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SUMMARY

Different approximate methods of determining the eigenvalues of the integro-differential matrix equation of a simple aeroelastic system are compared. It is shown that methods which use an approximate second order differential matrix equation with constant coefficients can give large errors in the values of complex eigenvalues, though the errors are usually small at airspeeds below the critical flutter speed, if the frequency parameter of each particular eigenvalue is lined-up with the value used to determine the aerodynamics. An improved method of solution using a finite series approximation to the indicial aerodynamics yielded in some cases an additional complex eigenvalue with a frequency of the same order as the other natural frequencies.

* Replaces R.A.E. Technical Report 68296 - A.R.C. 31379.

CONTENTS

1 INTRODUCTION

The primary purpose of a flutter calculation is to determine the critical flutter speed (if any), but the free oscillation characteristics at lower speeds are also of interest. In particular, when making flight flutter tests, or wind tunnel flutter tests on a model, a flutter speed may not be determined, and then all comparisons with theory will have to be made for the characteristics of the system at subcritical speeds. The different methods of flutter analysis commonly used agree as regards critical flutter speeds, provided the same basic data are used, but give different values of the decay rates at other speeds. Some assessment of the importance of these differences is therefore required*. Richardson[1] gives one example where the standard American approach (see section 2.2) is misleading and Clerc[2] has found that a method similar to the American approach can largely overestimate the magnitude of the relative damping ratio when compared with the traditional British approach with lined-up frequency parameter.

The present investigations are aimed at showing in particular how the traditional British approach (with and without lined-up frequency parameters) compares with the more rigorous approach of Richardson. Comparisons are also made with the American method of analysis.

2 METHODS OF SOLUTION

2.1 British approach

This is the standard approach in use in this country. The flutter equation is taken in the non-dimensional form

$$A \frac{d^2 q}{d\tau^2} + (vB + D) \frac{dq}{d\tau} + (v^2 C + E)\, q = 0 \qquad (1)$$

frequently with $D = 0$; where $\tau = V_o t/\ell$, $v = V/V_o$ and V_o and ℓ are a reference speed and length respectively. An exponential solution is postulated, leading to an eigenvalue problem to determine the complex eigenvalues λ for a solution in the form $q = \bar{q}\, e^{\lambda\tau}$. The system is unstable if, for any eigenvalue, $R(\lambda) > 0$ and a critical speed is defined by $R(\lambda) = 0$.

Strictly equation (1) applies only when the motion is simple harmonic, implying that λ is purely imaginary ($= i\omega$). The aerodynamic matrices B and C are functions of the frequency parameter $\nu = (\omega/v)$ and, in general, of the

*Since this was written we became aware of a recent paper by Natke[8] which contributes to such an assessment.

Mach number also, but in the present case the air is assumed to be incompressible, so there is no dependence on Mach number. However, a solution of equation (1) is usually obtained by assuming a value of ν for the calculation of the matrices B and C, and then solving equation (1) for λ. The value of λ so obtained will not, in general, be purely imaginary, nor will $\frac{1}{v} I(\lambda)$ be consistent with the assumed value of the frequency parameter ν. In order to achieve some measure of agreement and perfect agreement in the limiting case when λ is purely imaginary, the assumed value of ν is often lined-up with the derived value of $\frac{1}{v} I(\lambda)$ by the following procedure.

A graph is plotted of ω obtained from the eigenvalues λ $(= \mu + i\omega)$ against v and the intersections of each curve with the line $\omega = \nu v$ (where ν is the value of the frequency parameter assumed in the evaluation of B and C) give the lined-up values of frequency and speed. From the corresponding graphs of the relative damping ratio $= -\mu/\sqrt{\mu^2 + \omega^2}$ against v the appropriate values of this ratio can be found. From a series of such graphs for various ν graphs of the lined-up frequency and relative damping ratio can be plotted.

This method has the disadvantage that it is necessary to calculate eigenvalues for a large range of speeds without lining-up in order to obtain lined-up values for one value of ν; at least for the first few values of frequency parameter.

2.2 American approach

The system equation is taken in a rather different form from the above. The actual structural damping is ignored and a fictitious hysteretic structural damping $\frac{g}{\omega} E$ is introduced which is supposed just sufficient to maintain steady harmonic motion. The solution $q = \bar{q}\, e^{i\omega\tau}$ may then be taken which gives

$$(-\omega^2 A + i\omega v B + igE + v^2 C + E)\,\bar{q} = 0 \tag{2}$$

$$\left\{A - \frac{ivB}{\omega} - \left(\frac{v}{\omega}\right)^2 C - \frac{(1 + ig)}{\omega^2} E\right\}\bar{q} = 0 \; . \tag{3}$$

Since $\frac{v}{\omega} = \frac{1}{\nu}$, the problem reduces to a determination of the eigenvalues (in the usual algebraic sense) of the matrix

$$E^{-1}\left(A - \frac{iB}{\nu} - \frac{C}{\nu^2}\right) \; . \tag{4}$$

A chosen ν is used to determine B and C and so all the terms are known.

The complex eigenvalues of the matrix can then be found for a series of values of ν. Separation of real and imaginary parts enables a determination of g and ω separately; the velocity is obtained from ω and the assumed ν.

As g represents a damping which has to be introduced, a negative value which may be regarded as an excitation means that the system is intrinsically damped and therefore stable. The critical flutter speed is given by $g = 0$.

It will be seen that values of frequency and fictitious structural damping obtained by this method are accurate at all values of ν so that insofar as no lining-up is necessary the solution may be regarded as superior to that from the British approach.

There is however the problem of the relationship between g and the decay factor μ. In the limit as $\mu, v \to 0$ it can be shown that

$$g \to \frac{2\mu}{\omega} \simeq \frac{2\mu}{\sqrt{\mu^2 + \omega^2}}$$

where μ/ω is small. A more general relationship has been found by Zisfein and Frueh[3,4] but the introduction of the so-called base curve of the system is not very convenient for the present application. We have therefore used $-g/2$ as the relative damping factor in this case for comparison with the methods using $-\mu/\sqrt{\mu^2 + \omega^2}$ but it must be remembered that the comparison is close only for low values of the speed and decay factor.

A similar approach to the American method has been used in France[2]. The same equation (3) is solved but a different interpretation is put on the solution. It is assumed that equation (1) has a solution of the form $q = \bar{q}\, e^{i\omega(1+i\alpha)\tau}$ where the aerodynamic matrices B, C are determined for a frequency parameter $\nu = \omega/v$. This results in the equation

$$\left\{A - \frac{iB}{\nu\,(1 + i\alpha)} - \frac{C}{\nu^2\,(1 + i\alpha)^2} - \frac{E}{\omega^2\,(1 + i\alpha)^2}\right\} \bar{q} = 0 \quad . \tag{5}$$

It is then assumed that α can be neglected, i.e. put equal to zero, in the second and third terms. This is true at a critical flutter speed and is also a good approximation when v is small. Thus we again get equation (3) except that $(1 + ig)$ is replaced by $1/(1 + i\alpha)^2$ and α is here a measure of the decay rate. Equating imaginary parts of $(1 + i\alpha)^2$ and $1/1 + ig$ gives α in terms of g and hence

$$\frac{\alpha}{\sqrt{1 + \alpha^2}} = - \operatorname{sgn}(g) \frac{g}{2(1 + g^2)} \left\{1 + \frac{g^2}{4(1 + g^2)^2}\right\}^{-\frac{1}{2}} \tag{6}$$

$$\approx -\frac{g}{2} \quad \text{where } g \text{ is small} .$$

For any particular value of g, equating real parts of

$$\omega_F^2 (1 + i\alpha)^2 = \frac{\omega_A^2}{1 + ig}$$

gives

$$\left(\frac{\omega_A}{\omega_F}\right)^2 = (1 - \alpha^2)(1 + g^2) = 1 + g^2 - \frac{g^2}{4(1 + g^2)} = \left(\frac{v_A}{v_F}\right)^2 \tag{7}$$

where the subscripts F and A indicate the French and American interpretation respectively. The relative damping ratio is $\alpha/\sqrt{1+\alpha^2}$, but as Clerc[2] showed, it does not agree with the value obtained by the traditional British approach with lined-up frequency parameter except near $v = 0$ and at a critical flutter speed.

2.3 Richardson approach

For general motion the system equation has the form[5]

$$(A-A_1)\frac{d^2q}{d\tau^2} + Eq = v^2 \int_0^\tau K(\tau-\tau_o) \frac{dq(\tau)}{d\tau_o} d\tau_o \tag{8}$$

where $K(\tau)$ is the indicial aerodynamic matrix, and is related to the matrices B and C, already introduced, and the aerodynamic inertia matrix A, by the transform relationship

$$i\omega \int_0^\infty K(\tau_o)\, e^{-i\omega\tau_o}\, d\tau_o = -\left(\frac{i\omega}{v} B + C - A_1 \frac{\omega^2}{v^2}\right) . \tag{9}$$

This follows from (8) with $q = \bar{q}\, e^{i\omega\tau}$ and $\tau = \tau' + \frac{2N\pi}{\omega}$ when $N \to \infty$ (i.e. for simple harmonic motion of infinite duration) when compared with the equation for maintained sinusoidal oscillation (i.e. equation (1)).

The solution of the integro-differential equation (8) is not easy. Taking the Laplace transform of (8) the characteristic equation

$$\left|(A-A_1)\, p^2 + E - v^2 p\, \bar{K}(p)\right| = 0 \tag{10}$$

is obtained, where

$$\bar{K}(p) = \int_0^\infty K(\tau)\, e^{-p\tau}\, d\tau \tag{11}$$

$\bar{K}(p)$ is known only for purely imaginary values of p and in this case we have from (9)

$$\bar{K}(i\omega) = -\frac{1}{i\omega}\left(\frac{i\omega}{v} B + C - A_1 \frac{\omega^2}{v^2}\right) . \tag{12}$$

Milne[6] has examined this problem rigorously and suggests obtaining solutions of the characteristic equation by using power series expansions of $\bar{K}(p)$ about points on the imaginary axis.

A rather more simple approach to the solution of (8) has been suggested by Richardson[1]. His main idea was to approximate to the indicial aerodynamic matrix K by an expression which includes a power series in $v\tau$ multiplied by an exponential term

$$K(\tau) = A_o \frac{\delta(\tau)}{v} - A_1 \frac{\delta'(\tau)}{v^2} + K_\sigma + e^{-p_o v \tau} \sum_{r=0}^{m-1} \frac{K_r}{r!} (p_o v \tau)^r \tag{13}$$

where $\delta(\tau)$ is the right-hand Dirac delta function, i.e. the first differential of the function

$$H(\tau) = 0 \qquad \tau \leqslant 0$$

$$= 1 \qquad \tau > 0$$

The term $A_1 \dfrac{\delta'(\tau)}{v}$ represents the apparent mass effect of the air. The existence of a term proportional to $\delta(\tau)$ is well known and has been demonstrated for example by Milne (equation (2.11) of Ref.6).

The elements of $K(\tau)$ for a wing with heave and pitch freedoms, apart from the initial impulses and the constant terms, are proportional in the two-dimensional case to the Wagner function $k_1(\tau)$ (see Lomax[9]). A good approximation to $k_1(\tau)$ which has been suggested[9] is

$$k_1(\tau) \approx 2 - \left(\frac{1}{3} e^{-0 \cdot 09\tau} + \frac{2}{3} e^{-0 \cdot 6\tau}\right) .$$

It does not however have the right behaviour as τ tends to ∞ (cf. Milne[6]). This suggests that suitable values of p_o for our approximation should be in the range $0 \cdot 09 \to 0 \cdot 6$ and nearer the upper limit because of the doubts about the approximation for $k_1(\tau)$ for large τ.

The coefficients in equation (13) can be obtained from the matrices B and C by the use of equations (11) and (12) as will be shown later (see Appendix B).

Substitution of (13) in (8) gives

$$A \frac{d^2q}{d\tau^2} - vA_o \frac{dq}{d\tau} + (E - v^2 K_o) q$$

$$= v^2 \sum_{r=0}^{m-1} K_r \frac{(p_o v)^r}{r!} \int_0^\tau e^{-p_o v(\tau-\tau_o)} (\tau-\tau_o)^r \frac{dq}{d\tau_o}(\tau_o) \, d\tau_o \quad . \qquad (14)$$

If

$$I_r = \frac{1}{r!} \int_0^{\tau} e^{-p_o v(\tau-\tau_o)} (\tau-\tau_o)^r \frac{dq}{d\tau_o} (\tau_o) \, d\tau_o$$

$$\frac{\partial}{\partial\tau} (I_r) = I_{r-1} - p_o v \, I_r$$

$$\left(\frac{\partial}{\partial\tau} + p_o v\right)^r I_r = I_o \quad .$$

Hence following Richardson, we multiply equation (14) by the operator $\left(\frac{\partial}{\partial\tau} + p_o v\right)^m$ and obtain

$$\left(\frac{\partial}{\partial\tau} + p_o v\right)^m \left\{A \frac{d^2 q}{d\tau^2} - vA_o \frac{dq}{d\tau} + (E - v^2 K_\sigma) q\right\}$$

$$= v^2 \sum_{r=0}^{m-1} K_r' (p_o v)^r \left(\frac{\partial}{\partial\tau} + p_o v\right)^{m-r} I_o \quad . \qquad (15)$$

Assuming $q = \bar{q} \, e^{\lambda t}$ where $\lambda = \mu + i\omega$

$$I_o = \lambda \, e^{-p_o v\tau} \int_0^{\tau} e^{(\lambda+p_o v)\tau_o} \, \bar{q} \, d\tau_o$$

$$= \frac{e^{\lambda\tau} - e^{-p_o v\tau}}{\lambda + p_o v} \lambda \bar{q}$$

$$\left(\frac{\partial}{\partial\tau} + p_o v\right)^{m-r} I_o = \frac{\left(\frac{\partial}{\partial\tau} + p_o v\right)^{m-r} e^{\lambda\tau} \lambda \bar{q}}{\lambda + p_o v} \quad .$$

Hence $q = \bar{q}\, e^{\lambda t}$ is a solution if

$$(\lambda + p_o v)^m \{A\lambda^2 - vA_o\lambda + E - v^2 K_\sigma\} \bar{q} = v^2 \left\{ \sum_{r=0}^{m-1} K_r (p_o v)^r \lambda (\lambda + p_o v)^{m-r-1} \right\} \bar{q} \qquad \ldots (16)$$

The problem is now reduced to an eigenvalue problem in λ. However, the matrix involved is still not simple. Divide by $(\lambda + p_o v)^m$ and introduce m new variables defined by

$$\bar{q}_r = \frac{(p_o v)^r \lambda}{(\lambda + p_o v)^{r+1}} \bar{q} \qquad r=0,1\ldots m-1 \qquad (17)$$

to give

$$(A\lambda^2 - A_o v\lambda + E - v^2 K_\sigma) \bar{q} - \sum_{r=0}^{m-1} v^2 K_r \bar{q}_r = 0 \ . \qquad (18)$$

This can be reduced to a matrix form suitable for the same programme as was used for the British approach by the following:

It will be seen later that

$$\left.\begin{aligned} A_o &= - B_{\nu\to\infty} = - B_\infty \quad \text{(say)} \\ K_\sigma &= - C_{\nu=0} = - C_o \quad \text{(say)} \end{aligned}\right\} \qquad (19)$$

so that $-A_o v\lambda$ becomes $B_\infty v\lambda$ and $-v^2 K_\sigma$ becomes $v^2 C_o$ where B_∞, C_o are constants. Also we have the recurrence relations

$$\left.\begin{aligned} (\lambda + p_o v)\, \bar{q}_r &= p_o v\, \bar{q}_{r-1} \\ (\lambda + p_o v)\, \bar{q}_o &= \lambda\, \bar{q} \ . \end{aligned}\right\} \qquad (20)$$

These are not suitable as they stand, as the programme does not allow for terms linear in v, only terms in $v\lambda$. Multiply by λ

$$\left.\begin{aligned}(\lambda^2 + p_o v\lambda)\,\bar{q}_r &= p_o v\lambda\,\bar{q}_{r-1}\\ (\lambda^2 + p_o v\lambda)\,\bar{q}_o &= \lambda^2\,\bar{q}\quad .\end{aligned}\right\} \qquad (21)$$

This gives

$$\begin{pmatrix} A\lambda^2+B_\infty v\lambda+E+v^2C_o & -v^2K_o & -v^2K_1 & & -v^2K_{m-1} \\ -\lambda^2 I & (\lambda^2+p_o v\lambda)I & 0 & & 0 \\ 0 & -p_o v\lambda I & (\lambda^2+p_o v\lambda)I & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & -p_o v\lambda\, I & (\lambda^2+p_o v\lambda)I \end{pmatrix} \begin{pmatrix} \bar{q} \\ \bar{q}_o \\ \bar{q}_1 \\ \vdots \\ \bar{q}_{m-1} \end{pmatrix} = 0$$

... (22)

where I, 0 are unit and null matrices respectively.

This is the same type of eigenvalue problem as obtained by the British method (a second order, real lambda matrix) and it can be solved by the same computer programme.

If the matrices A etc. are of order n, this problem will give rise to $2n(m+1)$ eigenvalues instead of the usual $2n$. The meaning of an extra $2nm$ roots is discussed in some detail in Richardson's paper[1]. Of the $2n(m+1)$ roots, nm roots will be zero and are introduced to give equations in a suitable form for the computer programme, i.e. by the step from equations (20) to (21). The others will consist often of n complex pairs and nm real roots all approximately equal to $-p_o v$; (this is certainly the case when $v \to 0$). Such real roots are probably spurious, but the characteristic equation (10) does not necessarily have just $2n$ roots and it may well be that additional roots which are not approximately equal to $-p_o v$ are significant (see Appendix A). This can only be verified by seeing if the roots persist when an improved approximation is used for K.

3 COMPARATIVE APPLICATION

3.1 The system considered

A hypothetical two-dimensional system in incompressible flow, with freedoms in pitch about the leading edge, heave and control surface rotation was devised. With ℓ as the length of the wing chord, the control surface chord was $0{\cdot}24\ell$. The matrices A, B, C and E for this system are given by

$$A = \begin{pmatrix} 14{\cdot}767 & 7{\cdot}0154 & 0{\cdot}8796 \\ 7{\cdot}0154 & 4{\cdot}271 & 0{\cdot}7269 \\ 0{\cdot}8796 & 0{\cdot}7269 & 0{\cdot}927 \end{pmatrix}$$

$$B = 2\begin{pmatrix} \ell_{\dot z} & \ell_{\dot\alpha} & 10\ell_{\dot\beta} \\ -m_{\dot z} & -m_{\dot\alpha} & -10m_{\dot\beta} \\ -10h_{\dot z} & -10h_{\dot\alpha} & -100h_{\dot\beta} \end{pmatrix}$$

$$C = 2\begin{pmatrix} \ell_{z} & \ell_{\alpha} & 10\ell_{\beta} \\ -m_{z} & -m_{\alpha} & -10m_{\beta} \\ -10h_{z} & -10h_{\alpha} & -100h_{\beta} \end{pmatrix}$$

$$E = \begin{pmatrix} 2{\cdot}21 & 0{\cdot}7735 & 0 \\ 0{\cdot}7735 & 1{\cdot}3807 & 0 \\ 0 & 0 & 0{\cdot}79 \end{pmatrix}$$

where ℓ_z, $\ell_{\dot z}$ etc. are the two-dimensional aerodynamic derivatives, defined in Ref.7 and are functions of the frequency parameter $\nu = \omega/v$. The matrices B and C were evaluated for $\nu = 0{\cdot}1$, 0·28, 0·5, 0·6, 0·8, 1·0, 1·3, 1·6, 2·0, 2·2, 2·4, 2·6 and 5·0 using the formulae of Ref.7. The values are shown in Table 1 together with the values of C_0 and B_∞ required for the Richardson approach.

3.2 Results of the British approach

The responses of the system without lining-up were calculated using EMA programme R.A.E. 272/A to solve the eigenvalue equation (1) for a range of velocities ($v = 0$–$1{\cdot}1$) and for 8 values of frequency parameter in the range $\nu = 0{\cdot}5$ to $5{\cdot}0$. These results are tabulated in Table 2 and shown graphically in Figs.1-17. Fig.1 shows the critical conditions (obtained by interpolation) and the other Figures show the variation of the eigenvalues through the speed range.

The imaginary parts ω of the eigenvalue are plotted directly, but instead of the real parts μ, the relative damping ratios, $-\mu/\sqrt{\mu^2+\omega^2}$ are shown. Each pair of curves is labelled A, B or C according to the value of ω at the start of the curve ($v = 0$). It must however be borne in mind that what we have in the (ω, ν, v) space and similarly in the ($-\mu/\sqrt{\mu^2+\omega^2}$, ν, v) space for the whole set of results, is not necessarily three separate surfaces, but quite likely one surface which is triple-valued for each point (ν, v). Thus any point on this surface may be reached from different values of ω in the $v = 0$ plane, according to the route taken along the surface. When the lining-up of the values of ν was done as described in section 2.1 this was indeed found to be the case. The resulting curves of frequency and relative damping ratio are shown in Figs.18 and 19 and listed in Table 3. They are again labelled A, B or C according to the value of the frequency at $v = 0$. Some of the points on these A, B curves correspond to points on the B, C unlined-up curves, for the lowest frequency parameter. This complication made it necessary to obtain results for a large number of ν and v (see Table 2). The lined-up graph indicates a flutter speed of $v = 0{\cdot}79$ but no instability near $v = 0$ as is suggested by the unlined-up curves for the lower values of ν (cf. Figs.3, 5, 7 and 9). The unlined-up curves for $\nu = 1{\cdot}0$ upwards have the same character as the lined-up curves though the actual values can be considerably different. For example, the relative damping ratio for the least damped root is much larger at subcritical speeds on the $\nu = 1{\cdot}0$ curve than on the lined-up curve (cf. Figs.9 and 19). The critical speed from the unlined-up results does not vary very much from the true value except at the higher values of ν (see Fig.1).

3.3 Results of American approach

The matrix $E^{-1}\left(A - \frac{iB}{\nu} - \frac{C}{\nu^2}\right)$ required for the American method was evaluated for a range of values of ν ($\nu = 0{\cdot}5$ to $5{\cdot}0$) and are listed in

Table 4. The eigenvalues of this matrix were then obtained by inverse iteration using EMA Programme 622 and a purpose written calling routine by R.J. Davies. The programme required initial estimated values and the ones used were based either on the results of calculations from other values of ν or on ones from the British approach solutions. It was not necessary to obtain all three eigenvalues this way; when two had been found the third could be deduced by the following device.

For a matrix X of order n, the equation defining the eigenvalues - the characteristic equation is $|X - \lambda I|$ which is of degree n in λ. It may be written out

$$(-\lambda)^n + T_r(X)(-\lambda)^{n-1} + \dots + |X| = 0$$

where $T_r(X)$ is the trace of X (i.e. the sum of the diagonal elements). From the elementary theory of equations we have

$$\sum_{r=1}^{n} \lambda_r = - \text{coefficient } \lambda^{n-1}/\text{coefficient } \lambda^n$$

$$= T_r(X)$$

so that the sum of the eigenvalues is the trace of the matrix. Once two of them were known therefore the third could be calculated with relative ease.

The results of the calculations are plotted in Figs.19-20 and listed in Table 5. The eigenvalues obtained from the trace of the matrix are indicated in the table. $(-g/2)$ has been plotted as being comparable with the relative damping ratio obtained from the other methods.

3.4 Results of the Richardson approach

The K_r matrices occurring in the series approximation (13) to the indicial aerodynamic matrix K were obtained as described in Appendix B, for two values of p_o and two values of m viz.

$$p_o = 0{\cdot}4, \quad 0{\cdot}6$$

$$m = 2, \quad 3 .$$

(See section 2.3.)

In each case the values of the B and C matrices for the following values of ν_q were used to obtain the least squares solution.

$$\nu_q = 0{\cdot}1,\ 0{\cdot}28,\ 0{\cdot}5,\ 0{\cdot}6,\ 0{\cdot}8,\ 1{\cdot}0,\ 1{\cdot}3,\ 1{\cdot}6,\ 2{\cdot}6,\ 5{\cdot}0\ .$$

The values of C_o and B_∞ (see equations (B-9), (B-10), (B-13)) were also required and all these values of B and C are given in Table 1. The resultant K_r matrices are shown in Table 6.

Two checks were made to see how good were the approximations to the aerodynamic matrix K. Equation (B-7) was first used to obtain the first element on the leading diagonal of the matrices B and C (i.e. B_{11} and C_{11}). Some of these were obtained for the same values of ν_q as used in the calculation of the K_r matrices for direct comparison, and others at values of x which made calculation of equation (B-7) simple. The results are given in Table 7 and plotted against x in Figs.22-25. Secondly for one value of ν (1·0) and one value of p_o (0·6) the complete B and C matrices were evaluated from equation (B-7). The comparison with those originally calculated from the equations of Ref.7 is shown in Table 8.

All the results show that the approximation with $p_o = 0{\cdot}6$ and $m = 3$ is the best of the four cases considered. It gives results that lie almost always within 5% of the true value. An exception is the coefficient B_{12} in Table 8 where there is a 15% difference. This is however an unusual case in that $(B_\infty)_{12}$ is much larger than B_{12}, and the approximation to $\bar{B}_{12}$ (see equation (B-10)) is much better.

