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The Stability of Boundary Conditions in the Numerical Solution of the Time-Dependent Navier-Stokes Equations

- By -

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This paper investigates the stability of finite-difference schemes, including boundary conditions, for solving the time-dependent Navier-Stokes equations. The different types of boundary condition which may occur are listed and no-slip conditions are derived for a wall with suction. Stability analyses are completed for one-dimensional problems with various types of boundary conditions, using schemes suitable for two-dimensional problems. All the conditions introduced are shown to be stable if there is no flow across the boundary. For suction at a fixed or moving wall, it is shown that the mesh size must be restricted for both accuracy and stability.

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1. Introduction

Various proposals have been made for dealing with boundaries in finite-difference approximations to the two-dimensional, time-dependent, Navier-Stokes equations. The types of boundary which occur may be divided into three categories, those concerned with 'conditions at infinity', no-slip conditions at a fixed wall and no-slip conditions at a moving wall. The most common method of investigating 'numerical stability' of the finite-difference process is examination of the behaviour of amplitudes of Fourier components, assuming that the equations are approximately linear and that the region is infinite. This clearly does not take into account the effect of boundary conditions. In this paper, the method employed is equivalent to consideration of Fourier components with definite boundary conditions. The results, obtained here, are for one-dimensional problems using finite-difference schemes applicable to two dimensions. Of course, in practice, more efficient methods could be used to solve one-dimensional flows. So as to introduce linear parts of non-linear terms a steady 'cross-flow' is added. For example, for flow along an infinite fixed wall, suction is included at the wall. The boundary conditions are also extended to allow such such 'cross-flow'.

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*Replaces A.R.C.30 406

2. Numerical Method

The two-dimensional, time-dependent, Navier-Stokes equations may be written in terms of the stream function $\psi(x,y,t)$ and vorticity $\zeta(x,y,t)$ as

$$\frac{\partial\zeta}{\partial t} = \nu\nabla^2\zeta + J(\psi,\zeta) \qquad \dots (2.1)$$

and

$$\nabla^2\psi = -\zeta \qquad \dots (2.2)$$

where ν is the viscosity,

$$J(\psi,\zeta) \equiv \frac{\partial\psi}{\partial x}\frac{\partial\zeta}{\partial y} - \frac{\partial\psi}{\partial y}\frac{\partial\zeta}{\partial x} ,$$

and

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

The velocity of the fluid at any point is given by

$$u = \frac{\partial\psi}{\partial y} \qquad \dots (2.3)$$

and

$$v = -\frac{\partial\psi}{\partial x} , \qquad \dots (2.4)$$

where u and v are the velocity components in the x and y directions.

The numerical methods used by Fromm[3,5,6] and those concerned with weather prediction, e.g. Lilly[7], replace these equations by suitable finite-difference equations. (The viscosity ν is taken as zero in weather prediction). Vorticity values are first advanced over a time step using a difference approximation to (2.1). The vorticity values so obtained are used in a difference approximation to (2.2) and the resulting linear equations are solved to find values for the stream function. Various difference schemes have been tried and particular attention has been directed to finding satisfactory difference analogues of the non-linear term $J(\psi,\zeta)$. We shall assume that the operator ∇^2 in both equation (2.1) and (2.2) is replaced by the usual five-point difference formula so that (2.2) becomes

$$\psi_{i-1,j} + \psi_{i+1,j} + \psi_{i,j-1} + \psi_{i,j+1} - 4\psi_{i,j} = -h^2\zeta_{i,j} , \qquad \dots (2.5)$$

where $\psi_{i,j}$ and $\zeta_{i,j}$ are the values of the stream function and vorticity in the difference scheme at the point $x_i=x_o+ih$, $y_j=y_o+jh$. Thus the points included in the scheme are on a square grid of mesh size h .

3./

3. Boundary Conditions

3.1 We shall now consider numerical approximations to conditions on a boundary along the line $y = y_o$. We write ψ_o and ζ_o (instead of $\psi_{i,o}$ and $\zeta_{i,o}$) for the values of the stream function and vorticity at an arbitrary mesh point on this boundary, ψ_{-1} and ζ_{-1} for values one step outside the boundary and ψ_1 and ζ_1 for values one step inside the boundary (see Fig. 1).

3.2 Conditions at infinity

We consider first some types of conditions used away from the region of interest when the flow is assumed to take some steady form. The following boundary conditions may be used in the finite-difference equations:

Type (i) $\zeta_o = 0$ and $\psi_o = \psi_b$, where ψ_b is a given linear function of x.

The condition is used to obtain a steady flow across the boundary with zero vorticity at the boundary. As ψ_b is a linear function of x, the flow is independent of distance along the boundary. If ψ_b is a constant the boundary is a streamline.

One important use of the condition is at an upstream boundary at which there is a steady inflow with zero vorticity. It also applies along a line about which the flow is symmetric.

Type (ii) $\zeta_o = 0$ and $(\psi_o - \psi_{-1})/h = U$.

The condition may be used when the fluid has a specified velocity U along the boundary. The condition on the stream function comes from the simple difference form of equation (2.3).

Type (iii) $\zeta_o - \zeta_{-1} = \zeta_1 - \zeta_o$ and $\psi_o = \psi_b$, where ψ_b is a given linear function of x.

The vorticity is 'extrapolated' using a condition equivalent to taking $\partial^2\zeta/\partial y^2 = 0$.

Type (iv) $\zeta_o - \zeta_{-1} = \zeta_1 - \zeta_o$ and $(\psi_o - \psi_{-1})/h = U$.

This gives extrapolated vorticity and specified velocity U along the boundary.

Type (v) $\zeta_o - \zeta_{-1} = \zeta_1 - \zeta_o$ and $\psi_o - \psi_{-1} = \psi_1 - \psi_o$,

Both fields are extrapolated. This was used by Fromm[4] as a downstream boundary condition.

We note that Types (ii), (iv) and (v) do not exclude the possibility of flow across the boundary. Usually the opposite boundary condition would cause such a flow. For example, Type (i) could be used upstream and Type (v) downstream.

3.3 No-slip condition at a stationary wall

At a stationary wall or obstacle the velocity of the fluid is zero and the stream function is given. Vorticity is generated at such a boundary and this must be incorporated in the numerical method. The process, proposed and used by Fromm[3,6] is to first advance vorticity values at interior points using a finite-difference form of (2.1). Values of ψ at interior points may now be

found/

found using equations (2.5), as these involve the vorticity at only interior points and ψ is known on the boundary. Using the condition of zero velocity along the wall, it is possible to obtain 'hypothetical' values for ψ at points just outside the boundary. The values of ψ thus found may now be inserted in equations (2.5) for boundary points, to obtain the vorticity at these points. In the methods described below, the stream function at the wall is a linear function of x and therefore there is a suction velocity, which is constant along the wall. Fromm has used the first two of these methods.

<u>Type (vi)</u> $\psi_0 - \psi_{-1} = 0$ and $\psi_0 = \psi_b$, where ψ_b is a given linear function of x.

Values of ψ at interior points are found using (2.5) with $\psi_0 = \psi_b$ at the boundary. Reversing equation (2.5), we obtain for the boundary vorticity

$$\zeta_0 = -(\psi_1 - 2\psi_0 + \psi_{-1})/h^2$$

and thus, as $\psi_{-1} = \psi_0 = \psi_b$, we obtain

$$\zeta_0 = -(\psi_1 - \psi_b)/h^2 \qquad \cdots (3.1)$$

This condition may be derived by defining velocities at points,

$$y_{j-\frac{1}{2}} = y_0 + (j-\tfrac{1}{2})h, \text{ using}$$

$$u_{j-\frac{1}{2}} = \frac{\psi_j - \psi_{j-1}}{h}, \qquad \cdots (3.2)$$

a difference analogue of (2.3). At the boundary we choose ψ_{-1} so that $u_{-\frac{1}{2}} = 0$ (see Fig. 2(a)). Fromm[3] has shown that the condition is a more accurate approximation to flow with a wall along $y = y_0 - \frac{1}{2}h$ than with a wall along $y = y_0$.

<u>Type (vii)</u> $\psi_1 - \psi_{-1} = 0$ and $\psi_0 = \psi_b$, where ψ_b is a given linear function of x.

The method due to Thom[10] is similar to Type (vi) and the boundary vorticity is found using

$$\zeta_0 = -2(\psi_1 - \psi_b)/h^2 . \qquad \cdots (3.3)$$

We choose ψ_{-1} so that $u_{-\frac{1}{2}}$, as defined by (3.2), satisfies

$$u_{-\frac{1}{2}} = -u_{\frac{1}{2}}$$

and, if we define u_0 by

$$u_0 = \frac{1}{2}(u_{-\frac{1}{2}} + u_{\frac{1}{2}}),$$

we obtain (see Fig. 2(b))

$$u_0 = 0 .$$

<u>Type (viii)</u>/

and

Type (viii) $\psi_0 = \psi_b$, where ψ_b is a given linear function of x

$$\zeta_0 = -\frac{3}{h^2}(\psi_1 - \psi_b) - \tfrac{1}{2}\zeta_1 \; . \qquad \cdots (3.4)$$

This method, due to Woods[12], uses an approximation to equation (2.2) of higher degree than (2.5) when finding the boundary vorticity values. It has been employed successfully in several calculations of steady state solutions, e.g. Russell[9].

If the flow is dependent only on time and distance normal to the wall, the method may be obtained by considering the relation between the vorticity and the velocity of the fluid. From (2.2) and (2.3) we have

$$\zeta = -\frac{\partial u}{\partial y} \; . \qquad \cdots (3.5)$$

We approximate to (3.5) at $y = y_0$ by replacing $\partial u/\partial y$ by the derivative, at $y = y_0$, of the parabola through the points (y_0,u_0), $(y_{\frac{1}{2}},u_{\frac{1}{2}})$ and (y_1,u_1) where $u_0 = 0$ and $u_1 = \frac{1}{2}(u_{\frac{1}{2}} + u_{\frac{3}{2}})$. (See Fig. 2(c)).

We obtain

$$\zeta_0 = \frac{1}{h}(u_1 - 4u_{\frac{1}{2}}) \; . \qquad \cdots (3.6)$$

Now

$$\zeta_1 = -\frac{1}{h^2}(\psi_2 - 2\psi_1 + \psi_0)$$

$$= -\frac{1}{h}(u_{\frac{3}{2}} - u_{\frac{1}{2}}) \qquad \cdots (3.7)$$

and, on combining (3.7) and (3.6), we obtain (3.4).

4. Moving Wall

4.1 Moving Wall without suction

Another type of boundary condition is that of a wall (without suction) moving with constant velocity in its own plane. It was shown by Fromm[3] that, if there are no other factors influencing the flow, vorticity is 'conserved' for such a boundary, in the sense that the integral of the vorticity, over the region of the flow, is constant. In the discrete case, the integral is replaced by a summation. The methods used by Fromm[3,6] involve advancing vorticity values at both interior and boundary points using a difference form of equation (2.1) and boundary conditions consistent with conservation of vorticity. Stream function values are now obtained using equation (2.5) and a suitable velocity condition at the wall. However, it is necessary to ensure that the wall is a stream-line by making ψ constant along it and Fromm uses an averaging technique to obtain this. Equations (2.5) are solved iteratively and during each iteration the average of the ψ values on the wall is calculated. All values of ψ on the wall are replaced by this average before the next iteration is commenced. The final stream function values are used in equations (2.5),

along/

along the boundary, to obtain new vorticity values. This last step is necessary if other factors influencing the flow are to be accounted for. Other factors, e.g. an obstacle, modify the stream function and lead to the generation of vorticity at the wall. Two methods corresponding to Types (vi) and (vii) have been used by Fromm.

Type (ix) $\zeta_{-1} = \zeta_0$ and $(\psi_0 - \psi_{-1})/h = U$ where U is the wall velocity.

During the calculation of the stream function from equations (2.5), the averaging technique described above is used on the boundary. The vorticity on the wall is finally recalculated using the reverse of (2.5),

i.e.
$$\zeta_0 = -(\psi_1 - \psi_0)/h^2 + U/h$$

As in Type (vi) the wall is effectively one half-cell outside $y = y_0$.

Type (x) $\zeta_{-1} = \zeta_1$ and $(\psi_1 - \psi_{-1})/(2h) = U$.

