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APPROXIMATE SOLUTIONS OF THE THREE-DIMENSIONAL LAMINAR BOUNDARY LAYER MOMENTUM INTEGRAL EQUATIONS**

by

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SUMMARY

Six methods for the approximate solution of the three-dimensional laminar boundary layer momentum integral equations are presented and compared with three known exact solutions. These methods all involve the Pohlhausen technique of specifying velocity profiles in terms of one or two unknowns and substituting these expressions for the profiles into the two momentum integral equations to render them determinate.

Comparison of these methods with the exact solutions shows that the assumption of small cross-flow velocity in the boundary layer is generally adequate in cases involving favourable pressure gradients but introduces significant errors in cases involving adverse pressure gradients. In cases of moderate adverse pressure gradient the accuracy of the approximate solution may be improved to some extent by the adoption of an extension of the Luxton-Young technique. However, for large adverse gradients adequate accuracy may only be obtained by including the cross-flow terms in the momentum integral equations, and the method described here is then shown to lead to very satisfactory results in all the cases examined.

It appears that provided the maximum value of the angle β between the limiting and external streamlines is less than about 10° the small

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cross-flow assumption is of adequate accuracy for most engineering purposes.

1. INTRODUCTION

It is not intended that this paper should serve as a comprehensive review of the subject of three-dimensional laminar boundary layer theory, a full review and associated bibliography will be found in reference 1. However, a brief introduction to the subject is first presented as a preliminary to a discussion of the authors' work on approximate methods of solution to the three-dimensional, laminar, incompressible momentum integral equations. The boundary layer equations in curvilinear coordinates are initially presented and these lead to the momentum integral equations in streamline coordinates. The approximations associated with the assumption of small cross-flow velocity which lead to what hasbecome known as the "axially symmetric analogy"[2] are then developed and this leads in turn to a discussion of approximate methods of solution of the momentum integral equations. The Pohlhausen[3] type of approach is considered and a comparison is made between a method due to Cooke[4] and a method based upon Pohlhausen[3] quartic type velocity profiles devised by Young. The results given by both these methods for three cases involving favourable and unfavourable pressure gradients are compared with known exact solutions as are the results given by a method which involves an extension of the Luxton-Young[5] technique to the three-dimensional case. Finally, a method is presented which is not restricted to the case of small cross-flow velocity and includes all the terms in the momentum integral equations and its results are also compared with the exact solutions. It is shown that only the last method gives adequate accuracy for large adverse pressure gradients, but in general where the pressure gradients are less severe the small cross-flow assumption leads to results that are very satisfactory.

2. THE BOUNDARY LAYER AND MOMENTUM INTEGRAL EQUATIONS

A system of orthogonal curvilinear coordinates (ξ, η, ζ) is used. The surface on which the boundary layer lies is denoted by $\zeta=0$ and ζ measures the distance from the surface along a normal. On the surface $\zeta=0$ are two families of coordinate curves ξ = constant and η = constant orthogonal to one another. In this system an element of length (ds) within the boundary layer is given by

$$ds^2 = h_1^2 d\xi^2 + h_2^2 d\eta^2 + d\zeta^2$$

where h_1 and h_2 are length parameters which may be taken as functions of ξ and η only, provided that the surface curvature does not change abruptly and that the boundary layer thickness is small compared with the principal radii of curvature of the surface. Subject to these provisions the coordinate system can be taken as triply orthogonal within the boundary layer although it does not necessarily remain so further away from the surface.

In this coordinate system the boundary layer equations and the continuity equations are

$$\rho\left[\frac{u\partial u}{h_1\partial\xi} + \frac{v}{h_2}\frac{\partial u}{\partial\eta} + w\frac{\partial u}{\partial\zeta} - k_2 uv + k_1 v^2\right] = -\frac{1}{h_1}\frac{\partial P}{\partial\xi} + \frac{\partial}{\partial\zeta}\left(\mu\frac{\partial u}{\partial\zeta}\right), \qquad 2\cdot1$$

$$\rho\left[\frac{u\partial v}{h_1\partial\xi} + \frac{v}{h_2}\frac{\partial v}{\partial\eta} + w\frac{\partial v}{\partial\zeta} - k_1 uv + k_2 u^2\right] = -\frac{1}{h_2}\frac{\partial P}{\partial\eta} + \frac{\partial}{\partial\zeta}\left(\mu\frac{\partial v}{\partial\zeta}\right), \qquad 2\cdot2$$

$$0 = \frac{\partial P}{\partial\zeta}, \qquad 2\cdot3$$

$$\frac{\partial}{\partial\xi}\left(\rho h_2 u\right) + \frac{\partial}{\partial\eta}\left(\rho h_1 v\right) + \frac{\partial}{\partial\zeta}\left(\rho h_1 h_2 w\right) = 0, \qquad 2\cdot4$$

where u, v, w are the velocity components in the direction of the ξ, η, ζ axes respectively. P is the pressure, ρ the density, μ the viscosity and k_1, k_2 are the geodesic curvatures of the curves ξ = constant η = constant respectively, i.e.

$$k_1 = -\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial\xi}, \quad k_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial\eta}. \qquad 2\cdot5$$

The equations for incompressible flow were first given by Howarth[6] who used a system of coordinates which were triply orthogonal everywhere and hence were convenient for the direct application of vector analysis but strictly required that the coordinate system on the surface consisted of the lines of principal curvature. Squire[7] showed, however, that Howarth's boundary layer equations apply with the usual boundary layer approximations for the coordinate system used here. Timman[8] gave a derivation of the equations from an argument based upon first principles.