The eigenvalues of equation (22) were obtained with the same computer programme R.A.E. 272/A as for the British approach using the four sets of K_r matrices corresponding to the two chosen values of p_o and m, for a range of values of v from 0 to 1·0. The complex eigenvalues are given in Table 9. In every case three pairs of such eigenvalues were obtained and in a few cases a fourth pair of complex eigenvalues were found. In addition there were a number of zero real roots* and a number of real roots all approximately equal to $-p_o v$ (see section 2.3). The fourth pair of complex roots, when present, were very little different from $\lambda = -p_o v$ at low values of v, and it is impossible to decide where they become a pair of equal roots. Two almost

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*The large number of real roots (up to 18) at $v = 0$ were obtained without difficulty by the programme used, R.A.E. 272/A.

equal roots can be found as a complex pair with very small imaginary parts for the numerical accuracy can never be perfect. As in the other methods the curves have been labelled A, B, C, D according to the value of ω at $v = 0$.

The values of Table 9 are plotted in Figs.26-33. Apart from the extra root which is present for the cases where $m = 3$, all the approximations to K give very similar results. The $m = 2$ approximations give rather lower relative damping ratios for the least stable curve at subcritical speeds (cf. e.g. Figs.26 and 29). The critical speeds are to all intents and purposes the same in each case. The most noticeable difference is between the frequency curves for the two approximations where $m = 3$ (Figs.26 and 30). When $p_0 = 0{\cdot}6$ the frequency of the fourth eigenvalue (D) rises more rapidly than for $p_0 = 0{\cdot}4$ and this affects the form of the B curve at the higher speeds. In view of the comparison referred to above, one would expect the ($p_0 = 0{\cdot}6$, $m = 3$) results to be the best approximation to the true solution.

3.5 Comparisons between the different methods

The relevant comparison is that between the best approximation to the true solution for all speeds, as given by the Richardson approach using $m = 3$ and $p_0 = 0{\cdot}6$ (Figs.26 and 27), which we will call the 'true' solution, and the solutions obtained by the other methods.

(i) The British approach - the best solution for constant ν (Figs.8 and 9 for $\nu = 1{\cdot}0$).

(ii) The British approach - lined-up ν solution (Figs.18 and 19).

(iii) The American approach (Figs.20 and 21).

Inspection of these figures shows that the lined-up solution (ii) is quite a good approximation to the 'true' solution. The frequencies and relative damping ratios at subcritical speeds, the frequency peak at $\omega = 0{\cdot}6$, $v = 0{\cdot}8$ and the critical flutter speed all show good agreement* with the 'true values'. The rate of change of the relative damping ratio at the critical flutter speed is rather less than the 'true' value and there is an indication that one pair of complex roots (curve B in Figs.18 and 19) will become real at about $v = 0{\cdot}9$, which is quite different from the behaviour of curve B in Figs.26 and 27.

The values of the frequencies and fictitious structural damping obtained by the American approach (Figs.20 and 21) give fairly good indications of the

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*The critical flutter speeds and frequencies for all the cases are compared in Table 10 (see also Fig.1).

'true' frequencies and relative damping ratios when the latter are small ($< 0{\cdot}1$). The least stable root appears to be rather more unstable than it really is. The critical flutter speed is accurate, but the rate of change of $(-g/2)$ at this point does suggest a somewhat less violent onset of flutter than the 'true' results indicate. The American approach results are however of little value in predicting the free oscillation characteristics of the system.

Except for the value of the critical flutter speed the best unlined-up British approach solutions (Figs.8 and 9) are not in good agreement with the 'true' values. In particular the least stable root is much more stable between $v = 0{\cdot}4$ and $0{\cdot}8$ than is really the case.

4 CONCLUSIONS

The following points summarise the findings of this investigation. It would, of course, be desirable to repeat the investigation for an actual aircraft, using three-dimensional aerodynamics and larger values of m in the series approximations to the unsteady aerodynamic forces. Programme limitations made it impossible to take larger values of m in the present calculations.

(i) When using the British approach, it is important to line-up the assumed and calculated values of the frequency parameter ν. The method is then adequate for most purposes.

(ii) The American approach is of use when one is interested only in critical flutter speeds, for most of the information obtained is not what is required by the flutter analyst.

(iii) For the accurate determination of critical flutter speeds, the American approach is the simplest. The lining-up in the British approach is laborious and prone to error, though a good approximation may be obtained without lining-up provided the assumed frequency parameter is well chosen.

(iv) The Richardson approach is more straightforward than the lined-up British approach and is believed to give a truer solution. It might therefore be the best method to use in some cases from the point of view of accuracy and convenience. There is however the disadvantage of having a much larger eigenvalue problem to solve. Computing limitations may therefore make the Richardson approach unusable when a system with a large number of degrees of freedom is being considered. This problem may be minimised if, in the computing procedure, advantage is taken of the sparseness of the matrix in equation (22).

(v) The results of the Richardson approach show that as a consequence of aerodynamic effects, extra natural frequencies of a system may appear which are not present when the airspeed is zero. These are distinct from the rigid body natural frequencies - short period oscillations etc. This possibility should be borne in mind during flight flutter tests.

Appendix A

THE NATURE OF THE SYSTEM'S FREE MOTION

by D.L. Woodcock

An alternative approach perhaps clarifies the significance of the eigenvalues of the lambda matrix in (22). We will consider the response of the system to impulsive forces applied at an instant $\tau = \tau_1 > 0$, i.e. the solution of

$$(A-A_1)\frac{d^2q}{d\tau^2} + Eq = v^2\int_0^{\tau} K(\tau-\tau_o)\frac{dq(\tau_o)}{d\tau_o}\,d\tau_o + e^{-p_o v(\tau-\tau_1)}\sum_{s=0}^{\infty}\delta^{(s)}(\tau-\tau_1)\,f_s$$

(say) ... (A-1)

where f_s are arbitrary constant column matrices. Taking the Laplace transform of this we have, since $q(0) = \left(\frac{dq}{d\tau}\right)_{\tau=0} = 0$, assuming $R\ell(p) > 0$,

$$[(A-A_1)\,p^2 + E - v^2 p\,\bar{K}(p)]\,\bar{\bar{q}}(p) = e^{-p\tau_1}\sum_{s=0}^{\infty}(p+p_o v)^s f_s \qquad \text{(A-2)}$$

where $\bar{\bar{q}}(p)$ is the Laplace transform of $q(\tau)$ and $\bar{K}(p)$ is the Laplace transform of $K(\tau)$. With the approximation (13) for $K(\tau)$

$$\bar{K}(p) = \frac{A_o}{v} - \frac{pA_1}{v^2} + \frac{K_\sigma}{p} + \sum_{r=0}^{m-1}\frac{K_r(p_o v)^r}{(p+p_o v)^{r+1}} \qquad \text{(A-3)}$$

and (A-2) becomes

$$\left[Ap^2 - vp\,A_o + E - v^2 K_\sigma - v^2 p\sum_{r=0}^{m-1}\frac{K_r(p_o v)^r}{(p+p_o v)^{r+1}}\right]\bar{\bar{q}}(p) = e^{-p\tau_1}\sum_{s=0}^{\infty}(p+p_o v)^s f_s$$

... (A-4)

Now if the roots of

$$\left|(p+p_o v)^m\,(Ap^2 - vp\,A_o + E - v^2 K_\sigma) - v^2 p\sum_{r=0}^{m-1} K_r(p_o v)^r\,(p+p_o v)^{m-r-1}\right| = 0$$

... (A-5)

are

$$p = \lambda_i \qquad i = 1 \ldots t \tag{A-6}$$

where $i = t_o(=1) \ldots t_1-1$ indicates single roots

$= t_1 \quad \ldots t_2-1$ indicates double roots

etc.

then

$$\left[Ap^2 - v\,p\,A_o + E - v^2 K_\sigma - v^2 p \sum_{r=0}^{m-1} \frac{K_r(p_o v)^r}{(p+p_o v)^{r+1}}\right]^{-1}$$

$$= (p+p_o v)^m \sum_{j=0}^{n(m+2)-1} \sum_{i=t_j}^{t} \frac{R_{ij}}{(p-\lambda_i)^{j+1}} \tag{A-7}$$

where the R_{ij} are constant matrices.

From (A-7) and (A-4) we thus get an expression for $\bar{\bar{q}}(p)$ which we write as

$$\bar{\bar{q}}(p) = e^{-p\tau_1} \sum_{s=0}^{\infty} \bar{Q}_s(p)\, f_s \tag{A-8}$$

where

$$\bar{Q}_s(p) = \sum_{j=0}^{n(m+2)-1} \sum_{i=t_j}^{t} \frac{(p+p_o v)^{m+s}}{(p-\lambda_i)^{j+1}} R_{ij} \; . \tag{A-9}$$

Taking the inverse transform of (A-8) we have

$$q(\tau+\tau_1) = \sum_{s=0}^{\infty} Q_s(\tau)\, H(\tau)\, f_s \tag{A-10}$$

where $Q_s(\tau)$ is the inverse transform of $\bar{Q}_s(p)$ and is given by

$$Q_s(\tau) = \sum_{j=0}^{n(m+2)-1} \sum_{i=t_j}^{t} e^{\lambda_i \tau} \frac{e^{-(p_o v+\lambda_i)\tau}}{j!} \frac{\partial^{m+s}}{\partial \tau^{m+s}} \{e^{(p_o v+\lambda_i)\tau} \tau^j\} R_{ij} \quad . \tag{A-11}$$

Each term in $Q_s(\tau)$ is therefore a finite polynomial (of degree j) in τ, multiplied by $e^{\lambda_i \tau}$.

But (A-4) can be written, multiplying both sides by $(p+p_o v)^m$

$$\sum_{u=0}^{m+2} D_u (p+p_o v)^u \bar{\bar{q}}(p) = e^{-p\tau_1} \sum_{s=0}^{\infty} (p+p_o v)^{m+s} f_s \tag{A-12}$$

where the matrices D_u are simply related to the matrices A, A_o, E, K_σ and K_r of (A-4). Thus, since

$$(p+p_o v)^u \bar{Q}_s(p) = \bar{Q}_{s+u}(p) \tag{A-13}$$

then substituting for $\bar{\bar{q}}(p)$ from (A-8) in (A-12) gives, remembering that the f_s are arbitrary,

$$\sum_{u=0}^{m+2} D_u \bar{Q}_{s+u}(p) = (p+p_o v)^{m+s} I \tag{A-14}$$

for any $s \geqslant 0$.

Taking inverse transforms we have

$$\sum_{u=0}^{m+2} D_u Q_{s+u}(\tau) = e^{-p_o v\tau} \delta^{(m+s)}(\tau) I \tag{A-15}$$

$$= 0 \text{ for almost all } \tau \quad .$$

Consequently there are only (m+2) independent solutions and so we can rewrite (A-10) as

$$q(\tau+\tau_1) = \sum_{s=0}^{m+1} Q_s(\tau)\ H(\tau)\ f_s \qquad \text{(A-16)}$$

(the meaning of the column matrices f_s is here changed slightly). If all the roots of (A-5) are distinct then the expression for $Q_s(\tau)$ simplifies to (since $t_1 - 1 = t$, i.e. $j = 0$ only)

$$Q_s(\tau) = \sum_{i=1}^{t} (p_o v + \lambda_i)^{m+s}\ e^{\lambda_i \tau}\ R_{io} \ . \qquad \text{(A-17)}$$

The above expression for q (equation (A-16)) shows that each root of (A-5) represents, in general, genuine exponential behaviour of the solution of the equations of motion, when the approximation (13) to $K(\tau)$ is made. The exceptions are when all the coefficients of a particular $e^{\lambda_i \tau}$ in the equations (A-11) are zero. This may arise from a chance form of initial disturbance, and so is of no importance; or for other reasons such as:-

At $v = 0$ there will be an nm multiple root $\lambda_i = -p_o v\ (=0)$. For this root the coefficient of $R_{ij}\ e^{\lambda_i \tau}$ in (A-11) is zero for all $j < (m+s)$. Moreover comparison of the general form of the expansion (A-7) with the particular form for $r = 0$ (i.e. the expansion of $(Ap^2 + E)^{-1}$) shows that the matrices R_{ij} are null for all $j \geq m$. Consequently the coefficient of this $e^{\lambda_i \tau}$ in (A-11) is zero for each value of s. But these are isolated instances, and so we can say that none of the λ_i obtained from (A-5), or from (22), when the nm zero roots are deleted, are spurious solutions if the approximation (13) to $K(\tau)$ is correct. However we have evaluated the coefficients in this approximation by making the value of its transform $\bar{K}(p)$ (equation (A-3)) agree as closely as possible with the true value for purely imaginary values of p. Moreover $\bar{K}(p)$ actually has the form* (see Ref.6)

---

*Taken as single valued in the complex plane cut along the negative real axis.

$$\bar{K}(p) = \frac{1}{p}\left\{ \sum_{s=0}^{\infty} M_s\, p^s + p^2 \log p \sum_{s=0}^{\infty} N_s\, p^s \right\} \tag{A-18}$$

and so it follows that our approximation (A-3) cannot be very good at points near the negative real axis. Thus one would expect values of λ_i, which are roots of (A-5), to be good approximations to the systems eigenvalues, with the true $K(\tau)$, when they are complex with relatively not too large real parts. This suggests that the roots which we have obtained, which are approximately equal to $-p_o v$, are almost certainly spurious roots of the actual problem, but when such a root develops a sizeable imaginary part it may well not be spurious. Indeed Milne[6] has shown that with the true $K(\tau)$ the system cannot have any negative real roots. In addition, in the same paper, he shows that the solution of (A-1) has the form, for $\tau > 0$,

$$q(\tau) = \sum_{i=1}^{m} q_i\, e^{\lambda_i \tau} + \sum_{j=1} r_j \frac{1}{\tau^j} + O\left(e^{-\mu_m \tau}\right) \tag{A-19}$$

where λ_i $(i = 1 \ldots m)$ are all the roots, assumed distinct, of the characteristic equation (10) whose real parts are greater than $(-\mu_m)$, and the second term is an asymptotic expansion for an integral given in Ref.6. The leading non-zero term in this asymptotic expansion will theoretically dominate any decaying exponentials when τ is large, but the little experience there is[6] suggests that this does not occur until the value of τ is much too large to be of any practical interest.

Appendix B

DETERMINATION OF THE K_r MATRICES

Substitution of the series (13) for $K(\tau)$ in equation (11) gives, assuming $R\ell(p) > 0$,

$$\bar{K}(p) = \frac{A_o}{v} - p\frac{A_1}{v^2} + \frac{K_\sigma}{p} + \sum_{r=0}^{m-1} \frac{K_r(p_o v)^r}{(p + p_o v)^{r+1}} \quad . \tag{B-1}$$

If we now go to the limit $p \to i\omega$ we obtain, remembering $\nu = \omega/v$,

$$-i\omega\bar{K}(i\omega) = -iv\nu\left[\frac{A_o}{v} - \frac{i\nu A_1}{v} + \frac{K_\sigma}{iv\nu} + \sum_{r=0}^{m-1} \frac{K_r(p_o v)^r}{v^{r+1}(p_o + i\nu)^{r+1}}\right] \quad . \tag{B-2}$$

Substituting for $\bar{K}(i\omega)$ from (12)

$$-\nu^2 A_1 + i\nu B + C = -iv\nu\left[\frac{A_o}{v} - \frac{i\nu A_1}{v} + \frac{K_\sigma}{iv\nu} + \sum_{r=0}^{m-1} \frac{K_r(p_o v)^r}{v^{r+1}(p_o + i\nu)^{r+1}}\right] \quad . \tag{B-3}$$

When $\nu = \infty$

$$B = B_\infty = -A_o \tag{B-4}$$

When $\nu = 0$, $\nu B = 0$

$$C = C_o = -K_\sigma \quad . \tag{B-5}$$

Thus with the substitution

$$x = \nu^2/(\nu^2 + p_o^2) \tag{B-6}$$

$$
\begin{aligned}
i\nu(B-B_\infty) + (C-C_o) &= -\sum_{r=0}^{m-1} iK_r(1-x)^{r/2}\, x^{\frac{1}{2}}\, (\sqrt{1-x} - i\sqrt{x})^{r+1} \\
&= -i\sqrt{x(1-x)}\ \{K_o + K_1(1-2x) + K_2(1-x)\,(1-4x) + \ldots\} \\
&\quad - \{K_o x + K_1 2x(1-x) + K_2 x(1-x)(3-4x) \ldots\}
\end{aligned}
\tag{B-7}
$$

B and C are evaluated for ℓ values of ν (and hence x) $\ell \geqslant m$, and the resulting equations solved by the least squares method.

If δ_{ij} is the modulus of the error in the satisfaction of this equation for the ij^{th} term of the matrix then

$$
\begin{aligned}
\delta_{ij}^2 &= \{\bar{C}_{ij} + x\,[(K_{ij})_o + 2(1-x)\,(K_{ij})_1 + (1-x)\,(3-4x)\,(K_{ij})_2 + \ldots]\}^2 \\
&\quad + \{\nu\bar{B}_{ij} + \sqrt{x(1-x)}\,[(K_{ij})_o + (1-2x)\,(K_{ij})_1 + (1-x)\,(1-4x)\,(K_{ij})_2 + \ldots]\}^2 \\
&= \left\{\bar{C}_{ij} + \sum_{r=0}^{m-1} \alpha_r(x)\,(K_{ij})_r\right\}^2 + \left\{\nu\bar{B}_{ij} + \sum_{r=0}^{m-1} \beta_r(x)\,(K_{ij})_r\right\}^2 \quad \text{(say)}
\end{aligned}
\tag{B-8}
$$

where

$$C-C_o = \bar{C} = [\bar{C}_{ij}] \tag{B-9}$$

$$B-B_\infty = \bar{B} = [\bar{B}_{ij}] \tag{B-10}$$

$$K_r = [(K_{ij})_r]\ . \tag{B-11}$$

Let S_{ij} be the sum of $(\delta_{ij})^2$. The required $(K_{ij})_r$ are then given by the solution of the set of simultaneous equations

$$\frac{\partial S_{ij}}{\partial (K_{ij})_r} = 0 \qquad \text{for } r = 0 \text{ to } (m-1)$$

i.e.

$$0 = \sum_{q=0}^{\ell-1} \left[\alpha_r(x_q) \left\{ \bar{C}_{ij}(\nu_q) + \sum_{s=0}^{m-1} \alpha_s(x_q)\ (K_{ij})_s \right\} \right.$$

$$\left. + \beta_r(x_q) \left\{ \nu_q\ \bar{B}_{ij}(\nu_q) + \sum_{s=0}^{m-1} \beta_s(x_q)\ (K_{ij})_s \right\} \right] \quad . \qquad \text{(B-12)}$$

These equations for the different values of r and ij can be combined to give the single matrix equation.

$$\begin{bmatrix} \gamma_{00} I & \gamma_{01} I & \gamma_{02} I & \cdots & \gamma_{0,m-1} I \\ \gamma_{10} I & \gamma_{11} I & \gamma_{12} I & \cdots & \gamma_{1,m-1} I \\ \gamma_{20} I & \gamma_{21} I & \gamma_{22} I & \cdots & \gamma_{2,m-1} I \\ \vdots & \vdots & \vdots & & \vdots \\ \gamma_{m-1,0} I & \gamma_{m-1,1} I & \gamma_{m-1,2} I & \cdots & \gamma_{m-1,m-1} I \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \\ \vdots \\ K_{m-1} \end{bmatrix} = -\sum_{q=0}^{\ell=1} \left\{ \begin{bmatrix} \alpha_0(x_q)\ \bar{C}(\nu_q) \\ \alpha_1(x_q)\ \bar{C}(\nu_q) \\ \\ \\ \end{bmatrix} + \begin{bmatrix} \beta_0(x_q)\ \nu_q\ \bar{B}(\nu_q) \\ \beta_1(x_q)\ \nu_q\ \bar{B}(\nu_q) \\ \\ \\ \end{bmatrix} \right\}$$

(B-13)

where

$$\gamma_{rs} = \sum_{q=0}^{\ell-1} \{\alpha_r(x_q)\ \alpha_s(x_q) + \beta_r(x_q)\ \beta_s(x_q)\} \quad . \qquad \text{(B-14)}$$

This equation can easily be solved to give the matrices K_r. The γ_{rs} turn out to be surprisingly simple.

$$\left.\begin{aligned} \gamma_{00} &= \sum_{q=0}^{\ell-1} x_q \\ \gamma_{01} = \gamma_{10} = \gamma_{11} &= \sum_{q=0}^{\ell-1} x_q(1-x_q) \\ \gamma_{02} = \gamma_{20} &= \sum_{q=0}^{\ell-1} x_q(1-x_q)\ (1-2x_q) \\ \gamma_{12} = \gamma_{21} = \gamma_{22} &= \sum_{q=0}^{\ell-1} x_q(1-x_q)^2 \end{aligned}\right\} \qquad \text{(B-15)}$$

etc.

Table 1

OSCILLATORY AERODYNAMIC MATRICES

	B_{∞}			C_{o}		
	3·14159 0·78540 0·15912	3·92699 1·76715 0·60969	5·95689 3·97733 2·97667	0 0 0	6·28319 1·57080 0·31825	37·5622 15·8822 5·41081
ν	B			C		
0·1	5·71147 1·42787 0·28929	−2·35420 0·19685 0·29154	−40·61437 −7·66548 0·61777	0·08209 0·02052 0·00416	5·77304 1·44326 0·29241	34·22431 15·04774 5·24176
0·28	4·92207 1·23052 0·24931	1·11343 1·06376 0·46718	−17·11312 −1·79017 1·80814	0·32528 0·08132 0·01648	5·16603 1·29151 0·26167	29·74182 13·92712 5·01471
0·5	4·35144 1·08786 0·22041	2·50648 1·41202 0·53774	−6·78208 0·79259 2·33142	0·58197 0·14549 0·02948	4·78792 1·19698 0·24251	26·58033 13·13675 4·85458
0·6	4·17814 1·04453 0·21163	2·82657 1·49204 0·55395	−4·26015 1·42308 2·45916	0·67602 0·16900 0·03424	4·68515 1·17129 0·23731	25·63583 12·90062 4·80674
0·8	3·92684 0·98171 0·19890	3·22015 1·59043 0·57389	−1·02519 2·23182 2·62302	0·82930 0·20733 0·04201	4·54882 1·13720 0·23040	24·28273 12·56235 4·73820
1·0	3·75694 0·93924 0·19029	3·44156 1·64579 0·58510	0·89489 2·71184 2·72027	0·94694 0·23673 0·04796	4·46715 1·11679 0·22627	23·38156 12·33705 4·69256
1·3	3·58938 0·89734 0·18181	3·62527 1·69172 0·59441	2·58135 3·13345 2·80569	1·07747 0·26937 0·05458	4·39748 1·09937 0·22274	22·50687 12·11838 4·64825
1·6	3·48181 0·87045 0·17636	3·72465 1·71656 0·59944	3·55303 3·37637 2·85491	1·17121 0·29280 0·05932	4·36021 1·09005 0·22085	21·95505 11·98043 4·62030
2·0	3·38937 0·84734 0·17168	3·79781 1·73485 0·60315	4·31486 3·56683 2·89350	1·26007 0·31502 0·06382	4·33442 1·08361 0·21954	21·48893 11·86390 4·59669
2·2	3·35657 0·83914 0·17001	3·82089 1·74062 0·60431	4·56797 3·63011 2·90632	1·29392 0·32348 0·06554	4·32701 1·08175 0·21917	21·32581 11·82312 4·58843
2·4	3·32981 0·83245 0·16866	3·83853 1·74503 0·60521	4·76740 3·67996 2·91642	1·32262 0·33066 0·06699	4·32178 1·08044 0·21890	21·19377 11·79011 4·58174
2·6	3·30770 0·82693 0·16754	3·85229 1·74847 0·60590	4·92725 3·71993 2·92452	1·34716 0·33679 0·06824	4·31807 1·07952 0·21872	21·08547 11·76303 4·57626
5·0	3·19653 0·79913 0·16191	3·90876 1·76259 0·60877	5·65508 3·90188 2·96138	1·48588 0·37147 0·07526	4·31094 1·07774 0·21836	20·55591 11·63064 4·54943

Table 2

UNLINED-UP BRITISH METHOD SOLUTIONS

$\nu = 0{\cdot}5$	ω			$\frac{-\mu}{\sqrt{\mu^2 + \omega^2}}$		
v	A	B	C	A	B	C
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·3026	0·3831	0·8598	−0·0335	0·0263	0·1955
0·2	1·3455	0·3970	0·8295	−0·0355	0·0456	0·3454
0·3	1·3837	0·4152	0·8122	−0·0181	0·0611	0·4533
0·4	1·4153	0·4366	0·8053	0·0149	0·0788	0·5241
0·5	1·4412	0·4634	0·8000	0·0656	0·1004	0·5648
0·6	1·4684	0·5020	0·7769	0·1384	0·1224	0·5845
0·7	1·5215	0·5675	0·6888	0·2290	0·1219	0·6148
0·8	1·6233	0·6478	0·5119	0·3063	0·0360	0·7361
0·9	1·7540	0·6810	0·2901	0·3575	−0·0792	0·9065
1·0	1·8957	0·6838		0·3917	−0·1776	
1·1	2·0424	0·6720		0·4159	−0·2634	
$\nu = 0{\cdot}6$						
0	1·274	0·3776	0·8839	0	0	0
0·1	1·2950	0·3832	0·8669	−0·0238	0·0300	0·1835
0·2	1·3271	0·3974	0·8455	−0·0230	0·0521	0·3292
0·3	1·3535	0·4159	0·8378	−0·0043	0·0696	0·4347
0·4	1·3701	0·4374	0·8437	0·0302	0·0889	0·5019
0·5	1·3748	0·4641	0·8583	0·0848	0·1130	0·5345
0·55	1·3725	0·4811	0·8657	0·1232	0·1269	0·5376
0·6	1·3694	0·5024	0·8681	0·1725	0·1412	0·5307
0·65	1·3739	0·5310	0·8534	0·2344	0·1541	0·5147
0·7	1·4035	0·5730	0·7970	0·2983	0·1583	0·5057
0·75	1·4586	0·6340	0·6916	0·3469	0·1268	0·5440
0·8	1·5241	0·6821	0·5825	0·3807	0·0501	0·6346
0·85	1·5928	0·7028	0·4889	0·4057	−0·0243	0·7294
0·9	1·6626	0·7099	0·3940	0·4252	−0·0875	0·8197
0·95	1·7332	0·7101	0·2795	0·4409	−0·1424	0·9077
1·0	1·8041	0·7060	0·5997	0·4540	−0·1918	0·9957
1·05	1·8755	0·6991		0·4651	−0·2371	
1·1	1·9472	0·6899		0·4746	−0·2797	
$\nu = 0{\cdot}8$						
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2849	0·3833	0·8761	−0·0103	0·0345	0·1671
0·2	1·3010	0·3978	0·8688	−0·0520	0·0603	0·3065
0·3	1·3091	0·4165	0·8764	0·0144	0·0801	0·4094
0·4	1·3019	0·4381	0·9035	0·0484	0·1014	0·4739
0·5	1·2688	0·4643	0·9546	0·1023	0·1285	0·5005
0·6	1·1726	0·5005	1·0572	0·1941	0·1646	0·4782
0·7	0·9338	0·5649	1·2680	0·2585	0·2166	0·4641
0·8	0·7573	0·6111	1·4233	0·0492	0·4848	0·4955
0·9	0·7525	0·4819	1·5615	−0·1105	0·6966	0·5207
1·0	0·7363	0·3244	1·6961	−0·2169	0·8655	0·5400