The condition is otherwise the same as Type (ix) except that the final boundary vorticity is calculated from

$$\zeta_0 = -2(\psi_1 - \psi_0)/h^2 + 2U/h \ .$$

The wall is effectively along $y = y_0$.

4.2 Moving wall with suction

If there is suction at a moving wall, conditions of Type (ix) and (x) are no longer valid, as they would yield some transport of vorticity into the wall. However, vorticity is still conserved for such a wall when no other factors influence the flow, as is now shown.

We consider a flow dependent only on time and distance from an infinite wall, across which there is a constant suction velocity $-V$. If the wall is $y = y_0$, with fluid in the region $y > y_0$, equation (2.4) becomes

$$V = -\frac{\partial\psi}{\partial x}$$

and hence (2.1) reduces to

$$\frac{\partial\zeta}{\partial t} = \nu\frac{\partial^2\zeta}{\partial y^2} - V\frac{\partial\zeta}{\partial y} \ . \qquad \dots (4.1)$$

The Navier-Stokes equation for the velocity u becomes

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} - V\frac{\partial u}{\partial y} \qquad \dots (4.2)$$

and
$$\zeta = -\frac{\partial u}{\partial y} \ .$$

Integrating/

Integrating (4.1) and assuming that the vorticity becomes zero away from the wall, we obtain

$$\frac{d}{dt}\int_{y_0}^{\infty} \zeta \, dy = \left[-\nu \frac{\partial \zeta}{\partial y} + V\zeta\right]_{y=y_0} \qquad \cdots (4.3)$$

$$= \left[\nu \frac{\partial^2 u}{\partial y^2} - V \frac{\partial u}{\partial y}\right]_{y=y_0}$$

$$= \left[\frac{\partial u}{\partial t}\right]_{y=y_0} , \quad \text{from (4.2)} ,$$

$$= 0 , \quad \text{as the wall moves with constant velocity.}$$

To derive conservative boundary conditions we replace the space derivatives of ζ in (4.1) by finite-differences and obtain

$$\frac{d\zeta_j}{dt} = \frac{\nu}{h^2}\left(\zeta_{j+1} - 2\zeta_j + \zeta_{j-1}\right) - \frac{V}{2h}\left(\zeta_{j+1} - \zeta_{j-1}\right) , \qquad \cdots (4.4)$$

where $\zeta_j(t)$ is the vorticity at y_j at time t. In one dimension, any of the methods discussed by Lilly[7] for replacing $J(\psi,\zeta)$ reduce to the transport term in (4.4). We shall not be concerned with time integration here but the method used for this should be conservative. If non-conservative time difference methods are used, extra errors will be introduced although these may not be serious, as has been shown by Fromm[3,6] for the method devised by Dufort and Frankel[2].

We now sum equations (4.4) over all points to obtain the rate of change of total vorticity. This summation may be considered a numerical integration of (4.1) with respect to y. There are two results corresponding to boundary conditions of Types (ix) and (x).

4.3 <u>Boundary along $y = y_0 - \frac{1}{2}h$</u>

If we perform simple summation of equations (4.4) we obtain

$$\frac{d}{dt}\left(\sum_{j=0}^{N} \zeta_j\right) = \frac{\nu}{h^2}\left(-\zeta_0 + \zeta_{-1}\right) - \frac{V}{2h}\left(-\zeta_0 - \zeta_{-1}\right) ,$$

assuming $\zeta_j = 0$ for $j > N$.

Thus/

Thus

$$\frac{d}{dt}\left(h \sum_{j=0}^{N} \zeta_j\right) = -\nu\left(\frac{\zeta_0 - \zeta_{-1}}{h}\right) + V\left(\frac{\zeta_0 + \zeta_{-1}}{2}\right) \qquad \dots (4.5)$$

and this finite-difference analogy of (4.3). By equating the right-hand side of (4.5) to zero, we obtain the following conservative boundary condition.

<u>Type (xi)</u>

$$\zeta_{-1} = \left(\frac{\frac{\nu}{h} - \frac{V}{2}}{\frac{\nu}{h} + \frac{V}{2}}\right) \cdot \zeta_0 \qquad \dots (4.6)$$

and

$$\frac{\psi_0 - \psi_{-1}}{h} = U ,$$

where U is the velocity of the wall. In two-dimensional flow, the stream function must also be restricted so that it is linear along the wall, using a technique similar to that for Type (ix). The vorticity along the wall is then recalculated using

$$\zeta_0 = -(\psi_1 - \psi_0)/h^2 + U/h . \qquad \dots (4.7)$$

When $V = 0$, the method reduces to Type (ix). For non-zero V, we shall show that the method can only be successful if h is sufficiently small. The method gives a boundary effectively one half-cell below $y = y_0$ and thus, when performing numerical integration of the vorticity by summation, we assume each vorticity value, including boundary values, is constant over one cell.

4.4 <u>Boundary along $y = y_0$</u>

In this case, in the integration of the vorticity, the boundary values extend over only a half-cell in the interior of the region. We consider therefore the quantity:

$$\frac{d}{dt}\left(\sum_{j=1}^{N} \zeta_j + \tfrac{1}{2}\zeta_0\right) = \frac{\nu}{h^2}\left(-\zeta_1 - \zeta_0\right) - \frac{V}{2h}\left(-\zeta_0 - \zeta_1\right) + \frac{\nu}{2h^2}\left(\zeta_1 - 2\zeta_0 + \zeta_{-1}\right) - \frac{V}{4h}\left(\zeta_1 - \zeta_{-1}\right)$$

from (4.4).

Thus

$$\frac{d}{dt}\left(h \sum_{j=1}^{N} \zeta_j + \tfrac{1}{2}h\zeta_0\right) = -\nu\left(\frac{\zeta_1 - \zeta_{-1}}{2h}\right) + V\left(\frac{\zeta_1 + 2\zeta_0 + \zeta_{-1}}{4}\right) \qquad \dots (4.8)$$

which is a finite-difference analogy of (4.3). On equating the right-hand side to zero we obtain the following boundary conditions.

Type (xii)/

Type (xii)

$$\zeta_{-1} = \left(-V\zeta_o + \left[\frac{\nu}{h} - \frac{V}{2}\right]\zeta_1\right) \Big/ \left(\frac{\nu}{h} + \frac{V}{2}\right) \qquad \cdots (4.9)$$

and

$$\frac{\psi_1 - \psi_{-1}}{2h} = U \; .$$

Again the stream function must be restricted so that it is linear along the wall and the vorticity finally corrected using

$$\zeta_o = -2(\psi_1 - \psi_o)/h^2 + 2U/h \; , \qquad \cdots (4.10)$$

as in Type (x). The condition reduces to Type (x) when $V = 0$.

4.5 Impulsive start

If the wall starts impulsively from rest, with the fluid at rest, initially both the stream function and vorticity are zero everywhere, except the vorticity at the wall, which is determined by (4.7) or (4.10), i.e.

$$\zeta_o(0) = \frac{U}{h} \text{ for Type (xi)} \qquad \cdots (4.11)$$

and

$$\zeta_o(0) = \frac{2U}{h} \text{ for Type (xii)} \; . \qquad \cdots (4.12)$$

At any time the vorticity satisfies the one-dimensional form of (2.5) namely

$$\psi_{j+1} - 2\psi_j + \psi_{j-1} = -h^2\zeta_j \qquad \cdots (4.13)$$

and either (4.7) or (4.10) at the boundary.
Assuming ψ becomes zero away from the wall we obtain at all times for Type (xi)

$$\sum_{j=o}^{N} \zeta_j = \frac{U}{h} \qquad \cdots (4.14)$$

and for Type (xii)

$$\sum_{j=1}^{N} \zeta_j + \tfrac{1}{2}\zeta_o = \frac{U}{h} \; . \qquad \cdots (4.15)$$

This shows that both conditions are valid even initially for such flow. This result was obtained by Fromm[5] for Type (xi).

5./

5. Numerical Stability

5.1 We consider the stability of finite-difference methods of solving equations (2.1) and (2.2) when applied to the one-dimensional equation (4.1), with various types of boundary conditions. Thus we shall assume that vorticity values are advanced using a complete difference form of equations (4.4), and that stream function values are subsequently found, for each time step, using equations (4.13).

The ordinary differential equations (4.4) may be written in the form

$$\frac{d\underline{\zeta}}{dt} = M\,\underline{\zeta} + \underline{b} \qquad \ldots (5.1)$$

where $\underline{\zeta}$ is a vector with components ζ_j, M is a square matrix and $\underline{b}$ is a vector whose elements depend on the boundary conditions. We shall now consider the stability of equation (5.1). If λ_r, for $r = 1,2..n$, are the eigenvalues of M, we shall say that equation (5.1) is

unstable	if $\max_r \mathcal{R}\ell(\lambda_r) > 0$
neutrally stable	if $\max_r \mathcal{R}\ell(\lambda_r) = 0$
stable	if $\max \mathcal{R}\ell(\lambda_r) < 0$.

In general, a single perturbation in the solution of equation (5.1) causes an error whose magnitude increases with time for the first case, becomes constant for the second and decreases for the third. Clearly we cannot expect to obtain meaningful numerical results for an unstable equation. Even if (5.1) is stable in the sense given here, it does not follow that finite-difference solutions will necessarily converge to the solution of the differential equations as mesh sizes are decreased. Parter[8] has shown this for a first-order equation.

For a steady flow equation (5.1) becomes

$$M\,\underline{\zeta} = -\underline{b}$$

and this equation may be solved iteratively as is usual for two-dimensional problems, see e.g. Russell[9]. Most of the common iterative methods will not converge unless $\max_r \mathcal{R}\ell(\lambda_r) < 0$.

5.2 Eigenvalues of M

If $\underline{x}$ is an eigenvector of M, corresponding to an eigenvalue λ_r, the components x_j of $\underline{x}$ satisfy the difference equations

$$(\alpha + \beta)x_{j-1} - (2\alpha + \lambda_r)x_j + (\alpha - \beta)x_{j+1} = 0 \qquad \ldots (5.2)$$

where/

where $\alpha = \nu/h^2$ and $\beta = V/(2h)$. Homogeneous boundary conditions on the x_j are derived from those applied to ζ_j .

The general solution of equation (5.2) (for $\theta_r \neq 0$ or π) is given by

$$x_j = a^j(A \cos j\theta_r + B \sin j\theta_r) , \qquad \cdots (5.3)$$

where*

$$\lambda_r = 2\alpha\left(\frac{\cos \theta_r}{\text{ch } b} - 1\right) , \qquad \cdots (5.4)$$

$$a = \frac{(\alpha + \beta)^{\frac{1}{2}}}{(\alpha - \beta)^{\frac{1}{2}}} \qquad \cdots (5.5)$$

and

$$b = \ln a . \qquad \cdots (5.6)$$

We seek values of θ_r , A and B which give non-zero eigenvectors. If a solution is given by, $\theta_r = 0$ or π, the general solution (5.3) is no longer valid and we seek a solution containing terms of the form $j(\pm a)^j$. The following cases regarding α and β will be considered. (The cases $\alpha = \pm\beta$ are ignored as for these (5.2) reduces to a first-order equation).

Case A $\alpha > 0, \ \beta = 0$

This corresponds to $V = 0$. We obtain from equations (5.5) and (5.6), $a = 1$ and $b = 0$.

Case B $\alpha > |\beta| > 0$

This corresponds to $\frac{-2\nu}{h} < V < \frac{2\nu}{h}$. Both a and b are real.

We have

$$\mathcal{Re}(\lambda_r) = 2\alpha\left(\frac{\mathcal{Re}(\cos \theta_r)}{\text{ch } b} - 1\right) . \qquad \cdots (5.7)$$

Case C $0 < \alpha < \beta$

This corresponds to $V > 2\nu/h$. We let $a = -ik$ where $k > 1$, when $\text{ch } b = -i \text{ sh } c$, where $c = \ln k > 0$.

From (5.4) we find

$$\lambda_r = 2\alpha\left(\frac{i \cos \theta_r}{\text{sh } c} - 1\right)$$

and

$$\mathcal{Re}(\lambda_r) = -2\alpha\left(\frac{\text{Im}(\cos \theta_r)}{\text{sh } c} + 1\right) . \qquad \cdots (5.8)$$

Case/

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*The hyperbolic functions cosh and sinh are written as ch and sh for brevity.

