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An Approximate Method of Deriving the Transient Response of a Linear System from the Frequency Response

By

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SUMMARY

There are several published methods of deriving the transient response of a linear system from the known frequency response, but these are all rather lengthy. Another method is described, which is much more rapid, although less accurate. It is based on the calculation of the response of the system to a square wave as expressed by a Fourier Series.

For any system there is an optimum square wave frequency, and the process of selection of this fundamental frequency is described. It is shown that consideration of responses up to the eleventh harmonic only can give transient response curves which are in error by less than 2%. A description is given of a circular computor which speeds the calculations, and two tables of values are included for use with the computor.

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1 Introduction

The analysis and design of a servo system, based on the steady state response to a sinusoidal input, is called the Frequency Approach, and the characteristics of the system are most usually portrayed by curves of gain in decibels against log-frequency, and phase change in degrees against log-frequency. These curves can be drawn either from data obtained experimentally, or from the expression for the transfer function of the system, for example by using an asymptote approximation method (Ref.1). By examination of these two curves, examples of which are shown in Fig.3, a considerable amount of information may be deduced about the response characteristics, e.g. damping ratio and undamped natural frequency, and about the stability of the system.

Alternatively, the Transient Approach to the analysis of a system is based on its response, expressed in terms of output or error, when the system is subjected to a step or impulse function. The result is portrayed by curves of output or error against time, and from such curves details concerning the initial rate of response, overshoot, damping etc. can be obtained.

Whilst the major part of the design of a servo system can be carried out by using relationships based upon the frequency response, it is often important that the transient response of the proposed system be known during the design period. Consequently it would be of great assistance to have a method of calculating the transient response of a system being given its frequency response. Such a method could be used to obtain a transient response from gain and phase curves obtained either by plotting from the transfer function (without the need for solving polynomials) or from experimental results.

Several such methods have already been developed, apart from the exact mathematical one which requires analytical knowledge of the transfer function of the system (Ref.2). If the real part of the frequency response is plotted against the angular frequency, the response of the system to a unit impulse may be derived by a graphical integration (Ref.3); or if the real part of the frequency response divided by the angular frequency is plotted against the angular frequency, a harmonic analyser can be used to obtain the response to a step input (Ref.4). Another method uses special charts for the summation of the terms of a Fourier Series to give the response to a square wave input which is of fixed frequency (Ref.5).

All of these are rather tedious procedures, and for some time now a more rapid if less accurate method has been sought. It is thought that the present method is of some merit because of its speed, simplicity and flexibility, combined with reasonable accuracy.

2 Square Wave Response

A clue to a rapid method of performing the transition from frequency to transient response is given by the nature of the frequency response curves. These curves express the attenuation and change of phase suffered by each single-frequency signal, or by each component frequency of a complex signal, as it passes through the system, so that by expressing an input signal as a series of single-frequency components, the system output can be calculated.

For the normal transient response curves, the input is a step function, which unfortunately cannot be expressed by a Fourier Series. But a square wave, which can be represented by such a series, can also be regarded as a series of step functions, provided that the half-period of the wave is sufficiently long that the system output has settled to a stationary value before the input changes to its other stationary value. In other words, the response to each half-wave of a square wave signal will closely represent the transient response of a system which is not too lightly damped.

Now the Fourier Series for a square wave $F_i(t)$ with equal mark-to-space ratio, minimum value zero, maximum value unity and fundamental angular frequency ω_F, can be written as

$$F_i(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin (2n-1)\, \omega_F\, t \tag{1}$$

If this signal is fed into a given servo system, the amplitude and phase of the various components will be changed, and the resultant output wave will be expressible as

$$F_o(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{A_{2n-1}}{2n-1} \sin \left[(2n-1)\, \omega_F\, t + B_{2n-1}\right] \tag{2}$$

where

A_{2n-1} is the gain of the system at an angular frequency $(2n-1)\omega$;

B_{2n-1} is the phase change at that same frequency.

Consequently an approximate transient response curve can be obtained by evaluating equation (2) for values of t from zero to slightly less than a half-period of the fundamental frequency ω_F, using values of A_{2n-1} and B_{2n-1} obtained from the frequency response curves, and plotting the results against t.

3 The Approximate Transient Response

In order to reduce this square wave concept to a practical method of obtaining transient response from frequency response curves, the summation of equation (2) must be curtailed at a reasonably low number of terms, and a suitable fundamental frequency must be selected for the square wave. If the selected square wave frequency is too low, the method becomes rather laborious because a large number of terms of the series have to be summed; on the other hand, if the selected frequency is too high, accuracy may be sacrificed because of possible violation of the restriction that the system transient must die out during each half cycle. The method of selecting the square wave frequency will be outlined for various types of frequency response.

It has been found by experience that it is possible to achieve a satisfactory compromise between labour and accuracy by considering up to the eleventh harmonic of the square wave frequency; that is, considering six terms of the summation of equation (2). On this basis, a suitable fundamental frequency ω_F for the square wave is approximately one fifth of the lowest undamped natural frequency of the system.

Because the systems for which this method was primarily developed have a predominant quadratic factor in the transfer function, it is appropriate to consider firstly a system with a quadratic frequency transfer function. An example is

$$G(j\omega) = \frac{1}{(\omega_n^2 - \omega^2) + 2\, j\omega\, \zeta\, \omega_n} \qquad (3)$$

where $\omega_n/2\pi$ is the natural frequency and ζ the damping ratio. The gain/frequency curve of this system will be asymptotic to two lines, intersecting at an angular frequency ω_n, the lines being the 0 db axis for low values of ω, and a line at -12 db/octave for high values of ω. The transition section will vary in shape according to the value of ζ, tending towards a high peak for low ζ (see Fig.1).

When the gain curve of the system exhibits a definite peak, (as for $\zeta < 0.7$ in Fig 1, and System 1, Fig.3) the peak occurs near the resonant frequency ω_o, and selection of ω_F at one fifth of ω_o has been found to be satisfactory. As the resonant frequency ω_o is less than the natural frequency ω_n by an amount governed by ζ, it is possible to make ω_F slightly greater than $\omega_o/5$ and still have it approximately equal to $\omega_n/5$. Selection of ω_F anywhere within the range from $\omega_o/5$ to $(\omega_o/5 + 20\%)$ produces consistent results, whereas, selection of ω_F as less than $\omega_o/5$ introduces error into the result. The reason for the error is that with a low square wave frequency ω_F the system gain at the eleventh harmonic frequency $11\omega_F$, is not sufficiently low compared to that at the fundamental frequency, so that the series summation of equation (2) has not converged sufficiently. The errors in such a case could be reduced by considering higher harmonics than the eleventh.