Table 2 (Contd)

ν = 1·0	ω			$\frac{-\mu}{\sqrt{\mu^2 + \omega^2}}$		
v	A	B	C	A	B	C
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2789	0·3833	0·8817	-0·0016	0·0370	0·1567
0·2	1·2838	0·3979	0·8844	0·0065	0·0648	0·2919
0·3	1·2786	0·4168	0·9037	0·0257	0·0860	0·3939
0·4	1·2545	0·4384	0·9461	0·0560	0·1083	0·4599
0·5	1·1980	0·4642	1·0199	0·0973	0·1368	0·4937
0·6	1·0911	0·4984	1·1349	0·1371	0·1766	0·5094
0·7	0·9433	0·5517	1·2676	0·1398	0·2429	0·5304
0·8	0·8103	0·5867	1·3951	0·0108	0·4311	0·5535
0·9	0·7781	0·5092	1·5197	-0·1330	0·6268	0·5732
1·0	0·7544	0·3974	1·6441	-0·2360	0·7859	0·5891
1·1	0·7288	0·2231	1·7694	-0·3223	0·9349	0·6017
ν = 1·3						
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2737	0·3834	0·8866	0·0065	0·0391	0·1471
0·2	1·2672	0·3981	0·8995	0·0178	0·0685	0·2780
0·3	1·2481	0·4171	0·9312	0·0353	0·0908	0·3805
0·4	1·2090	0·4388	0·9874	0·0574	0·1137	0·4522
0·5	1·1439	0·4642	1·0692	0·0777	0·1433	0·5011
0·6	1·0547	0·4967	1·1683	0·0853	0·1851	0·5386
0·7	0·9443	0·5407	1·2740	0·0641	0·2559	0·5692
0·8	0·8424	0·5671	1·3826	-0·0356	0·4045	0·5936
0·9	0·7986	0·5224	1·4939	-0·1578	0·5726	0·6129
1·0	0·7698	0·4430	1·6076	-0·2556	0·7183	0·6280
1·1	0·7420	0·3286	1·7235	-0·3394	0·8535	0·6400
ν = 1·6						
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2707	0·3834	0·8894	0·0113	0·0402	0·1414
0·2	1·2568	0·3982	0·9091	0·0248	0·0705	0·2695
0·3	1·2287	0·4174	0·9490	0·0404	0·0933	0·3732
0·4	1·1824	0·4391	1·0116	0·0548	0·1166	0·4509
0·5	1·1186	0·4644	1·0917	0·0623	0·1464	0·5091
0·6	1·0395	0·4960	1·1817	0·0578	0·1889	0·5536
0·7	0·9438	0·5359	1·2776	0·0280	0·2598	0·5874
0·8	0·8552	0·5593	1·3780	-0·0624	0·3923	0·6132
0·9	0·8092	0·5277	1·4823	-0·1739	0·5446	0·6330
1·0	0·7785	0·4641	1·5899	-0·2681	0·6809	0·6483
1·1	0·7498	0·3730	1·7004	-0·3501	0·8074	0·6604

Table 2 (Contd)

ν = 2·6	ω			$\frac{-\mu}{\sqrt{\mu^2+\omega^2}}$		
v	A	B	C	A	B	C
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2665	0·3834	0·8932	0·0186	0·0416	0·1330
0·2	1·2409	0·3984	0·9238	0·0355	0·0730	0·2566
0·3	1·1984	0·4179	0·9770	0·0460	0·0964	0·3643
0·4	1·1468	0·4400	1·0437	0·0459	0·1198	0·4537
0·5	1·0894	0·4654	1·1169	00·0382	0·1497	0·5224
0·6	1·0220	0·4962	1·1962	0·0216	0·1925	0·5730
0·7	0·9422	0·5321	1·2817	−0·0172	0·2616	0·6101
0·8	0·8682	0·5533	1·3726	−0·0994	0·3763	0·6377
0·9	0·8226	0·5355	1·4683	−0·1988	0·5079	0·6585
1·0	0·7906	0·4909	1·5679	−0·2876	0·6298	0·6745
1·1	0·7616	0·4260	1·6709	−0·3665	0·7429	0·6869
ν = 5·0						
0	1·2747	0·3776	0·8839	0	0	0
0·1	1·2643	0·3835	0·8953	0·0225	0·0421	0·1284
0·2	1·2316	0·3986	0·9325	0·0417	0·0740	0·2494
0·3	1·1810	0·4185	0·9931	0·0476	0·0976	0·3609
0·4	1·1295	0·4410	1·0590	0·0390	0·1209	0·4571
0·5	1·0763	0·4669	1·1277	0·0250	0·1506	0·5299
0·6	1·0140	0·4977	1·2022	0·0032	0·1931	0·5827
0·7	0·9408	0·5325	1·2831	−0·0393	0·2607	0·6212
0·8	0·8732	0·5537	1·3697	−0·1191	0·3676	0·6496
0·9	0·8291	0·5418	1·4613	−0·2135	0·4893	0·6709
1·0	0·7974	0·5061	1·5569	−0·2995	0·6030	0·6873
1·1	0·7688	0·4532	1·6560	−0·3766	0·7083	0·7001

Table 3

LINED-UP BRITISH METHOD SOLUTIONS

	A			B			C		
ν	v	ω	$\frac{-\mu}{\sqrt{\mu^2+\omega^2}}$	v	ω	$\frac{-\mu}{\sqrt{\mu^2+\omega^2}}$	v	ω	$\frac{-\mu}{\sqrt{\mu^2+\omega^2}}$
0·5				0·844	0·422	0·804			
0·6	1·14	0·684	−0·312	0·842	0·504	0·71			
0·8	0·935	0·748	−0·154	0·78	0·624	0·42			
1·0	0·808	0·808	−0·020	0·45	0·45	0·122			
1·3	0·714	0·928	0·052	0·326	0·424	0·095			
1·6	0·632	1·011	0·05	0·255	0·408	0·084	0·984	1·576	0·646
2·6	0·434	1·128	0·045	0·15	0·39	0·058	0·402	1·045	0·454
5·0	0·242	1·21	0·048	0·076	0·38	0·032	0·184	0·925	0·23

Table 4

VALUES OF MATRIX $E^{-1}\left(A - \frac{iB}{\nu} - \frac{C}{\nu^2}\right)$

ν	Real part			Imaginary part		
0·5	4·97277	-6·66793	-43·00800	-4·21237	-1·93107	8·13445
	1·87368	3·36114	-13·43767	0·78406	-0·96354	-5·70521
	0·96416	-0·30780	-23·40673	-0·55799	-1·36136	-5·90234
0·6	5·19061	-3·69731	-28·51554	-3·37051	-1·86745	4·74427
	1·83313	2·80821	-9·45273	0·62736	-0·75488	-4·37567
	0·99302	0·08571	-15·72792	-0·44647	-1·16867	-5·18811
0·8	5·47233	-0·83830	-14·90035	-2·37584	-1·63870	1·60096
	1·78070	2·27605	-5·34247	0·44222	-0·52184	-2·91744
	1·03034	0·46442	-8·19803	-0·31471	-0·90805	-4·15035
1·0	5·64118	0·43969	-9·00432	-1·81844	-1·41814	0·35141
	1·74927	2·03818	-3·36446	0·33847	-0·39752	-2·16097
	1·05270	0·63371	-4·76653	-0·24088	-0·74063	-3·44338
1·3	5·79093	1·34244	-4·96893	-1·33641	-1·15927	-0·35759
	1·72140	1·87015	-1·88330	0·24875	-0·29306	-1·54541
	1·07254	0·75329	-2·30816	-0·17703	-0·57878	-2·73193
1·6	5·87808	1·77750	-3·08558	-1·05329	-0·97197	-0·58449
	1·70517	1·78917	-1·13439	0·19605	-0·23251	-1·20093
	1·08408	0·81092	-1·11115	-0·13952	-0·47424	-2·25863
2·0	5·94704	2·07740	-1·82265	-0·82026	-0·79528	-0·65196
	1·69234	1·73335	-0·60060	0·15268	-0·18271	-0·92643
	1·09322	0·85065	-0·28123	-0·10866	-0·38174	-1·93133
2·2	5·97012	2·16917	-1·44387	-0·73848	-0·72806	-0·64838
	1·68804	1·71626	-0·43388	0·13745	-0·16516	-0·83184
	1·09628	0·86281	-0·02661	-0·09782	-0·34771	-1·67222
2·4	5·98837	2·23872	-1·15969	-0·67154	-0·67095	-0·63457
	1·68464	1·70332	-0·30635	0·12500	-0·15073	-0·75504
	1·09870	0·87202	0·16653	-0·08896	-0·31920	-1·53820
2·6	6·00306	2·29271	-0·94105	-0·61577	-0·62190	-0·61551
	1·68191	1·69327	-0·20663	0·11461	-0·13866	-0·69142
	1·10064	0·87917	0·31650	-0·08157	-0·29499	-1·42382
5·0	6·07075	2·51842	-0·05022	-0·30944	-0·32885	-0·39052
	1·66931	1·65126	0·21766	0·05760	-0·07109	-0·34642
	1·10961	0·90907	0·94307	-0·04099	-0·15412	-0·74972

Table 5

AMERICAN METHOD SOLUTIONS

Frequency parameter		Speed		Eigenvalue	
				Real	Imaginary
ν	ω	-g/2	v	$1/\omega^2$	g/ω^2
0·5	0·61072	-0·29496	1·2214	2·68112	1·58164
	0·47955	0·55225	0·9591	4·34852	-4·80293
	-			-22·10245	-7·85694 T
0·6	0·63319	-0·22137	1·0553	2·49424	1·10429
	0·49399	0·44777	0·8233	4·09797	-3·66984
	-			-14·32131	-6·74955 T
0·8	0·69755	-0·07660	0·8719	2·05517	0·31482
	0·48152	0·25480	0·6019	4·31287	-2·19779
	-			-6·81769	-5·16506 T
1·0	0·45178	0·16086	0·4518	4·89935	-1·57618
	0·80645	-0·0004289	0·8064	1·53760	0·0013190
	-			-3·52413	-4·08448
1·3	0·93615	0·02804	0·7201	1·14107	-0·063988
	0·42547	0·11635	0·3273	5·52413	-1·28538
	-			-1·31228	-3·01203
1·6	1·01633	0·03304	0·6352	0·96812	-0·063975
	0·41095	0·09645	0·2568	5·92129	-1·14215
	-			-0·33331	-2·33831
2·0	1·07977	·0·03422	0·5399	0·85771	-0·058705 T
	0·39776	0·07950	0·1999	6·25755	-0·994975
	1·87679	0·13596	0·9384	0·28390	-1·78062
2·2	1·10095	0·03480	0·5004	0·82502	-0·057330 T
	0·39611	0·07306	0·18005	6·37336	-0·931281
	1·47218	1·72004	0·6692	0·46140	-1·58725
2·4	1·11786	0·03545	0·4658	0·80025	-0·056739 T
	0·39327	0·06754	0·1639	6·46557	-0·873348
	1·29925	1·20728	0·5414	0·59240	-1·43038
2·6	1·13174	0·03632	0·4353	0·78073	-0·056710
	0·39103	0·06276	0·1504	6·53989	-0·820828
	1·20194	0·93955	0·4623	0·69220	-1·30071
5·0	1·21409	0·04242	0·2428	0·67842	-0·057550
	0·38132	0·03350	0·0763	6·87734	-0·460743
	0·94945	0·27583	0·1899	1·10931	-0·611957

T indicates the solution is obtained by the 'trace method' see section 3.3.

Table 6

K_r MATRICES

	$p_o = 0{\cdot}6$, m = 3			$p_o = 0{\cdot}4$, m = 3		
K_o	-1·51095 -0·37773 -0·07654	2·02132 0·50534 0·10238	17·38758 4·34687 0·88069	-1·44934 -0·36233 -0·07342	2·00797 0·50200 0·10170	17·09156 4·27287 0·86570
K_1	0·46925 0·11730 0·02380	0·65061 0·16264 0·03296	2·24242 0·56063 0·11357	0·95872 0·23967 0·04859	0·38477 0·09618 0·01949	-1·06499 -0·26623 -0·05396
K_2	-0·59545 -0·14885 -0·03019	1·33257 0·33315 0·06749	10·05638 2·51405 0·50935	-0·80506 -0·20126 -0·04080	0·88432 0·22109 0·04478	8·11242 2·02807 0·41089
	m = 2			m = 2		
K_o	-1·40817 -0·35204 -0·07133	1·79130 0·44783 0·09073	15·65171 3·91291 0·79277	-1·34863 -0·33716 -0·06832	1·89735 0·47434 0·09610	16·07675 4·01917 0·81430
K_1	0·11651 0·02912 0·00592	1·44002 0·36000 0·07294	8·19979 2·04994 0·41531	0·57042 0·14260 0·02891	0·81131 0·20282 0·04109	2·84788 0·71197 0·14423

Table 7

COMPARISON OF AERODYNAMIC COEFFICIENTS

$p_o = 0 \cdot 6$		B_{11}			C_{11}		
ν	X	Calculated from Ref.7	Approximations from eqn. (28) m = 3	Approximations from eqn. (28) m = 2	Calculated from Ref.7	Approximations from eqn. (28) m = 3	Approximations from eqn. (28) m = 2
0	0	6·2832	5·8702	5·2944	0	0	0
0·1	0·0270	5·7115	5·7099	5·2464	0·0821	0·0614	0·0319
0·2	0·1		5·3272	5·1140		0·2060	0·1199
0·28	0·1788	4·9221	4·9875	4·9664	0·3253	0·3322	0·2176
0·3	0·2		4·9078	4·9259		0·3616	0·2444
0·3928	0·3		4·5881	4·7301		0·4813	0·3735
0·4899	0·4		4·3443	4·5265		0·5792	0·5073
0·5	0·4098	4·3514	4·3236	4·5060	0·5820	0·5882	0·5208
0·6	0·5	4·1781	4·1526	4·3151	0·6760	0·6697	0·6458
0·7349	0·6		3·9892	4·0959		0·7671	0·7890
0·8	0·64	3·9268	3·9264	4·0061	0·8293	0·8111	0·8475
0·9165	0·7		3·8301	3·8690		0·8856	0·9368
1·0	0·7353	3·7569	3·7706	3·7870	0·9469	0·9351	0·9901
1·2	0·8		3·6518	3·6343		1·0396	1·0893
1·3	0·8244	3·5894	3·6026	3·5759	1·0775	1·0841	1·1271
1·6	0·8767	3·4818	3·4869	3·4490	1·1712	1·1906	1·2094
1·8	0·9		3·4302	3·3918		1·2436	1·2464
2·6	0·9494	3·3077	3·2974	3·2691	1·3472	1·3667	1·3258
5·0	0·9858	3·1965	3·1875	3·1776	1·4859	1·4685	1·3849
∞	1·0	3·1416	3·1416	3·1416	1·5708	1·5110	1·4082
$p_o = 0 \cdot 4$							
0		6·2832	6·3808	5·0871			
0·1	0·0588	5·7115	5·9247	5·1306	0·0821	0·1023	0·0162
0·1333	0·1		5·6551	5·1493		0·1608	0·0322
0·2	0·2		5·1474	5·1544		0·2665	0·0871
0·2619	0·3		4·8096	5·1024		0·3365	0·1650
0·28	0·3289	4·9221	4·7368	5·0768	0·3253	0·3528	0·1917
0·3266	0·4		4·5932	4·9934		0·3901	0·2657
0·4	0·5		4·4501	4·8274		0·4466	0·3891
0·4899	0·6		4·3319	4·6043		0·5254	0·5354
0·5	0·6098	4·3514	4·3198	4·5795	0·5820	0·5349	0·5509
0·6	0·6923	4·1781	4·2030	4·3478	0·6760	0·6345	0·6906
0·6110	0·7		4·1902	4·3242		0·6457	0·7045
0·8	0·8	3·9268	3·9768	3·9870	0·8293	0·8269	0·8964
1·0	0·8621	3·7569	3·7870	3·7491	0·9469	0·9785	1·0270
1·2	0·9		3·6433	3·5928		1·0884	1·1111
1·3	0·9135	3·5894	3·5864	3·5352	1·0775	1·1309	1·1419
1·6	0·9412	3·4818	3·4599	3·4139	1·1712	1·2238	1·2061
2·6	0·9769	3·3077	3·2751	3·2510	1·3472	1·3560	1·2917
5·0	0·9936	3·1965	3·1794	3·1720	1·4859	1·4230	1·3328
∞	1·0	3·1416	3·1416	3·1416	1·5708	1·4493	1·3486

Table 8

COMPARISON OF AERODYNAMIC COEFFICIENTS ($\nu = 1 \cdot 0$, $p_o = 0 \cdot 6$)

i	j	B_{ij} Calculated from Ref.7	B_{ij} eqn. (28) m = 3	B_{ij} eqn. (28) m = 2	C_{ij} Calculated from Ref.7	C_{ij} eqn. (28) m = 3	C_{ij} eqn. (28) m = 2
1	1	3·7569	3·7706	3·7870	0·94694	0·93514	0·99006
1	2	3·4416	3·4724	3·4357	4·4671	4·5284	4·4055
1	3	0·8949	1·0312	0·7541	23·3816	23·7892	22·8618
2	1	0·93924	0·94265	0·94676	0·23673	0·23378	0·24752
2	2	1·6458	1·6535	1·6443	1·11679	1·13210	1·10137
2	3	2·7118	2·7459	2·6766	12·3370	12·4389	12·2071
3	1	0·19029	0·19098	0·19182	0·04796	0·04736	0·05014
3	2	0·58510	0·58666	0·58481	0·22627	0·22937	0·22314
3	3	2·7203	2·7272	2·7131	4·6926	4·7132	4·6666

Table 9

RICHARDSON METHOD SOLUTIONS

$p_o = 0{\cdot}6$ $m = 3$	ω				$-\frac{\mu}{\sqrt{\mu^2+\omega^2}}$			
v	A	B	C	D	A	B	C	D
0	1·2746	0·3776	0·8839	0	0	0	0	
0·1	1·2635	0·3835	0·8961	0·0067	0·0239	0·0417	0·1272	0·9931
0·2	1·2303	0·3984	0·9335	0·0182	0·0420	0·0711	0·2517	0·9868
0·3	1·1867	0·4177	0·9863	0·0316	0·0460	0·0913	0·3708	0·9853
0·4	1·1465	0·4389	1·0384	0·0731	0·0442	0·1119	0·4711	0·9618
0·5	1·1014	0·4635	1·0919	0·1256	0·0470	0·1390	0·5459	0·9337
0·6	1·0396	0·4956	1·1501	0·1907	0·0558	0·1760	0·6001	0·9007
0·7	0·9436	0·5465	1·2132	0·2721	0·0653	0·2315	0·6397	0·8599
0·8	0·8034	0·6124	1·2807	0·3830	-0·0030	0·3884	0·6690	0·8011
0·9	0·7452	0·5067	1·3523	0·5846	-0·1428	0·6037	0·6911	0·7272
1·0	0·7051	0·4762	1·4273	0·6943	-0·2471	0·6382	0·7081	0·7611
$m = 2$								
0	1·2746	0·3776	0·8839		0	0	0	
0·1	1·2643	0·3837	0·8951		0·0243	0·0421	0·1266	
0·2	1·2320	0·3986	0·9315		0·0453	0·0730	0·2472	
0·3	1·1826	0·4170	0·9910		0·0536	0·0944	0·3598	
0·4	1·1341	0·4367	1·0547		0·0494	0·1144	0·4575	
0·5	1·0851	0·4593	1·1203		0·0443	0·1379	0·5316	
0·6	1·0267	0·4891	1·1914		0·0410	0·1666	0·5854	
0·7	0·9495	0·5356	1·2687		0·0376	0·2011	0·6247	
0·8	0·8334	0·6213	1·3515		0·0103	0·2585	0·6537	
0·9	0·7440	0·6836	1·4392		-0·1225	0·4046	0·6756	
1·0	0·6978	0·7066	1·5309		-0·2354	0·5093	0·6925	

Table 9 (Contd)

$p_o = 0{\cdot}4$, m = 3	ω				$-\frac{\mu}{\sqrt{\mu^2+\omega^2}}$			
v	A	B	C	D	A	B	C	D
0	1·2746	0·3776	0·8839		0	0	0	
0·1	1·2636	0·3835	0·8959		0·0240	0·0418	0·1270	
0·2	1·2303	0·3982	0·9337		0·0427	0·0715	0·2504	
0·3	1·1847	0·4172	0·9891	0·0291	0·0470	0·0914	0·3682	0·9750
0·4	1·1430	0·4384	1·0446	0·0549	0·0433	0·1110	0·4683	0·9526
0·5	1·0984	0·4635	1·1019	0·0816	0·0428	0·1363	0·5434	0·9248
0·6	1·0404	0·4974	1·1642	0·1279	0·0477	0·1701	0·5979	0·8913
0·7	0·9540	0·5539	1·2320	0·1726	0·0564	0·2151	0·6378	0·8515
0·8	0·8085	0·6673	1·3049	0·2172	0·0097	0·3235	0·6674	0·8065
0·9	0·7450	0·7076	1·3821	0·2508	−0·1407	0·5109	0·6899	0·7680
1·0	0·7057	0·7373	1·4631	0·2691	−0·2478	0·6174	0·7072	0·7519
m = 2								
0	1·2746	0·3776	0·8839		0	0	0	
0·1	1·2641	0·3836	0·8953		0·0242	0·0419	0·1268	
0·2	1·2312	0·3985	0·9321		0·0446	0·0721	0·2482	
0·3	1·1833	0·4174	0·9902		0·0518	0·0929	0·3622	
0·4	1·1365	0·4382	1·0513		0·0481	0·1133	0·4604	
0·5	1·0879	0·4624	1·1144		0·0447	0·1391	0·5346	
0·6	1·0279	0·4942	1·1831		0·0436	0·1728	0·5884	
0·7	0·9452	0·5440	1·2576		0·0409	0·2172	0·6276	
0·8	0·8247	0·6285	1·3377		−0·0043	0·3059	0·6567	
0·9	0·7516	0·6654	1·4225		−0·1351	0·4644	0·6786	
1·0	0·7079	0·6741	1·5113		−0·2403	0·5762	0·6954	

Table 10

Method	Critical flutter speed v_{crit}	Frequency ω_{crit}	νv_{crit}
British unlined-up			
$\nu = 0{\cdot}5$	0·83	0·66	0·415
$\nu = 0{\cdot}6$	0·832	0·695	0·499
$\nu = 0{\cdot}8$	0·828	0·755	0·662
$\nu = 1{\cdot}0$	0·795	0·812	0·795
$\nu = 1{\cdot}3$	0·77	0·865	1·001
$\nu = 1{\cdot}6$	0·735	0·908	1·176
$\nu = 2{\cdot}6$	0·66	0·974	1·716
$\nu = 5{\cdot}0$	0·61	1·005	3·05
British lined-up	0·792	0·82	
American	0·805	0·81	
Richardson method			
$p_o = 0{\cdot}6,\ m = 3$	0·8	0·805	
$p_o = 0{\cdot}6,\ m = 2$	0·805	0·83	
$p_o = 0{\cdot}4,\ m = 3$	0·81	0·80	
$p_o = 0{\cdot}4,\ m = 2$	0·795	0·83	

SYMBOLS

A	inertia matrix, structural and aerodynamic
A_o	see equation (13)
A_1	aerodynamic inertia matrix
B	aerodynamic damping matrix
B_∞	$(B)_{\nu=\infty}$
$\bar{B}$	$B - B_\infty$
$\bar{B}_{ij}$	ij^{th} element of $\bar{B}$
C	aerodynamic stiffness matrix
C_o	$(C)_{\nu=0}$
$\bar{C}$	$C - C_o$
$\bar{C}_{ij}$	ij^{th} element of $\bar{C}$
D	structural damping matrix
E	structural stiffness matrix
$H(\tau)$	unit step function, see section 2.3
I	unit matrix
K	indicial aerodynamic matrix
$\bar{K}(p)$	Laplace transform of K
K_σ	see equation (13) $(= -C_o)$
K_r	see equation (13)
S_{ij}	$\Sigma(\delta_{ij})^2$ see Appendix B
V	airspeed
V_o	reference airspeed
g/ω	ratio of fictitious hysteretic structural damping matrix to matrix E
$h_z, h_{\dot{z}}, h_\alpha, h_{\dot{\alpha}}, h_\beta, h_{\dot{\beta}}$	hinge moment derivatives, see Ref.7
ℓ	reference length; number of values of ν used to determine K_r matrices
$\ell_z, \ell_{\dot{z}}, \ldots$	lift force derivatives, see Ref.7
m	see equation (13)
$m_z, m_{\dot{z}}, \ldots$	pitching moment derivatives, see Ref.7