Case D $\alpha = 0, \beta > 0$

This corresponds to $\nu = 0$ and $V > 0$. We obtain $a = -i$ and $\mathcal{R}e(\lambda_r) = -2\beta . \mathrm{Im}(\cos\theta_r)$. ... (5.9)

Case E $\beta < -\alpha < 0$

This corresponds to $V < -2\nu/h$. We let $a = ik$ where $0 < k < 1$ when $\mathrm{ch}\, b = -i\, \mathrm{sh}\, c$ where $c = -\ln k > 0$.

From (5.4) we find

$$\lambda_r = 2\alpha\left(\frac{i\cos\theta_r}{\mathrm{sh}\, c} - 1\right)$$

$$\mathcal{R}e(\lambda_r) = -2\alpha\left(\frac{\mathrm{Im}(\cos\theta_r)}{\mathrm{sh}\, c} + 1\right). \qquad \text{... (5.10)}$$

Case F $\alpha = 0 \quad \beta < 0$

This corresponds to $\nu = 0$ and $V < 0$. We obtain $a = i$ and $\mathcal{R}e(\lambda_r) = -2\beta . \mathrm{Im}(\cos\theta_r)$.

6. Particular Boundary Conditions

6.1 We now consider the effects of particular boundary conditions on the eigenvalues of M. Two conditions are required, one at each boundary. It should be remembered that if ψ is specified at both boundaries a fixed mass flow must occur across any line joining them. In the cases discussed below, V is positive in the direction from the first to second boundary. Table 1 shows values of $\underset{r}{\mathrm{Max}}\, \mathcal{R}e(\lambda_r)$ for numerical cases involving most of the different boundary conditions discussed and Table 2 a summary of the results of the stability investigations made in this Section.

6.2 Type (i) and Type (i)

This is the simplest combination. The boundary conditions on the eigenvector are

$$x_o = x_{n+1} = 0$$

and the solution is given by

$$A = 0 \quad \text{and} \quad \theta_r = \frac{r\pi}{n+1}, \; r = 1, 2, \ldots n.$$

Thus $\mathcal{R}e(\lambda_r) < 0$ for all r and equation (5.1) is stable.

6.3 Type (iii), (iv) or (v) and Type (iii), (iv) or (v)

We require

$$x_0 = 2x_1 - x_2$$

$$x_{n+1} = 2x_n - x_{n-1}$$

Using (5.3) we obtain

$$[a^2 \cos 2\theta - 2a \cos \theta + 1]A + [a^2 \sin 2\theta - 2a \sin \theta]B = 0$$

$$[a^2 \cos(n+1)\theta - 2a \cos n\theta + \cos(n-1)\theta]A + [a^2 \sin(n+1)\theta - 2a \sin n\theta + \sin(n-1)\theta]B = 0$$

and the θ_r are values of θ for which these equations yield non-zero A and B. The determinant of the matrix of coefficients of A and B is zero when

$$[2a^2 \cos 2\theta - 4a.(1+a^2) \cos \theta + 1 + 4a^2 + a^4] \sin(n-1)\theta = 0 .$$

This has $(n-2)$ roots

$$\theta_r = \frac{r\pi}{n-1} \qquad r = 1,2, \ldots (n-2)$$

and 2 roots when

$$\cos \theta_r = \frac{1}{2}\left(a + \frac{1}{a}\right) = \text{ch } b \text{ }, r = n-1, n .$$

For the first $(n-2)$ roots we have $\mathcal{Re}(\lambda_r) < 0$, for the last two $\mathcal{Re}(\lambda_r) = 0$ and therefore equation (5.1) is neutrally stable.

6.4 Type (i) or (ii) and Type (iii), (iv) or (v)

We require

$$x_0 = 0$$

$$x_{n+1} - 2x_n + x_{n-1} = 0 .$$

Thus $A = 0$ and the θ_r are roots of

$$a^{n+1} \sin(n+1)\theta - 2a^n \sin n\theta + a^{n-1} \sin(n-1)\theta = 0 .$$

Let

$$f(\theta) \equiv a^2 \sin(n+1)\theta - 2a \sin n\theta + \sin(n-1)\theta , \qquad \ldots (6.1)$$

when the θ_r are roots of $f(\theta) = 0$.

Case A

The roots are $\theta_r = r\pi/n$, for $r = 1, \ldots (n-1)$ together with $\theta_n = 0$, with corresponding eigenvector components $x_j = j$.

Case B/

Case B

First note that

$$f(0) = 0$$

$$f'(0) = (a-1)[(n+1)a - (n-1)] > 0, \quad \text{for } a > 1, \text{ i.e. } V > 0$$

$$\left.\begin{aligned} &< 0, \quad \text{for } \frac{n-1}{n+1} < a < 1 \\ &> 0, \quad \text{for } 0 < a < \frac{n-1}{n+1} \end{aligned}\right\} \text{ i.e. } V < 0.$$

Now

$$f\left(\frac{r\pi}{n}\right) = (a^2-1)(-1)^r \sin\frac{r\pi}{n}$$

which oscillates in sign for $r = 1,2 \ldots (n-1)$, therefore there are at least $(n-2)$ real roots.

Also

$$f\left(\frac{\pi}{n}\right) < 0, \quad \text{if } a > 1,$$
$$> 0, \quad \text{if } a < 1.$$

We therefore have a further real root for $a > (n-1)/(n+1)$ in the interval $(0,\pi/n)$.

We can similarly show that for $a > 1$ there is a root in the interval $([n-1]\pi/n,\pi)$.

For $a < 1$ i.e. $V < 0$ we can show that there is at least one root of the form $\theta_r = iz_r$ where z_r is real. Let

$$F(z) = -if(iz) = a^2 \operatorname{sh}(n+1)z - 2a \operatorname{sh} nz + \operatorname{sh}(n-1)z.$$

Now, $b = \log a < 0$, as $a < 1$, and

$$F(b) = \frac{1}{2}a^{n-1}(a^2-1)^2 > 0.$$

Also

$$F(z) \sim -\frac{a^2}{2} e^{-(n+1)z} \quad \text{as } z \to -\infty$$
$$< 0.$$

Thus there is a root z in the region $(-\infty,b)$ when

λ_r /

$$\lambda_r = 2\alpha\left(\frac{\text{ch } z_r}{\text{ch } b} - 1\right) > 0 ,$$

as $\text{ch } z_r > \text{ch } b$.

For $a < (n-1)/(n+1)$, there is a further root of the form $\theta_r = iz_r$ and, if $a = (n-1)/(n+1)$, there is a root $\theta_r = 0$.

We therefore deduce that, for this case, equation (5.1) is unstable for $V < 0$ and stable for $V > 0$.

Case C

We show that $\mathcal{R}e(\lambda_r) < 0$, for all r, and therefore that equation (5.1) is stable. The roots are complex and we let $\theta_r = \rho_r + i\sigma_r$ when

$$\begin{aligned} \mathcal{R}e(\cos\theta_r) &= \cos\rho_r \text{ ch } \sigma_r , \\ \text{Im}(\cos\theta_r) &= -\sin\rho_r \text{ sh } \sigma_r . \end{aligned} \qquad \dots (6.2)$$

Since $\text{Im}(\lambda_r) = 2\alpha\, \mathcal{R}e(\cos\theta_r)/\text{sh } c$ and complex eigenvalues occur in conjugate pairs, if ρ_r is a root then so is $\pi-\rho_r$. We can also assume $\sigma_r \geqslant 0$, without restriction.

From (6.1), using $a = -ik$, we have

$$\mathcal{R}e(f(\theta)) = -k^2 \sin(n+1)\rho \text{ ch}(n+1)\sigma - 2k\cos n\rho \text{ sh } n\sigma + \sin(n-1)\rho \text{ ch}(n-1)\sigma , \qquad \dots (6.3)$$

$$\text{Im}(f(\theta)) = -k^2 \cos(n+1)\rho \text{ sh}(n+1)\sigma + 2k\sin n\rho \text{ ch } n\sigma + \cos(n-1)\rho \text{ sh}(n-1)\sigma , \qquad \dots (6.4)$$

where $\theta = \rho + i\sigma$.

We suppose that there is a root $\rho_r + i\sigma_r$, with $0 \leqslant \rho_r \leqslant \pi$ and $\sigma_r \geqslant 0$. We can also assume, without loss, that $0 \leqslant \rho_r \leqslant \pi/2$ as $\pi-\rho_r$ is also a root.

If we further suppose $n\rho_r$ is in the first quadrant, i.e.

$$2m\pi \leqslant n\rho_r < (2m+\tfrac{1}{2})\pi$$

for some integer m, then we have the inequalities

$$\cos n\rho_r \geqslant 0, \sin(n+1)\rho_r \geqslant 0 \text{ and } \sin(n+1)\rho_r \geqslant \sin(n-1)\rho_r ,$$

together with,

$$k > 1 \text{ and } \text{ch}(n+1)\sigma_r \geqslant \text{ch}(n-1)\sigma_r .$$

We find, therefore, from (6.3), that $\mathcal{R}e(f(\theta_r)) \leqslant 0$ with equality for only

trivial/

trivial cases, We similarly obtain, $\mathcal{Re}(f(\theta_r)) \geq 0$, for $n\rho_r$ in the third quadrant. For $n\rho_r$ in the second or fourth quadrants, we can show using (6.4) that $\mathrm{Im}(f(\theta))$ is non-zero. We conclude that there is no root with $0 \leq \rho_r \leq \pi$. Thus, using equation (6.2), we deduce that $\mathrm{Im}(\cos\theta_r) \geq 0$ and, from equation (5.8), that $\mathcal{Re}(\lambda_r) < 0$.

Case D

We can show that this is stable by extending the method for Case C.

Case E

As for Case B with $V < 0$, the problem is unstable for Case E. It is not sufficient, as in Case C, to only consider the sign of $\mathrm{Im}(\cos\theta_r)$, as may be seen from (5.10), and therefore a different method must be adopted. For odd values of n , Case E may be shown unstable using a method similar to that of Case C for boundary conditions of Types (i) and (vi) (Section 6.7). This restriction to odd values of n is justified in Section 6.7.

Case F

This may be shown to be unstable using a method similar to that of Case C above.

6.5 Conclusions on extrapolation boundary condition

It has been shown that for outflow, e.g. as a downstream 'infinity' condition, the use of extrapolation boundary conditions (Types (iii), (iv) and (v)) may yield numerically stable equations. For inflow, e.g. as an upstream 'infinity' condition they yield unstable equations, except possibly, when a similar condition is used for the corresponding outflow. Since in any two-dimensional flow there will be other boundaries it is highly probable that extrapolation will always give unstable equations when used at an inflow boundary.

6.6 Type (vi) and Type (vi)

The boundary conditions for the vorticity involve stream function values and these must first be related to the vorticity. Equations (4.13) may be written in the form

$$H\,\underline{\psi} = -h^2\underline{\zeta} - \underline{c} \quad , \qquad \cdots (6.5)$$

where the components of $\underline{\psi}$ and $\underline{\zeta}$ are the values of the stream function and vorticity respectively at interior grid points, H is the $n \times n$ tridiagonal matrix

$$H = \begin{bmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{bmatrix} \quad \text{and} \quad \underline{c} = \begin{bmatrix} \psi_0 \\ 0 \\ \vdots \\ 0 \\ \psi_{n+1} \end{bmatrix}$$

ψ_0 /

ψ_0 and ψ_{n+1} are the specified boundary values of the stream function.
Now H^{-1} has i,j th element,

$$-\frac{(n+1-j)i}{n+1} \quad \text{for} \quad j \geq i ,$$

$$-\frac{(n+1-i)j}{n+1} \quad \text{for} \quad j < i ,$$

and therefore,

$$\psi_1 = h^2 \sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) \zeta_j + \frac{n\psi_0 + \psi_{n+1}}{n+1} \qquad \cdots (6.6)$$

and

$$\psi_n = h^2 \sum_{j=1}^{n} \left(\frac{j}{n+1}\right) \zeta_j + \frac{\psi_0 + n\psi_{n+1}}{n+1} . \qquad \cdots (6.7)$$

The boundary condition given by equation (3.1) becomes, for the lower boundary,

$$\zeta_0 = - \sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) \zeta_j + \frac{\psi_0 - \psi_{n+1}}{(n+1)h^2}$$

and, for the upper boundary,

$$\zeta_{n+1} = - \sum_{j=1}^{n} \left(\frac{j}{n+1}\right) \zeta_j + \frac{\psi_{n+1} - \psi_0}{(n+1)h^2} .$$

The boundary conditions for the eigenvectors of M are, therefore

$$x_0 = - \sum_{j=1}^{n} \left(\frac{n+1-j}{n+1}\right) x_j \quad \text{and} \quad x_{n+1} = - \sum_{j=1}^{n} \left(\frac{j}{n+1}\right) x_j ,$$

i.e.

$$\sum_{j=0}^{n} (n+1-j)\, x_j = 0 \qquad \cdots (6.8)$$

and

$$\sum_{j=1}^{n+1} j x_j = 0 . \qquad \cdots (6.9)$$

If/

If we substitute the general solution (5.3) in these equations and seek values of θ_r for which A and B are non-zero, we can show, after some considerable manipulation using the identities (A.3) - (A.6) of Appendix A, that θ_r are roots of

$$\{\sin\theta\ [\mathrm{ch}(n+2)b - \cos(n+2)\theta] - (n+2)[\mathrm{ch}\ b - \cos\theta]\sin(n+2)\theta\}/(\mathrm{ch}\ b - \cos\theta)^2 = 0. \qquad \ldots (6.10)$$

We shall only consider in detail cases when $\alpha = 0$ or $\beta = 0$, since for these the roots of (6.10) may easily be shown to be real. For other cases the roots are complex and no simple method has been found of determining the sign of the corresponding $\mathcal{Re}(\lambda_r)$.