The values $\partial P/\partial \xi, \partial P/\partial \eta$ are obtained from the flow at the edge of the boundary layer. Denoting values at the edge by the subscript "e" we find from 2.1 and 2.2

$$\rho_e\left[\frac{u_e}{h_1}\frac{\partial u_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial u_e}{\partial \eta}-K_2 u_e v_e+K_1 v_e^2\right]=-\frac{1}{h_1}\frac{\partial P}{\partial \xi} \qquad 2\cdot 6$$

$$\rho_e\left[\frac{u_e}{h_1}\frac{\partial v_e}{\partial \xi}+\frac{v_e}{h_2}\frac{\partial v_e}{\partial \eta}-K_1 u_e v_e+K_2 u_e^2\right]=-\frac{1}{h_2}\frac{\partial P}{\partial \eta} \quad . \qquad 2\cdot 7$$

The momentum integral equations are obtained by integrating 2.1 and 2.2 term by term across the boundary layer and using 2.4, 2.6 and 2.7 to eliminate w and P. If we write $U_e^2 = u_e^2 + v_e^2$ and, restricting ourselves to incompressible flow, define the various momentum and displacement thicknesses

$$\delta_1 = \int_0^\infty \frac{(u_e-u)}{U_e}\,d\zeta,\ \delta_2 = \int_0^\infty \frac{(v_e-v)}{U_e}\,d\zeta,\ \theta_{11} = \int_0^\infty \frac{(u_e-u)u}{U_e^2}\,d\zeta$$

$$\theta_{21} = \int_0^\infty \frac{(v_e-v)u}{U_e^2}\,d\zeta,\ \theta_{12} = \int_0^\infty \frac{(u_e-u)v}{U_e^2}\,d\zeta,\ \theta_{22} = \int_0^\infty \frac{(v_e-v)v}{U_e^2}\,d\zeta\,, \qquad 2\cdot 8$$

where

$$\tau_{01} = \left(\mu\frac{\partial u}{\partial \zeta}\right)_{\zeta=0} \quad \text{and} \quad \tau_{02} = \left(\mu\frac{\partial v}{\partial \zeta}\right)_{\zeta=0}\,,$$

the momentum integral equations become

$$\frac{1}{h_1 U_e^2}\frac{\partial}{\partial \xi}\left(U_e^2\theta_{11}\right)+\frac{1}{h_2 U_e^2}\frac{\partial}{\partial \eta}\left(\theta_{12}U_e^2\right)+\frac{1}{h_1 U_e}\frac{\partial u_e}{\partial \xi}\delta_1$$
$$+\frac{1}{h_2 U_e}\frac{\partial u_e}{\partial \eta}\delta_2-K_1\left(\theta_{11}-\theta_{22}-\frac{v_e}{U_e}\delta_2\right)-K_2\left[(\theta_{12}+\theta_{21})+\frac{v_e}{U_e}\delta_1\right]=\frac{\tau_{01}}{\rho_e U_e^2} \qquad 2\cdot 9$$

$$\frac{1}{h_1 U_e^2}\frac{\partial}{\partial\xi}(\theta_{21}U_e^2) + \frac{1}{h_2 U_e^2}\frac{\partial}{\partial\eta}(\theta_{22}U_e^2) + \frac{1}{h_1 U_e}\frac{\partial v_e}{\partial\xi}\delta_1$$

$$+\frac{1}{h_2 U_e}\frac{\partial v_e}{\partial\eta}\delta_2 - K_1\left(\theta_{21}+\theta_{12}+\frac{u_e}{U_e}\delta_2\right) - K_2\left(\theta_{22}-\theta_{11}-\frac{u_e}{U_e}\delta_1\right) = \frac{\tau_{02}}{\rho_e U_e^2} \qquad 2\cdot10$$

3. THE MOMENTUM INTEGRAL EQUATIONS IN STREAMLINE COORDINATES

If the curves η = constant ξ = constant on the surface $\zeta=0$ are taken to be the projection of the external streamlines on to the surface and their orthogonal trajectories respectively, we then have $v_e=0$ and $U_e=u_e$. The momentum and displacement thicknesses are then given by

$$\delta_1=\int_0^\infty\left(1-\frac{u}{u_e}\right)d\zeta,\ \delta_2=-\int_0^\infty\frac{v}{u_e}d\zeta,\ \theta_{11}=\int_0^\infty\left(1-\frac{u}{u_e}\right)\frac{u}{u_e}d\zeta$$

$$\theta_{12}=\int_0^\infty\left(1-\frac{u}{u_e}\right)\frac{v}{u_e}d\zeta,\ \theta_{21}=-\int_0^\infty\frac{uv}{u_e^2}d\zeta,\ \theta_{22}=-\int_0^\infty\frac{v^2}{u_e^2}d\zeta\ , \qquad 3\cdot1$$

and the momentum equations become

$$\frac{1}{h_1 u_e^2}\frac{\partial}{\partial\xi}(\theta_{11}u_e^2) + \frac{1}{h_2 u_e^2}\frac{\partial}{\partial\eta}(\theta_{12}u_e^2) + \frac{1}{h_1 u_e}\frac{\partial u_e}{\partial\xi}\delta_1$$

$$+\frac{1}{h_2 u_e}\frac{\partial u_e}{\partial\eta}\delta_2 - K_1(\theta_{11}-\theta_{22}) - K_2(\theta_{12}+\theta_{21}) = \frac{\tau_{01}}{\rho u_e^2} \qquad 3\cdot2$$

$$\frac{1}{h_1 u_e^2}\frac{\partial}{\partial\xi}(\theta_{21}u_e^2) + \frac{1}{h_2 u_e^2}\frac{\partial}{\partial\eta}(\theta_{22}u_e^2) - 2K_1\theta_{21} - K_2(\theta_{22}-\theta_{11}-\delta_1) = \frac{\tau_{02}}{\rho u_e^2}\ . \qquad 3\cdot3$$

If the external flow is irrotational a velocity potential exists which may be put equal to ξ so that $h_1=1/u_e$. Then the momentum integral equations become

$$\frac{\partial\theta_{11}}{\partial\xi} + \frac{1}{h_2 u_e}\frac{\partial}{\partial\eta}(\theta_{12}) + \frac{1}{u_e}\frac{\partial u_e}{\partial\xi}(2\theta_{11}+\delta_1) - \frac{1}{u_e}K_1(\theta_{11}-\theta_{22}) = \frac{\tau_{01}}{\rho u_e^3} \qquad 3\cdot4$$

$$\frac{\partial\theta_{21}}{\partial\xi} + \frac{1}{h_2 u_e}\frac{\partial}{\partial\eta}(\theta_{22}) + \frac{2}{u_e}\frac{\partial u_e}{\partial\xi}\theta_{21} + \frac{1}{h_2 u_e^2}\frac{\partial u_e}{\partial\eta}(\theta_{11}+\theta_{22}+\delta_1) - \frac{2}{u_e}K_1\theta_{21} = \frac{\tau_{02}}{\rho u_e^3} \qquad 3\cdot5$$

4. CROSS-FLOWS

The component of the flow in the boundary layer which is at right angles to the direction of the external streamlines is defined as a cross flow. Along a normal to the surface the cross-wise velocity component varies in magnitude from zero at the surface to some maximum and then to zero at the edge of the boundary layer. In streamline coordinates the cross-wise velocity is v.