SYMBOLS (Contd)

n	number of degrees of freedom of the system
p	Laplace transform parameter
p_o	see equation (13)
q	column matrix of generalised coordinates
$\bar{q}$	$q\,e^{-\lambda\tau}$
$\bar{q}_r$	see equation (17)
t	time
v	V/V_o
x	$\nu^2/(\nu^2 + p_o^2)$
x_q	see Appendix B
$\alpha_r(x)$	see equation (B-8)
$\beta_r(x)$	see equation (B-8)
$\gamma_{rs}(x)$	see equation (B-14)
$\delta(\tau)$	right hand Dirac delta function
δ_{ij}	see Appendix B
λ	complex eigenvalue
μ	real part of λ
ν	frequency parameter = ω/v
ν_q	see equation (B-12)
τ	$V_o t/\ell$
ω	imaginary part of λ
$-\frac{\mu}{\sqrt{\mu^2+\omega^2}}$	relative damping ratio

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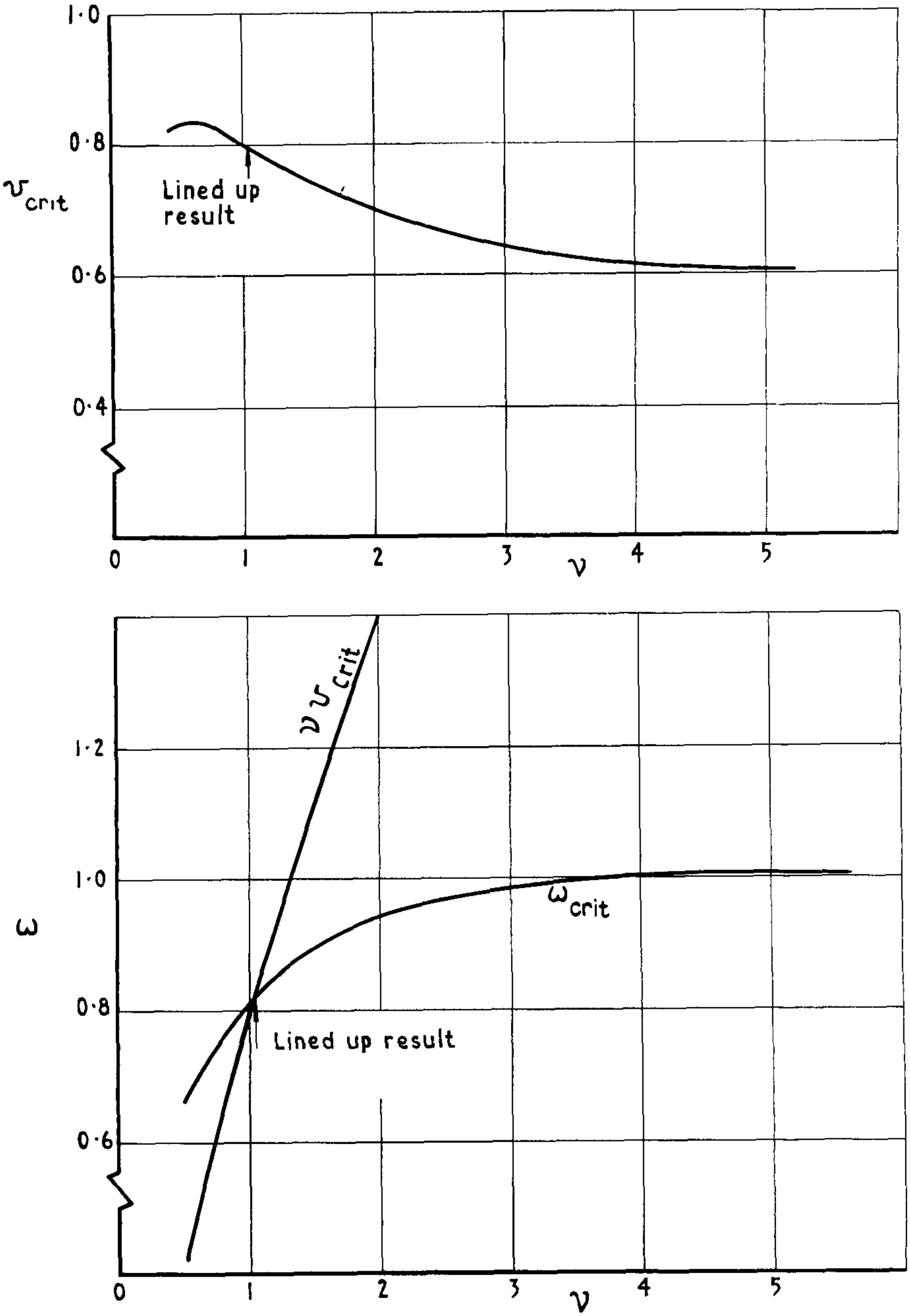