Case A

As $b = 0$ equation (6.10) becomes

$$f(\theta)/(1-\cos\theta)^2 = 0,$$

where

$$f(\theta) \equiv \sin\theta\ [1-\cos(n+2)\theta] - (n+2).[1-\cos\theta].\sin(n+2)\theta .$$

Now

$$f\left(\frac{r\pi}{n+2}\right) = \sin\left(\frac{r\pi}{n+2}\right).\ [1-(-1)^r] \quad \text{for } r = 1,2 \ldots n+1,$$

$$= 0 \quad \text{for even } r,$$

$$> 0 \quad \text{for odd } r,$$

and

$$f(0) = f(\pi) = 0 .$$

Also we have $f'\left(\frac{r\pi}{n+2}\right) < 0$, for even r, and $f'(\pi) < 0$, for even n.

If n is even, there are, therefore, $n/2$ roots

$$\theta_r = r\pi/(n+2) \quad \text{for } r = 2,4, \ldots n,$$

together with $n/2$ roots satisfying

$$\frac{(r-1)\pi}{n+2} < \theta_r < \frac{(r+1)\pi}{n+2} \quad \text{for } r = 3,5 \ldots n+1 .$$

If n is odd, there are $(n+1)/2$ roots,

$$\theta_r = r\pi/(n+2) \quad \text{for } r = 2,4, \ldots (n+1),$$

together with $(n-1)/2$ roots,

$$\frac{(r-1)\pi}{n+2} < \theta_r < \frac{(r+1)\pi}{n+2} \quad \text{for } r = 3,5, \ldots n .$$

Thus there are n real roots and equation (5.1) is stable.

Case D

$a = -i$ and equation (6.10) becomes

$$f(\theta)/(\cos\theta)^2 = 0 , \qquad \cdots (6.11)$$

where

$$f(\theta) \equiv \begin{cases} -\sin\theta[1 + \cos(n+2)\theta] + (n+2)\cos\theta\sin(n+2)\theta, & \text{for } n \equiv 0 \pmod 4 \\ -\sin\theta\cos(n+2)\theta + (n+2)\cos\theta\sin(n+2)\theta, & \text{for } n \equiv 1 \text{ or } 3 \pmod 4 \\ \sin\theta[1-\cos(n+2)\theta] + (n+2)\cos\theta\sin(n+2)\theta, & \text{for } n \equiv 2 \pmod 4. \end{cases}$$

By examining the behaviour $f(\theta)$ and $f'(\theta)$ at points $r\pi/(n+2)$ for $r = 0,1, \ldots (n+2)$, it can be seen that for all values of n there are n real roots of (6.11). Care needs to be taken over zeros of $f(\theta)$ at $\pi/2$ as the denominator is also zero at this point.

Using equation (5.9) we deduce that for all eigenvalues λ_r

$$\mathcal{Re}(\lambda_r) = 0$$

and equation (5.1) is neutrally stable.

This case is clearly not practicable in any physical problem, as a no-slip condition will not occur for $\nu = 0$, however it should provide some insight into the behaviour of the method as β/α is increased.

Case F

Due to the symmetry of the boundary conditions this is the same as Case D.

Cases B, C and E

As mentioned earlier the roots of equation (6.10) are complex for these cases. However, we might expect, on the evidence of Cases A, D and F, that equation (5.1) is at worst neutrally stable. This has been found true for all of several numerical cases, some of which are given in Table 1.

6.7 Type (i) and Type (vi)

Since the matrix H of (6.5) is unchanged, the second boundary condition is again (6.9) and we have boundary conditions

$$x_0 = 0$$

$$\sum_{j=1}^{n+1} jx_j = 0 .$$

The first gives $A = 0$ and using the identity (A.3), we deduce that we require the θ_r to be roots of

$$\{a(1-a^2)\sin\theta - (n+2)a^{n+4}\sin n\theta + [2(n+2) + (n+1)a^2]a^{n+3}\sin(n+1)\theta$$
$$- [2(n+1)a^2 + (n+2)]a^{n+2}\sin(n+2)\theta + (n+1)a^{n+3}\sin(n+3)\theta\}/$$
$$(1+a^2 - 2a\cos\theta)^2 = 0 . \qquad \cdots (6.12)$$

Case A/

Case A

Putting $a = 1$ we obtain for equation (6.12)

$$[(n+1)\sin(n+2)\theta - (n+2)\sin(n+1)\theta]/(1-\cos\theta) = 0 .$$

The numerator alternates in sign for points $r\pi/(n+\frac{3}{2})$, $r = 1,2 \ldots (n+1)$, and thus there are n real roots θ_r and equation (5.1) is stable.

Case C

This case corresponds to flow towards a fixed wall at which there is suction. If n is odd, there must be a real eigenvalue of M as complex eigenvalues occur in conjugate pairs. We shall examine the sign of this real eigenvalue and show that, if β/α is sufficiently large, the method is unstable. This restriction to odd values of n is justified, in that we are concerned with seeking indications of the stability of the method in two dimensions.

If λ_r is real, we deduce from (5.8), that $\cos\theta_r$ is purely imaginary and we therefore let

$$\theta_r = \frac{\pi}{2} + i\phi_r ,$$

when (5.8) becomes

$$\lambda_r = 2\alpha(\text{sh}\,\phi_r/\text{sh}\,c - 1) .$$

We have $\mathcal{Re}(\lambda_r) > 0$ if ϕ_r is real and $\phi_r > c > 0$.

Putting $\theta = \frac{\pi}{2} + i\phi$ in (6.12), we deduce (remembering n is odd) that ϕ_r are the roots of

$$F(\phi)/(\text{sh}\,c - \text{sh}\,\phi)^2 = 0 , \qquad \ldots (6.13)$$

where

$$F(\phi) = k(1+k^2)\text{ch}\,\phi - (n+2)k^{n+4}\text{ch}\,n\phi + [2(n+2) - (n+1)k^2]k^{n+3}\text{sh}(n+1)\phi$$
$$+ [2(n+1)k^2 - (n+2)]k^{n+2}\text{ch}(n+2)\phi - (n+1)k^{n+3}\text{sh}(n+3)\phi ,$$

$$a = -ik \text{ and } c = \ln k .$$

Notice first, that $F(c) = F'(c) = 0$, but this does not give a root $\phi = c$ of equation (6.13) as the denominator also has a double zero at this point.

If $F''(c)$ is positive, $F(\phi)$ has a minimum at $\phi = c$ and, since

$F(\phi)/$

$$F(\phi) \sim -\frac{(n+1)}{2} k^{n+3} e^{(n+3)\phi} \quad \text{as} \quad \phi \to \infty ,$$
$$< 0 ,$$

there will be a real root ϕ_r with $\phi_r > c$.

Now

$$F''(c) = \tfrac{1}{2}(n+1)(n+2)+k^2 (n^2 + 3n+3)+\tfrac{1}{2} k^4 (n+1)(n+2)-(n+2)k^{2n+4}-(n+1)k^{2n+6}$$

and $F''(c) > 0$, if $k < k_c$, where k_c is the only root of

$$\tfrac{1}{2}(n+1)(n+2)+k^2(n^2 + 3n+3)+\tfrac{1}{2}k^4 (n+1)(n+2)-(n+2)k^{2n+4}-(n+1)k^{2n+6} = 0 \qquad \ldots (6.14)$$

with $k_c > 1$.

Thus if n is odd and $k < k_c$, the method is unstable.

Since

$$k^2 = -\frac{\alpha + \beta}{\alpha - \beta}$$

the condition, $k < k_c$, is equivalent to

$$\beta/\alpha > (k_c^2+1)/(k_c^2-1) = (\beta/\alpha)_c \quad \text{(say)} \qquad \ldots (6.15)$$

i.e.

$$\frac{Vh}{2\nu} > (k_c^2+1)/(k_c^2-1) \quad .$$

A graph of $(\beta/\alpha)_c$ against n , for odd n , is shown in Fig. 3. By computing the eigenvalues of M it has been found, in all of several numerical cases with n odd, that the real eigenvalue derived from (6.13) is the eigenvalue with largest real part (e.g. Table 3 gives eigenvalues of M for $n = 15, \alpha = 1, \beta = 11$). Thus, for odd n , equation (5.1) has been proved unstable for $(\beta/\alpha) > (\beta/\alpha)_c$ and on the basis of numerical results it is probably stable for $(\beta/\alpha) < (\beta/\alpha)_c$. If n is even, direct calculation of the eigenvalues of M suggests that equation (5.1) becomes unstable, as (β/α) is increased, at points just above a smooth curve (Fig. 3) joining the critical values $(\beta/\alpha)_c$ for odd n . Critical values of (β/α) are shown in Fig. 3 for n = 16, 30. Approximately therefore, the region above the curve represents unstable ratios β/α and the region below the curve, stable ratios.

Case D.

By putting $k = 1$ in Case C, this case is proved unstable for odd n . Numerical evidence suggests it is also unstable for even n .

Case B/

Case B

Numerical evidence suggests that Case B is always stable. Table 1 gives some results.

Cases E and F

These cases are of little interest since they represent flow out of a fixed wall. Numerical evidence suggests that equation (5.1) is stable for all values of α and β for both cases.

6.8 Type (ii) and Type (vi)

The use of the boundary condition of Type (i) with Type (vi) implied a fixed mass flow across the region between the boundaries as ψ is specified at each. The boundary condition of Type (ii) fixes only the velocity along the boundary and allows the stream function to adjust according to this and other factors influencing the flow. For the condition of Type (vi) we again need to determine the stream function in terms of the vorticity. We express equation (4.13) as

$$L \underline{\psi} = -h^2 \underline{\zeta} + \underline{d} \quad ,$$

where the components of $\underline{\psi}$ and $\underline{\zeta}$ are the stream function and vorticity at interior grid points, L is the $n \times n$ tridiagonal matrix,

$$L = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix} \quad \text{and } \underline{d} = \begin{bmatrix} Uh \\ 0 \\ \vdots \\ 0 \\ -\psi_{n+1} \end{bmatrix}$$

U is the velocity at the first boundary and ψ_{n+1} the specified stream function at the second boundary.

The last row of L^{-1} has -1 for each element, as may be verified by premultiplying L by such a row vector. Thus

$$\psi_n = h^2 \sum_{j=1}^{n} \zeta_j - Uh + \psi_{n+1} \qquad \cdots (6.16)$$

and, on using equation (3.1) modified for an upper boundary, we obtain

$$\zeta_{n+1} = - \sum_{j=1}^{n} \zeta_j + \frac{U}{h} \quad . \qquad \cdots (6.17)$$

The/

The boundary conditions on the eigenvalues of M are thus

$$x_0 = 0$$

and

$$\sum_{j=1}^{n+1} x_j = 0 \ .$$

The first condition implies $A = 0$ and from the second, on using the identity (A.1), we deduce that the θ_r must be roots of

$$[\sin\theta = a^{n+2}\sin(n+1)\theta - a^{n+1}\sin(n+2)\theta]/(\text{ch}\, b - \cos\theta) = 0 \ . \qquad \ldots (6.18)$$

<u>Cases A and B</u>

As we are only interested in suction at the wall, we assume V is non-negative and thus $a \geqslant 1$, for these cases. It is easily seen that the numerator in (6.18) alternates in sign for points $r\pi/(n + \frac{3}{2})$ and therefore there are n real roots and equation (5.1) is stable.