If in the system with a quadratic transfer function, ζ is greater than 0.7, there is no peak in the gain curve (Fig.1). The undamped natural frequency in such a case is selected by using the phase curve, since the phase shift at the natural frequency is 90^o. There again, selection of ω_F greater than one fifth of ω_n is to be preferred to a lower value.

For systems having factors like $\dfrac{1}{1 + j\omega\tau}$ in the frequency transfer function, the gain/frequency curve will be of a different form, and no such direct method of selection of ω_F exists. If however, by neglecting trends in the high-frequency portion of the gain curve, it becomes similar to that for a purely quadratic system, then the method given above may be used to get a first estimate of ω_F. If on the other hand, the gain curve is of a complicated form, then a first estimate of ω_F may be made by selection of the eleventh harmonic at a frequency such that the gain has fallen by 15 db below that at very low frequency. In these cases the calculation is carried through, as described later, with this estimated ω_F, and the maximum contribution of the 11th harmonic to the final result is noted. If this contribution is approximately 2% or less of the maximum contribution of the fundamental, then the result should be accurate within the capabilities of this method. If it is more than 2% then the estimate of ω_F is unsatisfactory, and the calculation should be repeated with a revised higher value of ω_F (or the calculation can be carried on to the 13th harmonic). On the other hand, if the maximum contribution of the 7th harmonic to the final result is less than 2% of that of the fundamental, and the 9th and 11th harmonics contribute negligible amounts, the value of ω_F as originally estimated was too high, and the calculation should be repeated with a lower value of ω_F.

The selection of the fundamental frequency of the square wave, ω_F, as outlined above is all based on obtaining a frequency which is appropriate to curtailing the summation of terms of equation (2) after considering six terms, i.e. after the 11th harmonic.

Having thus selected the square wave frequency, the appropriate values of A_{2n-1} and B_{2n-1} are read from the frequency response curves of the system, and the evaluation of equation (2) carried out for 6 terms for a suitable number of values of t. To facilitate this calculation, tables of values and a computor, as described in Section 4, have been developed.

As a check on the accuracy of this approximate method, it has been applied in a number of cases where there existed an exact transient response curve and corresponding frequency response curves. Once familiarity with the procedure is gained, particularly appreciation of the selection of the ω_F frequency, it was found that the resulting curves were in error by less than 5%, and often less than 2%.

4 Aids to Calculation

To reduce substantially the time taken in evaluating the expression of equation (2), two special tables of values and a circular computor have been developed.

4.1 The $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$ Table - Table I

In the expression for the output $F_o(t)$ of a system subjected to a square wave input, which is repeated here for convenience,

$$F_o(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{A_{2n-1}}{2n-1} \sin\left[(2n-1)\,\omega_F\, t + B_{2n-1}\right] \qquad (2)$$

A_{2n-1} is the absolute gain of the system, and since the gain curve is usually plotted in terms of decibels against log-frequency, a conversion must be carried out. Decibel to power ratio or voltage ratio conversion tables are available, but in this work it has been found convenient to prepare one to give values of the composite term $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$, showing the value of this for various db values, and $n = 1, 2, \ldots\ldots 7$. The values are obtained from the expression

$$\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1} = \frac{2}{\pi} \cdot \frac{1}{2n-1} \text{ antilog}_{10} \frac{A_{2n-1}\,(\text{db})}{20} \qquad (4)$$

and are shown in Table I. In using this table it is not necessary to interpolate between the db values given.

Both this and Table II have been extended to give figures for calculating the 13th harmonic if it proves necessary to obtain greater accuracy, and this is more expedient than repeating the whole calculation with a revised value of ω_F .

4.2 <u>The "φ" Table - Table II</u>

Without some change being made in the form of equation (2), it will be necessary in making each separate evaluation to select suitable values of t . If however a symbol T defined as $1/\omega_o$ is introduced, then equation (2) may be rewritten as

$$F_o(t) = \frac{1}{2} + \frac{2}{\pi} \sum \frac{A_{2n-1}}{2n-1} \sin\left[(2n-1)\frac{\omega_F}{\omega_o} \cdot \frac{t}{T} + B_{2n-1}\right] \qquad (5)$$

In future $(2n-1)\dfrac{\omega_F}{\omega_o} \cdot \dfrac{t}{T}$ will be denoted by φ .

Now as t/T is non-dimensional, a standard set of values for this parameter may be used, and a table could be prepared giving the value of φ for various t/T . But, because the system phase change B_{2n-1} is usually expressed in degrees it has been found expedient to make this table so that φ can be read off in degrees for suitable values of t/T and $n = 1, 2, 3 \ldots\ldots 7$. Since the angular frequency of the fundamental component of the square wave (ω_F) has been selected as $1/5$ of the angular frequency of the peak of the gain curve (ω_o), then $\omega_F/\omega_o = 0.2$, and the "φ - t/T" table gives values of

$$\varphi = (2n-1)\ 0.2 \cdot \frac{t}{T} \cdot \frac{360}{2\pi}$$

$$= (2n-1) \cdot \frac{36}{\pi} \cdot \frac{t}{T} \qquad (6)$$

Further, to facilitate the use of a special circular computor, the values of φ obtained from equation (6) have been reduced by multiples of 360^o, so that Table II gives the equivalent angle in the range 0^o to 360^o.

4.3 <u>The Circular Computor</u>

The final aid to the computation of the transient response curves by this approximate method is a circular computor for rapidly evaluating the expression

$$f_o(t) = \frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1} \sin(\varphi + B_{2n-1}) \qquad (7)$$

This computor consists of a circular base graduated both around the periphery from 0^o to 360^o and across the face from -1.2 through 0 to +1.2 (or -0.6 to 0 to +0.6) as shown in Fig.2. Across this graduated circle swing two arms which can be set with any angle between them and along one of these a cursor can be set at any radius from 1.2 down to 0.1 (or 0.6 to 0.05).

Assume that the computor is temporarily set up as shown in Fig.2a, when OA and OB are the two arms, and the cursor C on OB is set (on the scale along OE) to be equal to $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$ as given by Table I. The angle AOB between the two arms is the phase angle B_{2n-1} corresponding to the particular value of $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$; generally B_{2n-1} will be a negative angle, but the method applies equally well if it is positive. If now OA is set to a value of φ as given by Table II, then the length CD represents $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1} \sin (\varphi + B_{2n-1})$, and its value is obtained from the scale along EOF , with due regard to sign.

Thus if OC is set to the value of $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$ for $n = 1$, and the angle AOB is set to the appropriate B_{2n-1} , then as OA is advanced to the values of φ as shown on the φ - t/T table, the various values of $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1} \sin (\varphi + B_{2n-1})$ may be read off the linear scale. This can be repeated for $n = 2, 3 \ldots\ldots 6$ and the necessary summation carried out to get the points to plot for the transient response. These values can be either plotted against t/T , or should a correct time scale be required, the corresponding values may be obtained, remembering that $T = 1/\omega_o$.

