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by

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A RESULT CONCERNING THE SUPERSONIC FLOW BELOW A PLANE DELTA WING

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SUMMARY

If a plane delta wing in inviscid supersonic flow supports an attached shockwave, the surface pressure distribution is uniform near the leading edges, and non-uniform near the centreline. In this Report an exact expression is given for the pressure gradient at the junction of the uniform and non-uniform regions.

* Replaces RAE Technical Report 72077 - ARC 33828

CONTENTS

1 INTRODUCTION

One of the classical problems of supersonic aerodynamics is the inviscid flow past the lower surface of a plane delta wing supporting an attached shockwave. This problem, together with the type of solution usually assumed, is shown in Fig.1. The flow near the leading edges is uniform, bounded by a plane shockwave, and may easily be calculated by the exact shock relationships. This uniform flow terminates as soon as the influence of the apex is felt, that is, on the Mach cone drawn from the apex. The construction of this cone follows at once from knowing the direction and Mach number of the uniform flow. The inner region of the flow is non-uniform. Computation of the flow in this region is difficult, although by now numerical solutions have been presented by many authors. One of the difficulties is that near the boundary of the non-uniform region, flow quantities change rather rapidly, and although the effect of this on the accuracy of numerical solutions has been discussed, the true local behaviour does not seem to have been determined.

In the present note an expression is derived for the spanwise pressure gradient on the wing surface on the inboard side of the boundary between the uniform and non-uniform flows. Ability to predict this gradient accurately should be a good test of any proposed numerical method.

2 ANALYSIS

If the flow is as drawn in Fig.1, and contains no embedded shocks, then all flow properties will be continuous across OBC, and in a region which includes that surface the flow will be irrotational. For the present purpose we only require that these two properties of continuity and irrotationality hold near OB. That is, the analysis will be unaffected by any embedded shocks which may exist, provided they do not extend to the surface. Inboard of OBC, we shall assume that all flow variables are twice differentiable functions of position.

Through O draw OA parallel to the velocity vector in the uniform flow (region OTCB) and take OA to be the axis of spherical polar coordinates. Specifically, for any required point P, let r be distance from O, θ be the angle POA, and ϕ be the angle between the plane POA and the plane of the wing. Let (u,v,w) be corresponding velocity components. The equations of inviscid conical flow of an ideal gas are, in these coordinates, as follows[1].

<u>Continuity</u>

$$2u + v \cot \theta + \frac{\partial v}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial w}{\partial \phi} + \frac{1}{\rho}\left(v \frac{\partial \rho}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial \rho}{\partial \phi}\right) = 0 \quad . \qquad (1)$$

<u>r-momentum</u>

$$v \frac{\partial u}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial u}{\partial \phi} - v^2 - w^2 = 0 \quad . \qquad (2)$$

<u>θ-momentum</u>

$$v \frac{\partial v}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial v}{\partial \phi} + uv - w^2 \cot \phi = -\frac{1}{\rho} \frac{\partial p}{\partial \theta} \quad . \qquad (3)$$

<u>φ-momentum</u>

$$v \frac{\partial w}{\partial \theta} + \frac{w}{\sin \theta} \frac{\partial w}{\partial \phi} + uw + vw \cot \phi = -\frac{1}{\rho \sin \theta} \frac{\partial p}{\partial \phi} \quad . \qquad (4)$$

<u>Energy</u>

$$\frac{(\gamma - 1)}{2} (u^2 + v^2 + w^2) + a^2 = \text{constant} \quad . \qquad (5)$$

The flow near the leading edge may be written in terms of the magnitude q of the uniform velocity vector, thus

$$u = q \cos \theta \qquad (6a)$$

$$v = -q \sin \theta \qquad (6b)$$

$$w = 0 \qquad (6c)$$

$$a = q \sin \mu \qquad (6d)$$

where μ is the Mach angle in the uniform flow. The boundary of the uniform flow is $\theta = \mu$.