Fig.1 Critical flutter conditions from British method solutions

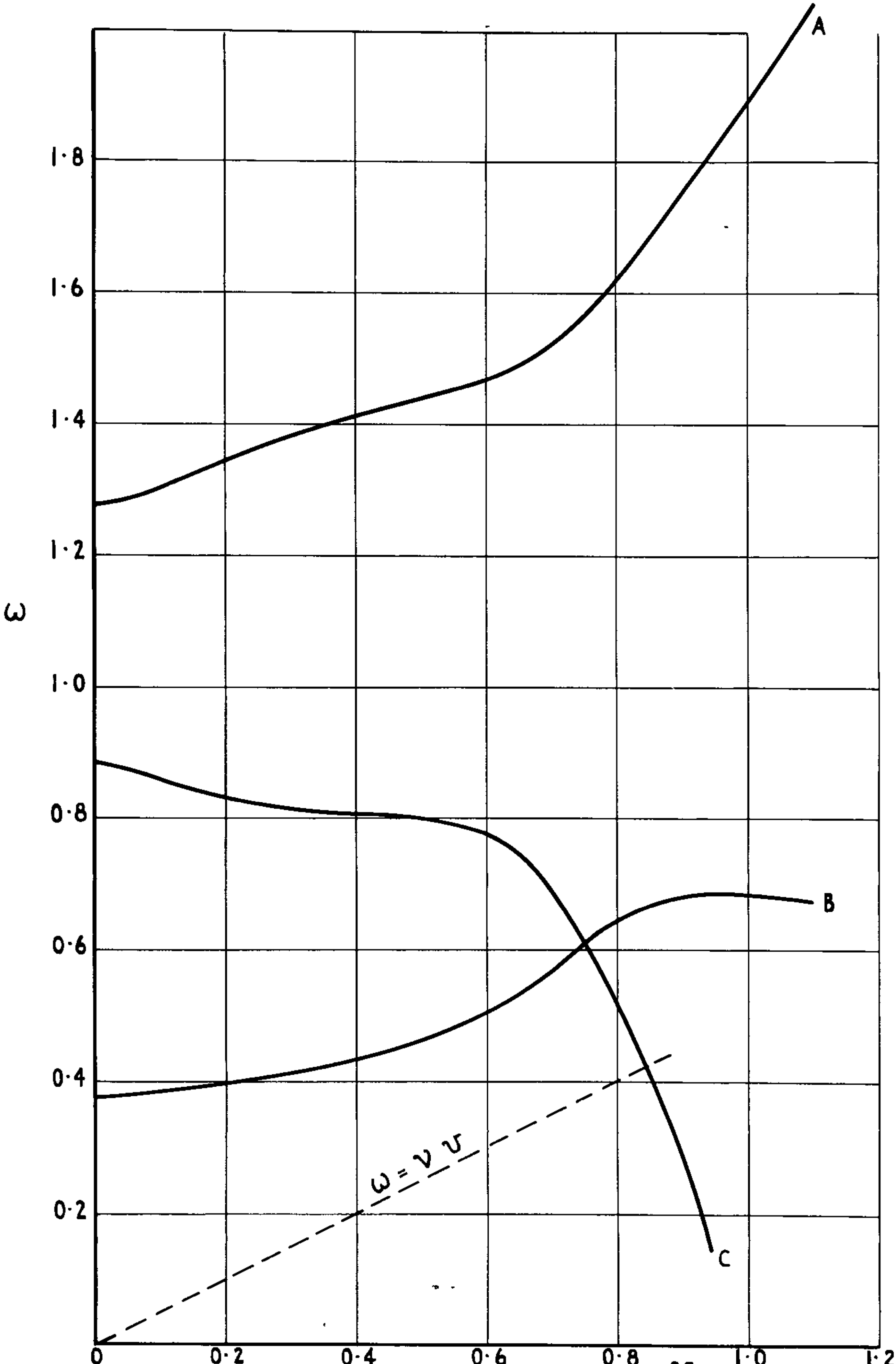

Fig 2 British method solution (ν=0·5) – frequency

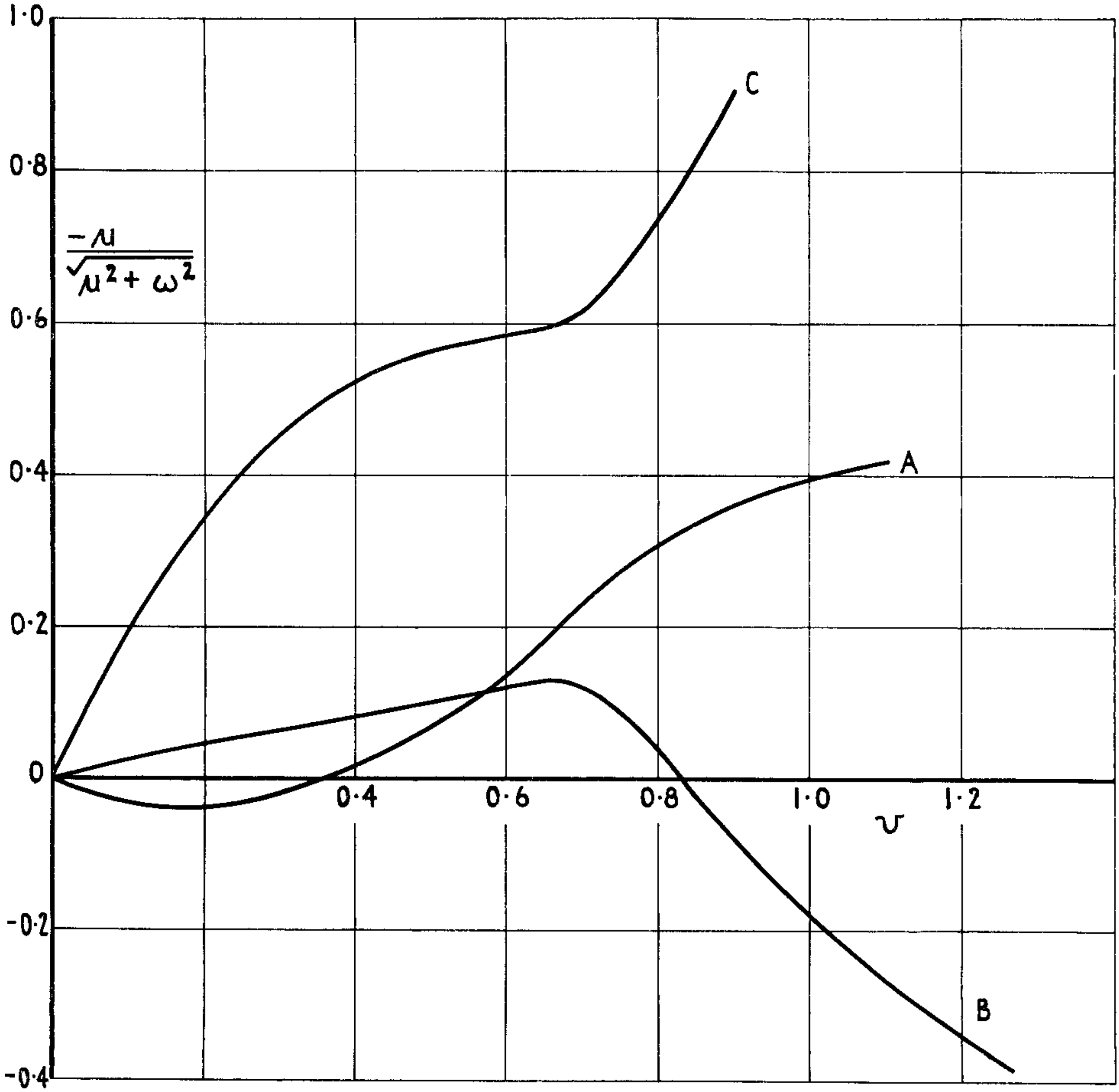

Fig. 3 British method solution ($\nu = 0{\cdot}5$)
– relative damping ratio

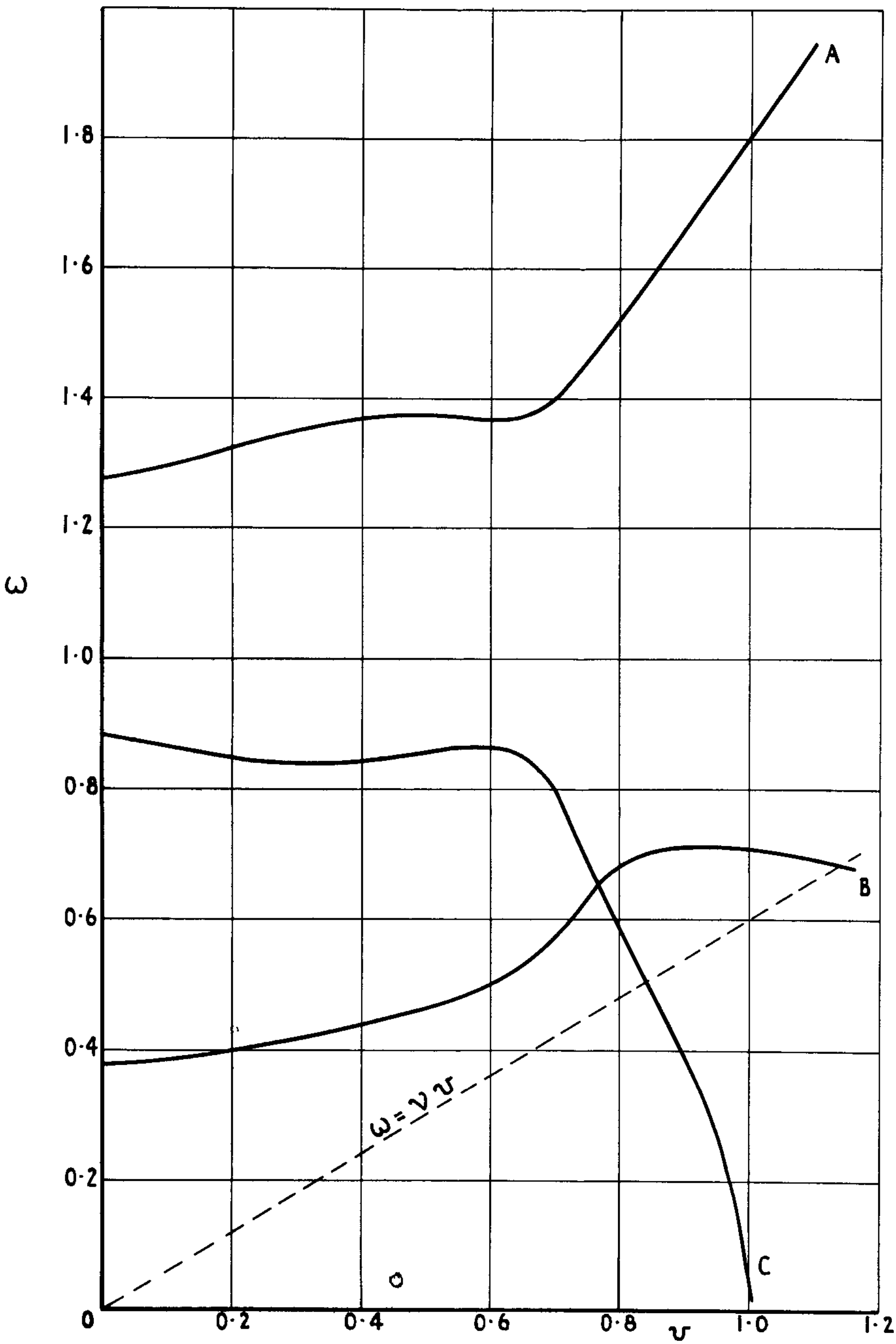

Fig. 4 British method solution $(\nu = 0\cdot6)$ – frequency

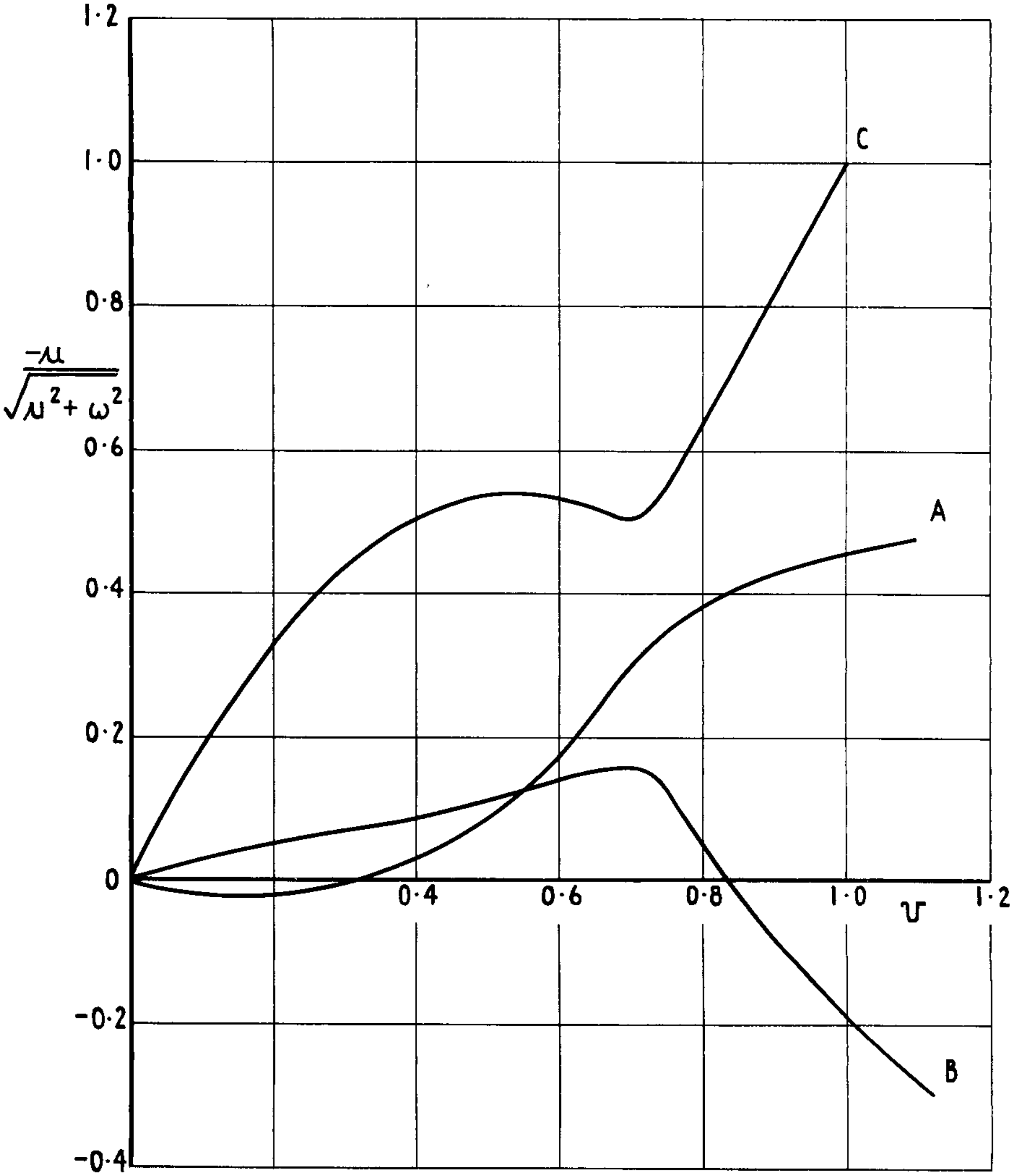

Fig. 5 British method solution ($\nu = 0{\cdot}6$)
– relative damping ratio

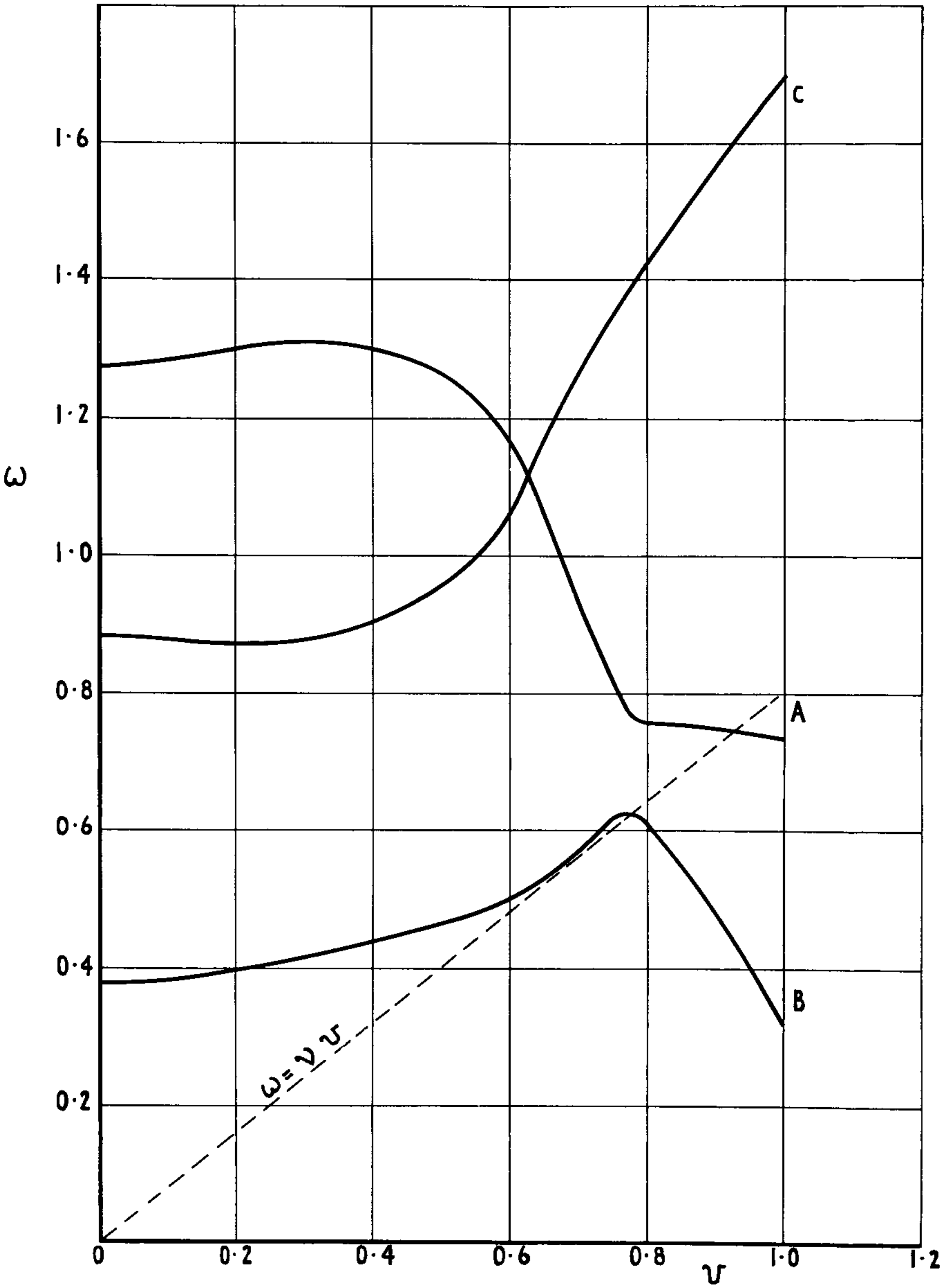

Fig.6 British method solution $(\nu = 0{\cdot}8)$-frequency

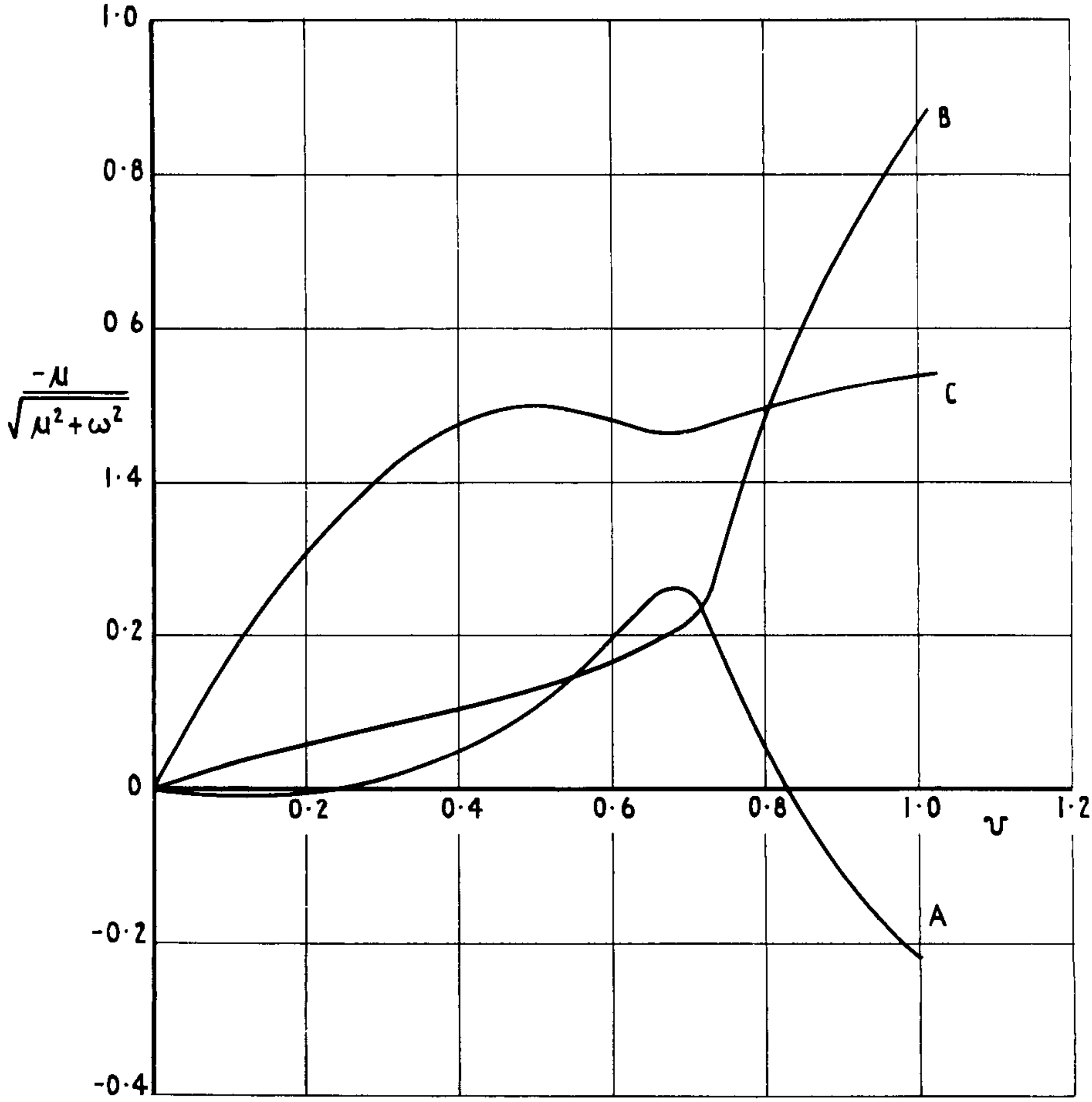

Fig.7 British method solution $(\nu = 0\cdot 8)$
-relative damping ratio

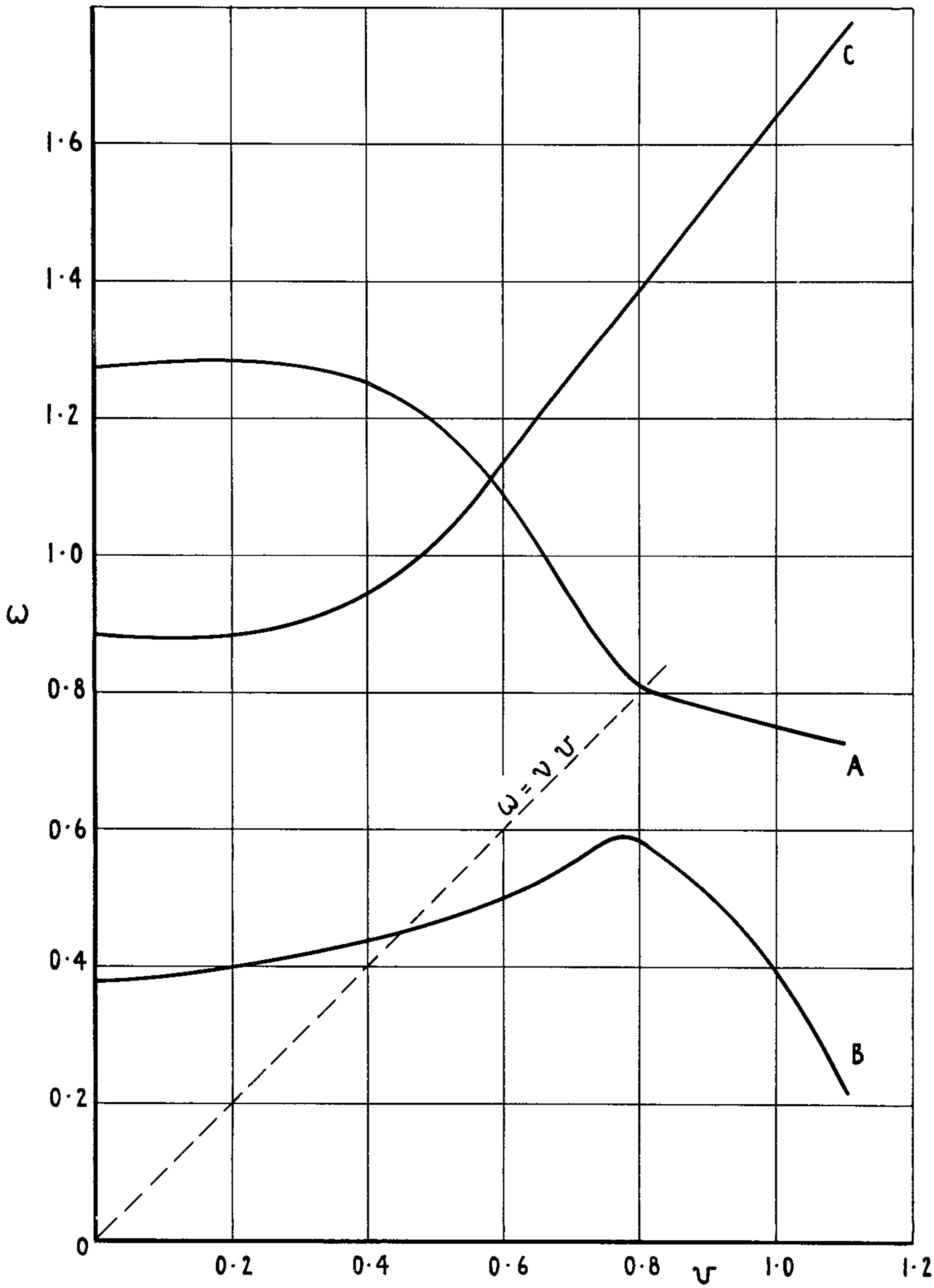

Fig. 8 British method solutions (ν = 1·0) – frequency

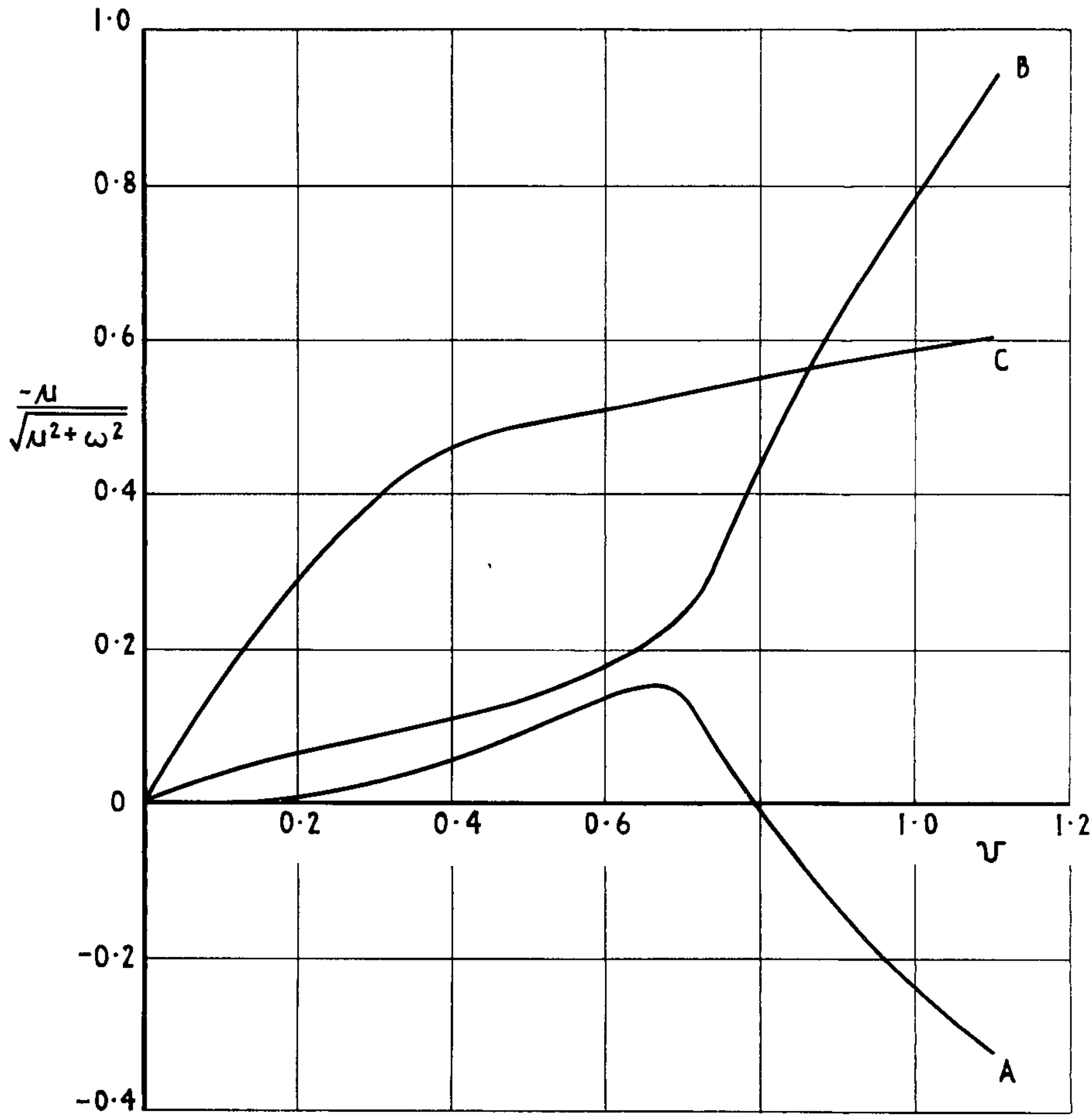

Fig. 9 British method solutions $(\nu = 1 \cdot 0)$
— relative damping ratio