5 Steps in Obtaining an Approximate Transient Response Curve

The various steps for obtaining a transient response curve by this method will now be set out in logical sequence.

(1) Select the most suitable value of ω_F by referring to the given frequency response curves. If the gain/frequency curve is of the same general shape as the $\zeta = 0.1$ and $\zeta = 0.5$ curves of Fig.1, select ω_F so that the angular frequency at the maximum gain is $5\omega_F$. If the gain curve falls off monotonically, as for $\zeta = 1.0$ in Fig.1, select ω_F so that at $5\omega_F$ the phase-shift is -90^o. For more complex gain curves, estimate ω_F so that at $11\omega_F$ the gain of the system has fallen by 15 db below the very low frequency value. Although the selection of ω_F is fairly critical, where the choice is not obvious it is best to err (up to 20%) towards a higher value.

Having selected ω_F , read off from the gain and phase curves the data to complete the following schedule, in which ω_{2n-1} is equivalent to $(2n-1)\,\omega_F$.

n	1	2	3	4	5	6
ω_{2n-1}						
A_{2n-1}						
B_{2n-1}						

(2) Referring to Fig.2, set the cursor C so that OC corresponds on the scale of EOF to the value of $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$ as given by Table I, for $n = 1$ and the gain as shown in the schedule above; it is not necessary to interpolate between the db values given in Table I. Then with OB along the 0° line, set OA on $-B_{2n-1}$; i.e. at an angle equal in magnitude but opposite in sign to the phase change at the frequency given by $n = 1$.

(3) Set OA on the first value of φ from Table II ($n = 1$ column), read off the value of $\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1} \sin(\varphi - B_{2n-1})$ on the appropriate linear scale along EOF , and record it. Repeat this for sufficient values of φ , putting the resulting figures in a tabular form as shown below.

Seconds						
t/T n	0.5	1.0	1.5			
1						
2						
3						
4						
5						
6						
Σ + 0.5						

The "Seconds" row gives the time values corresponding to the t/T , which if required are obtained by using the relation $T = \frac{1}{\omega_o} = \frac{1}{5\omega_F}$.

(4) Repeat the procedure given in steps 2 and 3 for values of $n = 2, 3, 4, 5$ and 6, so completing the body of the table of step 3.

(5) Check the suitability of the value of ω_F used. The maximum value in the $n = 6$ row should be approximately 2% of the maximum value in the $n = 1$ row. If these values are negligible and the maximum value in the $n = 5$ or even $n = 4$ row are about 2% of the maximum in the $n = 1$ row, then the selected ω_F is too high, and the calculation should be repeated with a lower ω_F ; if these values are much greater than 2% then ω_F is too low, and either the calculation should be repeated with a higher ω_F , or the 13th harmonic should be calculated. Should this additional harmonic not produce sufficient accuracy the calculation will have to be repeated with a higher ω_F .

(6) Sum each of the columns of six (or seven) values, and to each add 0.5. This gives figures which when plotted against t/T, or time, give the required transient response curve. The 0.5 is the first term of equation (2).

A more rapid plot of the transient response curve may be made by using only selected values of t/T, to give only the general form of the curve.

6 Typical Examples

Typical frequency response curves for two different systems are shown in Figs.3 and 4. System 1 (Fig.3) is a simple proportional control system with an inertia and friction load, so that it has been possible to compute exactly the transient response of the system from the transfer function which is

$$G(p) = \frac{k}{Jp^2 + fp + k}$$

with $k = 1$, $J = 1$, and $f = 0.6$.

The sets of figures used in the calculation are shown in Fig.4. The values of gain and phase change as read from the curves are set out in Fig.4a, - this is the "schedule" of Step 1, Section 5 - and the figures obtained from the circular computor are set out in Fig.4b. These results are plotted in Fig.7. Superimposed on the curve for System 1, can be seen several additional points, which are the values obtained from the exact solution for the transient response of the system.

System 2 is a practical servo system, the frequency response curves shown in Fig.4 being derived from the transfer function of the system. The calculations are in this case shown in Fig.6, and once again the resulting values are plotted in Fig.7 to give the transient response curve.