Along the surface of wing ($\phi = 0$) we have the boundary condition $w = 0$, and so the following equations, valid on the surface, can be derived from equations (1) to (3).

$$2u + v \cot\theta + \frac{\partial v}{\partial\theta} + \frac{1}{\sin\theta}\frac{\partial w}{\partial\phi} = -\frac{v}{\rho}\frac{\partial\rho}{\partial\theta} \tag{7}$$

$$v\frac{\partial u}{\partial\theta} - v^2 = 0 \tag{8}$$

$$v\frac{\partial v}{\partial\theta} + uv = -\frac{1}{\rho}\frac{\partial p}{\partial\theta} \tag{9}$$

In equation (7) we can substitute $a^{-2}\partial\rho/\partial\theta$ for $\partial p/\partial\theta$ (because the entropy is constant) and if we also use equation (9) to eliminate $\partial v/\partial\theta$ we obtain

$$u + v\cot\theta + \frac{1}{\sin\theta}\frac{\partial w}{\partial\phi} = \frac{1}{\rho v}\left[1 - \frac{v^2}{a^2}\right]\frac{\partial p}{\partial\theta} \tag{10}$$

which we rewrite to give an explicit equation for the pressure gradient on the wing surface, thus

$$\frac{\partial p}{\partial\theta} = \rho v \frac{u + v\cot\theta + \frac{1}{\sin\theta}\frac{\partial w}{\partial\phi}}{1 - \frac{v^2}{a^2}} \tag{11}$$

On the boundary of the non-uniform region we can substitute into this expression the values of u, v, a, θ, taken over from the uniform flow. The numerator and the denominator are both then zero, but we can still obtain an answer by applying l'Hospitals rule; replacing each of them by their derivative with respect to θ. We obtain

$$\frac{\partial p}{\partial\theta} = \rho v \frac{\frac{\partial u}{\partial\theta} + \cot\theta\frac{\partial v}{\partial\theta} - v\,\mathrm{cosec}^2\theta - \frac{\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^2 w}{\partial\theta\partial\phi}}{\frac{2v^2}{a^3}\frac{\partial a}{\partial\theta} - \frac{2v}{a^2}\frac{\partial v}{\partial\theta}} \tag{12}$$

This can be simplified by substituting v for $\partial u/\partial\theta$ (equation (8)) and noting that at the point of interest $v = -a$. Thus

$$\frac{\partial p}{\partial\theta} = -\tfrac{1}{2}\rho v^2 \frac{\cot\theta\frac{\partial v}{\partial\theta} - v\cot^2\theta - \frac{\cos\theta}{\sin^2\theta}\frac{\partial w}{\partial\phi} + \frac{1}{\sin\theta}\frac{\partial^2 w}{\partial\theta\partial\phi}}{\frac{\partial a}{\partial\theta} + \frac{\partial v}{\partial\theta}} \tag{13}$$

We shall now show that on the line OB the terms involving derivatives with respect to ϕ are both zero. We are assuming that all flow quantities are continuous across the surface OBC. Since this surface coincides with $\theta = \mu$ = constant, derivatives involving only r or ϕ will also be continuous across it. In particular $\partial w/\partial\phi$ is identically zero outboard of OBC, (see equation (6c)), and is therefore zero immediately inboard of OBC.

We cannot directly apply this argument to the term with $\partial^2 w/\partial\theta\partial\phi$, because in general derivatives involving θ will not be continuous across OBC. However, we can show that this particular term is zero by appealing (for the only time in the course of the argument) to the locally irrotational nature of the flow. The r-component of vorticity can be written[1] as:-

$$\xi = \frac{1}{r}\left[\frac{\partial w}{\partial\theta} + w \cot\theta - \frac{1}{\sin\theta}\frac{\partial v}{\partial\phi}\right] \tag{14}$$

and this is zero in a region including the line OB. Therefore we also have at that point

$$\frac{\partial\xi}{\partial\phi} = 0 = \frac{1}{r}\left[\frac{\partial^2 w}{\partial\theta\partial\phi} + \frac{\partial w}{\partial\phi}\cot\theta - \frac{1}{\sin\theta}\frac{\partial^2 v}{\partial\phi^2}\right] . \tag{15}$$