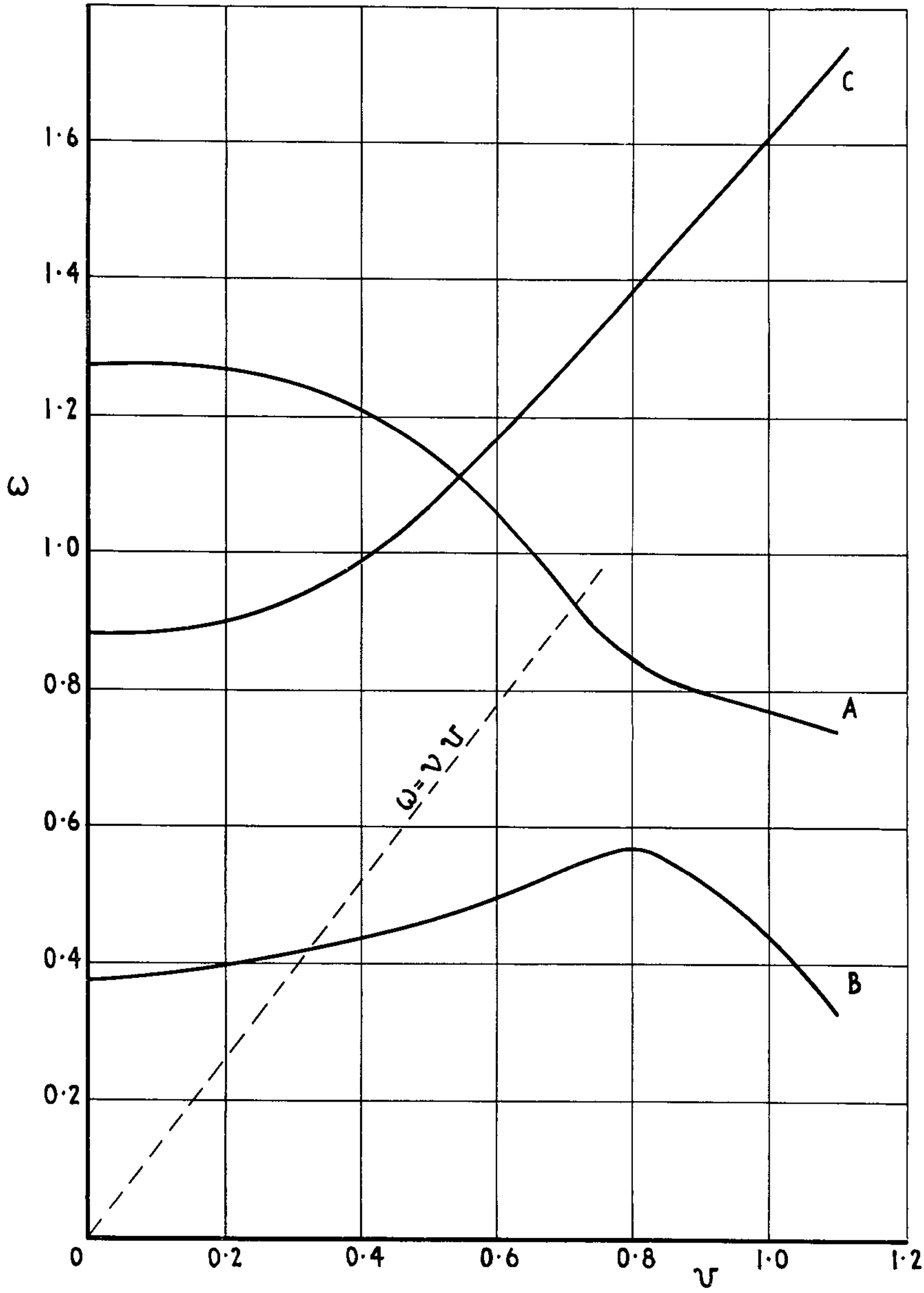

Fig.10 Bitish method solutions $(\nu = 1\cdot3)$ –frequency

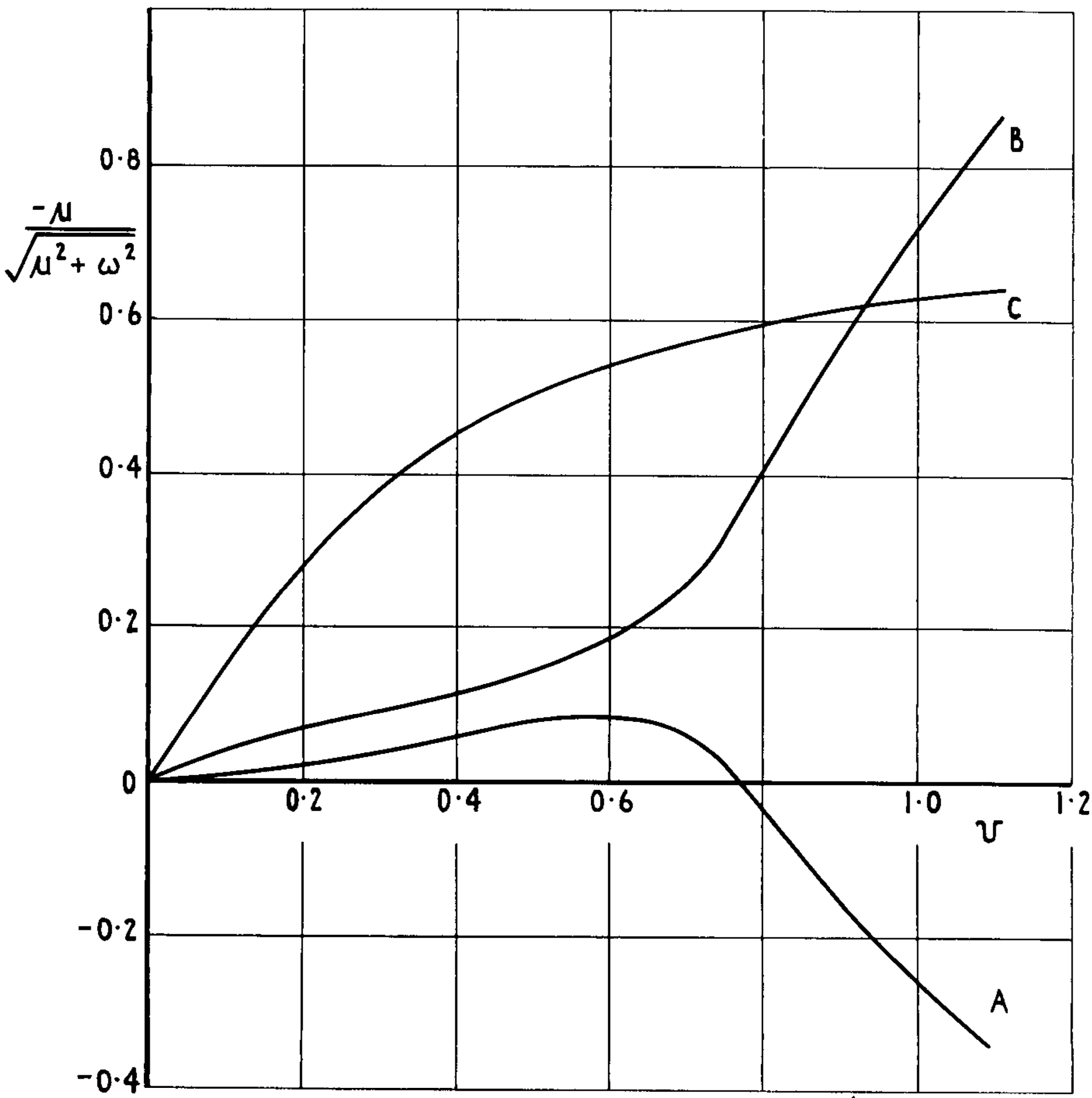

Fig.11 British method solutions $(\nu = 1\cdot 3)$
– relative damping ratio

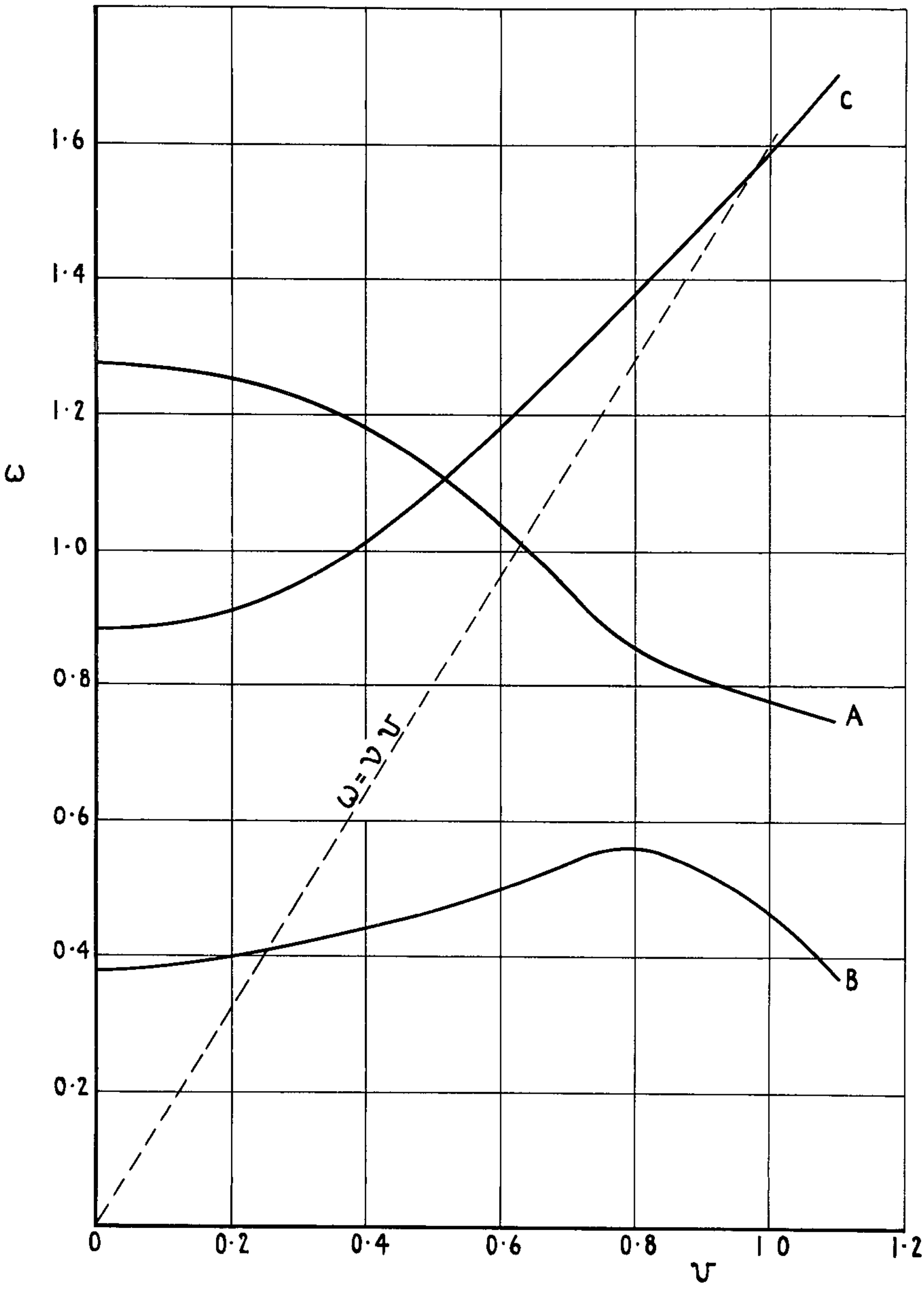

Fig. 12 British method solutions $(\nu = 1 \cdot 6)$-frequency

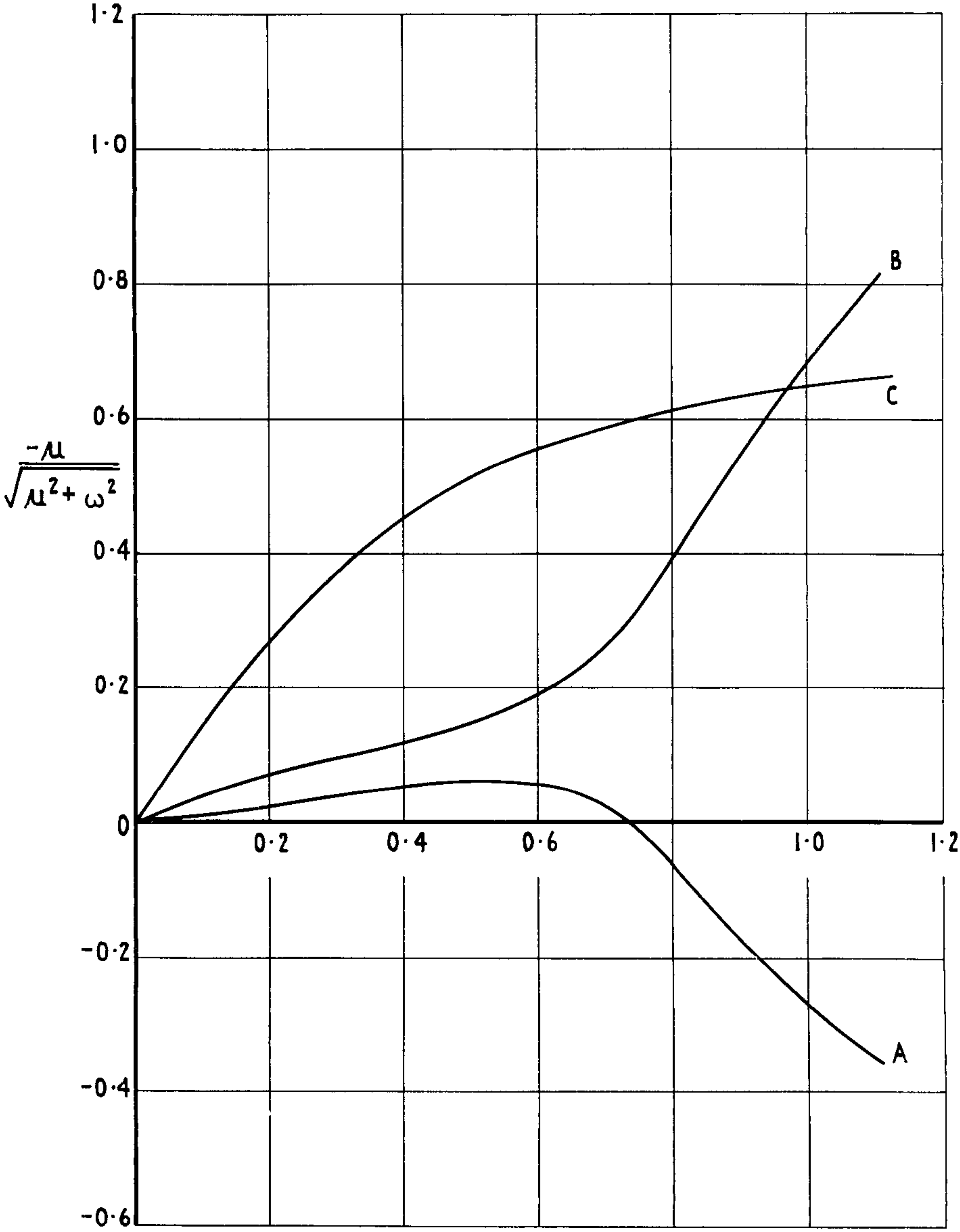

Fig.13 British method solutions (ν=1·6)
–relative damping ratio

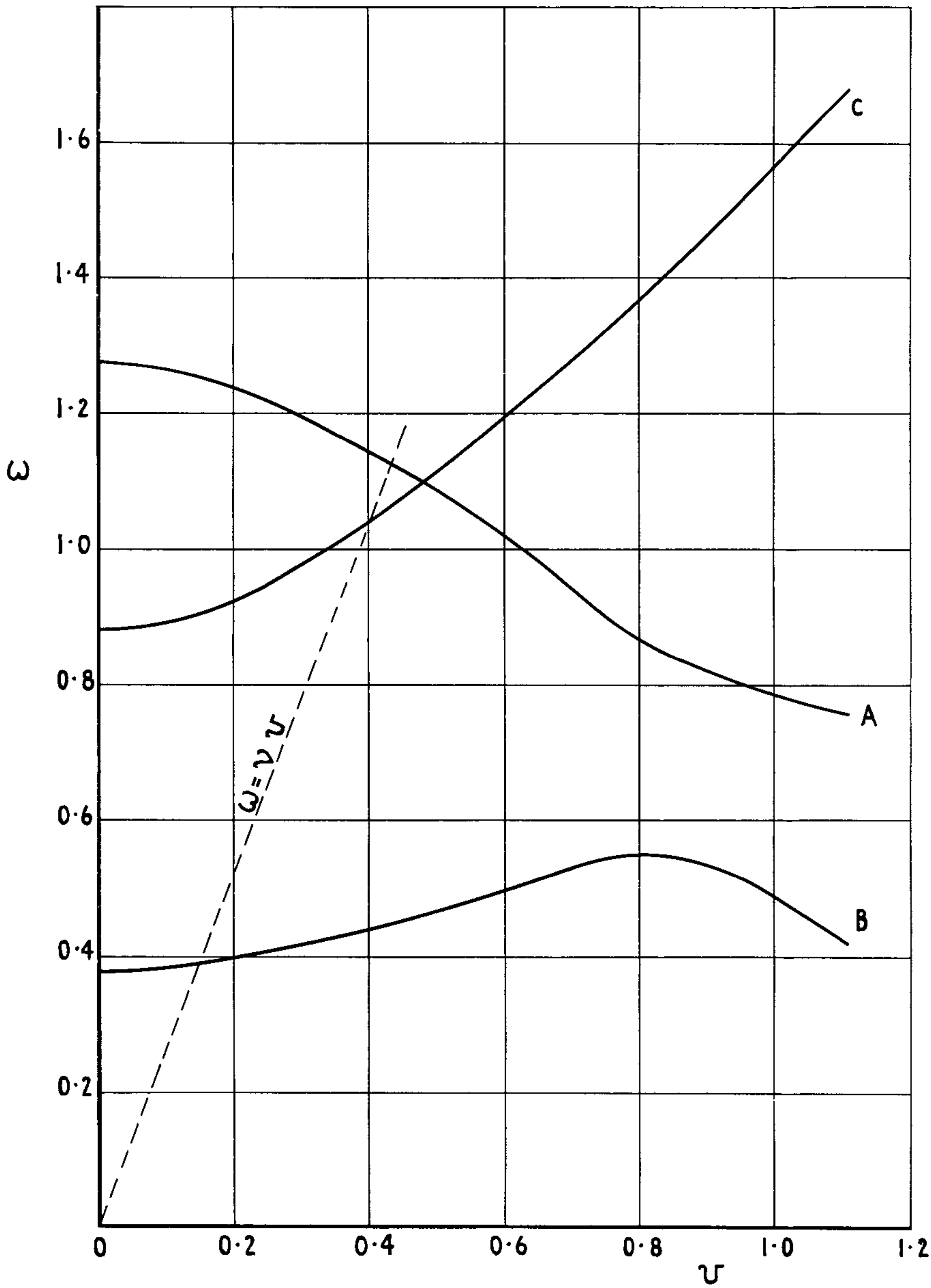

Fig.14 British method solutions ($\nu = 2\cdot 6$) -frequency

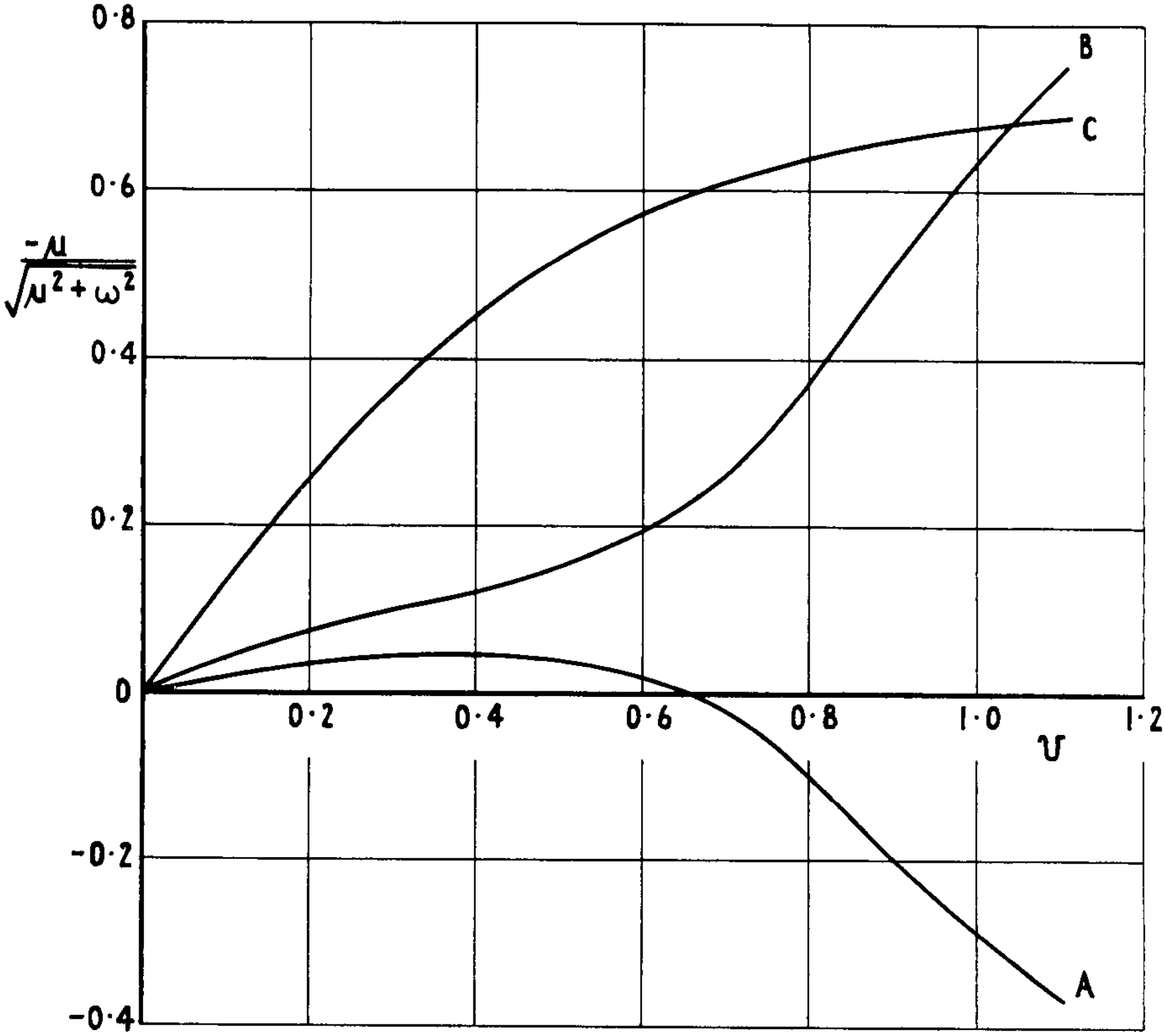

Fig.15 British method solutions ($\nu = 2\cdot6$)
—relative damping ratio

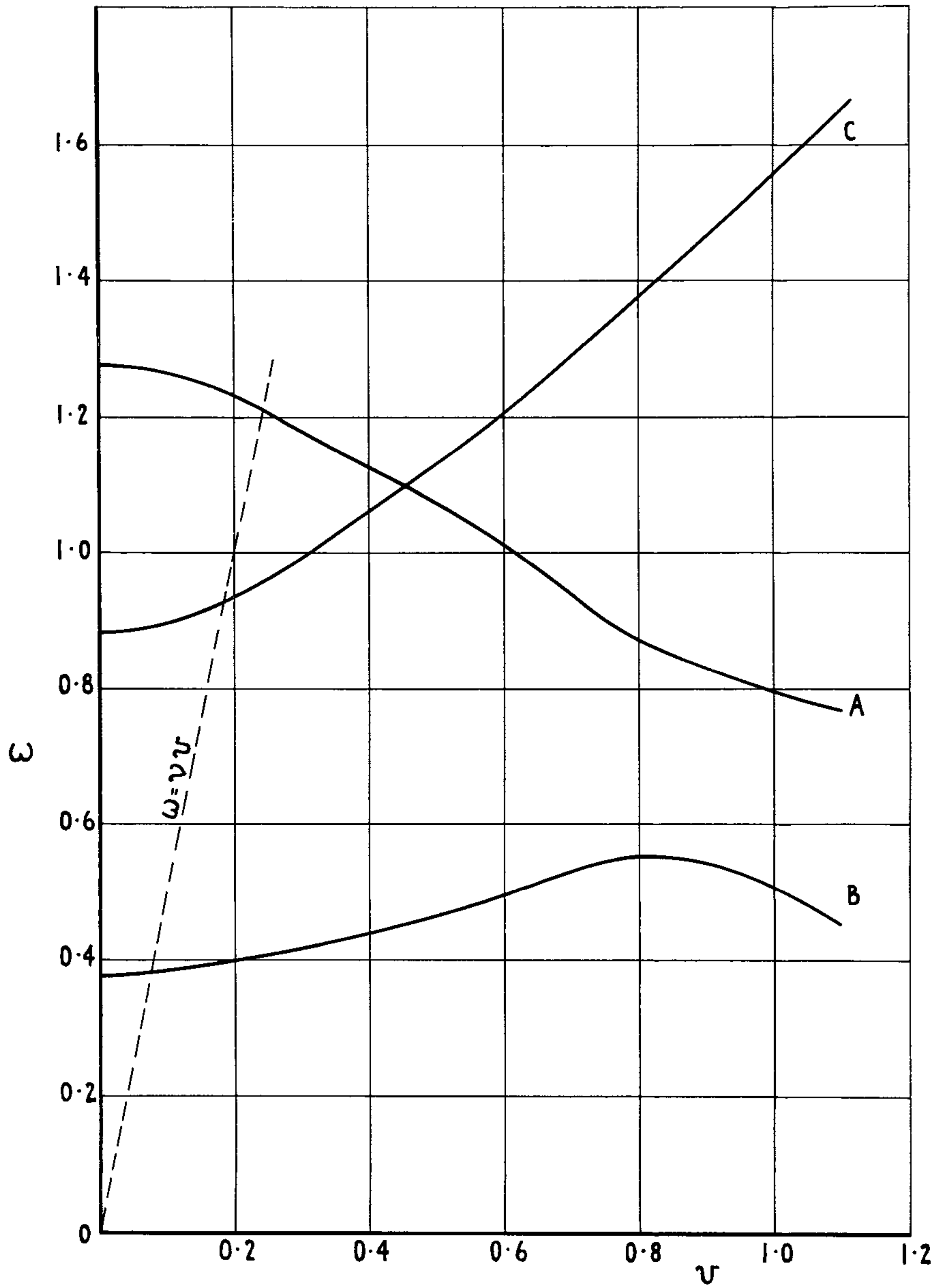

Fig. 16 British method solutions (ν=5·0)–frequency

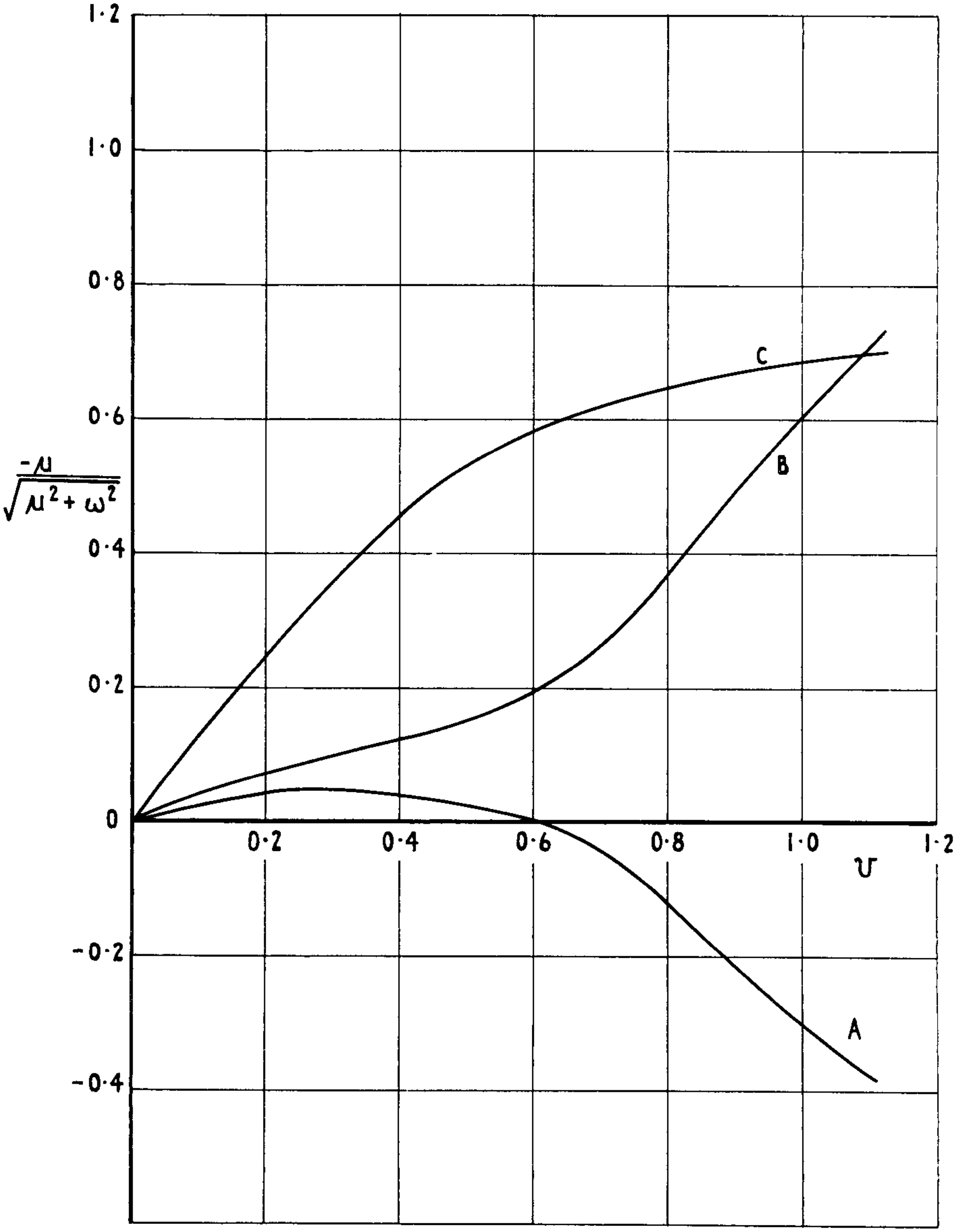

Fig. 17 British method solutions ($\nu = 5 \cdot 0$)
–relative damping ratio

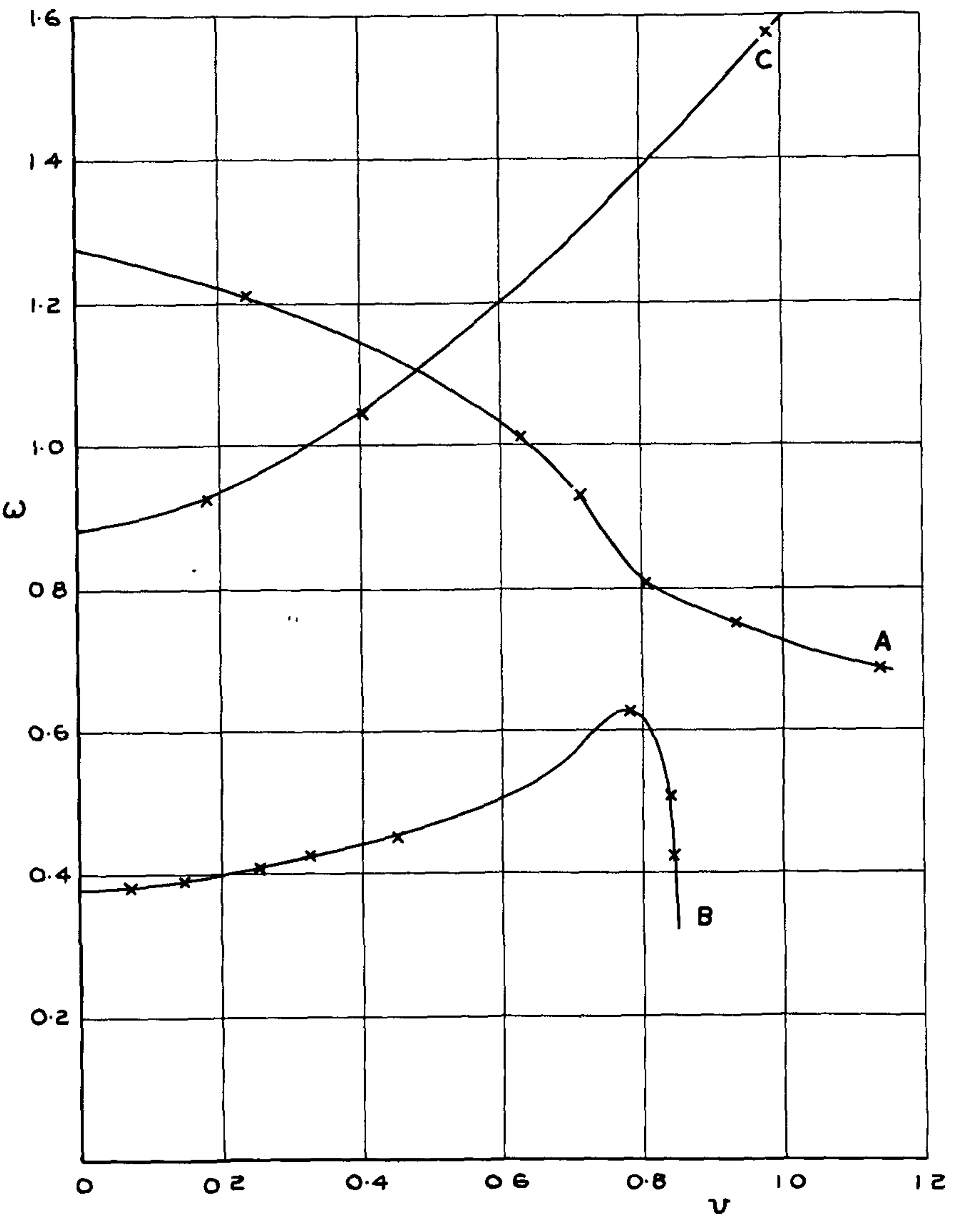

