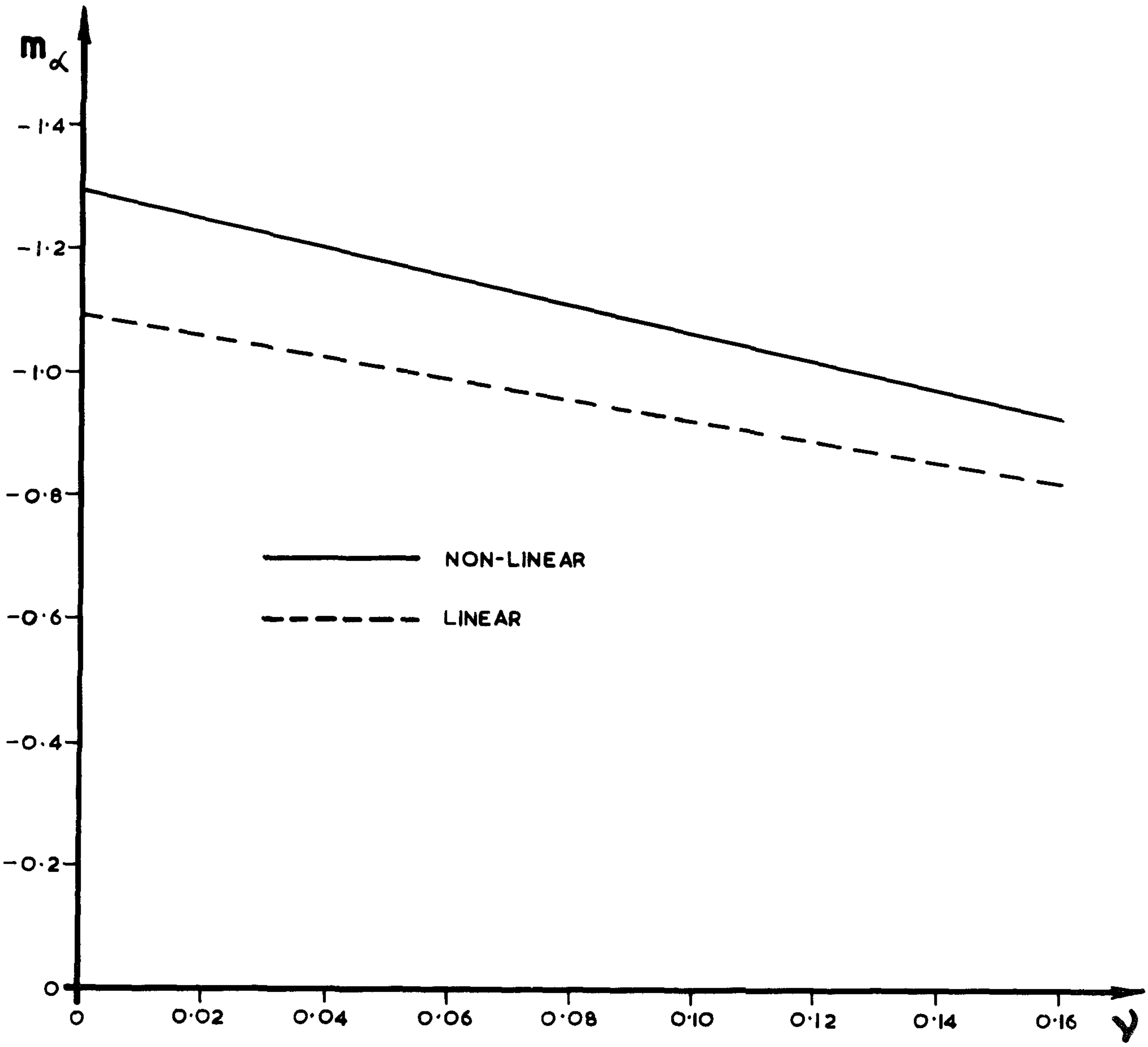
VARIATION OF L_α WITH ν ; NACA 0012 , $M_\infty = 0{\cdot}7$, $X_0 = 0{\cdot}0$