But we have just shown that $\partial w/\partial\phi = 0$, and by a similar argument $\partial^2 v/\partial\phi^2 = 0$, so we must also have $\partial^2 w/\partial\theta\partial\phi = 0$, and equation (13) reduces to

$$\frac{\partial p}{\partial\theta} = -\tfrac{1}{2}\rho v^2 \, \frac{\cot\theta\,\dfrac{\partial v}{\partial\theta} - v\cot^2\theta}{\dfrac{\partial a}{\partial\theta} + \dfrac{\partial v}{\partial\theta}} . \tag{16}$$

This equation now involves derivatives with respect to θ only, and of these we eliminate all except the pressure derivative, which is our primary interest. From equation (9) we can write

$$\frac{\partial v}{\partial\theta} = -u - \frac{1}{\rho v}\frac{\partial p}{\partial\theta} . \tag{17}$$

Also we can write the energy equation (5) in terms of derivatives along the wing, e.g.

$$a\frac{\partial a}{\partial\theta} = -\frac{(\gamma - 1)}{2}\left[u\frac{\partial u}{\partial\theta} + v\frac{\partial v}{\partial\theta}\right] \tag{18}$$

and by combining this with equation (8) for $\partial u/\partial\theta$ and equation (17) for $\partial v/\partial\theta$ we get

$$a\frac{\partial a}{\partial\theta} = \frac{(\gamma - 1)}{2\rho}\frac{\partial p}{\partial\theta} \quad . \tag{19}$$

Putting (17) and (19) into equation (16) and setting $\theta = \mu$, $u = q\cos\mu$, $v = -q\sin\mu$, we have finally

$$\frac{\partial p}{\partial\theta} = \frac{\frac{1}{2}q\cos\mu\frac{\partial p}{\partial\theta}}{q\cos\mu - \frac{\gamma + 1}{2\rho q\sin\mu}\frac{\partial p}{\partial\theta}} \quad . \tag{20}$$

This equation allows two possible pressure gradients, either

$$\frac{\partial p}{\partial\theta} = 0 \tag{21}$$

or

$$\frac{\partial p}{\partial\theta} = \frac{\rho q^2\sin\mu\cos\mu}{\gamma + 1} \quad . \tag{22}$$

The second solution can be written more concisely by putting $\rho q^2 = \gamma M^2 p$, $\sin\mu = 1/M$, $\cos\mu = (M^2 - 1)^{\frac{1}{2}}/M$, so that equation (22) becomes

$$\frac{1}{p}\frac{\partial p}{\partial\theta} = \frac{\gamma}{\gamma + 1}(M^2 - 1)^{\frac{1}{2}} \quad . \tag{23}$$

Thus, if the flow is of the type shown in Fig.1, the initial surface pressure gradient in the non-uniform flow must either take this value or zero. Nothing in the above shows which value will be taken, but it is likely that the question can be settled by examining a special case. If the angle of incidence is very small, we can set, approximately, in equation (22), $\rho = \rho_\infty$, $q = u_\infty$, $\sin\mu = 1/M_\infty$, $\cos\mu = (M_\infty^2 - 1)^{\frac{1}{2}}/M_\infty$, and get

$$\frac{\partial p}{\partial\theta} = \frac{\rho_\infty u_\infty^2}{(\gamma + 1)}\frac{(M_\infty^2 - 1)^{\frac{1}{2}}}{M_\infty^2} \quad . \tag{24}$$

This result was given by Lighthill[2], as applying at any small angle of incidence, and so we can assume that the non-zero solution is the one which generally applies inboard of B.

Note that the above analysis applies without modification to any conical wing having a sufficiently extensive plane region at its tips. In particular, it applies to any wing of diamond or caret section, provided the non-uniform flow is expansive rather than compressive, in which case an inner shockwave may result.

An attempt was made to determine the second derivative, $\partial^2 p/\partial\theta^2$, by a similar procedure, but could not be carried through because of the appearance in the analysis of a term $\partial^3 w/\partial\phi\partial\theta^2$ which could not be evaluated. The second and higher derivatives are probably not determined until some further account has been taken of the entire boundary conditions governing the non-uniform flow.