Fig. 18 British method solutions - (lined-up υ) - frequency

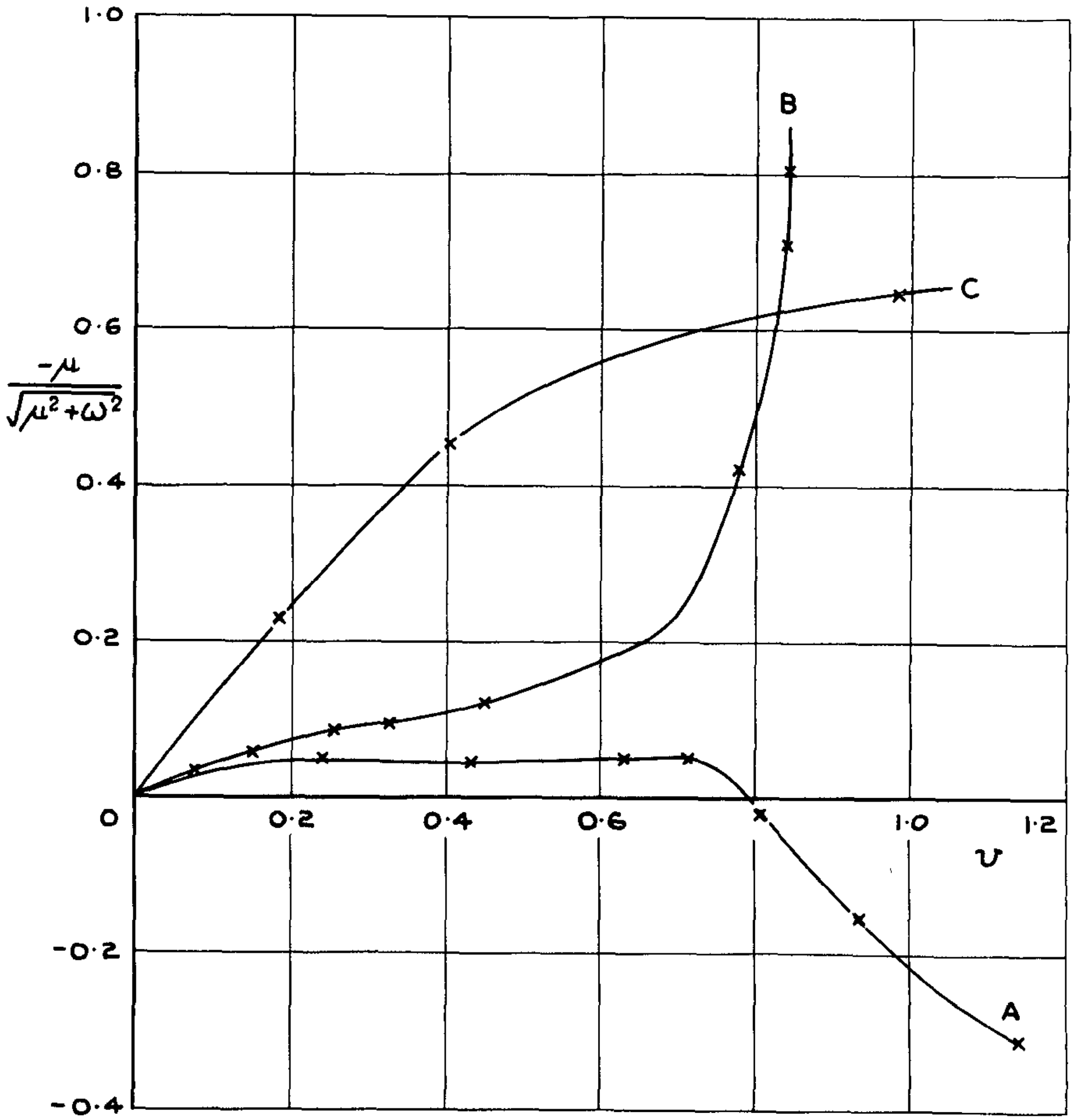

Fig. 19 British method solutions (lined-up v)
-relative damping ratio

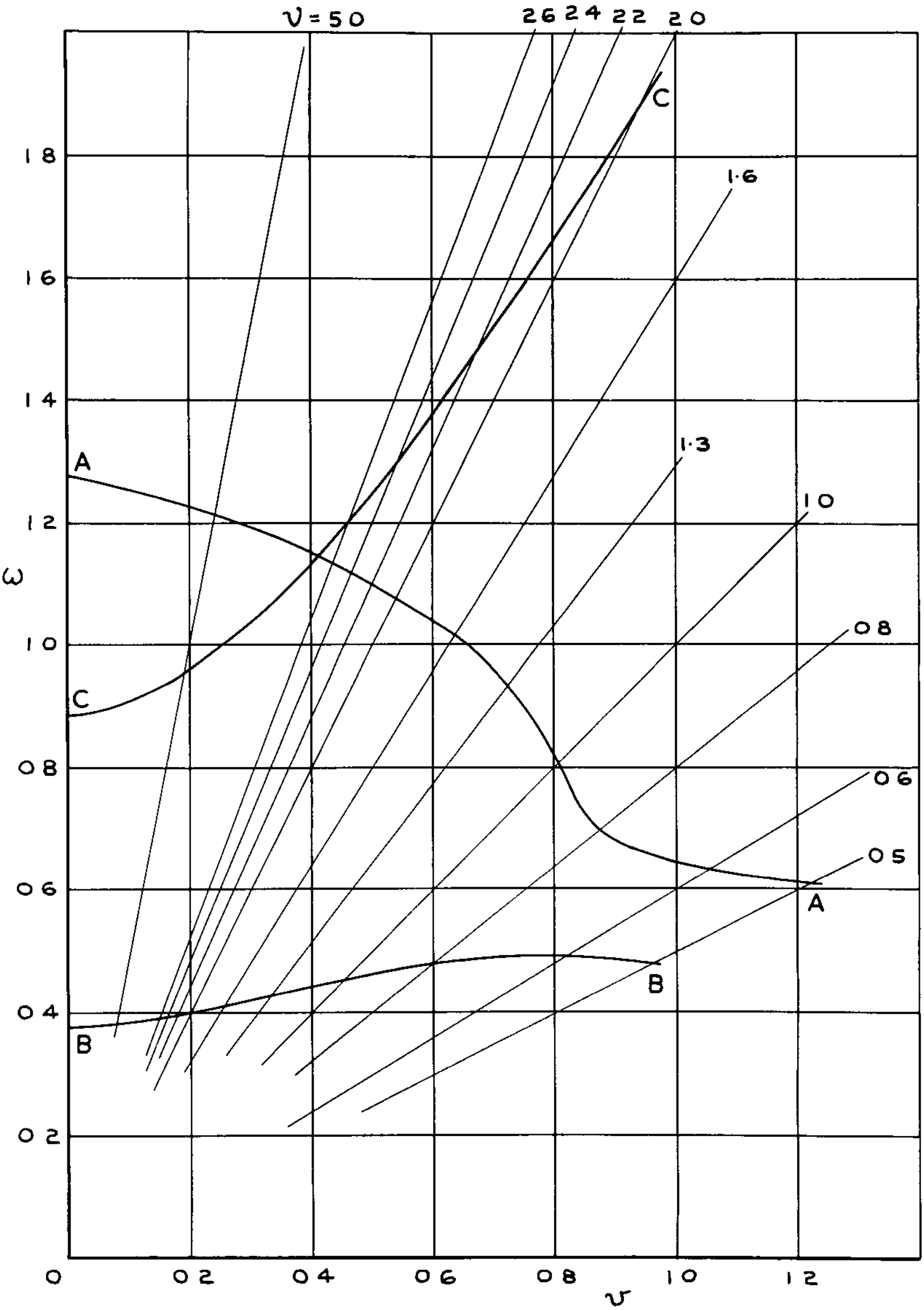

Fig. 20 American method solutions-frequency

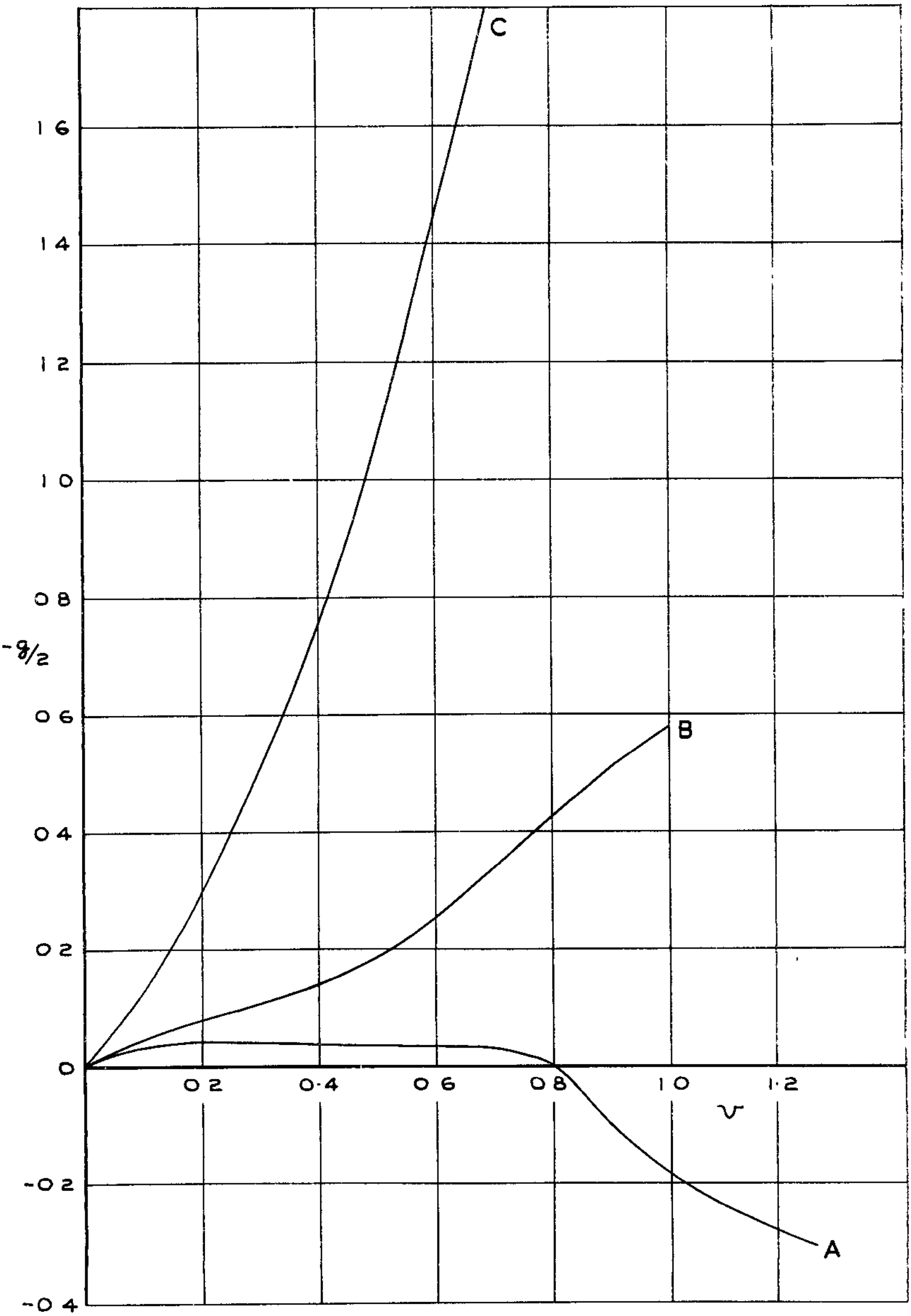

Fig. 21 American method solutions-fictitious structural damping factor

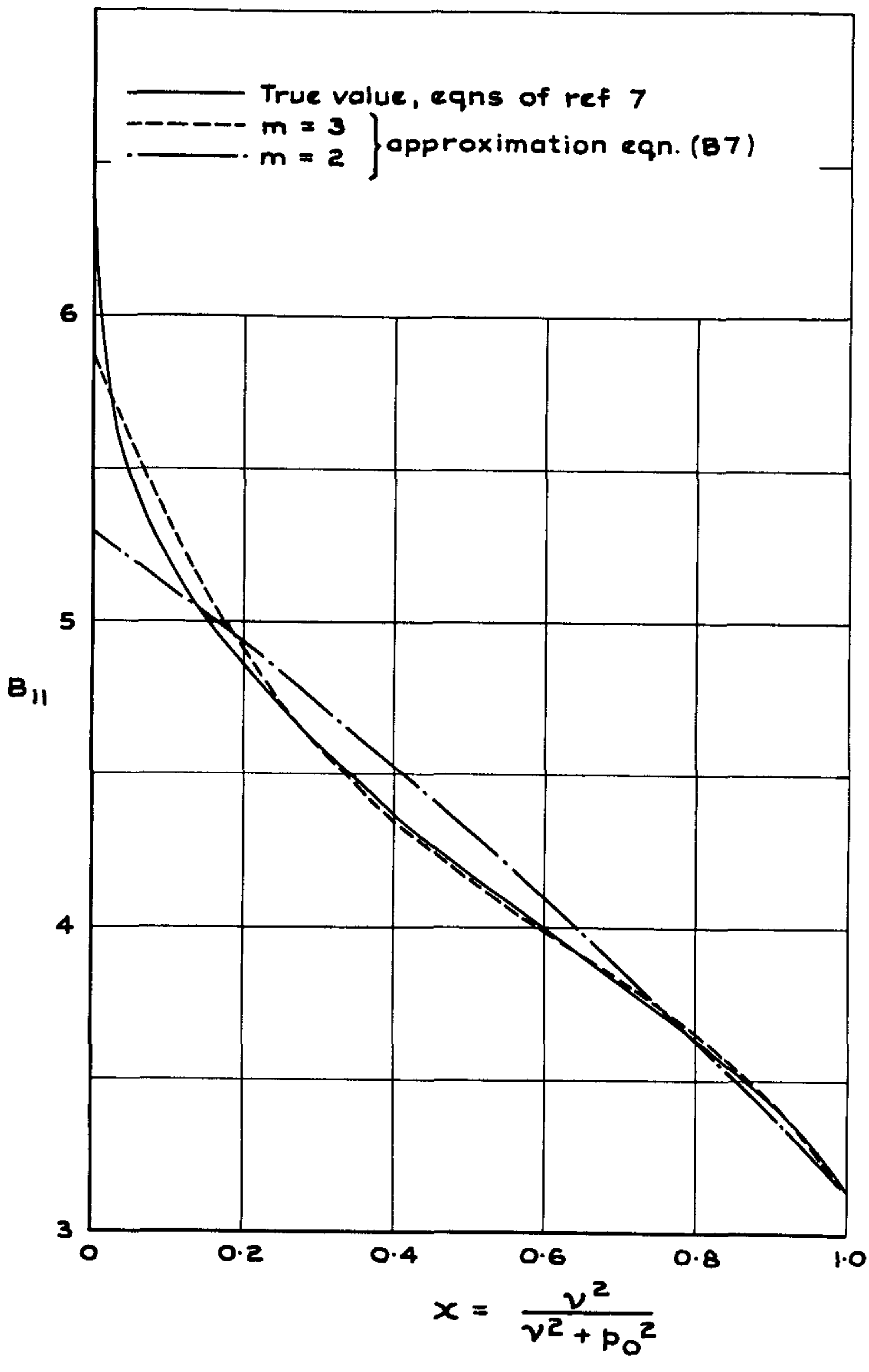

Fig. 22 Check for K_r matrices, $p_o = 0{\cdot}6$ - values of B_{11}

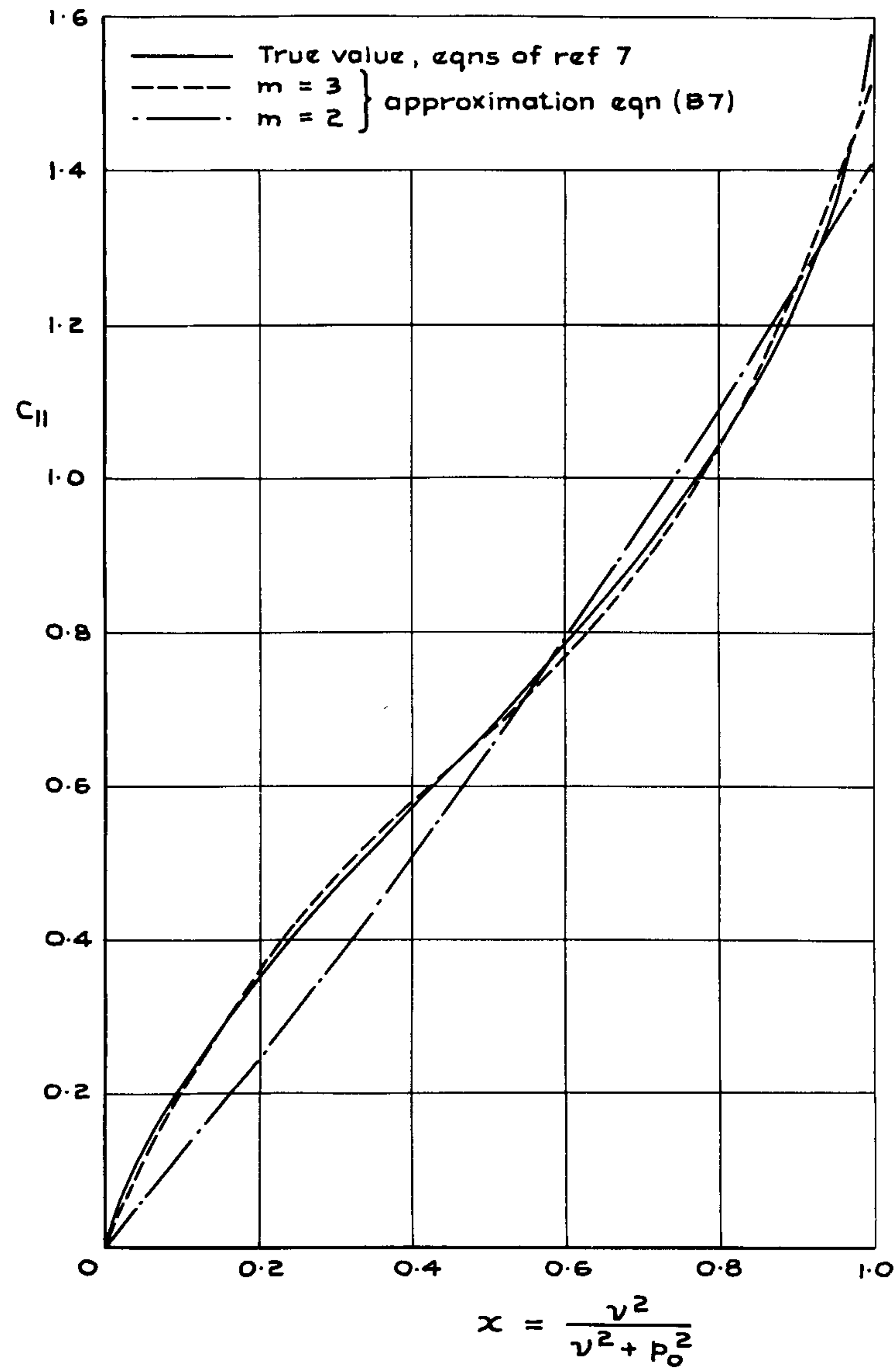

Fig. 23 Check for K_r matrices, $p_0 = 0{\cdot}6$ – values of C_{11}

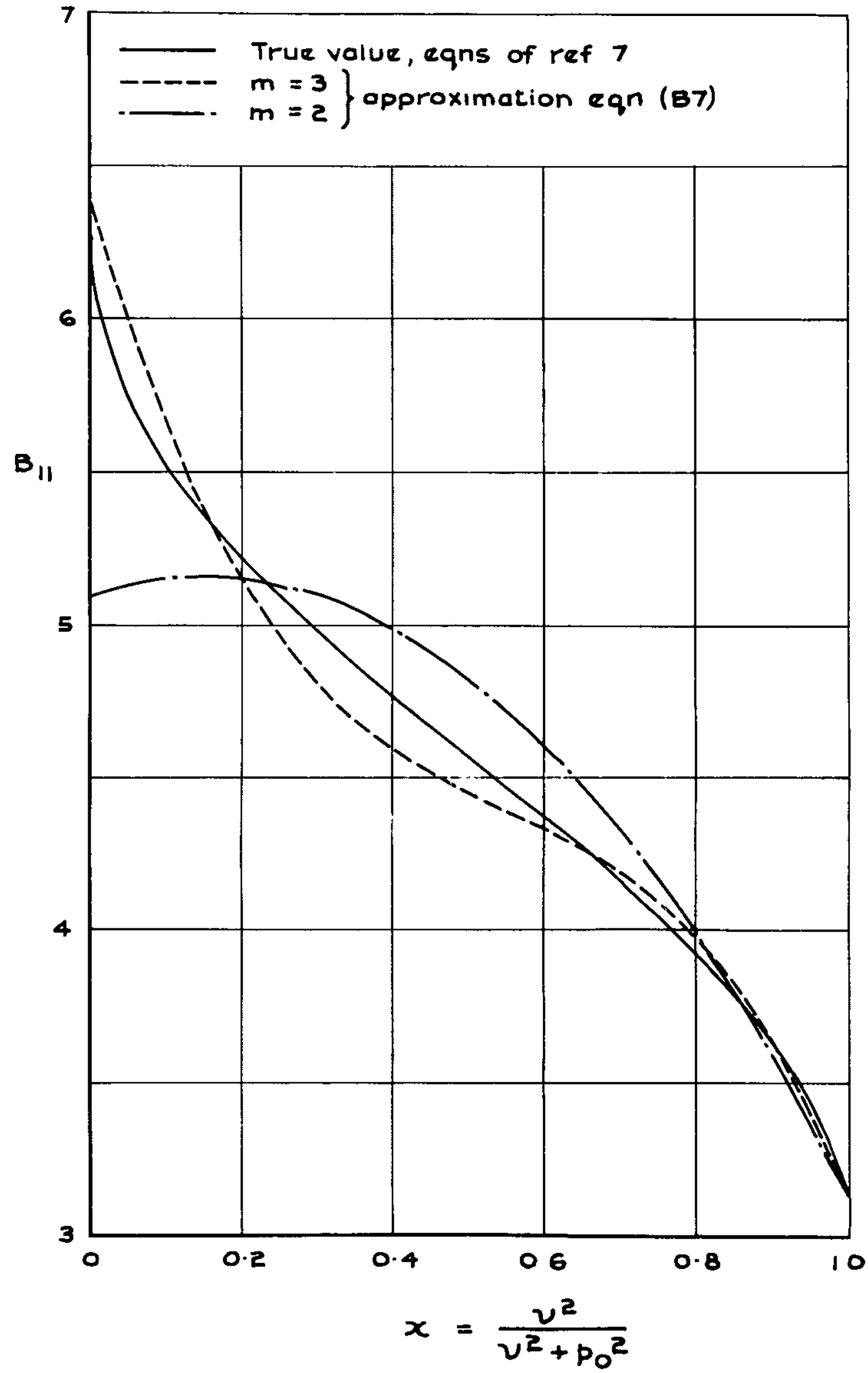

Fig. 24 Check for K_r matrices, $p_o = 0{\cdot}4$ - values of B_{11}

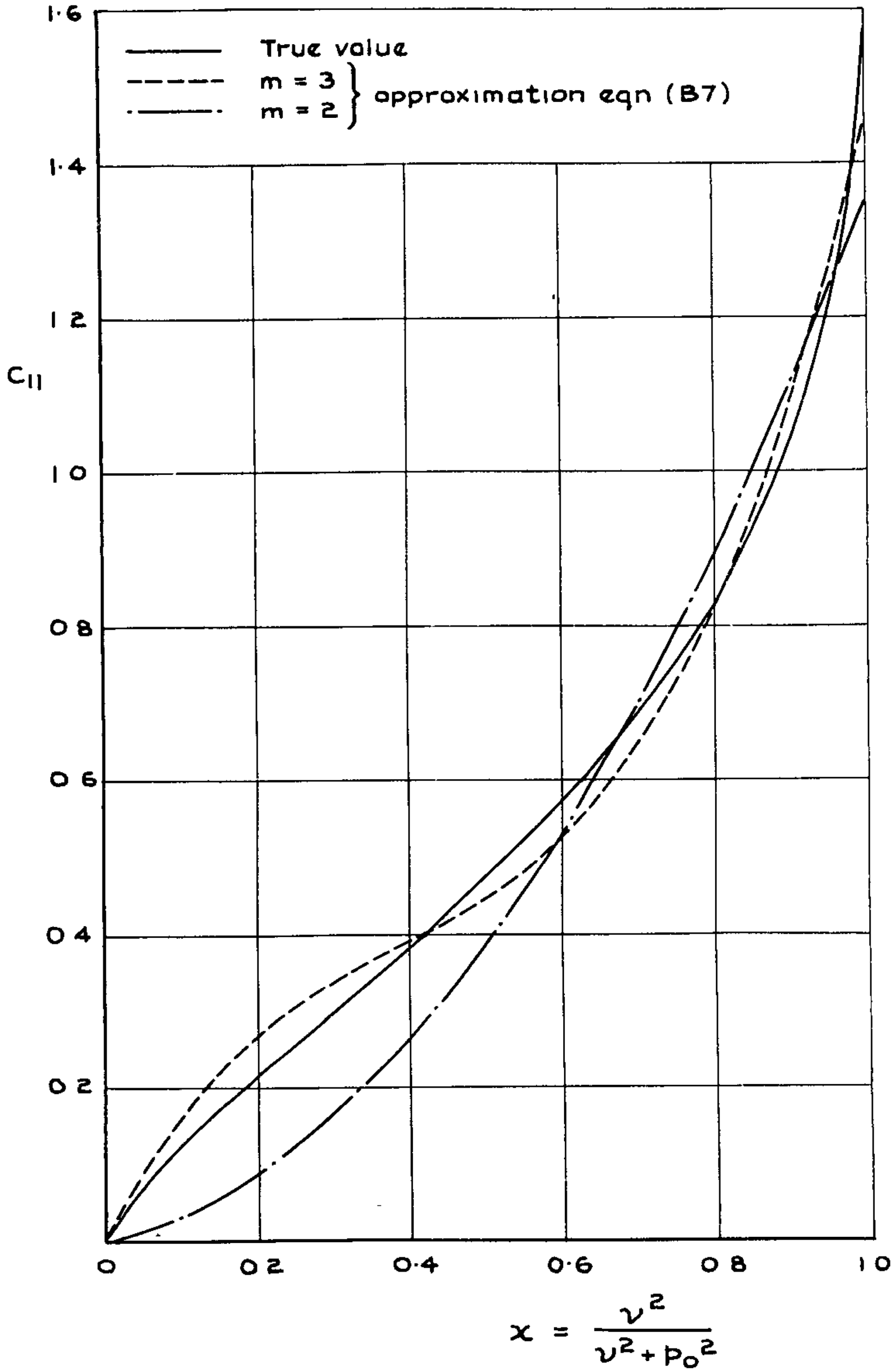

Fig. 25 Check for K_r matrices, $p_0 = 0{\cdot}4$ – values of C_{11}

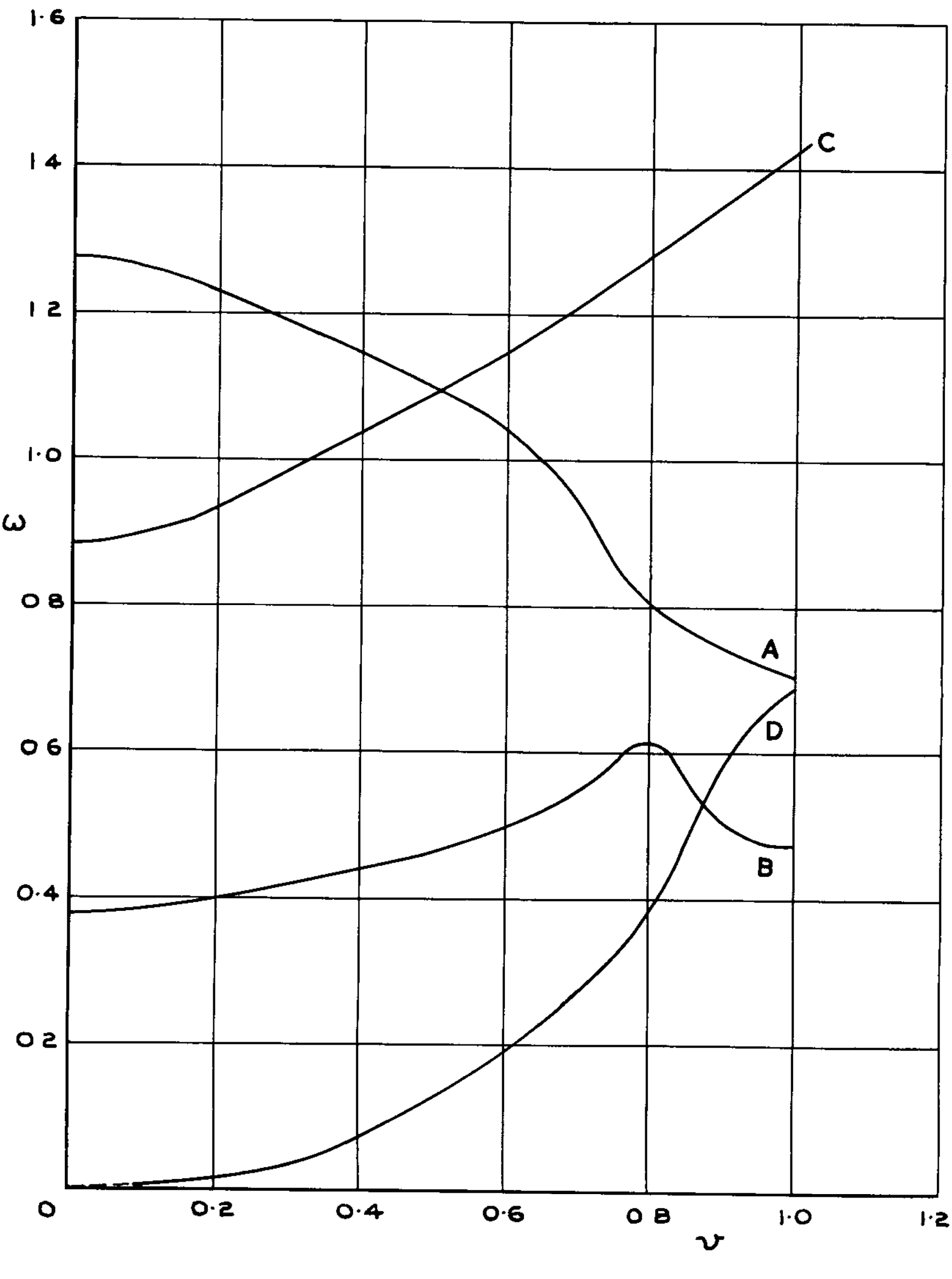

Fig. 26 Richardson method solutions
(p_o = 0·6, m = 3) - frequency