The physical explanation for the existence of cross-flows is described in Reference 1. Briefly if the streamlines at the edge of the boundary layer are curved there must be a cross-wise pressure gradient to balance the centrifugal force. Now 2.3 shows that this pressure gradient will not vary along a normal to the surface so that in the boundary layer where the fluid elements have been retarded by viscosity they must, to provide the same centrifugal force, follow a more highly curved path than that of the element at the outer edge of the boundary layer. The resultant direction of the flow will clearly then be different at different levels in the boundary layer. The limit of this direction as the surface is approached is known as the direction of the limiting streamline. The angle β between the external streamline and the limiting streamline may be defined as

$$\tan\beta = \lim_{\zeta\to 0}\frac{v}{u} = \frac{\tau_{02}}{\tau_{01}} \quad . \qquad 4.1$$

With the sudden imposition of a cross-wise pressure gradient the cross-flow will immediately start to grow until the cross-wise viscous forces balance the cross-wise pressure and centrifugal forces. When the pressure gradient is removed the cross flow does not immediately disappear but because of the cross-wise shear stresses its reduction to zero is gradual.

5. THE AXIALLY SYMMETRIC ANALOGY

It has been long established that if the cross-wise velocities and cross-wise gradients are small the streamwise flow may be calculated independently of the cross-flow. Having done this the cross-flow may then be calculated from a linear first order differential equation. Eichelbrenner and Oudart[9] pointed out that this simplification leads for the streamwise flow to an analogy with axially symmetric flow. This is readily demonstrated for the equations of motion but we shall confine our attention to the momentum integral equation.

Consider equation 3.2, neglecting the cross-flow terms we have

$$\frac{1}{h_1 u_e^2}\frac{\partial}{\partial \xi}(\theta_{11} u_e^2) + \frac{1}{h_1 u_e}\frac{\partial u_e}{\partial \xi}\delta_1 - \kappa_1 \theta_{11} = \frac{\tau_{01}}{\rho u_e^2} \qquad 5.1$$

Writing $(1/h_1)(\partial/\partial\xi)$ as $\partial/\partial s$ and $h_2 = r$ so that $\kappa_1 = -(1/h_1 h_2)(\partial h_2/\partial\xi) = -(1/r)(\partial r/\partial s)$ we find 5.1 becomes

$$\frac{\partial\theta_{11}}{\partial s} + \theta_{11}\left[(H+2)\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\right] = \frac{\tau_{01}}{\rho u_e^2} \qquad 5.2$$

where $H = \delta_1/\theta_{11}$. This is the momentum integral equation for the boundary layer flow over an axially symmetric body of cross-sectional radius r. Now $\kappa_1 = -(1/r)(\partial r/\partial s)$ is the geodesic curvature of the orthogonal trajectories of the streamlines. It is thus a measure of the amount these streamlines diverge or converge. If $\partial r/\partial s$ is positive the streamlines diverge just as in axially symmetric flow.

With the assumption of small cross flow velocity the cross-wise momentum equation 3.3 becomes

$$\frac{\partial\theta_{21}}{\partial s} + 2\theta_{21}\left(\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\right) + \kappa_2(\theta_{11} + \delta_1) = \frac{\tau_{02}}{\rho u_e^2} \qquad 5.3$$

where

$$\kappa_2 = -\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial\eta} = \frac{1}{u_e r}\frac{\partial u_e}{\partial\eta}$$

in irrotational flow if we put $h_1 = 1/u_e$.

6. <u>THE DETERMINATION OF r</u>

The parameter r is a function of the geometry of the body and of the external flow. Cooke[2] has shown how r may be determined. If the equation of the surface in Cartesian coordinate is $z = z(x, y)$ and if U

and V are velocity components parallel to the axes x and y then r is given by

$$u_e \frac{\partial}{\partial s}\left(\log \frac{u_e^2 r^2}{g}\right) = 2\left(\frac{\delta U}{\delta x} + \frac{\delta V}{\delta y}\right)$$

where

$$u_e \frac{\partial}{\partial s} = \frac{U\delta}{\delta x} + \frac{V\delta}{\delta y} \qquad 6{\cdot}1$$

$$\frac{\delta}{\delta x} = \frac{\partial}{\partial x} + \frac{\partial z}{\partial x}\frac{\partial}{\partial z}, \quad \frac{\delta}{\delta y} = \frac{\partial}{\partial y} + \frac{\partial z}{\partial y}\frac{\partial}{\partial z}$$

$$g = 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 \quad .$$

If $\partial z/\partial x$ and $\partial z/\partial y$ are small (i.e. if the surface is nearly flat) equations 6.1 simplify to

$$u_e \frac{\partial}{\partial s}\left(\log u_e r\right) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \qquad 6{\cdot}2$$

$$u_e \frac{\partial}{\partial s} = \frac{U\partial}{\partial x} + \frac{V\partial}{\partial y} \quad .$$

It should perhaps be mentioned, as r arose in connection with the axially symmetric analogy, that r is a function of the body geometry and the external flow and in no sense is small cross flow implied in 6.1 and 6.2 above.

7. APPROXIMATE SOLUTION OF THE MOMENTUM INTEGRAL EQUATIONS

As argued in reference 1 the choice of a streamline coordinate system is encouraged by the result obtained from various studies of the velocity profiles of the known exact solutions for three-dimensional laminar boundary layers that the streamwise velocity profiles are virtually identical to the velocity profiles in corresponding two-dimensional boundary layers. Turning to approximate methods of solution and dealing only with the Pohlhausen[3] type of approach one has two momentum integral equations and hence two parameters may be introduced into the description of the velocity profiles

$$u/u_e = f(M_1, N, \xi, \eta, \zeta)$$
$$v/u_e = g(M_2, N, \xi, \eta, \zeta) \qquad 7.1$$

where M_1 and M_2 are not independent but are related by the external flow. It has been found that the streamwise flow cannot be adequately represented by a singly infinite family of velocity profiles when the streamwise pressure gradients are large or rapidly changing; a two-dimensional boundary layer requires a doubly infinite family in corresponding circumstances. However, in what follows we shall adopt the usual approach of representing the streamwise profiles by a single parameter M_1.