Fig. 3



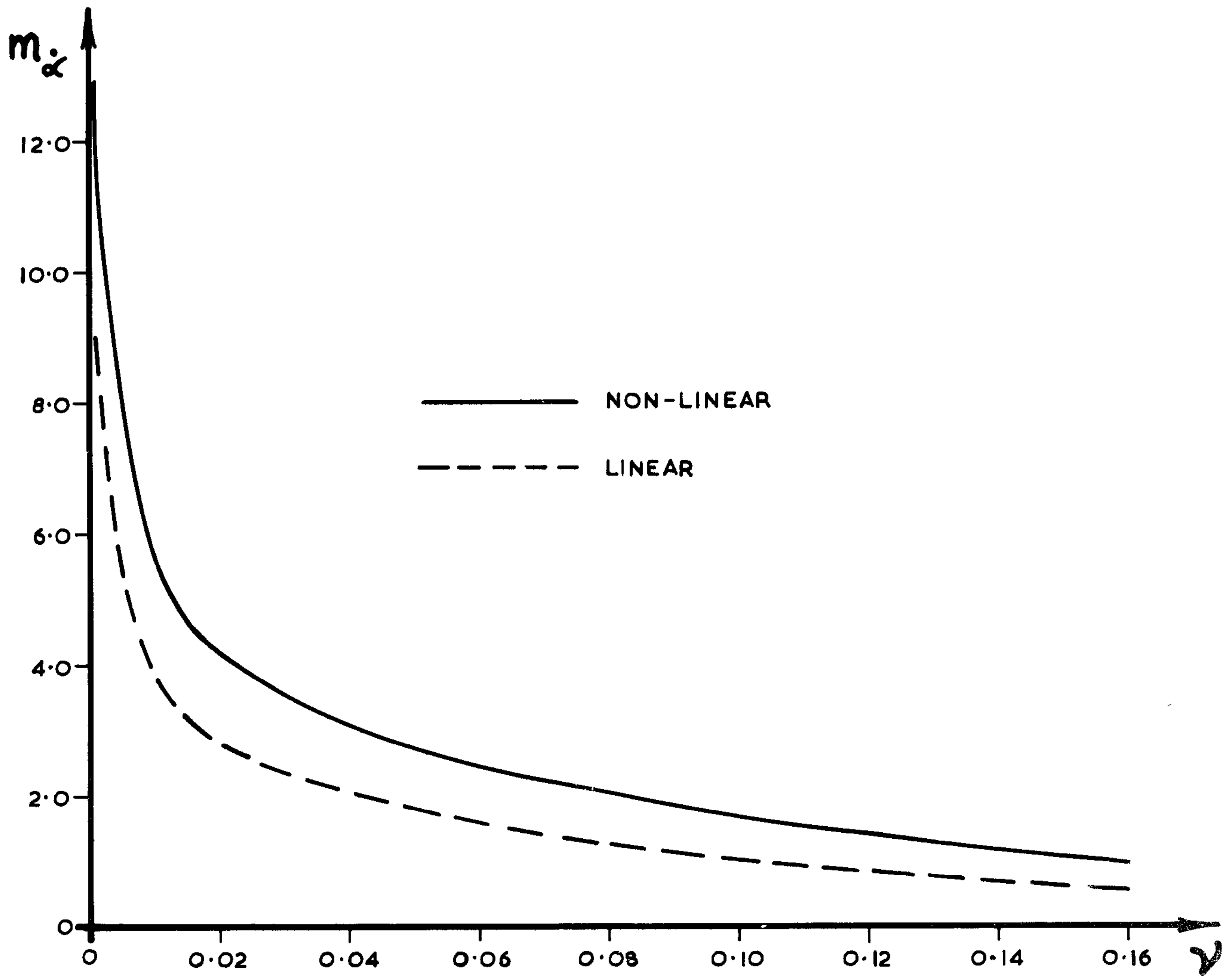
VARIATION OF $L_{\dot{\alpha}}$ WITH ν ; NACA 0012, $M_\infty = 0{\cdot}7$ $X_0 = 0{\cdot}0$