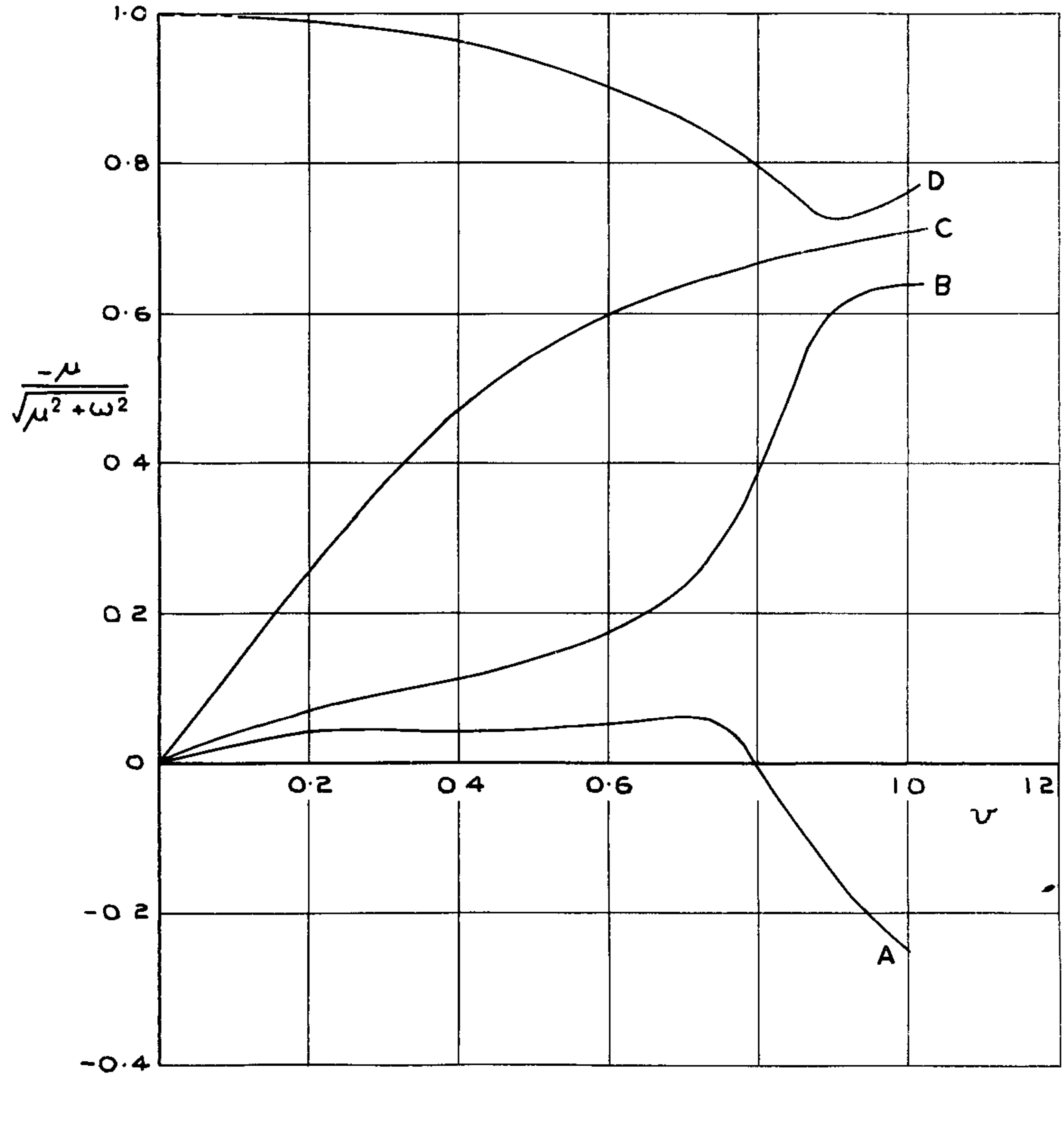

Fig. 27 Richardson method solutions (p_0 = 0·6, m = 3) – relative damping ratio

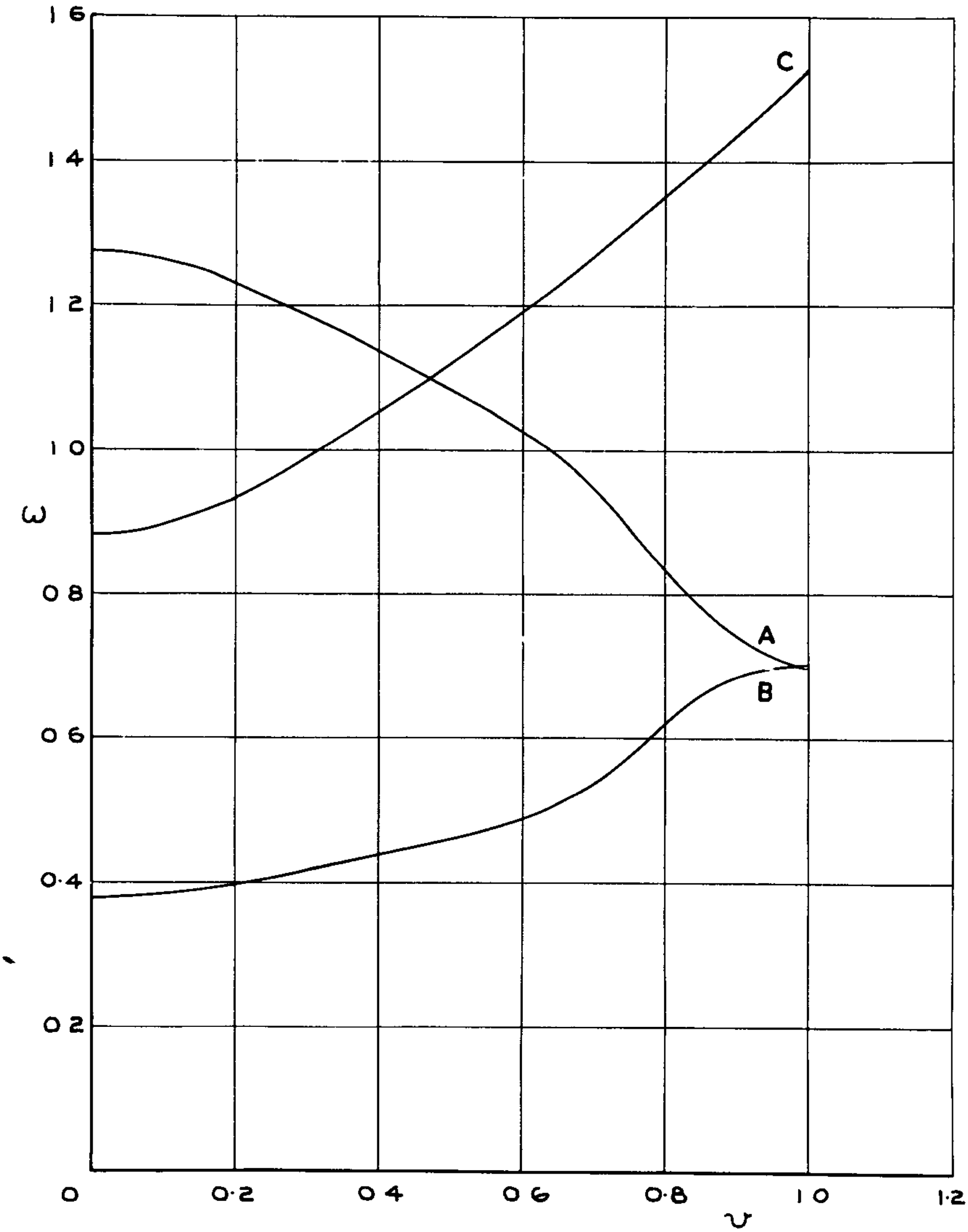

Fig 28 Richardson method solutions
(p_0 = 0 6, m = 2) – frequency

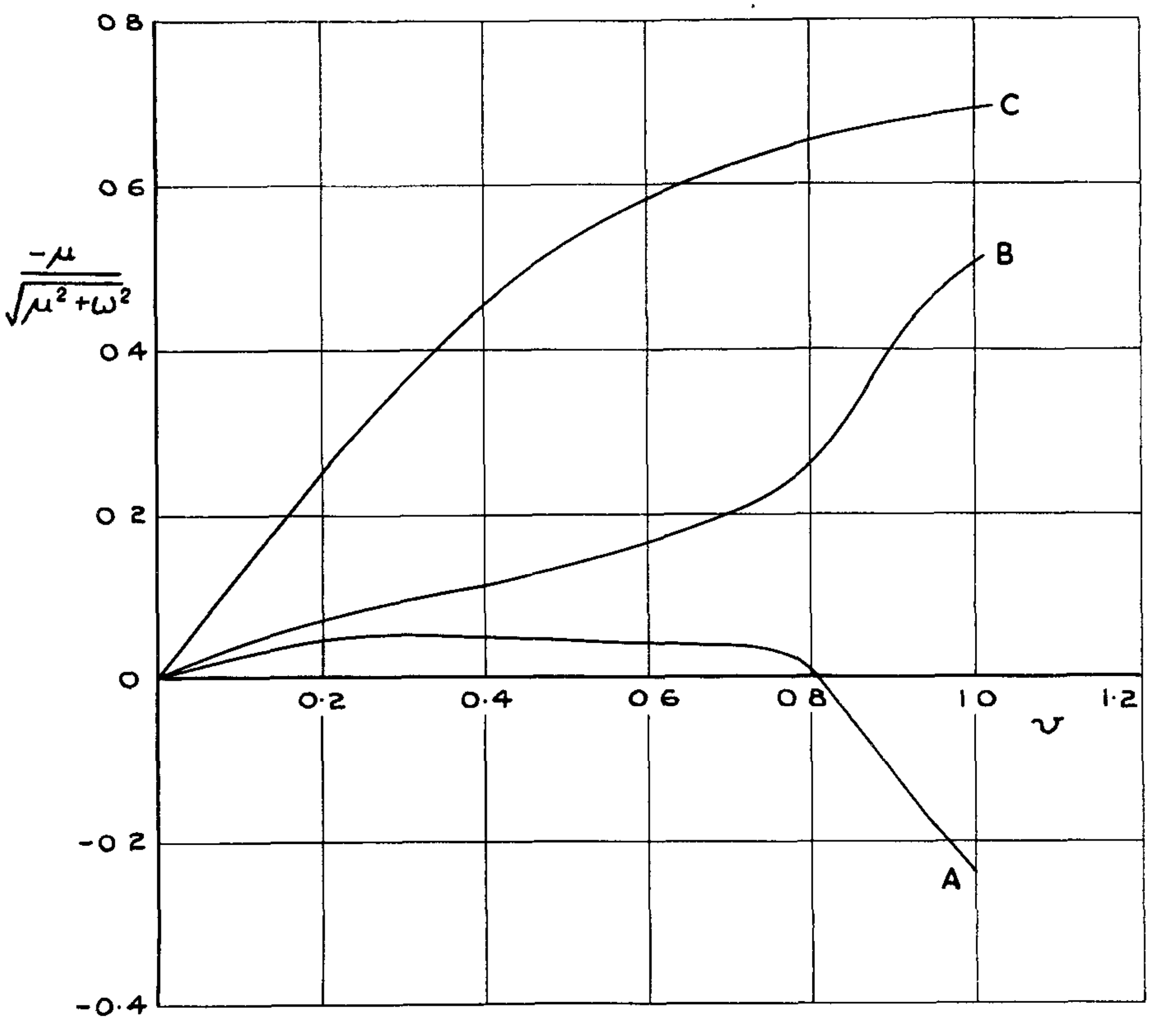

Fig. 29 Richardson method solutions (p_0 = 0·6, m = 2) – relative damping ratio

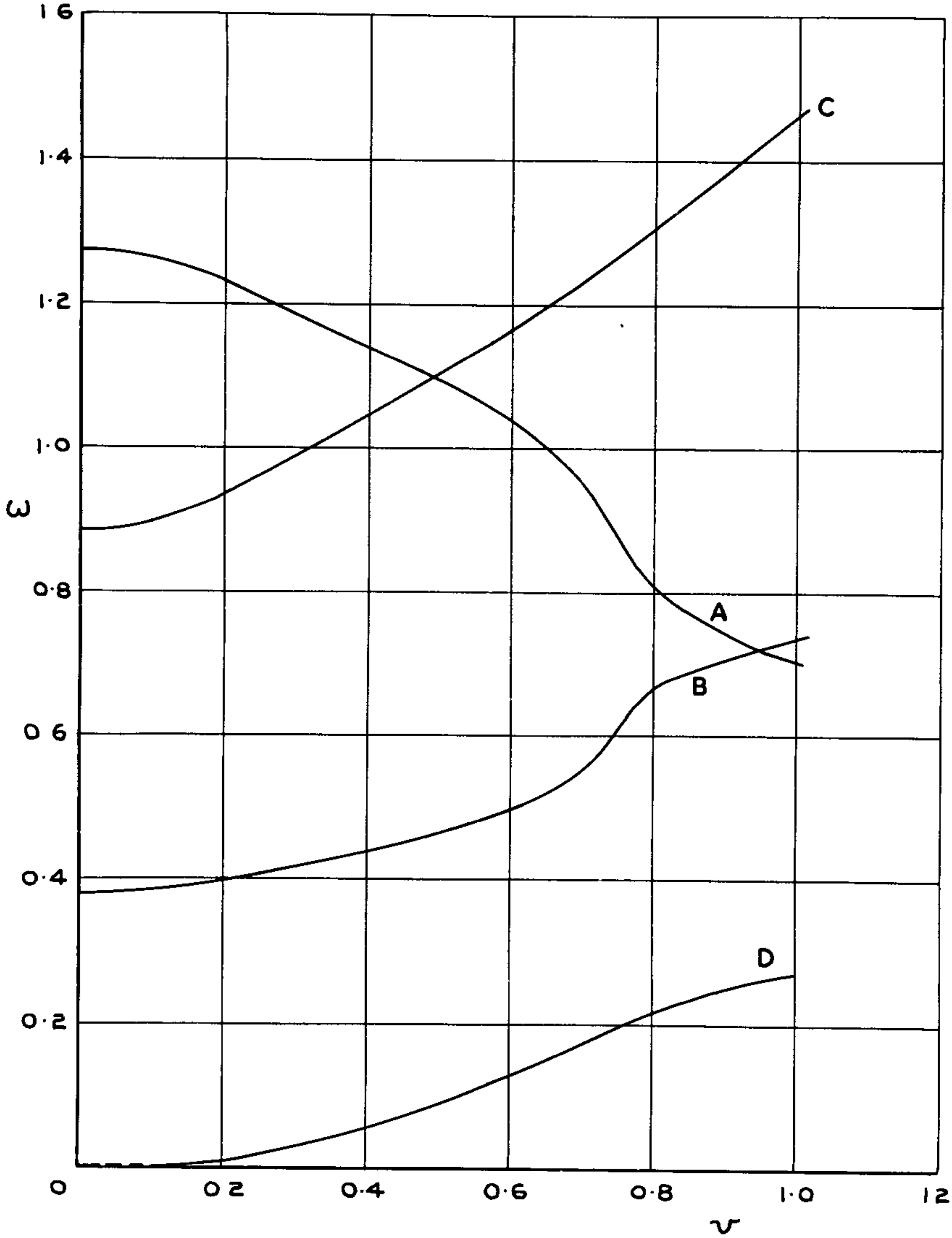

Fig. 30 Richardson method solution ($p_0 = 0{\cdot}4$, $m = 3$) – frequency

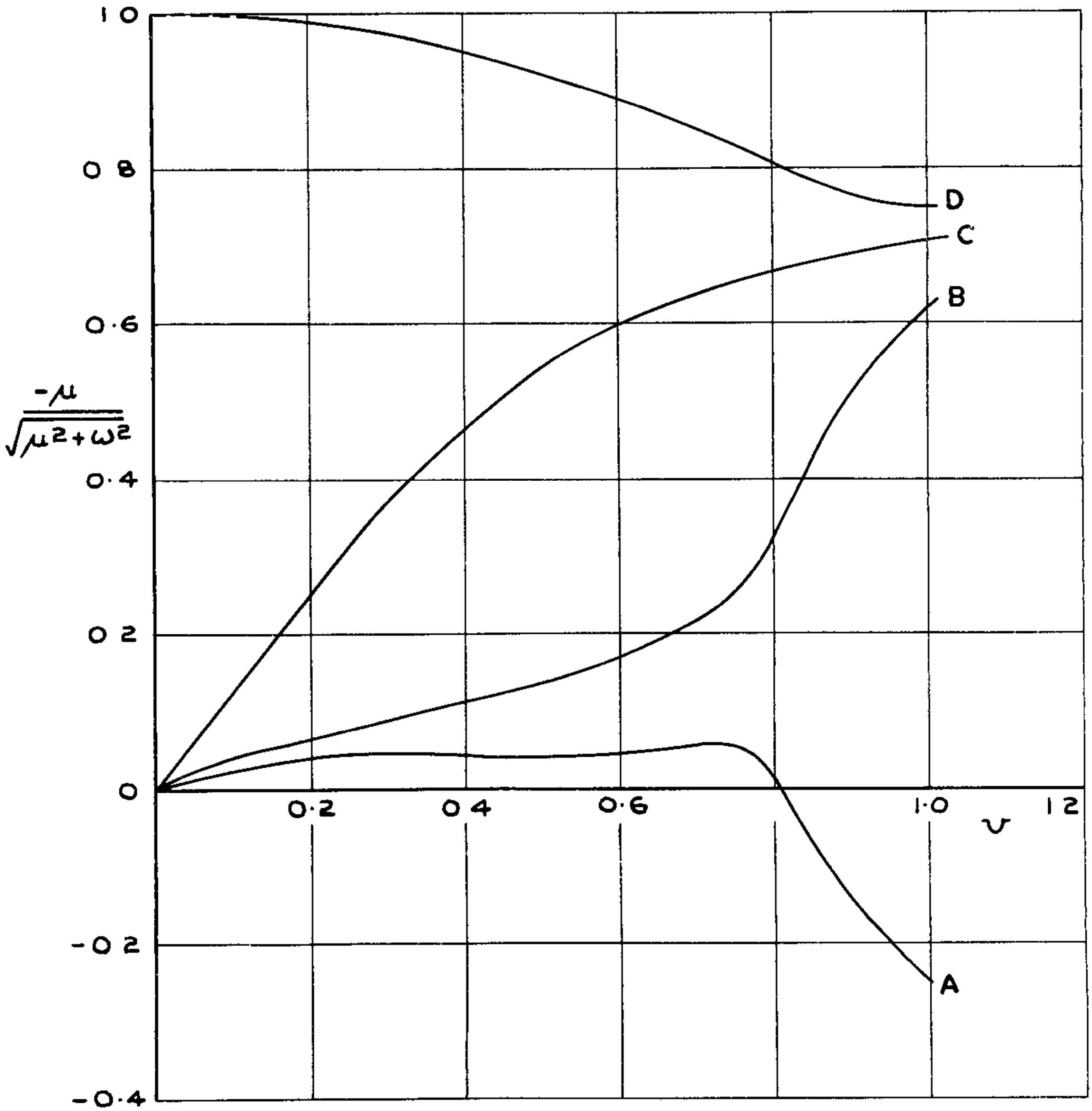

Fig. 31 Richardson method solution
(p_0= 0·4, m = 3) - relative damping ratio

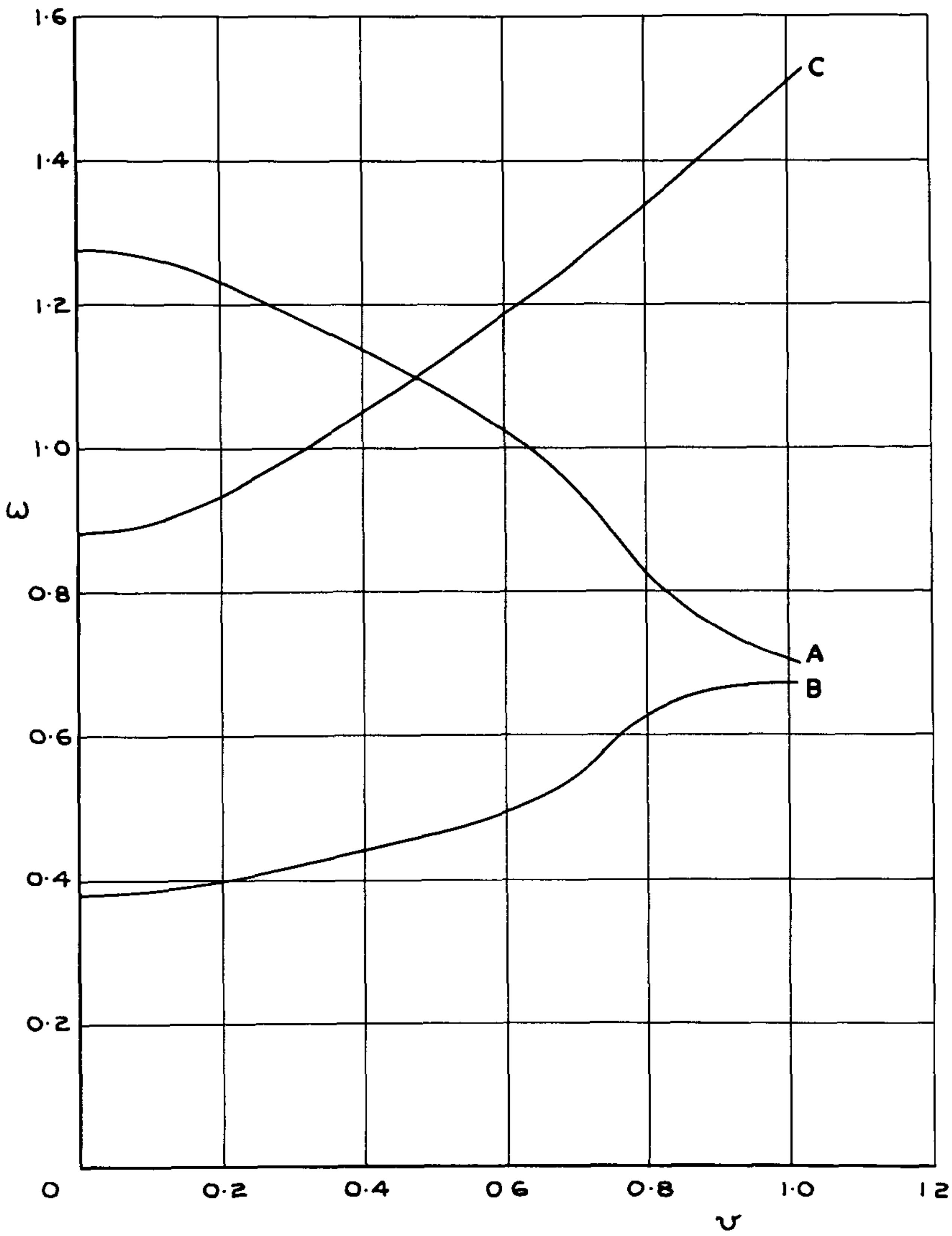

Fig. 32 Richardson method solution
(p_o = 0·4, m = 2) – frequency

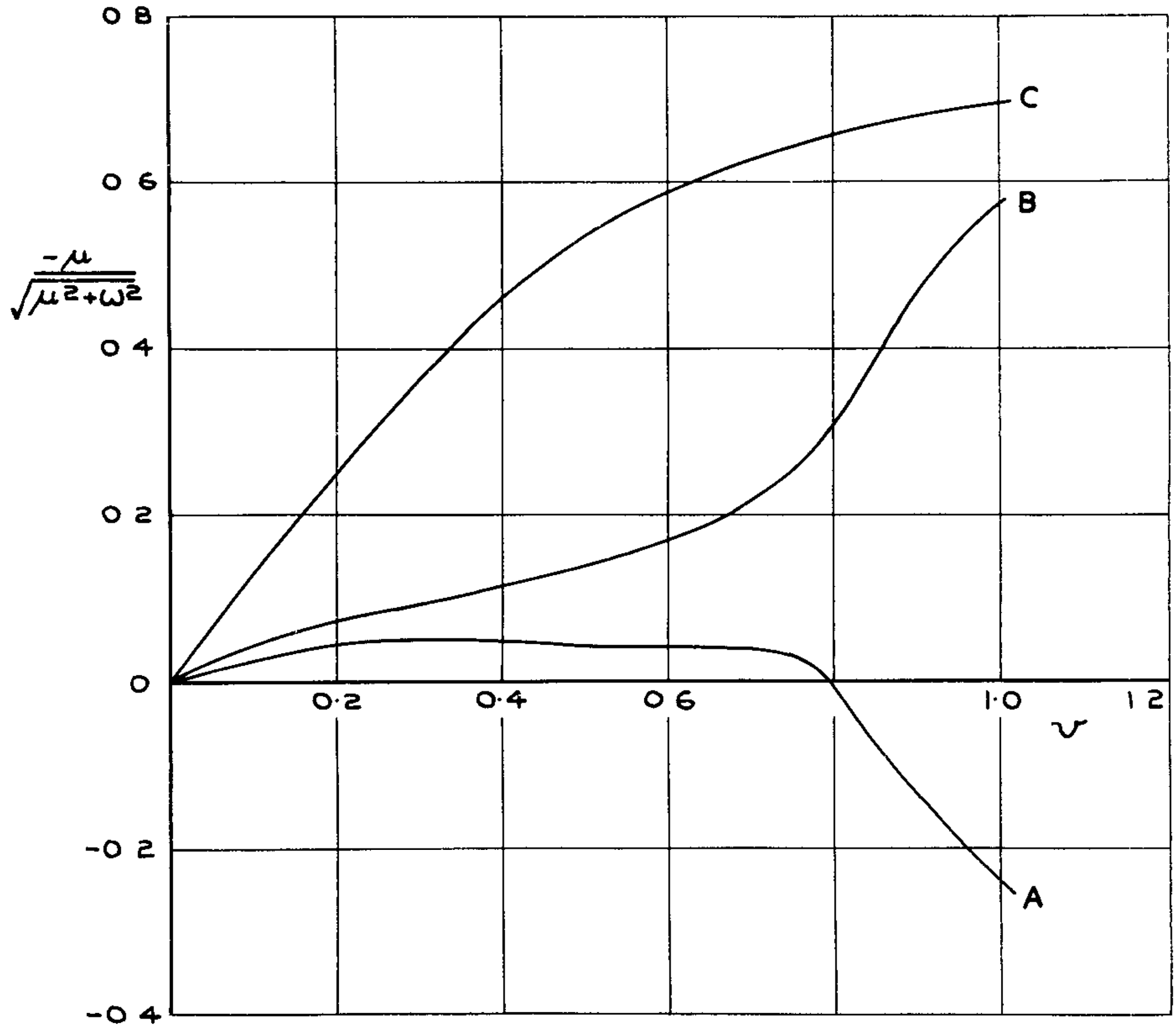

Fig. 33 Richardson method solutions (p_0 = 0·4, m = 2) –relative damping ratio

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December 1968

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