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<u>TABLE I</u>

A_{db}	$\log_{10} A = \frac{A_{db}}{20}$	A_{abs}	$\frac{2}{\pi} \cdot \frac{A_{2n-1}}{2n-1}$						
			n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7
+10.0	0.5000	3.162	2.013	0.673	0.403	0.288	0.224	0.183	0.155
+9.0	0.4500	2.818	1.794	0.598	0.359	0.256	0.199	0.163	0.138
+8.0	0.4000	2.512	1.599	0.533	0.320	0.229	0.178	0.145	0.123
+7.0	0.3500	2.239	1.425	0.475	0.285	0.204	0.158	0.130	0.109
+6.0	0.3000	1.995	1.270	0.423	0.254	0.182	0.141	0.116	0.098
+5.0	0.2500	1.778	1.132	0.377	0.226	0.162	0.126	0.103	0.087
+4.5	0.2250	1.679	1.069	0.356	0.214	0.153	0.119	0.097	0.082
+4.0	0.2000	1.585	1.009	0.336	0.202	0.144	0.112	0.092	0.078
+3.5	0.1750	1.496	0.952	0.328	0.191	0.136	0.106	0.087	0.073
+3.0	0.1500	1.413	0.900	0.300	0.180	0.129	0.100	0.082	0.069
+2.5	0.1250	1.334	0.849	0.283	0.170	0.122	0.094	0.077	0.065
+2.0	0.1000	1.259	0.802	0.267	0.160	0.115	0.089	0.073	0.062
+1.5	0.0750	1.189	0.757	0.252	0.151	0.108	0.084	0.069	0.058
+1.0	0.0500	1.122	0.714	0.235	0.143	0.102	0.079	0.065	0.055
+0.75	0.0375	1.090	0.694	0.231	0.139	0.099	0.077	0.063	0.053
+0.5	0.0250	1.059	0.674	0.225	0.135	0.097	0.075	0.061	0.052
+0.25	0.0125	1.029	0.655	0.219	0.131	0.094	0.073	0.060	0.050
0	0.0000	1.000	0.637	0.212	0.127	0.091	0.071	0.058	0.049
-0.25	-1.9875	0.9716	0.619	0.206	0.124	0.088	0.069	0.056	0.048
-0.5	-1.9750	0.9441	0.601	0.200	0.120	0.086	0.067	0.055	0.046
-0.75	-1.9625	0.9173	0.584	0.195	0.117	0.083	0.065	0.053	0.045
-1.0	-1.9500	0.8913	0.567	0.189	0.114	0.081	0.063	0.052	0.044
-1.5	-1.9250	0.8414	0.536	0.179	0.107	0.077	0.060	0.049	0.041
-2.0	-1.9000	0.7943	0.506	0.169	0.101	0.072	0.056	0.046	0.039
-2.5	-1.8750	0.7499	0.477	0.159	0.096	0.068	0.053	0.043	0.037
-3.0	-1.8500	0.7079	0.451	0.150	0.090	0.064	0.050	0.041	0.035
-3.5	-1.8250	0.6683	0.426	0.142	0.085	0.061	0.047	0.039	0.033
-4.0	-1.8000	0.6310	0.402	0.134	0.080	0.058	0.045	0.036	0.031
-4.5	-1.7750	0.5957	0.379	0.126	0.076	0.054	0.042	0.034	0.029
-5.0	-1.7500	0.5623	0.358	0.119	0.072	0.051	0.040	0.033	0.028
-6.0	-1.7000	0.5012	0.319	0.106	0.064	0.046	0.035	0.029	0.025
-7.0	-1.6500	0.4467	0.284	0.095	0.057	0.041	0.032	0.026	0.022
-8.0	-1.6000	0.3981	0.253	0.085	0.051	0.036	0.028	0.023	0.019
-9.0	-1.5500	0.3548	0.226	0.075	0.045	0.032	0.025	0.021	0.017
-10.0	-1.5000	0.3162	0.201	0.067	0.040	0.029	0.022	0.018	0.015
-11	-1.4500	0.2818	0.179	0.060	0.036	0.026	0.020	0.016	0.014
-12	-1.4000	0.2512	0.160	0.053	0.032	0.023	0.018	0.015	0.012
-13	-1.3500	0.2239	0.143	0.048	0.029	0.020	0.016	0.013	0.011
-14	-1.3000	0.1995	0.127	0.042	0.025	0.018	0.014	0.012	0.010
-15	-1.2500	0.1778	0.113	0.038	0.023	0.016	0.013	0.011	0.009
-16	-1.2000	0.1585	0.101	0.034	0.020	0.014	0.011	0.010	0.008
-17	-1.1500	0.1413	0.090	0.030	0.018	0.013	0.010	0.008	0.007
-18	-1.1000	0.1259	0.080	0.027	0.016	0.011	0.009	0.007	0.006
-19	-1.0500	0.1122	0.071	0.024	0.014	0.010	0.008	0.007	0.005
-20	-1.0000	0.1000	0.064	0.021	0.013	0.009	0.007	0.006	0.005

TABLE II

The "$\varphi - \frac{t}{T}$" Table

$\frac{t}{T}$ \ n	1	2	3	4	5	6	7	n / $\frac{t}{T}$
0.50	5.7	17.2	28.6	40.1	51.6	63.0	74.5	0.50
0.75	8.6	25.8	43.0	60.2	77.3	94.5	111.7	0.75
1.00	11.5	34.4	57.3	80.2	103.1	126.1	149.0	1.00
1.25	14.3	43.0	71 6	100.3	128.9	157.6	186.2	1.25
1.50	17.2	51.6	85.9	120.3	154.7	189.1	223.5	1.50
1.75	20.1	60.2	100.3	140.4	180.5	220.6	260.7	1.75
2.00	22.9	68.8	114.6	160.4	206.3	252.1	297.9	2.00
2.25	25.8	77.3	128.9	180.5	232.0	283.6	335.2	2.25
2.50	28.6	85.9	143.2	200.5	257.8	315.1	12.4	2.50
2.75	31.5	94.5	157.6	220.6	283.6	346.6	49.7	2.75
3.00	34.4	103.1	171.9	240.6	309.4	18.2	86.9	3.00
3.25	37.2	111.7	186.2	260.7	335.2	49.7	124.1	3.25
3.50	40.1	120.3	200.5	280.7	1.0	81.2	161.4	3.50
3.75	43.0	128.9	214.9	300.8	26.7	112.7	198.6	3.75
4.00	45.8	137.5	229.2	320.9	52.5	144.2	235.8	4.00
4.25	48.7	146.1	243.5	340.9	78.3	175.7	273.1	4.25
4.50	51.6	154.7	257.8	1.0	104.1	207.2	310.4	4.50
4.75	54.4	163.3	272.2	21.0	129.9	238.7	347.6	4.75
5.00	57.3	171.9	286.5	41.1	155.7	270.3	24.8	5.00
5.50	63.0	189.1	315.1	81.2	207.2	333.3	99.3	5.50
6.00	68.8	206.3	343.8	121.3	258.8	36.3	173.8	6.00
6.50	74.5	223.5	12.4	161.4	310.4	99.3	248.3	6.50
7.00	80.2	240.6	41.1	201.5	1.9	162.4	322.8	7.00
7.50	85.9	257.8	69.7	241.6	53.5	225.4	37.3	7.50
8.00	91.7	275.0	98.4	281.7	105.1	288.4	111.7	8.00
8.50	97.4	292.0	127.0	321.8	156.6	351.4	186.2	8.50
9.00	103.1	309.4	155.7	1.9	208.2	54.5	260.7	9.00
9.50	108.9	326.6	184.3	42.0	259.8	117.5	335.2	9.50
10.0	114.6	343.8	213.0	82.1	311.3	180.5	49.7	10.0
10.5	120.3	1.0	241.6	122.2	2.9	243.5	124.1	10.5
11.0	126.0	18.2	270.3	162.4	54.9	306.6	198.7	11.0
11.5	131.8	35.3	298.9	202.5	106.0	9.6	273.1	11.5
12.0	137.5	52.5	327.5	242.6	157.6	72.6	347.6	12.0
12.5	143.2	69.7	356.2	282.7	209.1	135.6	62.1	12.5
13.0	149.0	86.9	24.8	322.8	260.7	198.7	136.6	13.0
13.5	154.7	104.1	53.5	2.9	312.3	261.7	211.1	13.5
14.0	160.4	121.3	82.1	43.0	3.8	324.7	285.6	14.0
14.5	166.2	138.5	110.8	83.1	55.4	27.7	0	14.5
15.0	171.9	155.7	139.4	123.2	107.0	90.8	74.5	15.0