If we consider the momentum integral equations in the case of small cross-flow (5.2 and 5.3) and in the description of the streamwise velocity profile take M_1 to be the usual Pohlhausen parameter $\lambda = (\delta^2/\nu)(\partial u_e/\partial s)$, the streamwise equation may be solved in a manner which follows closely the Pohlhausen[3] technique in two dimensions. Here δ is a parameter related to the boundary layer thickness. We then obtain all the unknowns in equation 5.2 as known functions of λ, i.e.

$$\frac{\theta_{11}}{\delta} = f_1(\lambda), \ \frac{\delta_1}{\delta} = f_2(\lambda), \ \frac{\tau_{01}}{\rho u_e^2} = \frac{\nu}{u_e \delta}\left[f_3(\lambda)\right],$$

for then u/u_e is expressed as a specified function of ζ/δ plus λ times another specified function of ζ/δ, these two functions being determined by an appropriate number of boundary conditions and their specified form. If we now substitute these expressions in 5.2 we get

$$\frac{\partial(\delta f_1)}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\,\delta f_1 + \frac{1}{u_e}\frac{\partial u_e}{\partial s}\,\delta\left(f_2 + 2f_1\right) = \frac{\nu}{u_e \delta}\, f_3 \,,$$

which after a little algebra may be written

$$\left(\frac{1}{2}f_1 + \lambda\frac{\partial f_1}{\partial \lambda}\right)\frac{\partial\left(\lambda\Big/\frac{\partial u_e}{\partial s}\right)}{\partial s} = \frac{1}{u_e}\left(f_3 - \lambda f_2 - 2\lambda f_1\right) - \frac{\partial f_1}{\partial \lambda}\left(\frac{\delta^2}{\nu}\right)^2\frac{\partial^2 u_e}{\partial s^2} - f_1\left(\frac{\delta^2}{\nu}\right)\frac{1}{r}\frac{\partial r}{\partial s} \qquad 7.2$$

and this apart from the last term on the right-hand side is identical with the equation obtained by Pohlhausen in two dimensions.

For the cross flow momentum integral equation with small cross flow, 5.3, we take $M_2 = -(\delta^2/\mu)(1/u_e)(\partial P/r\,\partial\eta)$. The choice of M_2 in this form arises naturally, as will be shown below, from the boundary condition imposed upon the cross-flow velocity profile by the second equation of motion (2.2) at the wall. In the case of irrotational flow it will be seen from equations 2.7 and 2.5 that M_2 becomes $M_2 = (\delta^2/\nu)(\partial u_e/r\,\partial\eta)$ but we retain the more general form here as we wish to consider comparisons with exact solutions in which the external flow is rotational. If the assumed velocity profile for v/u_e is also in the form of a specified function of y/δ times M_2 plus another specified function of y/δ times N we then find as shown in the example below that $\theta_{21}/\delta = f_4(\lambda)N + f_5(\lambda)M_2$ and that

$$\tau_{o2}/\rho u_e^2 = \frac{\nu}{u_e\delta}\left(C_1 M_2 + C_2 N\right)$$

where $f_4(\lambda)$, $f_5(\lambda)$ and the constants C_1 and C_2 are determined by the specified forms of the functions chosen to describe the cross-flow velocity profile. Substituting in 3.3 and assuming small cross-flow yields

$$\frac{\partial\theta_{21}}{\partial s} + 2\theta_{21}\left(\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\right) + \frac{\nu M_2}{\delta u_e}\left(f_1 + f_2\right) = \frac{\nu}{u_e\delta}\left[C_1 M_2 + C_2 N\right]$$

but

$$N = \left(\theta_{21}/\delta - f_5 M_2\right)/f_4$$

and therefore

$$\frac{\partial\theta_{21}}{\partial s} + 2\theta_{21}\left(\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\right) + \frac{\nu M_2}{\delta u_e}\left(f_1 + f_2\right) = \frac{\nu}{u_e\delta}\left[C_1 M_2 + \frac{C_2}{f_4}\left(\frac{\theta_{21}}{\delta} - f_2 M_2\right)\right] \qquad 7.3$$

or

$$\frac{\partial\theta_{21}/\delta}{\partial s} + \frac{\theta_{21}}{\delta}\frac{\nu}{2\delta^2}\frac{\partial(\delta^2/\nu)}{\partial s} + 2\left(\frac{\theta_{21}}{\delta}\right)\left(\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{1}{r}\frac{\partial r}{\partial s}\right) + \left(f_1 + f_2\right)\frac{M_2\nu}{u_e\delta^2}$$

$$= \frac{\nu}{u_e\delta^2}\left[C_1 M_2 + \frac{C_2}{f_4}\left(\frac{\theta_{21}}{\delta} - f_5 M_2\right)\right] \qquad 7.4$$

Equation 7.4 is linear in θ_{21}/δ and may be solved by a step by step process once we have the streamwise solution and hence $\partial(\delta^2/\nu)\partial s$ and δ^2/ν

We now give two examples of this type of approximate solution. The first involves the use of Pohlhausen[3] quartics for the description of the velocity profiles and the second, which is Cooke's method[4], involves profiles suggested by Timman[10].

For the first example the streamwise velocity profile is represented by the well-known Pohlhausen quartic in ζ/δ i.e.

$$\frac{u}{u_e} = F(\zeta/\delta) + \lambda G(\zeta/\delta) \quad \text{where } F(\zeta/\delta) = 2\zeta/\delta - 2(\zeta/\delta)^3 + (\zeta/\delta)^4$$

$$G(\zeta/\delta) = \frac{1}{6}\frac{\zeta}{\delta}\left(1-\frac{\zeta}{\delta}\right)^3 .$$

And therefore

$$\frac{\theta_{11}}{\delta} = f_1(\lambda) = \frac{37}{315} - \frac{\lambda}{945} - \frac{\lambda^2}{9072}$$

$$\frac{\delta_1}{\delta} = f_2(\lambda) = \frac{3}{10} - \frac{\lambda}{120}$$

$$\frac{\tau_{01}}{\rho u_e^2} = \frac{\nu}{u_e\delta}\,(f_3\lambda) = \frac{\nu}{u_e\delta}\,(2 + \lambda/6) .$$

For the cross flow velocity profile we assume

$$\frac{v}{u_e} = a_2\bar{\zeta} + b_2\bar{\zeta}^2 + c_2\bar{\zeta}^3 + d_2\bar{\zeta}^4 \qquad 7{\cdot}5$$

where $\bar{\zeta} = \zeta/\delta$.