3 COMPARISON WITH NUMERICAL SOLUTIONS AND DISCUSSION

In Fig.2a we show calculations[3] for the compression side of a plane delta wing with 45° sweep angle at 4° incidence in a free stream of Mach number equal to 3.0. The numerical results are quite compatible with the present analysis, but they do seem to indicate that the local solution is only valid over a very limited part of the span.

The same conclusion is reached if we examine (Fig.3) some more recent calculations by several authors[4-6] for a wing with 50° sweep at 15° incidence in a Mach 4.0 stream. For this problem Voskresenkii[5] and South[6] have obtained extremely similar solutions by very different methods. The rather earlier work of Babaev[4] is now known[7] to contain some errors. None of these solutions takes account of the singular nature of the flow on OBC; indeed, all employ finite difference methods that might be expected to smear out discontinuities.

The situation shown in Fig.3 is a little unclear. It is probably best to disregard Babaevs results, especially as the effect of the errors discovered by Ganzer is not easy to assess. The points given by South could well be consistent with a curve having the predicted slope, especially if one accepts that the points nearest the boundary probably contain the worst errors. Voskresenskii presented his results as a continuous curve in his report, but must originally have obtained them as discrete points, rather like South's, so that his results also may not be incompatible with the present analysis.

Probably the most accurate prediction currently available would be an empirical fairing between South's or Voskresenkii's numerical values and the present local solution.

The present analysis thus supplements the current numerical methods by supplying some additional detail, which could be significant if, for example, a boundary layer calculation were to be carried out. For any future numerical methods it could provide in advance a possibly useful boundary condition, or it can be used as an *a posteriori* check on the accuracy of the computations.

4 IMPLICATIONS FOR THIN-SHOCK-LAYER THEORY

The present result has an important bearing on conical thin-shock-layer theory[8-12]. This theory is based on expanding the flow variables in powers of ε, where

$$\varepsilon = \frac{\rho_\infty}{\rho} .$$

ρ_∞ being the density in the free stream, and ρ a typical density in the disturbed flow. The theory is expected to apply to hypersonic flows. When first presenting the relevant equations, Messiter[8] noted that a difficulty occurred in matching the uniform and non-uniform regions of the flow. Squire[10], Woods[11], and Roe[12] proposed various devices for overcoming the difficulty, Squire and Roe both permitting a slight relaxation of the body boundary conditions and Woods proposing a solution with discontinuous shock strength.

The result in this Report offers a partial explanation of the difficulty. In assigning orders of magnitude to the various terms in the equations of motion, thin-shock-layer theory assumes that the spanwise pressure gradients are everywhere of order $\varepsilon^{\frac{1}{2}}$.

However, the true magnitude of the pressure gradient at the boundary of the non-uniform flow can be found from equation (23). For hypersonic flow we can replace $(M^2 - 1)^{\frac{1}{2}}$ in this equation by $M = u/a$. Since $a^2 = \gamma p/\rho$ we have

$$\frac{\partial p}{\partial \theta} = \frac{u}{(\gamma + 1)} \left(\frac{\rho p}{\gamma}\right)^{\frac{1}{2}}$$

in which all quantities are of order unity, except ρ, which is of order ε^{-1}. Thus, at this particular point, the pressure gradient should be of order $\varepsilon^{-\frac{1}{2}}$, and although the assumptions of thin-shock-layer theory are valid throughout most of the flow, the assumption regarding pressure gradients is not uniformly valid everywhere. This possibility, which was conjectured in Ref.12, is almost certainly connected with the above mentioned difficulty. However, it should be mentioned that this is not the only anomaly contained in thin-shock-layer theory: discussion of certain others will be found in Refs.8-12.

5 CONCLUSIONS

An expression has been derived for the pressure gradient on the surface of a plane delta wing in supersonic flow, at the point where the uniform and non-uniform regions meet. The result has been compared with numerical solutions worked out by several authors. It can be regarded as supplementing these in a significant point of detail. The result has also been used to explain a shortcoming of thin-shock-layer theory.