<u>Case C</u>

As with Types (i) and (vi) we consider the real root which must exist when n is odd. We again put $\theta_r = \frac{\pi}{2} + i\phi_r$ and seek a root ϕ_r with $\phi_r > c$ when the method is unstable. Equation (6.18) becomes

$$F(\phi)/(\text{sh}\, c - \text{sh}\,\phi) = 0 \qquad \ldots (6.19)$$

where

$$F(\phi) = \text{ch}\,\phi + k^{n+2}\,\text{sh}(n+1)\phi - k^{n+1}\,\text{ch}(n+2)\phi \ .$$

$F(c) = 0$, but $\phi = c$ is not a root of (6.19), as the denominator also vanishes.

Now

$$F'(c) = \tfrac{1}{2}\left[\, (n+1)\frac{1}{k} + (n+2)k - k^{2n+3} \,\right]$$

and

$$F(\phi) \sim -k^{n+1}e^{(n+2)} \quad \text{as} \quad \phi \to \infty$$
$$< 0 \ ,$$

Therefore there is at least one root ϕ_r, with $\phi_r > c$ if $F'(c) > 0$, which will be true if $k < k_c$, where k_c is the root of

(n+1)/

$$(n+1)\frac{1}{k} + (n+2)k - k^{2n+3} = 0 \qquad \cdots (6.20)$$

with $k_c > 1$.

A graph of the critical ratio

$$(\beta/\alpha)_c = (k_c^2 + 1)/k_c^2 - 1) \qquad \cdots (6.21)$$

is shown in Fig. 3. Numerical evidence again suggests that, for odd n, the real eigenvalue is larger than the real parts of other eigenvalues and that, for even n, the critical values are just above a smooth curve through the points given by (6.21). Approximately therefore the curve divides the region into stable and unstable cases.

Case D

By putting $k = 1$ in Case C this case is proved unstable for odd n. Numerical evidence suggests it is also unstable for even n.

Cases E and F

These cases are of little interest. Numerical evidence suggests they would be stable.

6.9 Conclusions on Type (vi)

For one-dimensional flow, equation (5.1) is stable if this condition is used at each boundary. If Type (i) or (ii) is used with Type (vi) a restriction on (Vh/ν) must be observed. In any two-dimensional flow, with a boundary of Type (vi), there must be other types of boundary and it is therefore probable that if the ratio (Vh/ν) is too large, the finite-difference method will be unstable. It is interesting that equation (5.1) should only be conditionally stable for suction, with a restriction on the magnitude of the suction velocity (for a given mesh length), whereas for fluid flow out of the wall equation (5.1) is always stable. This of course does not reflect the physical behaviour of flow at a wall with suction and emphasises the difficulty of comparing stability of the numerical method with stability of the actual flow.

We shall see in Section 7.3 that this restriction on Vh/ν, for stability, is not as severe as that required to obtain an accurate numerical solution.

6.10 Type (vii) and Type (vii)

The stream function values one grid point inside the boundaries are again given by (6.6) and (6.7) and thus we obtain, using (3.3), the boundary conditions

$$\zeta_0 = -2\sum_{j=1}^{n}\left(\frac{n+1-j}{n+1}\right)\zeta_j = \frac{2(\psi_0-\psi_{n+1})}{(n+1)h^2}$$

$$\zeta_{n+1} = -2\sum_{j=1}^{n}\left(\frac{j}{n+1}\right)\zeta_j + \frac{2(\psi_{n+1}-\psi_0)}{(n+1)h^2} \quad .$$

The/

The boundary conditions for the components of eigenvectors of the matrix M are therefore,

$$\sum_{j=0}^{n} (n+1-j)x_j - \tfrac{1}{2}(n+1)x_o = 0$$

and

$$\sum_{j=1}^{n+1} jx_j - \tfrac{1}{2}(n+1)x_{n+1} = 0 \ .$$

After some considerable manipulation using the identities (A.3) to (A.6), it can be shown that the θ_r are roots of

$$\{4\text{ch } b \sin\theta\,[\text{ch}(n+1)b - \cos(n+1)\theta] + (n+1)\sin(n+1)\theta\,[\cos 2\theta - \text{ch } 2b]\}/(\text{ch } b - \cos\theta)^2 = 0 \ . \qquad \cdots (6.22)$$

Again we only consider in detail the cases when $\alpha = 0$ or $\beta = 0$.

Case A

As $b = 0$, (6.22) becomes

$$\sin\theta[2 - 2\cos(n+1)\theta - (n+1)\sin(n+1)\theta\sin\theta]/(1-\cos\theta)^2 = 0 \ .$$

By considering the behaviour of the numerator at points $r\pi/(n+1)$, it can be shown that there are 'n real roots in the range $0 \leqslant \theta \leqslant \pi$. If n is odd, one root is $\theta = \pi$, the eigenvalue is -4 and the corresponding eigenvector has components $x_j = (-1)^j$. Equation (5.1) is therefore stable.

Cases D and F

$a = \pm i$ and (6.22) has n roots

$$\theta_r = r\pi/(n+1) \ , \quad r = 1,2, \ldots n,$$

and therefore, as $\mathcal{Re}(\lambda_r) = 0$ for all r , equation (5.1) is neutrally stable.

Cases B and E

The roots of (6.22) are complex. Numerical evidence (see Table 1) suggests that equation (5.1) is at worst neutrally stable for all α and β, as might be expected from the results for Cases A, D and F.

6.11 Type (i) and Type (vii)

The conditions on the components of eigenvectors of M are

$$x_o = 0$$

$$\sum_{j=1}^{n+1} jx_j - \tfrac{1}{2}x_{n+1} = 0$$

from/

from which we deduce using the identity (A.3) that the θ_r must be roots of

$$\{2a(1-a^2)\sin\theta - (n+1)a^{n+3}\sin(n-1)\theta + 2(n+1-a^2)a^{n+2}\sin n\theta$$
$$+ [(n+1)(a^4-1)+4a^2].\ a^{n+1}\sin(n+1)\theta - 2[1+(n+1)a^2]a^{n+2}\sin(n+2)\theta$$
$$+ (n+1)a^{n+3}\sin(n+3)\theta\}/(\text{ch } b-\cos\theta)^2 = 0 \ . \qquad \ldots (6.23)$$

Case A

$a = 1$ and consideration of the numerator in (6.23) for points $r\pi/(n+1)$ shows that there are n real roots in $(0,\pi)$.

Cases C and D

Assuming n is odd, analysis similar to that for Case C of Types (i) and (vi) (Section 6.7) may be made. The result is the same except that (6.14) is replaced by

$$(n+1)k^{2n+6} + 2k^{2n+4} - (n-1)k^{2n+2} - (n+1)^2k^4 - 2(n^2+2n+2)k^2 - (n+1)^2 = 0 \ .$$

Numerical results are similar to those for Types (i) and (vi). Critical values of $(\beta/\alpha)_c$ are shown in Fig. 3.

Cases B, E and F

Numerical evidence (Table 1) suggests these are stable cases.

6.12 Type (ii) and Type (vii)

The stream function value ψ_n is given by (6.16) and on using (3.3) we obtain conditions

$$x_0 = 0$$
$$\sum_{j=1}^{n+1} x_j - \tfrac{1}{2}x_{n+1} = 0 \ .$$

Using the identity (A.1), we deduce that the θ_r are roots of

$$\{2\sin\theta + (a^2-1)a^n\sin(n+1)\theta - 2a^{n+1}\sin\theta\cos(n+1)\theta\}/(\text{ch } b-\cos\theta) = 0 \ . \qquad \ldots (6.24)$$

Cases A and B

We are only interested in $V \geq 0$ for which we have $a \geq 1$. By examining the behaviour of the numerator in (6.24) for points $r\pi/(n+1)$, it can be seen that there are n real roots in $(0,\pi)$.

Cases C/

Cases C and D

Assuming n is odd, analysis similar to that for Case C of Types (ii) and (vi) (Section 6.8) may be made. The result is the same except that (6.20) is replaced by

$$k^{2n+4} - k^{2n+2} - (2n+3)k^2 - (2n+1) = 0 .$$

Numerical results are similar to those for Types (i) and (vi) and a graph of the critical values $(\beta/\alpha)_c$ is shown in Fig. 3.

Cases E and F

Numerical evidence (Table 1) suggests these cases are stable.

6.13 Conclusions on Type (vii)

The boundary condition of Type (vii) has similar stability properties for the one-dimensional problem to those of Type (vi), except that a more severe restriction on the ratio Vh/ν must be observed.

6.14 Type (i) and (viii)

The stream function ψ_n, one grid point from the wall, is given by equation (6.7) and using equation (3.4) at an upper boundary we obtain as conditions on components of eigenvectors of M

$$x_0 = 0$$

$$x_{n+1} = -\frac{3}{n+1}\sum_{j=1}^{n} jx_j - \tfrac{1}{2}x_n .$$

On using the identity (A.3), we deduce that the θ_r are roots of

$$\{6a(1-a^2)\sin\theta + (n+1)a^{n+2}\sin(n-2)\theta - 2(n+1)(1+3a^2)a^{n+1}\sin(n-1)\theta$$

$$+ [(n+1)(1+12a^2) + 3a^4(n-1)]a^n\sin n\theta + [2(n+1)(a^4-3) - 6a^2(n-1)]a^{n+1}\sin(n+1)\theta$$

$$+ [3(n-1) - 4a^2(n+1)]a^{n+2}\sin(n+2)\theta + 2(n+1)a^{n+3}\sin(n+3)\theta\}/(\text{ch } b - \cos\theta)^2 = 0 .$$

... (6.25)

Case A

$a = 1$ and (6.25) reduces to

$$f(\theta)/(1-\cos\theta) = 0 \qquad \text{... (6.26)}$$

where

$$f(\theta) = (n+1)\sin(n-1)\theta - 6(n+1)\sin n\theta + 3(n-1)\sin(n+1)\theta + 2(n+1)\sin(n+2)\theta .$$

Now $f(\theta)$ satisfies:

$$f\left(\frac{r\pi}{n+\frac{1}{2}}\right) \begin{cases} > 0 & r \text{ even} \\ < 0 & r \text{ odd} \end{cases} \quad r = 1,2, \ldots n$$

and therefore there are at least $(n-1)$ real roots of (6.26) in $(0,\pi)$. We show that the remaining root θ_n is of the form

$$\theta_n = \pi + i\, z_n \quad ,$$

where z_n is real.

From equation (5.4) with $b = 0$ we deduce that the corresponding eigenvalue λ_n is given by

$$\lambda_n = -2\alpha(\text{ch}\ z_n + 1)\ . \qquad \cdots (6.27)$$

We let $\theta = \pi + iz$ and obtain

$$(-1)^n if(\pi+iz) = (n+1)\text{sh}(n-1)z + 6(n+1)\text{sh}\ nz + 3(n-1)\text{sh}(n+1)z - 2(n+1)\text{sh}(n+2)z$$

$$= F(z) \qquad \text{(say)}.$$

$F(z)$ has two zeros apart from $z = 0$, as

$$F(z) \sim -(n+1)e^{(n+2)z} \quad \text{as} \quad z \to \infty$$

$$< 0\ ,$$

$$F(0) = 0\ ,$$

$$F'(0) > 0\ ,$$

showing that there is a zero z_n, with $z_n > 0$. Since $F(z)$ is an odd function there will be a corresponding zero $-z_n$.

The eigenvalue given by (6.27) will be negative and equation (5.1) is stable.

Cases C and D

Assuming n is odd, analysis similar to that for Case C of Types (i) and (vi) (Section 6.7) may be made. The result is the same except that equation (6.14) is replaced by

$$2(n+1)k^{2n+6} - 3(n-1)k^{2n+4} - 6(n+1)k^{2n+2} - (n+1)k^{2n} - 3(n+1)^2k^4 - 6(n^2+2n+2)k^2 - 3(n+1)^2 = 0\ .$$

Numerical results are similar to those for Types (i) and (vi) and a graph of the critical values $(\beta/\alpha)_c$ is shown in Fig. 3.