The boundary conditions for v/u_e are

$$\bar{\zeta} = 1, \quad \frac{v}{u_e} = 0 \quad \text{and} \quad \frac{d}{d\bar{\zeta}}\left(\frac{v}{u_e}\right) = 0 .$$

From the second equation of motion (2.2) at the wall

$$\mu\left(\frac{\partial^2 v}{\partial \zeta^2}\right)_0 = \frac{1}{h_2}\frac{\partial P}{\partial \eta} ,$$

and hence from 7.5

$$\frac{\mu u_e}{\delta^2}\,2b_2 = \frac{1}{h_2}\frac{\partial P}{\partial \eta} , \text{ therefore } b_2 = \frac{\delta^2}{2\mu u_e^2}\;\frac{1}{h_2}\frac{\partial P}{\partial \eta} = -\frac{M_2}{2} . \qquad 7{\cdot}6$$

From the boundary conditions

$$a_2 + b_2 + c_2 + d_2 = 0, \quad a_2 + 2b_2 + 3c_2 + 4d_2 = 0, \quad b_2 = -M_2/2$$

hence

$$a_2 = \frac{M_2}{4} + \frac{d_2}{2}, \quad c_2 = \frac{M_2}{4} - \frac{3d_2}{2}$$

i.e.

$$\frac{v}{u_e} = \frac{M_2}{4}(\bar{y} - 2\bar{y}^2 + \bar{y}^3) + \frac{d_2}{2}(\bar{y} - 3\bar{y}^3 + 2\bar{y}^4)$$

$$= M_2 F_2(\bar{y}) + d_2\, G_2(\bar{y}) \text{ say}$$

or writing $d_2 = N$

$$\frac{v}{u_e} = M_2 F_2(\bar{y}) + N G_2(\bar{y}) \quad . \qquad 7.7$$

Hence

$$\frac{\theta_{21}}{\delta} = -\int_0^1 (F_1 + \lambda G_1)(M_2 F_2 + N G_2)\, d\bar{y} = -\left[M_2 I_1 + \lambda I_2 + N(I_3 + \lambda I_4)\right]$$

where

$$I_1 = \int_0^1 F_1 F_2\, d\bar{y} = 3/224, \quad I_2 = \int_0^1 G_1 F_1\, d\bar{y} = 1/4032, \quad I_3 = \int_0^1 G_2 F_1\, d\bar{y} = \frac{263}{5040}$$

$$I_4 = \int_0^1 G_1 G_2\, d\bar{y} = 5/6048$$

i.e.

$$\frac{\theta_{21}}{\delta} = f_4 N + f_5 M_2 = N\left(-\frac{263}{5040} - \frac{5}{6048}\lambda\right) + M_2\left(-\frac{3}{224} - \frac{1}{4032}\lambda\right) \quad .$$

Also

$$\left(\frac{\partial v}{\partial y}\right)_{y=0} = \frac{u_e a_2}{\delta} = \frac{u_e}{\delta}\left(\frac{M_2}{4} + \frac{N}{2}\right) \text{ and } \frac{\nu}{u_e^2}\left(\frac{\partial v}{\partial y}\right)_{y=0} = \frac{\nu}{\delta u_e}\left(\frac{M_2}{4} + \frac{N}{2}\right)$$

i.e.

$$\frac{\tau_{02}}{\rho u_e^2} = \frac{\nu}{u_e \delta}(C_1 M_2 + C_2 N) = \frac{\nu}{\delta u_e}\left(\frac{M_2}{4} + \frac{N}{2}\right) \quad . \qquad 7.8$$

The second method uses the profiles suggested by Timman[13]

$$u/u_e = f(z) - \lambda g(z)$$

$$v/u_e = N K(z) - M_2 g(z)$$

where $K(z) = z e^{-z^2}$

$$1 - f(z) = 2g(z) + e^{-z^2} = \frac{2}{3\sqrt{\pi}}\, z e^{-z^2} + \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\, dt \qquad 7.9$$

$$z = \zeta/\delta_T \ .$$

Here λ, M_2 and N are formally as defined previously, but with δ_T replacing δ , and it should be noted that now δ_T is a scaling length, related to the boundary layer thickness, but not to be confused with the δ of the Pohlhausen method. Thus the upper limit of z is ∞ and not one as previously. Timman shows that for these profiles

$$\theta_{11}/\delta_T = f_1(\lambda) = \cdot 289430 + \cdot 007335\,\lambda - 0\,003798\lambda^2$$

$$\delta_1/\delta_T = f_2(\lambda) = \cdot 752253 - 0\,066987\,\lambda$$

$$\frac{\tau_{01}}{\rho u_e^2} = \frac{\nu}{u_e\delta_T}\left[f_3(\lambda)\right] = \frac{\nu}{u_e\delta_T}\ \frac{2}{3\sqrt{\pi}}(2+\lambda) = \frac{\nu}{u_e\delta_T}\ 0\,376127\,(2+\lambda) \qquad 7.10$$

$$\theta_{21}/\delta_T = f_4 N + f_5 M_2 = N(-0{\cdot}294628 - 0{\cdot}022314\,\lambda) + M_2(-0{\cdot}029826 - 0\,0037975\,\lambda)$$

$$\frac{\tau_{02}}{\rho u_e^2} = \frac{\nu}{u_e\delta_T}\,(c_1 M_2 + c_2 N) = \frac{\nu}{u_e\delta_T}\left(\frac{2}{3\sqrt{\pi}}M_2 + N\right) \cdot$$

Cooke[4] simplified the solution by making the approximations $\theta_{11}/\delta_T = 0{\cdot}293$ and

$$\frac{4}{3\sqrt{\pi}}\,\frac{\theta_{11}}{\delta_T}(2+\lambda) - 2\lambda\,\frac{\theta_{11}}{\delta_T}\,\frac{\delta_1}{\delta_T} \triangleq 0{\cdot}436 - 2(0{\cdot}293)^2\lambda \ . \qquad 7.11$$

These approximations were based on Zaat's work[14,15].