Values of φ in degrees for various n and $\frac{t}{T}$, calculated from the expression

$$\varphi = (2n - 1) \frac{36}{\pi} \frac{t}{T}.$$

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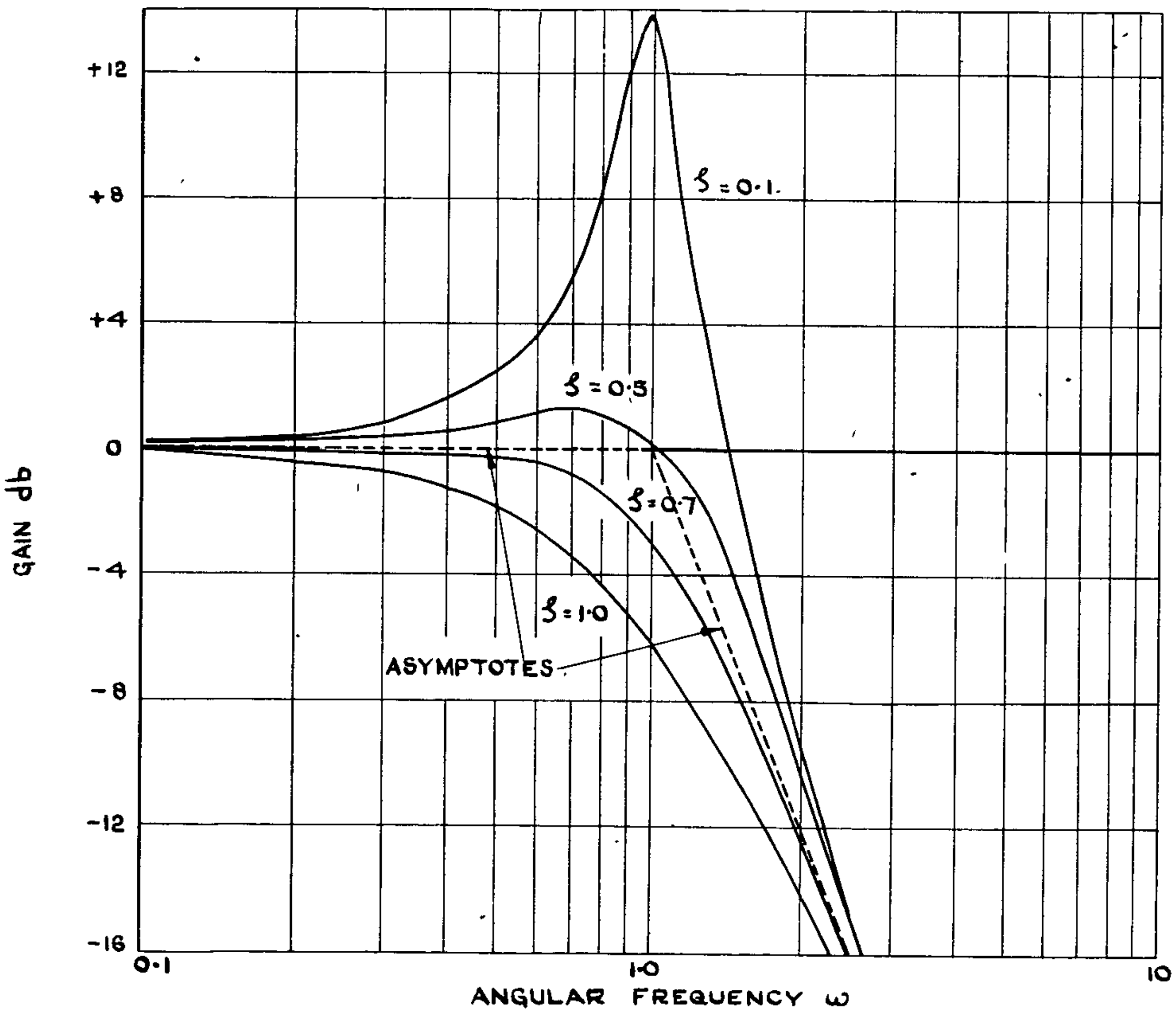

FIG.1. GAIN CURVES. SYSTEM TRANSFER FUNCTION $\dfrac{1}{(\omega_n^2-\omega^2)+2j\omega\zeta\omega_n}$

FIG.1.

FIG.2 &2a

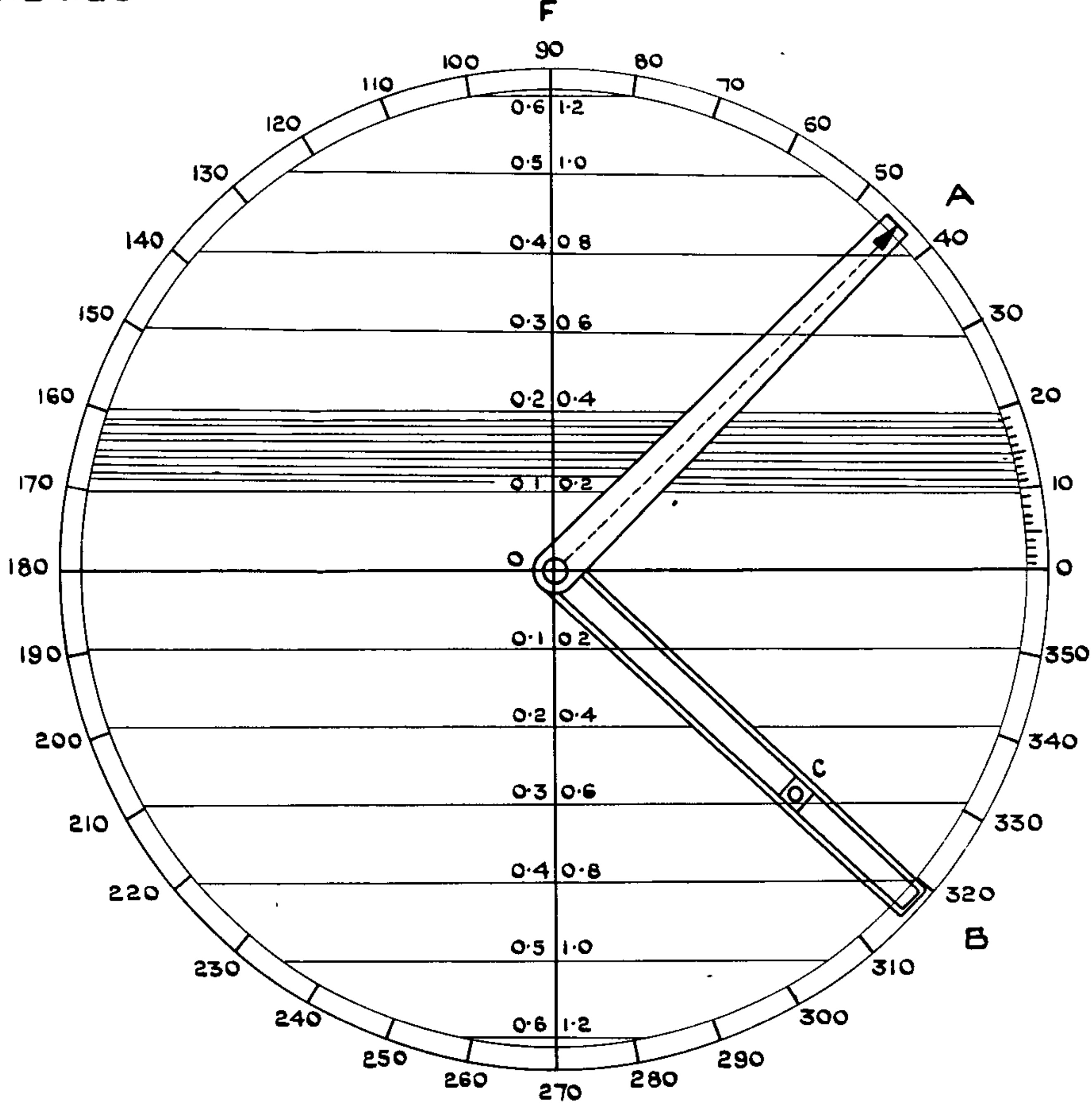

FIG.2.