Cases B,/

Cases B, E, F

Numerical evidence (Table 1) suggests these cases are stable.

6.15 Type (ii) and Type (viii)

Using equation (6.16) for the stream function, we obtain boundary conditions

$$x_0 = 0$$

$$x_{n+1} = -3 \sum_{j=1}^{n} x_j - \tfrac{1}{2} x_n \ .$$

We deduce with the aid of the identity (A.1), that the θ_r are roots of

$$\{6 \sin\theta - a^n \sin(n-1)\theta + (1+5a^2)a^{n-1} \sin n\theta + (2a^2-5)a^n \sin(n+1)\theta$$

$$- 2a^{n+1} \sin(n+2)\theta\}/(\text{ch } b - \cos\theta) = 0 \ . \qquad \ldots (6.28)$$

Case A

$a = 1$ and by examining the behaviour of the numerator of (6.28), it can be shown that there are $(n-1)$ real roots. The remaining eigenvalue takes the form of (6.27).

Cases C and D

Assuming n is odd, analysis similar to that for Case C of Types (ii) and (vi) (Section 6.8) may be made. The result is the same except that (6.20) is replaced by

$$2k^{2n+4} - 5k^{2n+2} - k^{2n} - 3(3+2n)k^2 - 3(1+2n) = 0 \ .$$

Numerical results are similar to those for Types (i) and (vi). The critical values of $(\beta/\alpha)_c$ are just below those for Type (i) and Type (viii) and for large n, $(\beta/\alpha)_c \approx 2{\cdot}186$.

Cases B, E and F

Numerical evidence (Table 1) suggests that these cases are stable.

6.16 Conclusions on Type (viii)

The restriction on the ratio Vh/ν , for a stable process, is more severe than for either Type (vi) or (vii). There will also be difficulties with choosing a suitable stable method of replacing time derivatives in equation (5.1) even for Case A. An eigenvalue of the form (6.27) will cause severe restriction on the size of the time step δt in the simple explicit method. We require (Varga[11], p.265),

$$\delta t \leq \min_{1 \leq r \leq n} \left\{ \frac{-2\,\mathcal{R}e(\lambda_r)}{|\lambda_r|^2} \right\}$$

$$= \frac{1}{\alpha(\text{ch } z + 1)}$$

This/

This is more severe than the restriction

$$\delta t \leq \frac{1}{2\alpha}$$

which is required if all θ_r are real.

The Dufort-Frankel[2] method is always unstable when an eigenvalue of the form of (6.27) occurs. For (5.1) the method is

$$\underline{\zeta}^{s+1} - \underline{\zeta}^{s-1} = \delta t \; [(M + 2\alpha \; I)\underline{\zeta}^{s} - \alpha\underline{\zeta}^{s+1} - \alpha\underline{\zeta}^{s-1} + \underline{b}^{s}] ,$$

where $\underline{\zeta}^s$ is the vorticity vector after s time steps and $\underline{b}^s$ is determined by the boundary conditions. We can write the method in the two stage form

$$\begin{bmatrix} \underline{\zeta}^{s+1} \\ \underline{\zeta}^{s} \end{bmatrix} = C \begin{bmatrix} \underline{\zeta}^{s} \\ \underline{\zeta}^{s-1} \end{bmatrix} + \begin{bmatrix} \underline{b}^{s} \\ \underline{0} \end{bmatrix}$$

where C is the $2n \times 2n$ matrix

$$C = \begin{bmatrix} (1 + \alpha.\delta t)^{-1}.\,\delta t.(M+2\alpha I) & (1+\alpha.\delta t)^{-1}(1-\alpha.\delta t)I \\ I & 0 \end{bmatrix} .$$

This recurrence relation is unstable if any eigenvalue of C has absolute value greater than unity. The eigenvalues of C are

$$\frac{(\lambda_r + 2\alpha)\delta t \pm \sqrt{(\lambda_r + 2\alpha)^2.\,\delta t^2 + 4(1+\alpha.\delta t)(1-\alpha.\delta t)}}{2(1+\alpha.\delta t)} \quad \text{for } r = 1,2 \ldots n$$

One of these, when λ_r is given by (6.27), is

$$\frac{-\delta t.\alpha.\mathrm{ch}\, z_r - \sqrt{1+\alpha^2.\delta t^2\, \mathrm{sh}^2\, z_r}}{1 + \alpha.\,\delta t}$$

which is less than -1.

6.17 Type (i) and Type (xi)

In one dimension, when using Type (xi) for a moving wall, there is no need to apply the correction (4.7) to the vorticity after the stream function has been found, and therefore (4.6) is the actual boundary condition. For a two-dimensional problem, it will still be important that the condition given by (4.6) yields a stable method, assuming that no correction of vorticity is required.

Using/

Using (4.6) at an upper boundary and (5.5) we obtain the boundary conditions

$$\zeta_0 = 0 ,$$

$$\zeta_{n+1} = a^2 \zeta_n ,$$

where ζ_n is the vorticity at the boundary grid point.

The θ_r must therefore be roots of

$$\sin(n+1)\theta - a \sin n\theta = 0 \quad . \qquad \cdots (6.29)$$

<u>Cases A and B</u>

If $f(\theta) = \sin(n+1)\theta - a \sin n\theta$

$f(\theta)$ alternates in sign for points $r\pi/(n+\frac{1}{2})$, $r = 1,2, \ldots n$, and there are at least $(n-1)$ roots in $(0,\pi)$.

Also we have $f\left(\dfrac{\pi}{n+\frac{1}{2}}\right) < 0$ and, if $a < 1 + \dfrac{1}{n}$, $f'(0) > 0$, in which case the remaining root lies in $(0,\pi/[n+\frac{1}{2}])$.

If $a = 1 + \dfrac{1}{n}$, one root is $\theta_n = 0$ and the corresponding eigen-vector has components $x_j = j.a^j$.

For $a > 1 + \dfrac{1}{n}$, we show there is a root of the form $\theta_n = iz_n$, where z_n is real. Putting $\theta = iz$, we obtain

$$if(iz) = -\mathrm{sh}(n+1)z + a\ \mathrm{sh}\ nz = F(z) \qquad \text{(say)}.$$

Now, $b = \ln a > 0$ and therefore,

$$2F(b) = \frac{1}{a^{n+1}} - \frac{1}{a^n} < 0 \ . \qquad \cdots (6.30)$$

Also, $F(0) = 0$ and $F'(0) > 0$ thus there is a zero z_n of $F(z)$, with $0 < z_n < b$, when

$$\lambda_n = 2\alpha\left(\frac{\mathrm{ch}\ z_n}{\mathrm{ch}\ b} - 1\right) < 0 \ . \qquad \cdots (6.31)$$

Thus equation (5.1) is stable for these cases.

Case C

Assuming n is odd, we consider the real eigenvalue which must exist. If $\theta = \frac{\pi}{2} + i\phi$, equation (6.29) becomes

$$F(\phi) = \text{sh}(n+1)\phi - k \text{ ch } n\phi = 0$$

and this has a root $\phi_r > c = \ln k$,

as

$$F(\phi) \sim \tfrac{1}{2}e^{(n+1)\phi} \quad \text{as} \quad \phi \to \infty$$

$$> 0$$

and

$$F(c) = -\tfrac{1}{2}(1+k^2)/k^{n+1} < 0 .$$

The corresponding eigenvalue, $\lambda_r = 2\alpha\ (\text{sh}\ \phi_r/\text{sh}\ c-1)$, is positive and equation (5.1) is unstable. For even n , we find by direct calculation of eigenvalues (Table 1) that the method may be stable for some values of $\beta/\alpha > 1$ but as β/α is increased it becomes unstable.

Cases D, E and F

Using a method similar to that used for Cases C and D of Section 6.4 it can be shown that equation (5.1) is always unstable for Case D and always stable for Cases E and F.

6.18 Type (i) and Type (xii)

As with Type (xi) the vorticity is not corrected after the stream function has been calculated. Using (4.9) at an upper boundary and (5.5), we obtain the boundary conditions

$$\zeta_0 = 0$$

$$\zeta_{n+1} = (a^2-1)\zeta_n + a^2\zeta_{n-1} ,$$

where ζ_n is the vorticity at the boundary grid point. Thus the θ_r must be roots of

$$\cos n\theta \sin\theta - \sin n\theta \text{ sh } b = 0 . \qquad \text{... (6.32)}$$

Cases A and B

By considering the behaviour of

$$f(\theta) = \cos n\theta \sin\theta - \sin n\theta \text{ sh } b$$

at points $r\pi/n$, for $r = 0,1 \ldots n$, it can be seen that there are at least (n-2) real roots of (6.32) in $(0,\pi)$. Using a method similar to that for

Case A/

Case A of Types (i) and (xi) (Section 6.17), we find the remaining roots θ_{n-1} and θ_n are as follows:

if $1-n \text{ sh } b > 0$ then $0 < \theta_{n-1} < \pi/n$

" $1-n \text{ sh } b = 0$ " $\theta_{n-1} = 0$ and the eigenvector components are $x_j = ja^j$

". $1-n \text{ sh } b < 0$ " $\theta_{n-1} = iz_{n-1}$ where $0 < z_{n-1} < b$

+ $1+n \text{ sh } b > 0$ " $(n-1)\pi/n < \theta_n < \pi$

" $1+n \text{ sh } b = 0$ " $\theta_n = \pi$ and the eigenvector components are $x_j = j(-a)^j$

" $1+n \text{ sh } b < 0$ " $\theta_n = \pi + iz_n$ with $z_n > 0$

In all cases the corresponding eigenvalues are real and negative and equation (5.1) is stable.

Cases C, D, E and F

Method and results are the same as those for Types (i) and (xi).

6.19 Conclusions on moving wall conditions

For Cases A and B, both Type (xi) and (xii) make equation (5.1) at worst neutrally stable. For Case C, both methods are unstable and thus, if there is suction at a moving wall, it is necessary to restrict the mesh size h so that we have $\beta/\alpha < 1$. In Section 7.3 we shall show that this restriction for a stable method is not, serious, as accuracy considerations are even more limiting.

7. A Numerical Example

7.1 Flow along a moving wall with suction

As a test of boundary conditions of Type (xi) and (xii) we consider the numerical solution of a problem for which the analytic solution may be derived. The motion, first introduced in Section 4.2, is that of flow along a wall which has started impulsively from rest at time $t = 0$. We seek a solution $u(y,t)$ of equation (4.2) satisfying the boundary conditions

$$u(y,0) = 0, \quad \text{for } y \geq 0, \qquad \cdots (7.1)$$
$$u(0,t) = U, \quad u(\infty,t) = 0, \quad \text{for } t > 0.$$

The solution, which may be found by integral transformation, is

$$u(y,t) = \tfrac{1}{2}U\left\{\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} - \frac{V}{2}\sqrt{\frac{t}{\nu}}\right) + e^{Vy/\nu}\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu t}} + \frac{V}{2}\sqrt{\frac{t}{\nu}}\right)\right\} \cdots (7.2)$$

where $\operatorname{erfc}(x)$ is the complimentary error function defined by

erfc(x)/

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-x^2}\, dx$$

and $-V$ is the suction velocity.

On substituting (7.2) in

$$\zeta = -\frac{\partial u}{\partial y}$$

we obtain

$$\zeta(y,t) = U\left[\frac{1}{\sqrt{\pi\nu t}} \exp\left\{-\left(\frac{y}{2\sqrt{\nu t}} - \frac{V}{2}\sqrt{\frac{t}{\nu}}\right)^2\right\} - \frac{V}{2\nu}\exp(Vy/\nu)\,\mathrm{erfc}\left\{\frac{y}{2\sqrt{\nu t}} + \frac{V}{2}\sqrt{\frac{t}{\nu}}\right\}\right]. \qquad \ldots (7.3)$$

The numerical solutions shown in Fig. 4 were obtained using the Crank-Nicholson[1] implicit method applied to equation (5.1). This method was chosen because it is conservative and is stable if equation (5.1) is stable in the sense defined here. This latter property is shown, for example, by Varga[11]. The implicit method would be difficult to apply to a two-dimensional problem, but is used to avoid inaccuracies introduced by methods which are not conservative, and difficulties with methods whose stability depends on the time step, δt .