Cooke then obtained the streamwise momentum integral equation in the form

$$\frac{\partial\left[r^2 u_e^6(\delta_T^2/\nu)\right]}{\partial s} = 5{\cdot}08\, r^2 u_e^5 \qquad 7.12$$

or

$$\frac{\partial(r^2 u_e^6\,\theta_{11}^2)}{\partial s} = 0{\cdot}436\, r^2 u_e^5\,\nu \qquad 7.13$$

and the cross-flow momentum integral equation as

$$\frac{\partial(r^2 u_e^2\,\theta_{21})}{\partial s} = r^2\,\frac{u_e\nu}{\delta_T}\left[N + M_2(0{\cdot}067\lambda - 0{\cdot}669)\right] \cdot \qquad 7.14$$

The method involving Pohlhausen quartic velocity profiles may be similarly simplified by a simple extension of a two-dimensional method due to Young[11]. Taking the streamwise momentum integral equation for small cross-flow

$$\frac{\partial \theta_{11}}{\partial s} + \theta_{11}(H+2)\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{\theta_{11}}{r}\frac{\partial r}{\partial s} = \frac{\tau_{01}}{\rho u_e^2} \qquad 5.2$$

and using the expression for $\tau_{01}/\rho u_e^2$ given by the assumption of the Pohlhausen quartic velocity profile, viz

$$\frac{\tau_{01}}{\rho u_e^2} = \frac{\nu}{u_e \delta}(2 + \lambda/6) ,$$

we find that 5.2 becomes, if we write $\delta/\theta_{11} = f$,

$$\frac{\partial \theta_{11}}{\partial s} + \theta_{11}(H+2)\frac{1}{u_e}\frac{\partial u_e}{\partial s} + \frac{\theta_{11}}{r}\frac{\partial r}{\partial s} = \frac{1}{u_e}\frac{\partial u_e}{\partial s}\frac{f\theta_{11}}{6} + \frac{2\nu}{f\theta_{11}u_e}$$

or

$$\theta_{11}\frac{\partial \theta_{11}}{\partial s} = \frac{1}{u_e}\frac{\partial u_e}{\partial s}\theta_{11}^2\left[f/6 - (H+2)\right] - \theta_{11}^2\frac{1}{r}\frac{\partial r}{\partial s} + \frac{2\nu}{f u_e} \quad . \qquad 7.15$$

This can be written

$$\frac{\partial \theta_{11}^2}{\partial s} + \frac{1}{u_e}\frac{\partial u_e}{\partial s}\theta_{11}^2 q + \theta_{11}\frac{1}{r}\frac{\partial r}{\partial s} = \frac{4\nu}{f u_e}$$

where

$$q = 2\left[(H+2) - f/6\right] \quad .$$

According to the Pohlhausen method the extremes of λ are about +7 and -12 for which the corresponding values of H are 2.31 and 2.74 and the corresponding values of δ/θ_{11} range from about 7 to 9.5. Thus q varies little over the range of interest and if we assume q to be constant we obtain

$$\frac{\partial(r^2\theta_{11}^2 u_e^q)}{\partial s} = \frac{r^2 4\nu}{f} u_e^{q-1} \qquad 7.16$$

If we then assume the flat plate values of $f = 9.072$ and H = 2.59 so that q = 6.16 and assume f also to be a constant we have

$$\frac{\partial(r^2\theta_{11}^2 u_e^{6.16})}{\partial s} = 0.441 \ \nu r^2 u_e^{5.16} \qquad 7.17$$

which is very similar to the form (7.13) due to Cooke above. Similarly the assumption of H constantly equal to 2.59 and $\delta/\theta_{11}^{\cdot} = 9.072$ simplifies the cross-flow momentum integral equation for the Pohlhausen quartic type method to the form

$$\frac{\partial(\theta_{21}\, r^2 u_e^2)}{\partial s} = \frac{r^2 u_e \nu}{\delta}\left[0.5N - 0.1457 M_2\right] \qquad 7.18$$

In both 7.18 above and 7.14 N is related to θ_{21}/δ (or θ_{21}/δ_T) by an equation of the form

$$N = \frac{1}{f_4}\left(\frac{\theta_{21}}{\delta} - f_5 M_2\right) \text{ or } \frac{1}{f_4}\left(\frac{\theta_{21}}{\delta_T} - f_5 M_2\right) \qquad 7.19$$

where f_4 and f_5 are functions of λ only.

8. SOME COMPARISONS WITH EXACT SOLUTIONS

The four methods so far described based on the small cross-flow assumption (i.e. that using Pohlhausen quartic velocity profiles, its associated approximate method due to Young, that using Timman's profiles and Cooke's approximation thereof) have been programmed in Mercury Autocode for use on the University of London Atlas Computer and have been compared with three known exact solutions used as tests by Cooke[4]. These called Examples 1, 2 and 3 respectively in Figures 1 to 12 have velocity components

$$\bar{U}_1 = U_1/u_\infty = 1$$

$$\bar{V}_1 = V_1/u_\infty = A_1\frac{X}{c} + A_2\left(\frac{X}{c}\right)^2 + A_3\left(\frac{X}{c}\right)^3$$

respectively where X, Y are cartesian coordinates and U_1, V_1 are velocities in the directions X increasing and Y increasing, c is a representative length and

$$\left.\begin{aligned} A_1 &= 2 \\ A_2 &= 1 \\ A_3 &= -1 \end{aligned}\right\} \text{ EXAMPLE 1}$$

$$\left.\begin{aligned} A_1 &= 4 \\ A_2 &= 4 \\ A_3 &= -4 \end{aligned}\right\} \text{EXAMPLE 2}$$

$$\left.\begin{aligned} A_1 &= 4 \\ A_2 &= -4 \\ A_3 &= 4 \end{aligned}\right\} \text{EXAMPLE 3}$$

Cooke shows that for these cases for which the streamlines are translates we may take

$$\frac{\partial}{\partial s} = \frac{1}{\bar{u}}\frac{\partial}{\partial x}, \quad \frac{\partial}{r\partial \eta} = \frac{\bar{V}_1}{\bar{u}}\frac{\partial}{\partial x}, \quad M_2 = -\frac{\lambda}{\bar{V}_1}$$

where $\quad \bar{u}^2 = \bar{u}_1^2 + \bar{V}_1^2 = (u_e/u_\infty)^2$.