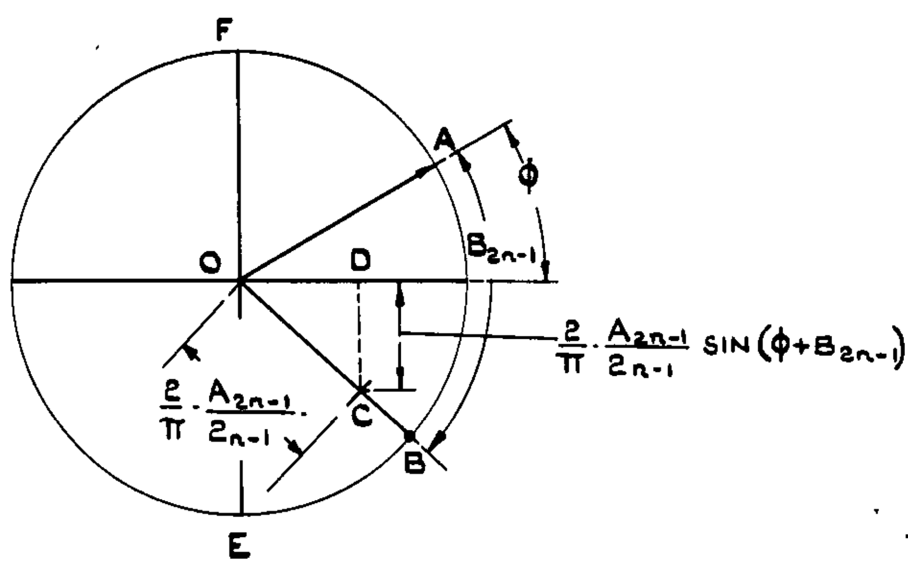

FIG. 2a.

FIG. 2&2a. THE CIRCULAR COMPUTER.

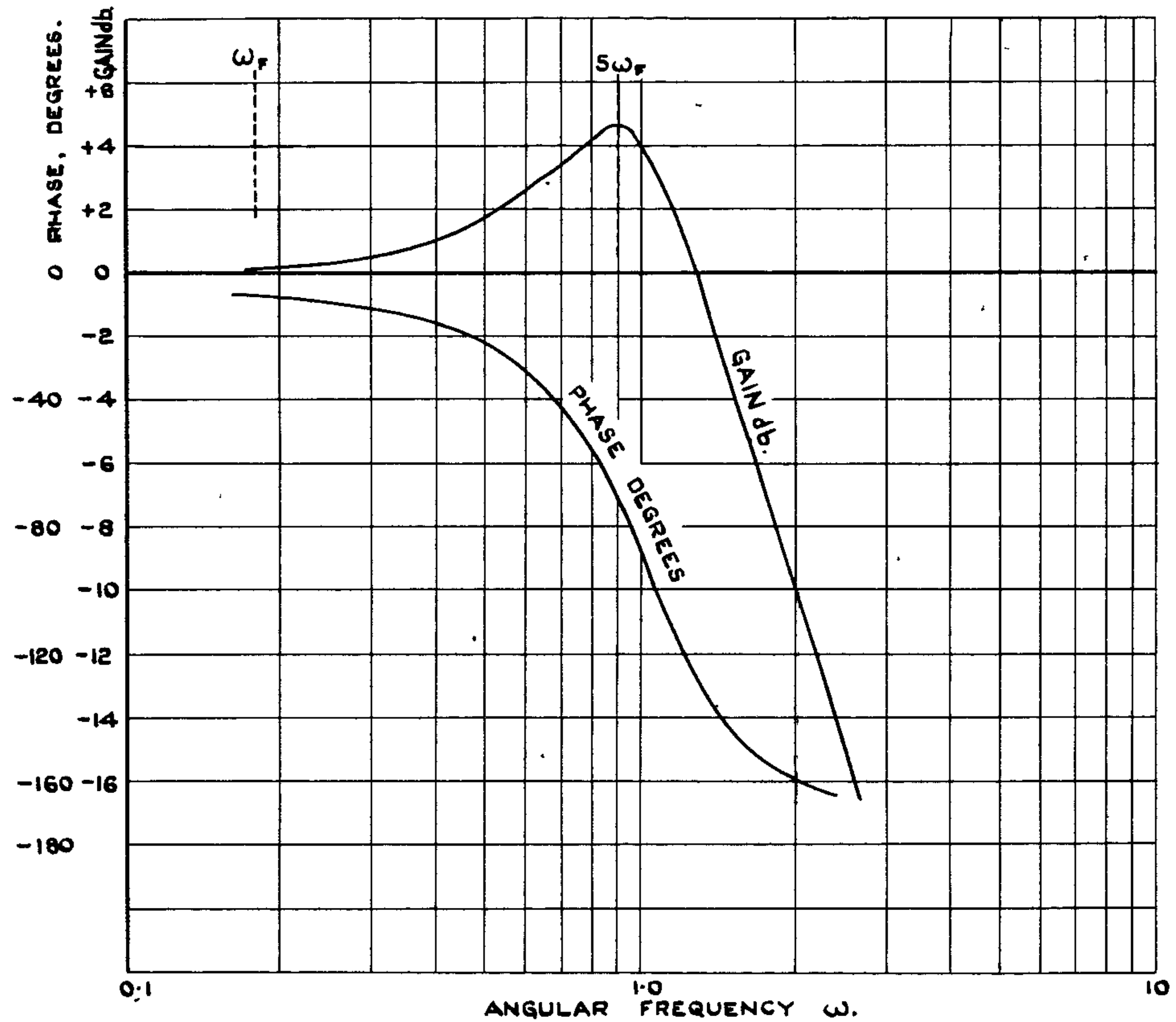

FIG. 3. FREQUENCY RESPONSE CURVES. SYSTEM I.