NOTATION

r, θ, ϕ	spherical polar coordinates, defined at the beginning of section 2.1
u, v, w	corresponding velocity components
a	velocity of sound
p	static pressure
q	velocity = $(u^2 + v^2 + w^2)^{\frac{1}{2}}$
z	spanwise distance
M	Mach number = q/a
γ	ratio of specific heats
μ	Mach angle = $\sin^{-1} 1/M$
ρ	density
ξ	radial component of vorticity

Subscript

$()_\infty$	denotes free stream values

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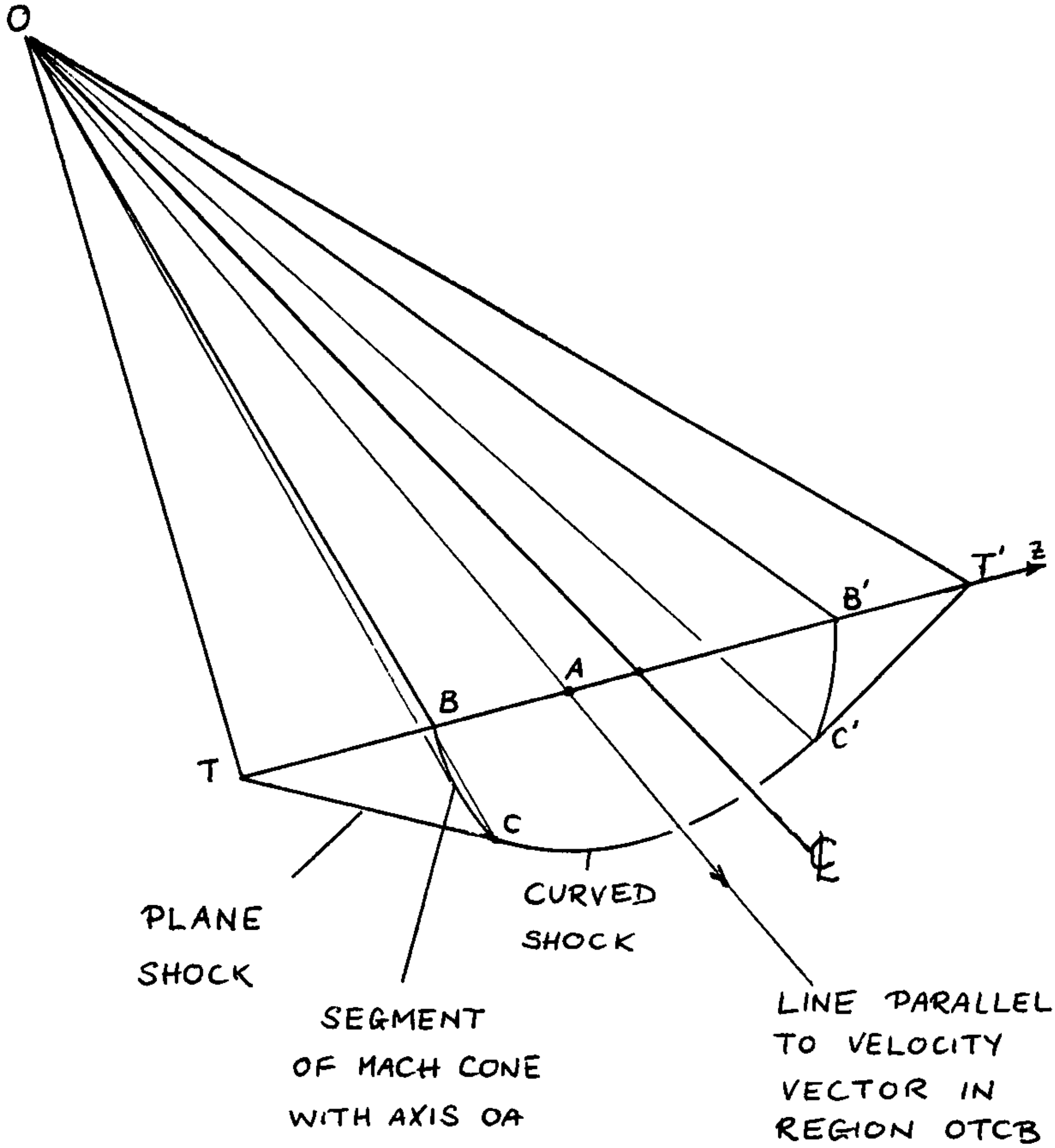