These three examples all have streamlines with a point of inflexion at $x/c = 0.5$. For Examples 1 and 2 the pressure gradient is initially favourable and changes to unfavourable at the point of inflexion. For Example 3 the reverse is the case, the pressure gradient is originally unfavourable and changes to favourable at the point of inflexion.

The computer programme tabulated the solution of the two simultaneous differential equations 7.2 and 7.4, the integration being performed by means of a library routine employing a Runga-Kutta-Merson technique.

Study of Figures 1 to 12, in which the results are presented reveals that, although there is little to choose between the four methods for the prediction of streamwise momentum thickness, Cooke's approximation of the Timman profiles method produces slightly more accurate answers for the streamwise skin friction than does the Young type approximation of the Pohlhausen quartic profiles method. This is, perhaps, to be expected as the Timman profiles satisfy all the boundary conditions at the outer edge of the boundary layer automatically. The predictions for $\tan\beta = \tau_{02}/\tau_{01}$ are good apart from the adverse pressure gradient for $x/c < 0.5$ in Example 3, Figure 12. The predictions for θ_{11} are not so good in the cases involving stronger adverse pressure gradients and larger cross-flows (Examples 2 and 3, Figures 5 and 9) but note should be taken of the false zeros of all the diagrams. The approximations made by Cooke appear to lead to the smallest errors for the cases examined, and Cooke's method seems therefore the best of the four tested. The only experimental checks[12,13] upon Cooke's method known to the present authors show comparisons of predictions for $\tan\beta$ by Cooke's method with values obtained from flow visualisation tests. These have confirmed that $\tan\beta$ is well predicted by Cooke's method.

9. TWO OTHER METHODS

It will be seen that in all cases and in particular in Example 3 significant errors in the streamwise momentum thickness predictions occur in the presence of adverse pressure gradients. In an attempt to improve the predictions for adverse pressure gradients the technique devised by Luxton and Young[5] for the case of the two-dimensional laminar compressible boundary layer with heat transfer has been adapted to the three-dimensional laminar boundary layer with small cross-flow.

The starting point for this method is equation 7.16

$$\frac{\partial(r^2\theta_{11}^2 u_e^g)}{\partial s} = \frac{r^2 4\nu u_e^{g-1}}{f}. \qquad 9.1$$

From an analysis of exact solutions Luxton and Young[5] derive expressions for the dependence of f and g upon λ which in the simple incompressible case with zero heat transfer considered here may be reduced to

$$\delta/\theta_{11} = f = 9.072(1 + D_1\lambda)$$

and

$$H = 2.59 + D_2\lambda \qquad 9.2$$

with

$$g = 2\left[(H+2) - f/6\right]$$

and $D_1 = -0.0198$, $D_2 = -0.0742$ for favourable pressure gradients and $D_1 = -0.0246$, $D_2 = -0.106$ for adverse pressure gradients. We have also

$$\lambda = \frac{\delta^2}{\nu}\frac{\partial u_e}{\partial s} = \frac{f^2}{\nu}\theta_{11}^2\frac{\partial u_e}{\partial s}. \qquad 9.3$$

The calculation proceeds in a series of small steps in s, q and f are held constant during each step but vary from step to step. The procedure may be summarised as follows:

(i) Find the values of H and f at $s = 0$ from equations 9.2. In most cases $\lambda_{s=0} = 0$ but if this is not so then $\lambda_{s=0}$ must be calculated from a known value of θ_{11} at $s=0$ by an iterative process through equations 9.3 and 9.2.

(ii) Integrate equation 9.1 over a small step in s to obtain a value of θ_{11} at s_1.

(iii) Using the value of $f_{s=0}$ in equation 9.3 find an approximate value of λ_{s_1}.

(iv) Using the approximate value of λ_{s_1} find f_{s_1}.

(v) Use this value of f_{s_1} in equation 9.3 to find a more accurate value of λ_{s_1}.

(vi) Substitute this more accurate value of λ_{s_1} into equations 9.2 to find values of f_{s_1}, H_{s_1} and hence g_{s_1}. The equation 9.1 may then be reintegrated over the step from $s = 0$ to $s = s_1$ using the mean values of f and q over that step. This procedure (ii) to (vi) may be repeated until the value of λ_{s_1} converges to a given tolerance.

(vii) Using the values of f_{s_1} and g_{s_1} repeat the procedure to find the solution at s_2.

This method has been applied to the three examples mentioned previously and as will be seen from Figures 1, 5 and 9 a definite improvement in the form of the distribution of θ_{11} is obtained, but the overall improvement for the larger adverse gradients is somewhat disappointing in the light of the results obtained in two dimensions (see Ref. 5).

For these large pressure gradients the question then arises as to the magnitude of the errors introduced by the assumption of small cross-flows, and we are led to consider the development of a method which does not involve this assumption. Here a difficulty is encountered since the momentum integral equations contain terms such as $(1/r)(\partial\theta_{12}/\partial\eta)$ and $(1/r)(\partial\theta_{22}/\partial\eta)$. In the general case these must be accounted for by a calculation procedure which first ignores these terms and solves the momentum integral equations along several streamlines and then repeats the process accounting for the derivatives with respect to η by means of the differences in θ_{12} and θ_{22} found upon neighbouring streamlines by the initial calculation. The whole calculation thus proceeds as an iterative process. For the particular cases considered here the process is however somewhat simpler since we may account for the derivatives in the η direction by the relation given above viz

$$\frac{1}{r}\frac{\partial}{\partial\eta} = \frac{\bar{V}_1}{\bar{u}}\cdot\frac{\partial}{\partial x}\ .$$