FIG.3.

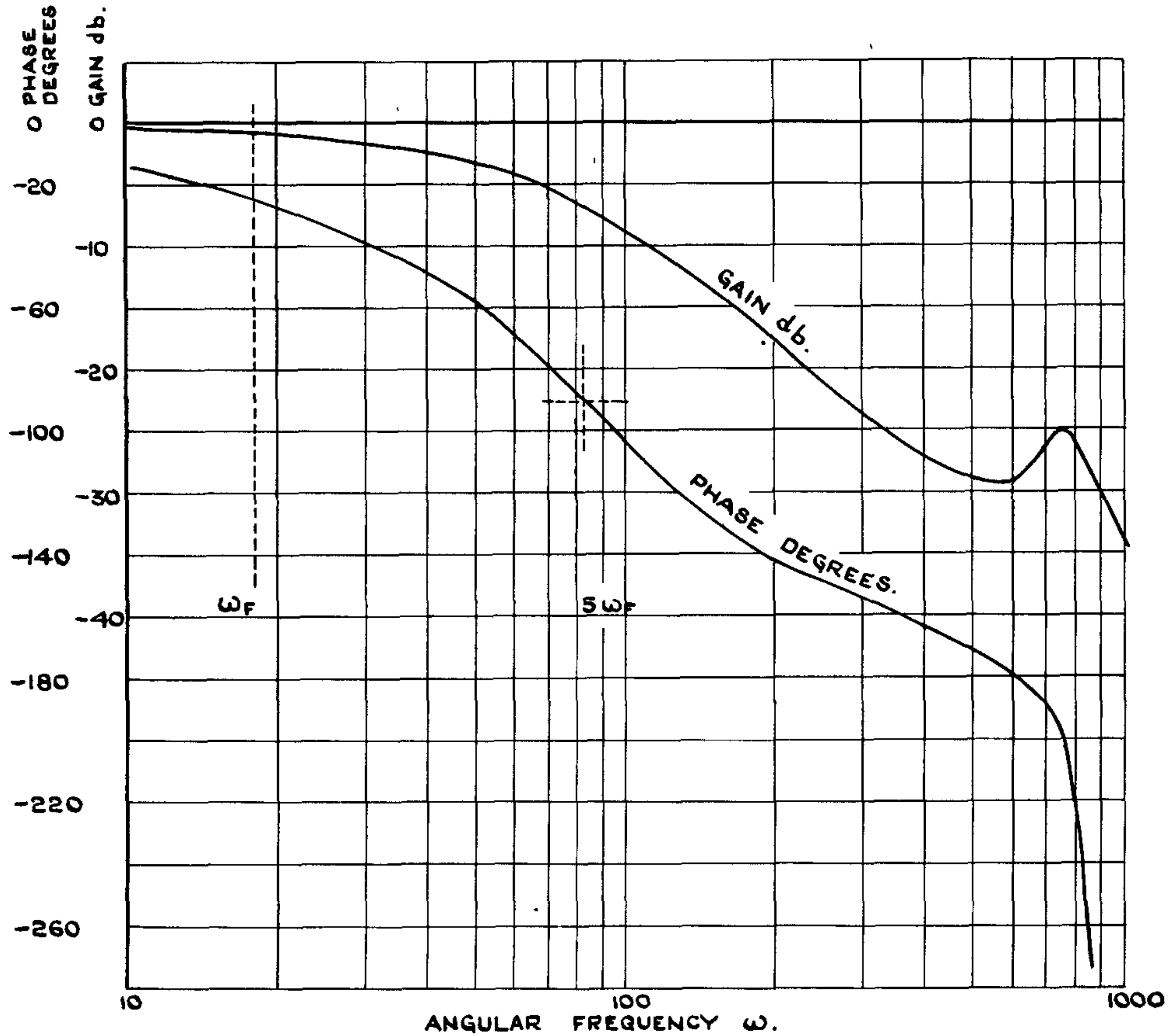

FIG.4. FREQUENCY RESPONSE CURVES. SYSTEM 2.

FIGURE 5

Calculation - System 1

Select $\omega_o = 0.90$ at the peak of the gain curve, so that $\omega_F = 0.18$

(a)

n	1	2	3	4	5	6
ω_{2n-1}	0.18	0.54	0.90	1.26	1.62	1.98
A_{2n-1}	+0.2	+2.1	+4.7	+0.3	-5.5	-10.0
B_{2n-1}	-6.3°	-24.6°	-70.5°	-127.8°	-150.1°	-158.4°

Since $\omega_o = 0.90$, $T = \frac{1}{\omega_o} = 1.11$

(b)

Secs	0.56	1.11	1.67	2.22	2.78	3.33	3.89	4.44	5.0	5.55	6.11	6.66	7.22	7.77	8.33	8.88	9.44	10.0
ω \ t/T	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
1	-0.01	+0.06	+0.12	+0.18	+0.25	+0.31	+0.36	+0.42	+0.46	+0.51	+0.55	+0.58	+0.61	+0.63	+0.65	+0.66	+0.66	+0.65
2	-0.04	+0.05	+0.12	+0.19	+0.24	+0.26	+0.26	+0.25	+0.21	+0.15	+0.07	-0.01	-0.09	-0.16	-0.21	-0.25	-0.27	-0.26
3	-0.15	-0.05	+0.06	+0.15	+0.21	+0.21	+0.17	+0.08	-0.03	-0.13	-0.19	-0.21	-0.18	-0.11	-0.01	+0.10	+0.18	+0.21
4	-0.09	-0.07	-0.01	+0.05	+0.09	+0.09	+0.04	-0.02	-0.08	-0.09	-0.07	-0.01	+0.05	+0.09	+0.09	+0.04	-0.02	-0.08
5	-0.03	-0.03	0	+0.03	+0.03	+0.01	-0.02	-0.03	-0.03	0	+0.03	+0.03	+0.01	-0.02	-0.03	-0.03	0	+0.03
6	-0.02	-0.01	+0.01	+0.02	+0.01	-0.01	-0.02	0	+0.01	+0.02	0	-0.02	-0.02	0	+0.02	+0.01	0	-0.02
Σ	-0.34	-0.05	+0.30	+0.62	+0.83	+0.87	+0.83	+0.70	+0.54	+0.46	+0.39	+0.36	+0.38	+0.43	+0.51	+0.53	+0.55	+0.53
+0.5	+0.16	+0.45	+0.80	+1.12	+1.33	1.37	+1.33	+1.20	+1.04	+0.96	+0.89	+0.86	+0.88	+0.93	+1.01	+1.03	+1.05	+1.03