Fig. 4



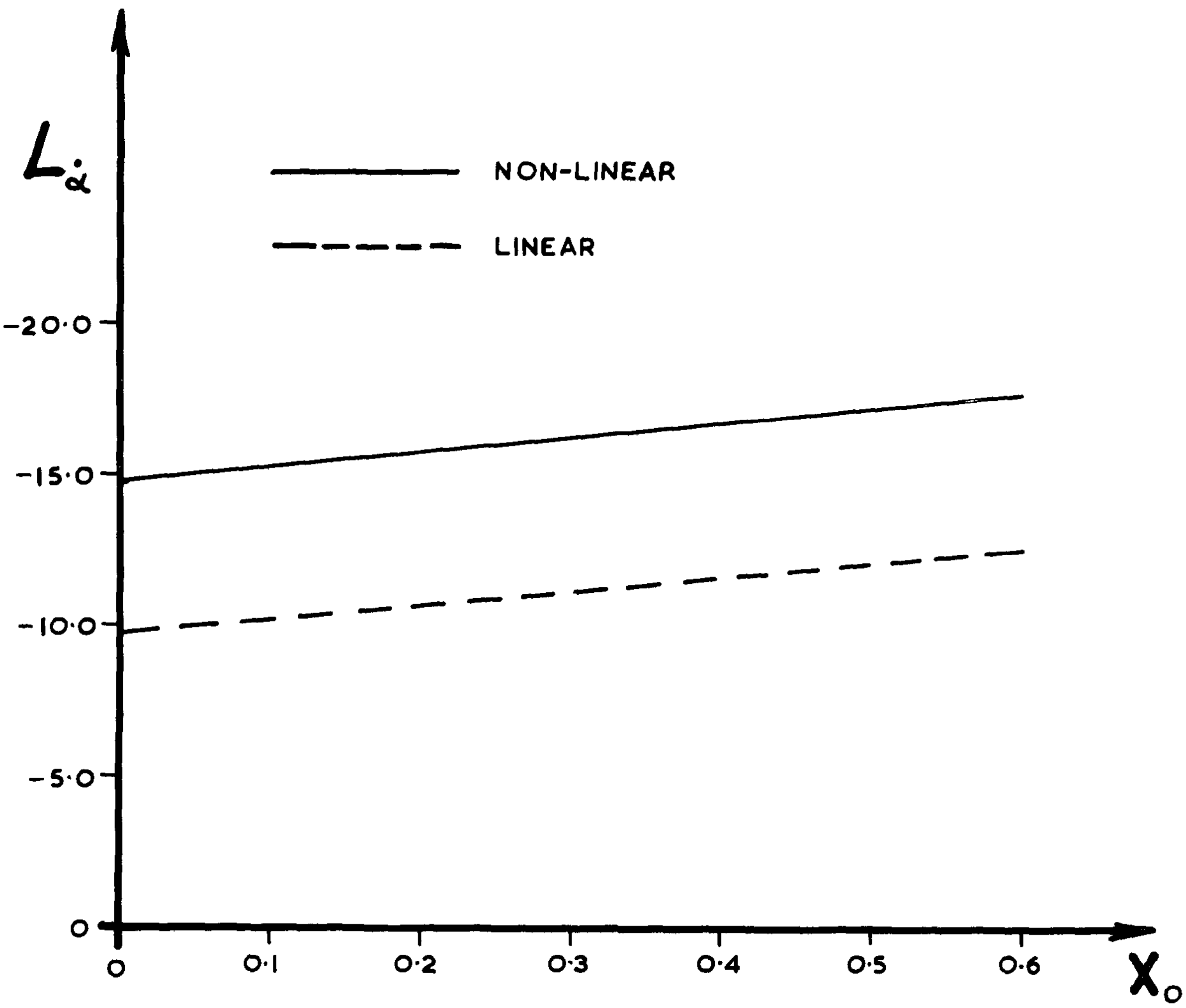
VARIATION OF m_α WITH ν; NACA 0012, M_∞=0·7, X_0=0·0