Fig.1 Supersonic flow past a plane delta wing with an attached shockwave

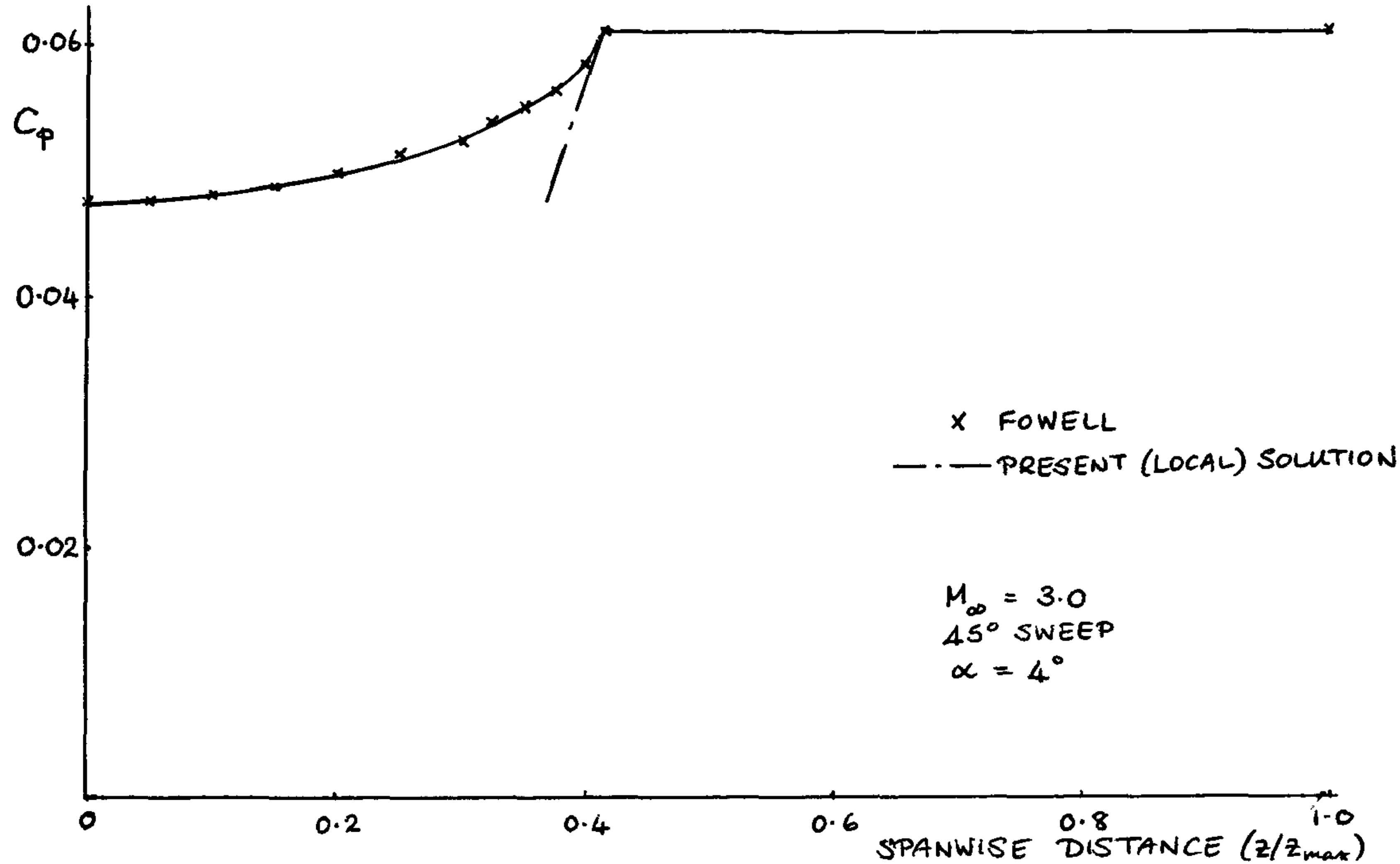

Fig.2 Comparison with numerical solution