FIGURE 6

Calculation - System 2

Neglect gain curve peak at $\omega = 750$ rad/sec.
Select ω_F by the -90° phase-shift criterion.
For -90°, $\omega \simeq 83$.
Choose $\omega_F = 18$ which is $\frac{83}{5} + 10\%$.
$\therefore \quad \omega_o = 90.$

n	1	2	3	4	5	6
ω_{2n-1}	18	54	90	126	162	198
A_{2n-1}	-0.75	-3.5	-8	-11	-15	-17
B_{2n-1}	-24.5	-63	-96	-120	-133	-140

Since $\omega_o = 90$, $T = \frac{1}{\omega_o} = 0.011$

Secs	0.0056	0.0111	0.0167	0.0222	0.0278	0 0333	0.0389	0.0448	0.050
n \ t/T	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5
1	-0.19	-0.13	-0.08	-0.02	+0.04	+0.10	+0.16	+0.21	+0.26
2	-0.10	-0.07	-0.03	+0.01	+0.06	+0.09	+0.12	+0.14	+0.14
3	-0.05	-0.03	-0.01	+0.02	+0.04	+0.05	+0.05	+0.04	+0.02
4	-0.03	-0.02	0	+0.02	+0.03	+0.02	+0.01	-0.01	-0.02
5	-0.01	-0.01	0	+0.01	+0.01	0	-0.01	-0.01	-0.01
6	-0.01	0	+0.01	+0.01	0	-0.01	-0.01	0	+0.01
Σ	-0.39	-0.26	-0.11	+0.05	+0.18	+0.25	+0.32	+0.37	+0.40
+0.5	+0.11	+0.24	+0.39	+0.55	+0.68	+0.75	+0.82	+0.87	+0.90

Secs	0.0555	0.0611	0.0666	0.0722	0.0777	0.0833	0.0888	0.0944	0.010
n \ t/T	5.0	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0
1	+0.32	+0.36	+0.41	+0.45	+0.48	+0.51	+0.54	+0.56	+0.57
2	+0.14	+0.12	+0.09	+0.05	+0.01	-0.04	-0.08	-0.11	-0.13
3	-0.01	-0.03	-0.05	-0.05	-0.04	-0.02	0	+0.03	+0.05
4	-0.03	-0.02	0	+0.02	+0.03	+0.02	+0.01	-0.01	-0.02
5	+0.01	+0.01	+0.01	0	-0.01	-0.01	-0.01	+0.01	+0.01
6	+0.01	0	-0.01	-0.01	0	+0.01	0	0	-0.01
Σ	+0.44	+0.44	+0.45	+0.46	+0.47	+0.47	+0.46	+0.48	+0.47
+0.5	+0.94	+0.94	+0.95	+0.96	+0.97	+0.97	+0.96	+0.98	+0.97

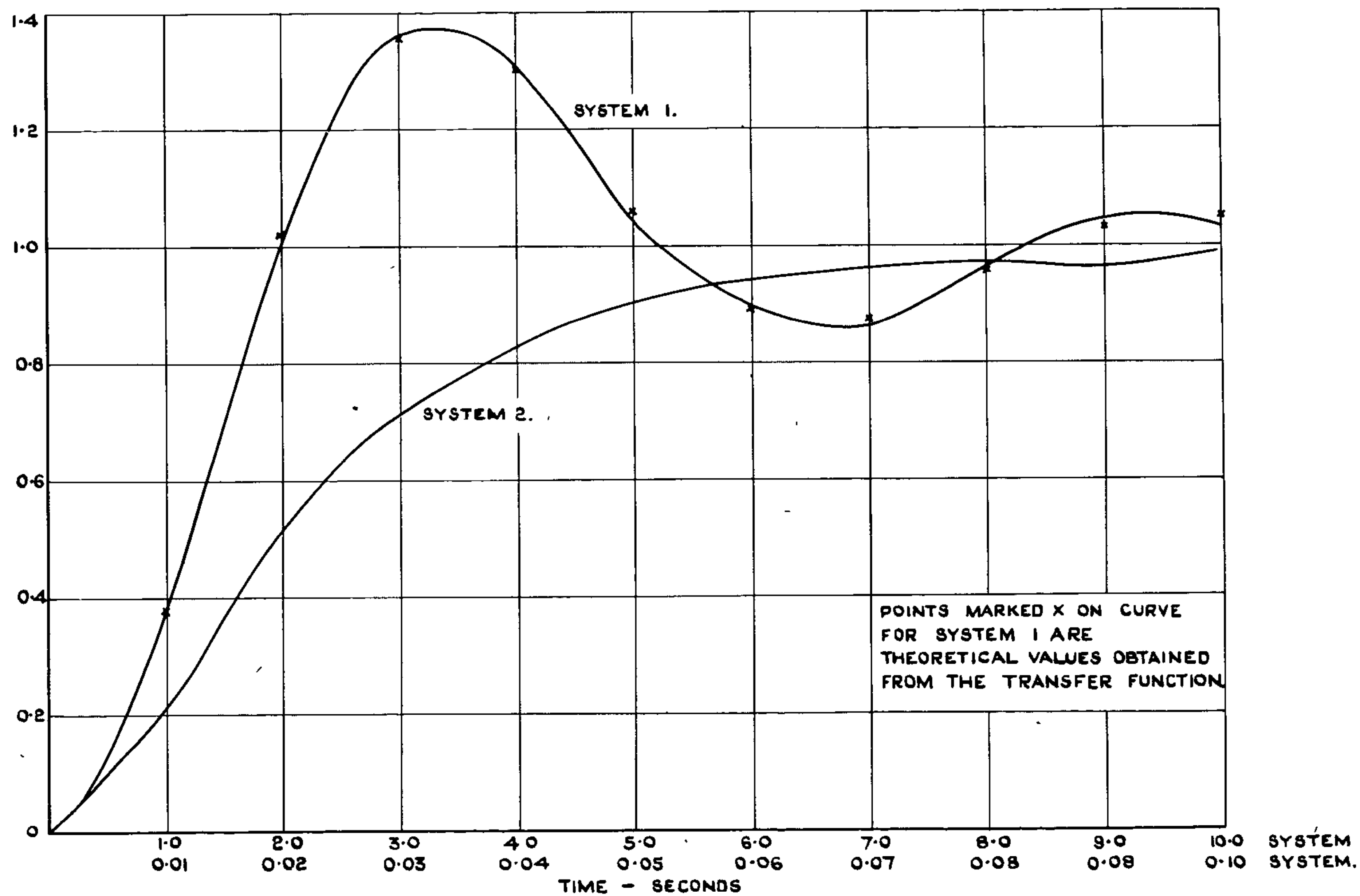

FIG. 7. TRANSIENT RESPONSE CURVES. SYSTEMS 1 AND 2.

FIG. 7.

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