Fig. 5



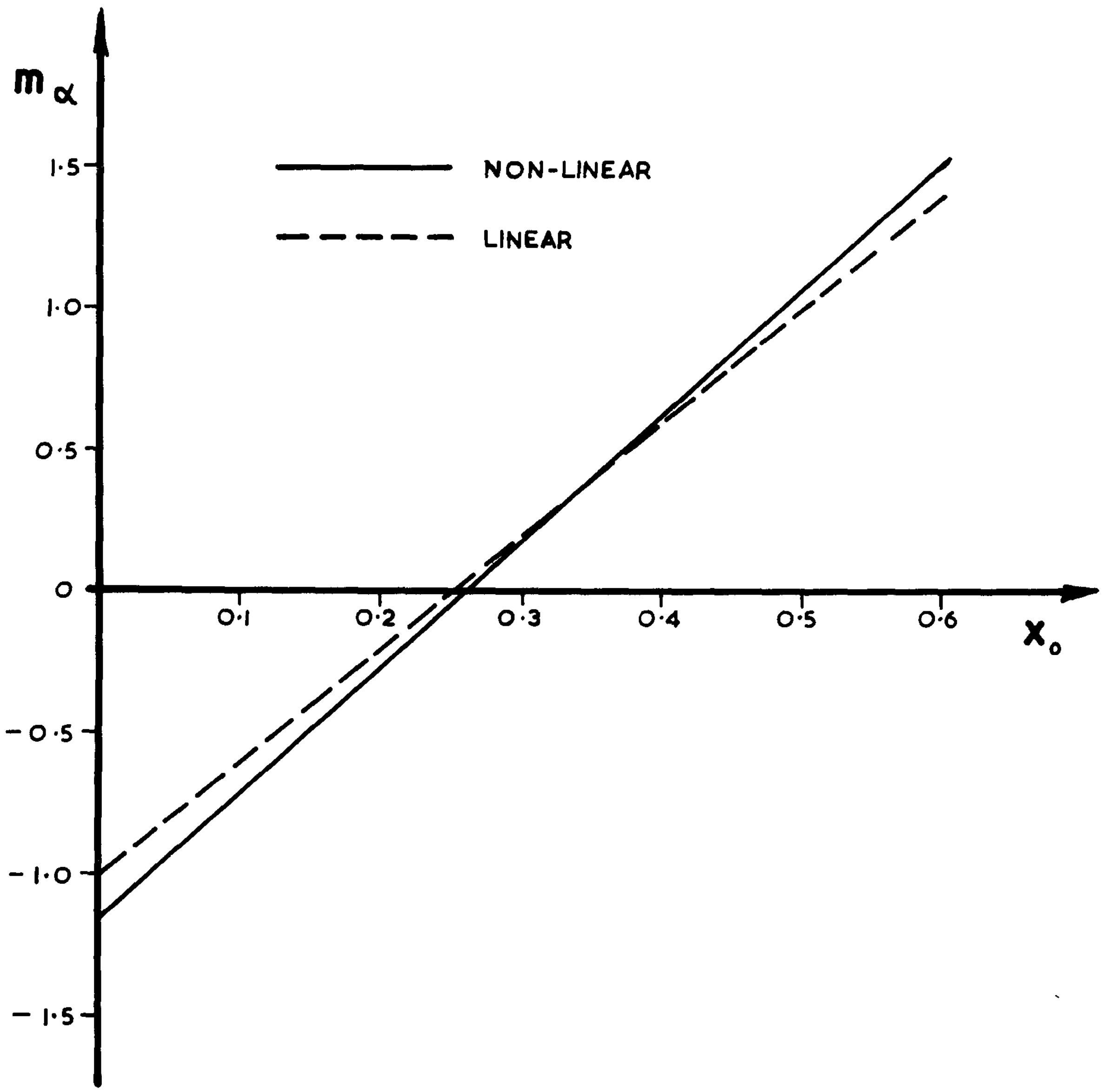
VARIATION OF $m_{\dot{\alpha}}$ WITH ν; NACA 0012, $M_{\infty}=0{\cdot}7$, $X_o=0{\cdot}0$