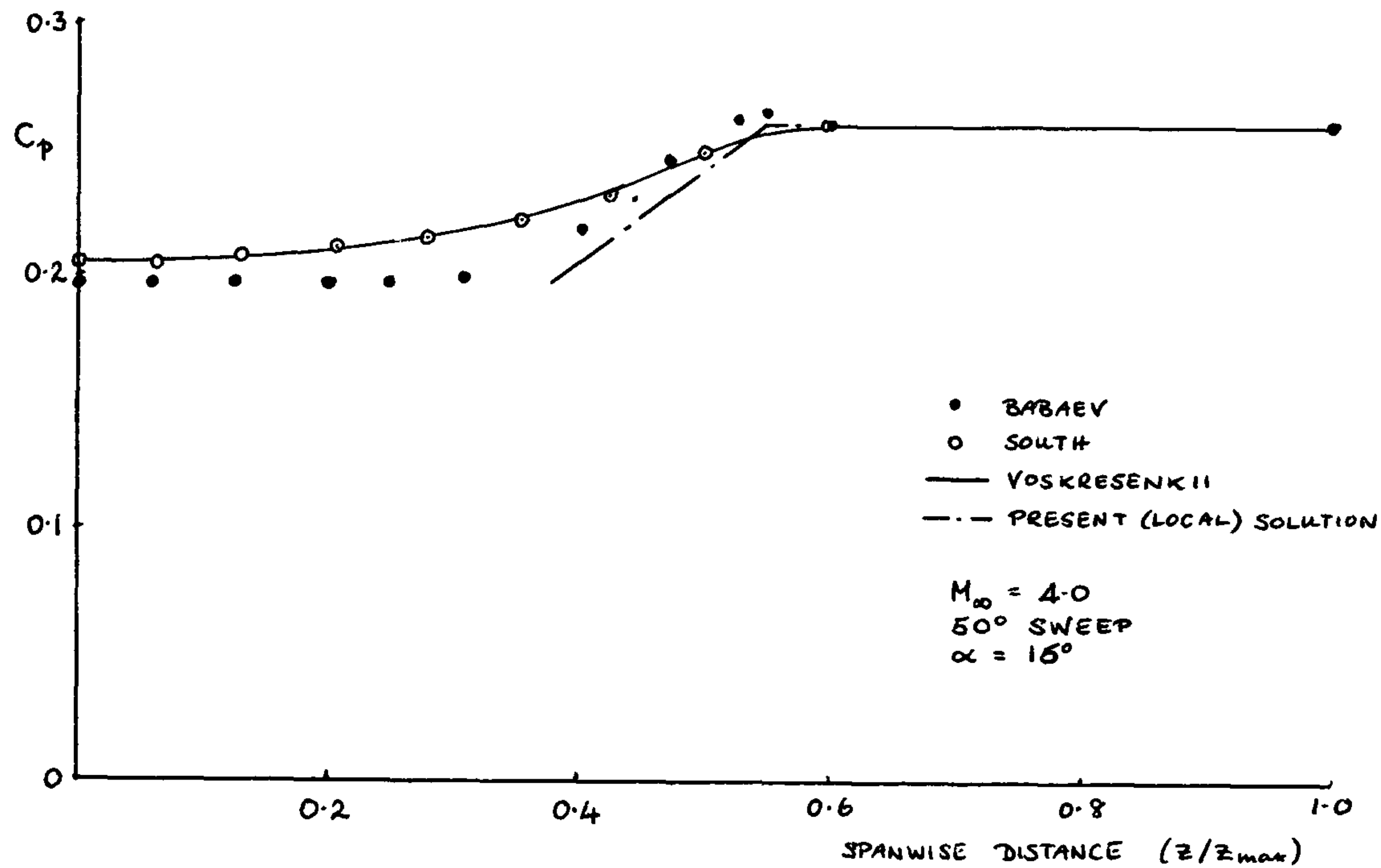

Fig.3 Further comparison

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