The results shown in Fig. 4 are for $\nu = \cdot 15$, $V = -\cdot 75$ and $-1\cdot 5$, $h = \cdot 1$, $n = 15$ and $\delta t = \cdot 01$. (Thus $\alpha = 15$ and $\beta = -3\cdot 75$ and $-7\cdot 5$) . The calculations were repeated with lower boundary conditions of Types (xi) and (xii). For Type (xi), the results were moved by a half-grid length as mentioned in Section 3.3. In all cases Type (i) was used for the upper 'infinity' boundary. The results appear satisfactory despite the fact that the stability of the boundary conditions depends on β/α , though of course the ratio β/α has been chosen so that we have Case B. (For these examples, β is negative as the wall is at the lower and not the upper boundary, as in Section 6).

As time increases, $\zeta(y,t)$ in equation (7.3) does not decay to zero but approaches the steady state solution

$$\zeta(y,\infty) = -\frac{UV}{\nu} \cdot e^{Vy/\nu} .$$

In equation (5.1) $\underline{b}$ is a zero vector and, if the finite difference solution is not to decay to zero, the equation must be neutrally stable. One eigenvalue of the matrix M for both Type (xi) and (xii) is in fact nearly zero. For Type (xi), $F(b)$ of (6.30) will be nearly zero for large n and thus $z_n \simeq b$ in (6.31). (In some cases the root is so near zero that it is recorded as such in Table 1). An exact zero does not occur because of the effect of Type (i) at the opposite boundary.

7.2/

7.2 A fixed wall

We similarly consider the flow along a wall (with suction), which has been impulsively brought to rest from moving in its own plane. Instead of (7.1) we have boundary conditions:

$$u(y,0) = U, \quad \text{for} \quad y > 0 ,$$

$$u(\infty,t) = U, \; u(0,t) = 0, \quad \text{for} \quad t > 0 .$$

The analytic solution $u(y,t)$ will only differ from that of (7.2) by a constant and a change of sign. Thus the vorticity will be given by (7.3) with a change of sign. In the numerical method, Type (vi) may be used for the boundary condition at the fixed wall and Type (ii) at 'infinity'. Initially the stream function satisfies

$$\psi_{j+1} = \psi_j = Uh \quad \text{for all} \quad j$$

and the vorticity is zero except at the boundary, where equation (3.1) applies, i.e.

$$\zeta_0 = -(\psi_1 - \psi_0)/h^2 = -U/h .$$

Now for an upper boundary, Type (vi) is equivalent to equation (6.17), which for a lower boundary becomes

$$\sum_{j=0}^{n} \zeta_j = -U/h$$

and this is precisely the equation used for conservation of vorticity at a lower boundary (equation (4.14)). The difference in sign occurs because the impulse at the wall is in the opposite direction.

Thus for this problem, the numerical method using Types (vi) and (ii) will produce the same results (apart from negation) as that using Types (xi) and (i). We can similarly show that Types (vii) and (ii) give the same results as Types (xii) and (i). The vector $\underline{b}$ in equation (5.1) is non-zero and the equation does not need to be neutrally stable. The finite-difference approximation to the steady state solution is $-M^{-1}\underline{b}$.

7.3 Restrictions on β/α

The form of the steady state solution indicates that the grid length h must be restricted. If h is too large, the vorticity will be concentrated into the region between the boundary and the first interior grid point and the finite-difference approximation will be completely inaccurate. The difference approximation

$$\frac{\zeta(h,\infty) - \zeta(0,\infty)}{h} = -\frac{UV}{\nu}\left(\frac{e^{Vh/\nu} - 1}{h}\right) ,$$

to the derivative

$\partial\zeta/$

$$\frac{\partial\zeta(\frac{1}{2}h,\infty)}{\partial y} = -\frac{UV^2}{\nu^2} \cdot e^{\frac{Vh}{2\nu}} ,$$

will have relative error E , where

$$E = \left| 1 - \left(\frac{\zeta(h,\infty) - \zeta(0,\infty)}{h} \right) \Big/ \left(\frac{\partial\zeta(\frac{1}{2}h,\infty)}{\partial y} \right) \right|$$

$$= \left| 1 - \frac{\nu}{Vh} \left(e^{\frac{Vh}{2\nu}} - e^{\frac{-Vh}{2\nu}} \right) \right|$$

$$= \left| 1 - \frac{\alpha}{\beta} \,\mathrm{sh}\left(\frac{\beta}{\alpha} \right) \right| ,$$

where $\frac{\beta}{\alpha} = \frac{Vh}{2\nu}$. (The change of sign is made because the wall is a lower and not an upper boundary). For example, if $\beta/\alpha = 1$, $E \simeq \cdot 18$ which represents quite a large error whereas for the numerical example of Section 7.1, $\beta/\alpha = 0{\cdot}25$ and $0{\cdot}5$ when $E \simeq 0{\cdot}010$ and $0{\cdot}042$. It is hence necessary for the ratio β/α to be restricted and most problems would probably at least require $\beta/\alpha < 1$. This restriction on β/α will also apply to two-dimensional flows. The instabilities, found in Section 6 for boundary conditions of Types (vi), (vii), (viii), (xi) and (xii), are not as likely to restrict the magnitude of β/α as the accuracy considerations presented here.

8. Conclusions

As a condition at 'infinity', extrapolation can only be successful at an outflow boundary and will give an unstable process if used at an upstream, i.e. an inflow boundary. The no-slip conditions for a fixed wall, which generate vorticity at the boundary by calculating it from the stream function, will be stable if there is no suction at the wall. If there is a suction velocity $-V$, then the mesh size h must be chosen so that the ratio Vh/ν (where ν is the viscosity) is restricted in magnitude. As Vh/ν is increased the finite difference method becomes firstly inaccurate and secondly unstable. At a moving wall vorticity is conserved and the conditions consistent with this will yield stable finite difference methods if there is no suction. If there is suction, then again Vh/ν must be restricted for accuracy and stability. If pressure calculations are included in the finite-difference method, it is possible to determine the vorticity at a fixed wall using pressures. A pressure term is included in equation (4.2) and using (4.3) we can determine the amount of vorticity generated. The stability of such a method would be the same as that of the corresponding method for a moving wall. For steady flow there will need to be similar restrictions on Vh/ν if iterative methods are used.

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Appendix A

The following identities are required:

$$\sum_{r=1}^{n} a^r \sin r\theta \equiv a\{\sin\theta + a^{n+1}\sin n\theta - a^n \sin(n+1)\theta\}/(1 - 2a\cos\theta + a^2) \qquad (A.1)$$

$$\sum_{r=0}^{n} a^r \cos r\theta \equiv \{1 - a\cos\theta + a^{n+2}\cos n\theta - a^{n+1}\cos(n+1)\theta\}/(1 - 2a\cos\theta + a^2) \qquad (A.2)$$

$$\sum_{r=1}^{n} ra^r \sin r\theta \equiv \{a(1-a^2)\sin\theta - (n+1)a^{n+3}\sin(n-1)\theta + [2(n+1)+na^2]a^{n+2}\sin n\theta$$
$$-[2na^2+(n+1)]a^{n+1}\sin(n+1)\theta + na^{n+2}\sin(n+2)\theta\}/(1 - 2a\cos\theta + a^2)^2 \qquad (A.3)$$

$$\sum_{r=1}^{n} ra^r \cos r\theta \equiv \{-2a^2 + a(1+a^2)\cos\theta - (n+1)a^{n+3}\cos(n-1)\theta + [2(n+1) + na^2]a^{n+2}\cos n\theta$$
$$- [2na^2+(n+1)]a^{n+1}\cos(n+1)\theta + na^{n+2}\cos(n+2)\theta\}/(1 - 2a\cos\theta + a^2)^2$$
$$(A.4)$$

$$\sum_{r=1}^{n} (n+1-r)a^r \sin r\theta \equiv \{[(n+2)a^2+n]a\sin\theta - (n+1)a^2\sin 2\theta + a^{n+4}\sin n\theta$$
$$- 2a^{n+3}\sin(n+1)\theta + a^{n+2}\sin(n+2)\theta\}/(1 - 2a\cos\theta + a^2)^2 \qquad (A.5)$$

$$\sum_{r=0}^{n} (n+1-r)a^r \cos r\theta \equiv \{(n+1) + 2(n+2)a^2 - [(3n+4) + (n+2)a^2]a\cos\theta + (n+1)a^2\cos 2\theta$$
$$+ a^{n+4}\cos n\theta - 2a^{n+3}\cos(n+1)\theta + a^{n+2}\cos(n+2)\theta\}/(1 - 2a\cos\theta + a^2)^2 .$$
$$(A.6)$$

Table 1/

Table 1

Values of $\max_r \mathcal{R}e(\lambda_r)$

Boundary Condition Types:	Case	B	B	C	E	C	E
Lower	Upper \ β	+0·5	-0·5	+5·0	-5·0	+15·0	-15·0
(i)	(iii)	-0·31439 -0·29253	+0·00458 +0·00030	-2·08718 -2·02671	+0·83971 +0·48082	-2·28262 -2·08537	+5·01251 +3·64840
(vi)	(vi)	-0·52417 -0·30225	-0·52417 -0·30225	-1·83496 -1·30190	-1·83496 -1·30190	-1·98636 -1·76528	-1·98636 -1·76528
(i)	(vi)	-0·35205 -0·30628	-0·49920 -0·29196	-1·16549 -1·34923	-1·98298 -1·34037	+1·22316 +0·66670	-2·08464 -2·04027
(ii)	(vi)	-0·35612 -0·30726	-0·30978 -0·15007	-0·95308 -1·20822	-0·38385 -0·17588	+2·41150 +1·62210	-0·53466 -0·23778
(vii)	(vii)	-0·57812 -0·32290	-0·57812 -0·32290	-2·00287 -1·39146	-2·00287 -1·39146	-2·00285 -1·94466	-2·00285 -1·94466
(i)	(vii)	-0·35777 -0·30810	-0·53267 -0;30497	-0·39633 -0·75668	-1·89068 -1·38054	+4·09456 +2·97769	-2·17906 -2·07605
(ii)	(vii)	-0·36270 -0·30926	-0·33426 -0·15829	-0·02129 -0·51676	-0·39133 -0·17651	+6·04077 +4·45470	-0·58447 -0·21402
(i)	(viii)	-0·36021 -0·30881	-0·53535 -0·30609	+5·62269 +5·89842	-1·84306 -1·41286	+26·6770 +27·5973	-2·12444 -2·04952
(ii)	(viii)	-0·36530 -0·30997	-0·34121 -0·16032	+6·50898 +6·51089	-0·43534 -0·19156	+29·6448 +29·6495	-0·75206 -0·24993
(i)	(xi)	-0·00001 -0·00000	-0·32461 -0·29642	-0·31928 +0·03425	-2·03950 -2·01244	+1·98124 +1·54669	-2·12545 -2·03923
(i)	(xii)	-0·0002 -0·0000	-0·33106 -0·29864	+0·27778 +0·14374	-2·09271 -2·02781	+4·43483 +3·11547	-2·28853 -2·08654

In all cases $\alpha = 1$ and for each set of boundary conditions and value of β the two entries are for $n = 10$ and 15 respectively.

Table 2/

Table 2

A summary of the results of Section 6

Boundary Condition Types: Lower	Upper	A	B		C	D	E	F
		$\alpha > 0$, $\beta = 0$	$\alpha > \lvert\beta\rvert > 0$: $\beta > 0$	$\beta < 0$	$\beta > \alpha > 0$	$\alpha = 0$, $\beta > 0$	$\beta < -\alpha < 0$	$\alpha = 0$, $\beta > 0$
(i)[1]	(i)[1]	S	S	S	S	S	S	S
(iii)[2]	(iii)[2]	N	N	N	N	N	N	N
(i)[1]	(iii)[2]	S	S	U	S	S	U	U
(vi)	(vi)	S	S[3]	S[3]	S[3]	N	S[3]	N
(i)	(vi)	S	S[3]	S[3]	C.S	U	S[3]	S[3]
(ii)	(vi)	S	S	S[3]	C.S	U	S[3]	S[3]
(vii)	(vii)	S	S[3]	S[3]	S[3]	N	S[3]	N
(i)	(vii)	S	S[3]	S[3]	C.S	U	S[3]	S[3]
(ii)	(vii)	S	S	S[3]	C.S	U	S[3]	S[3]
(i)	(viii)	S	S[3]	S[3]	C.S	U	S[3]	S[3]
(ii)	(viii)	S	S[3]	S[3]	C.S	U	S[3]	S[3]
(i)	(xi)	S	S	S	U[4]	U	S	S
(i)	(xii)	S	S	S	U[4]	U	S	S

The entries S , N and U indicate stable, neutrally stable and unstable cases respectively. C.S indicates cases which are conditionally stable, i.e. they are unstable for sufficiently large values of β/α .