Fig. 6



VARIATION OF $L_{\dot{\alpha}}$ WITH X_o ; NACA 0012 , M_∞=0·7 ν=0·06

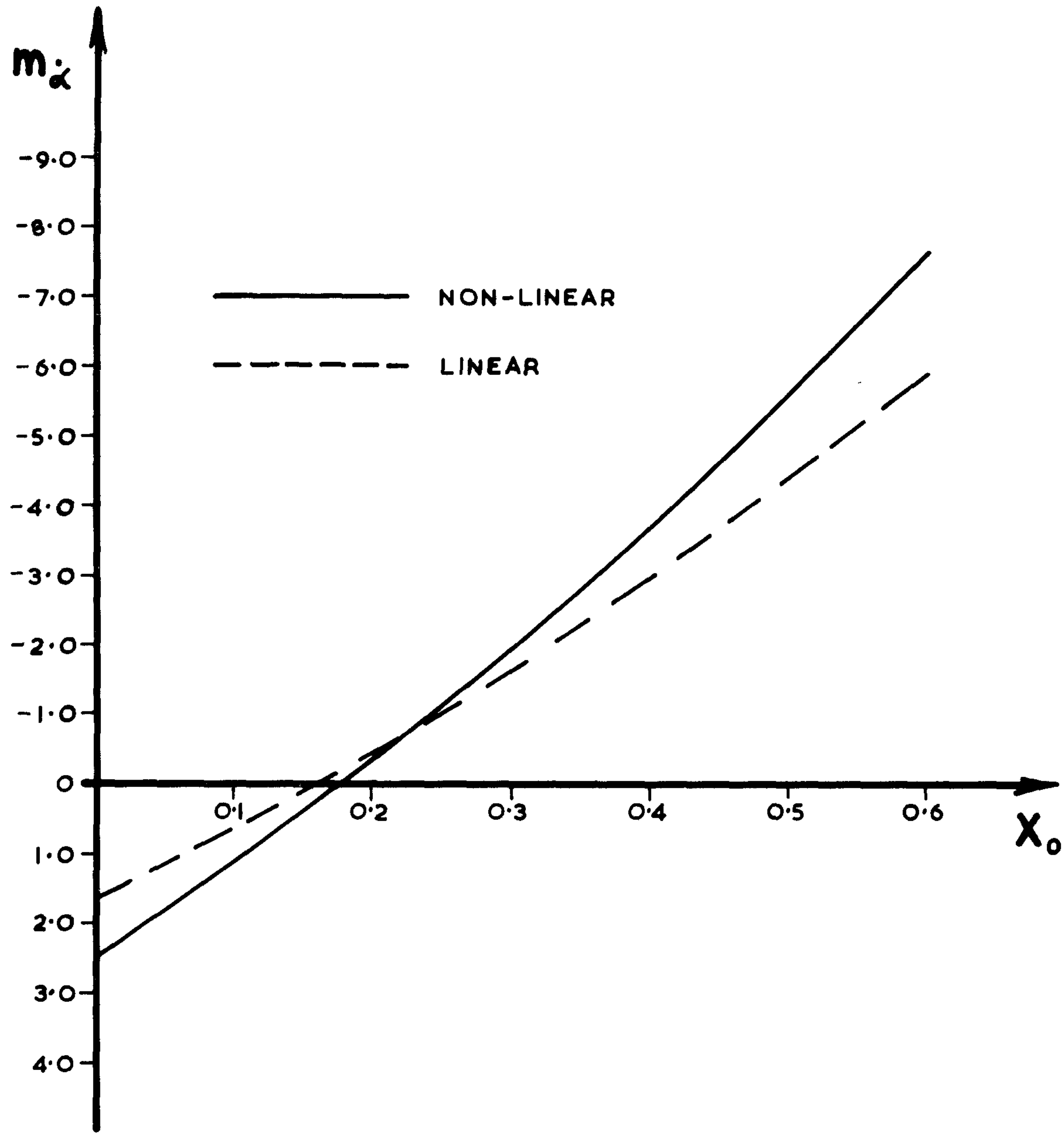
Fig. 7



VARIATION OF m_α WITH X_0 ; NACA 0012

$M_\infty = 0{\cdot}7$, $\nu = 0{\cdot}06$

Fig. 8



VARIATION OF $m_{\dot{\alpha}}$ WITH X_0; NACA 0012, M_∞=0·7
ν =0·06

Fig. 9

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A TWO-DIMENSIONAL AEROFOIL OSCILLATING AT LOW FREQUENCIES IN HIGH SUBSONIC FLOW

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