Notes:

1 or Type (ii).

2 or Type (iv) or (v)

3 result postulated from numerical evidence

4 except for even values of n , when, for $\beta - \alpha$ sufficiently small, numerical evidence suggests these cases are stable.

Table 3/

Table 3

Eigenvalues of M for $\alpha = 1$, $\beta = 11$ and boundary conditions of Types (i) and (vi)

-0·15780		
-0·49459	±	4·40293i
-1·17858	±	8·16412i
-1·42657	±	11·4562i
-1·65094	±	14·3632i
-1·79940	±	16·7140i
-1·91326	±	18·4651i
-1·97781	±	19·5373i

BMG

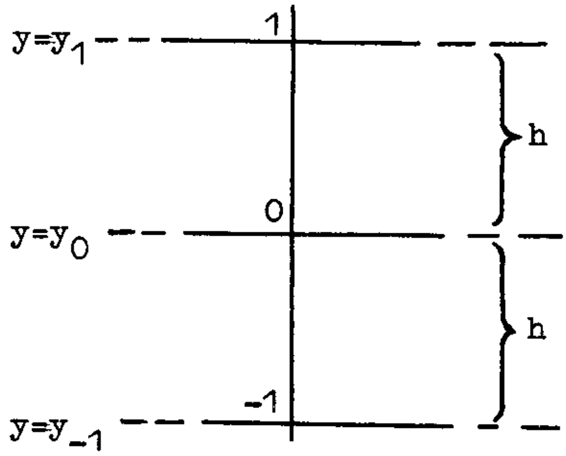

Fig. 1. Boundary points.

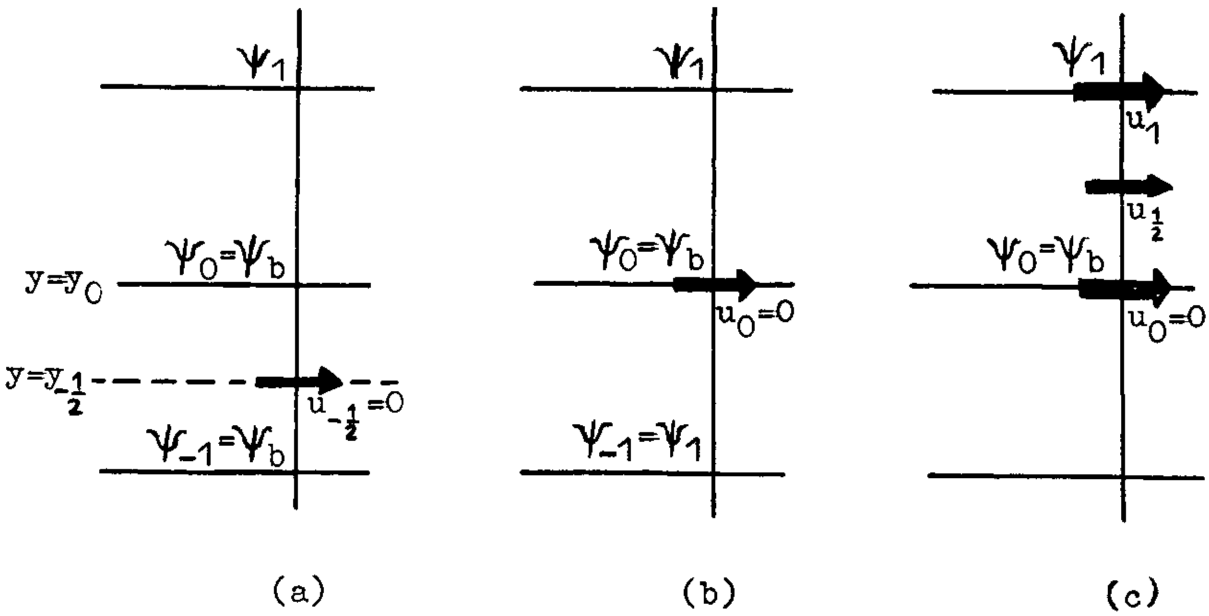

Fig. 2. Boundary conditions, (a) Type (vi), (b) Type (vii), (c) Type (viii).

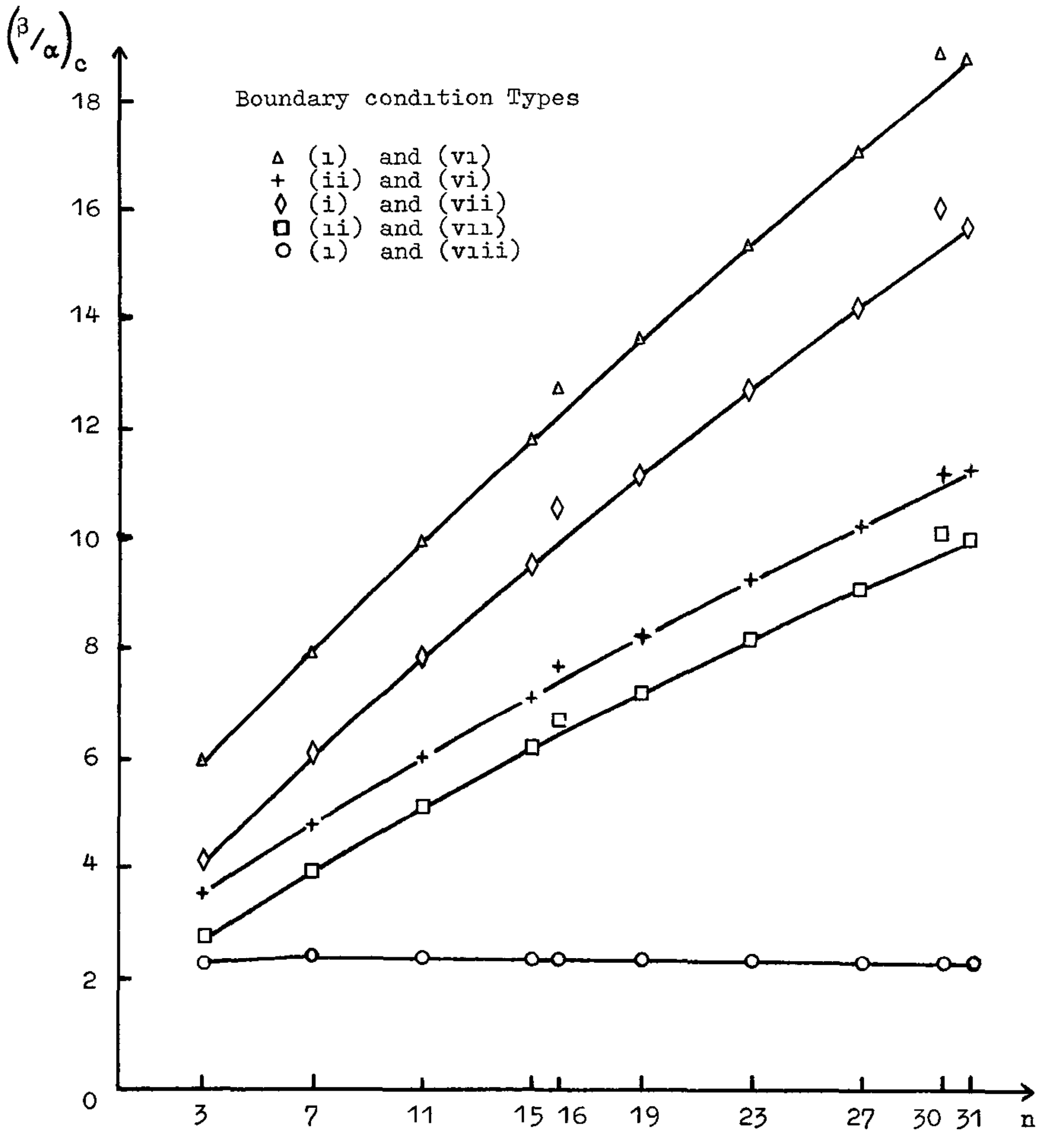

Fig. 3. Critical values of (β/α). For odd n, the values were obtained analytically, e.g. from equation (6.15) for Type (i) and (vi). For n = 16,30 the values were obtained by calculating eigenvalues of M.

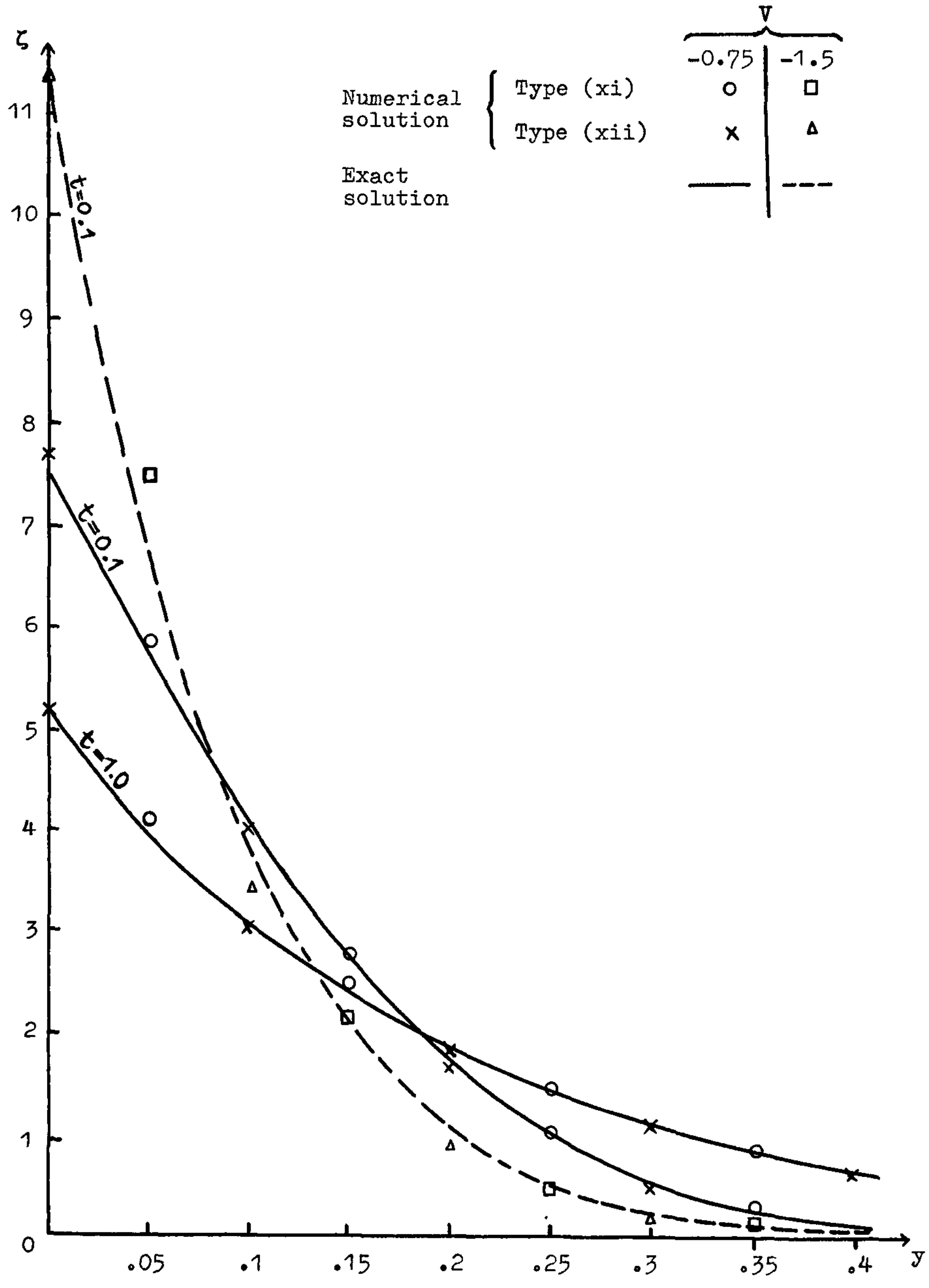

Fig. 4. Numerical and exact solutions for the problem of Section 7.1.

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