The method devised is as follows and as will be seen it includes all the terms in the momentum integral equations. Timman has shown for his profiles that

$$\theta_{12}/\delta_T = \cdot 205372N + \cdot 03161\,M_2 - 0\cdot 022314\,\lambda N - 0\cdot 003798\,\lambda M_2$$

$$\theta_{22}/\delta_T = -\cdot 156664N^2 - \cdot 044638\,M_2 N - \cdot 003798\,M_2^2$$

$$\delta_2/\delta_T = -\cdot 5N - \cdot 066987\,M_2\ .$$

Substituting the Timman profile expressions for

$$\frac{\theta_{12}}{\delta_T},\ \frac{\theta_{22}}{\delta_T},\ \frac{\delta_2}{\delta_T},\ \frac{\theta_{11}}{\delta_T},\ \frac{\delta_1}{\delta_T},\ \frac{\theta_{21}}{\delta_T},\ \frac{\tau_{01}}{\rho u_e^2},\ \frac{\tau_{02}}{\rho u_e^2}$$

into the momentum equations 3.2 and 3.3 and using

$$M_2 = -\frac{\lambda}{\bar{v}_1},\ K_2\,u_e^2 = -\frac{1}{\rho h_2}\frac{\partial P}{\partial\eta} = \frac{\nu}{\delta_T^2}M_2 u_e,\ \frac{1}{r}\frac{\partial}{\partial\eta} = \bar{v}_1\frac{\partial}{\partial s}$$

gives two simultaneous differential equations involving δ_T^2/ν, $\partial(\delta_T^2/\nu)/\partial s$, N, $\partial N/\partial s$, $\lambda = (\delta_T^2/\nu)(\partial u_e/\partial s)$ and functions of the external flow. These two equations were then rearranged by much lengthy but straightforward algebra into the form

$$\frac{\partial(\delta_T^2/\nu)}{\partial s} = f_1\,(\delta_T^2/\nu,\ N,\ s)$$

$$\frac{\partial(N)}{\partial s} = f_2\,(\delta_T^2/\nu,\ N,\ s)$$

which could be solved by means of the library routine mentioned above. The method was then programmed and the results are presented in Figures 1 to 12 in which it is termed Method 3.

It will be seen that the inclusion of the cross-flow terms in the momentum integral equations results in a marked improvement in the accuracy of the results particularly in the presence of strong adverse pressure gradients. The remaining relatively small discrepancies between the results given by Method 3 and the exact results can be ascribed to errors arising from the velocity profiles chosen.

10. CONCLUDING REMARKS

For the three-dimensional laminar boundary layer the use of the small cross-flow assumption together with a Pohlhausen type approximate solution of the momentum integral equations results in good agreement with exact solutions for cases involving favourable pressure gradients. Of the two types of velocity profiles, Pohlhausen quartics and Timman's profiles, tested in approximate solutions here, Cooke's approximation of the method involving the latter profiles yielded results which were marginally superior to those obtained by a Young type approximation of the method involving the former profiles. The results produced by these methods for the streamwise momentum thickness in adverse pressure gradients are by no means as good, however. This is thought to be due to the nature of the streamwise momentum integral equation which, with the assumption of small cross-flow velocity, we may rewrite as

$$\frac{\partial\theta_{11}}{\partial s} = \frac{\tau_{01}}{\rho u_e^2} - \frac{1}{u_e}\frac{\partial u_e}{\partial s}\left[2\theta_{11} + \delta_1\right] - \frac{\theta_{11}}{r}\frac{\partial r}{\partial s} \quad . \qquad 10{\cdot}1$$

For adverse pressure gradients $-(1/u_e)(\partial u_e/\partial s)$ is positive so that if at any stage the value of θ_{11} predicted by the approximate solution is too large compared with the exact solution the value of $\partial\theta_{11}/\partial s$ over the next step will in consequence also tend to be too large and the approximate solution will tend to diverge from the exact solution. Similarly, should the value of θ_{11} be too small $\partial\theta_{11}/\partial s$ will be too small and once more the approximate solution will diverge from the exact solution. For favourable pressure gradients ($-(1/u_e)(\partial u_e/\partial s)$ negative) this does not occur as a too large value of θ_{11} produces a too small value of $\partial\theta_{11}/\partial s$ and vice versa. The last term in the above equation $-(\theta_{11}/r)(\partial r/\partial s)$ tends to act in the opposite sense but it is generally dominated by the second term as far as the net effect of errors in θ_{11} are concerned. For the adverse pressure gradient case, Example 3, shown in Figure 9, neglect of the cross-flow terms in the streamwise momentum integral equation and the assumption that θ_{11}/δ is a constant both have the effect of producing a value of θ_{11} which is too large when compared with the exact solution. This results in the divergence mentioned above and the consequent inaccuracy of this type of approximate method. The assumption that θ_{11}/δ is a constant may be removed by the adoption of the extension of the Luxton-Young technique presented here and results in some improvement of accuracy for favourable and small adverse gradients. However for large adverse gradients the assumption of small cross-flows leads to significant errors and must be discarded to achieve adequate accuracy.

The approximate method involving the full momentum integral equations developed here produces for the cases considered very satisfactory results but at the expense of greater computational complexity which would be even more marked in the general case where an iterative procedure would be required.

As a rough tentative guide as to when the pressure gradients and the cross-flows are such as to call for the inclusion of the cross-flow terms we may note that for Examples 1 and 2 where the small cross-flow methods are for most purposes of acceptable accuracy the maximum value of β was of the order of 10° whilst for Example 3 the maximum value of β was about 20°.

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NOTATION

ξ, η, ζ	Orthogonal curvilinear coordinates with ζ measured normal to the surface.
h_1, h_2	Metrics in the ξ, η, ζ coordinate system. $(h_3 = 1)$.
u, v, w	Velocities in the ξ, η, ζ directions respectively.
K_1	$= -\dfrac{1}{h_1 h_2}\dfrac{\partial h_2}{\partial \xi}$ geodesic curvature of the curve ξ = constant.
K_2	$= -\dfrac{1}{h_1 h_2}\dfrac{\partial h_1}{\partial \eta}$ geodesic curvature of the curve η = constant.
s, n	Directions along and normal to an external streamline respectively.
r	$= h_2$
$\delta_1, \delta_2, \theta_{11}$ $\theta_{21}, \theta_{12}, \theta_{22}$	Displacement and momentum thicknesses defined by equation 3.1.
τ_{01}, τ_{02}	Skin friction components in the ξ, η directions respectively.
H	$= \delta_1/\theta_{11}$ the streamwise shape parameter.
M_1, M_2, N	Parameters used in description of velocity profiles.
λ	$= (\delta^2/\nu)(\partial u_e/\partial s)$, the Pohlhausen velocity profile parameter.
ρ	The density of the fluid.
μ	The viscosity of the fluid.
ν	μ/ρ the kinematic viscosity of the fluid.
P	The static pressure in the fluid.
Suffix e	External to the boundary layer.

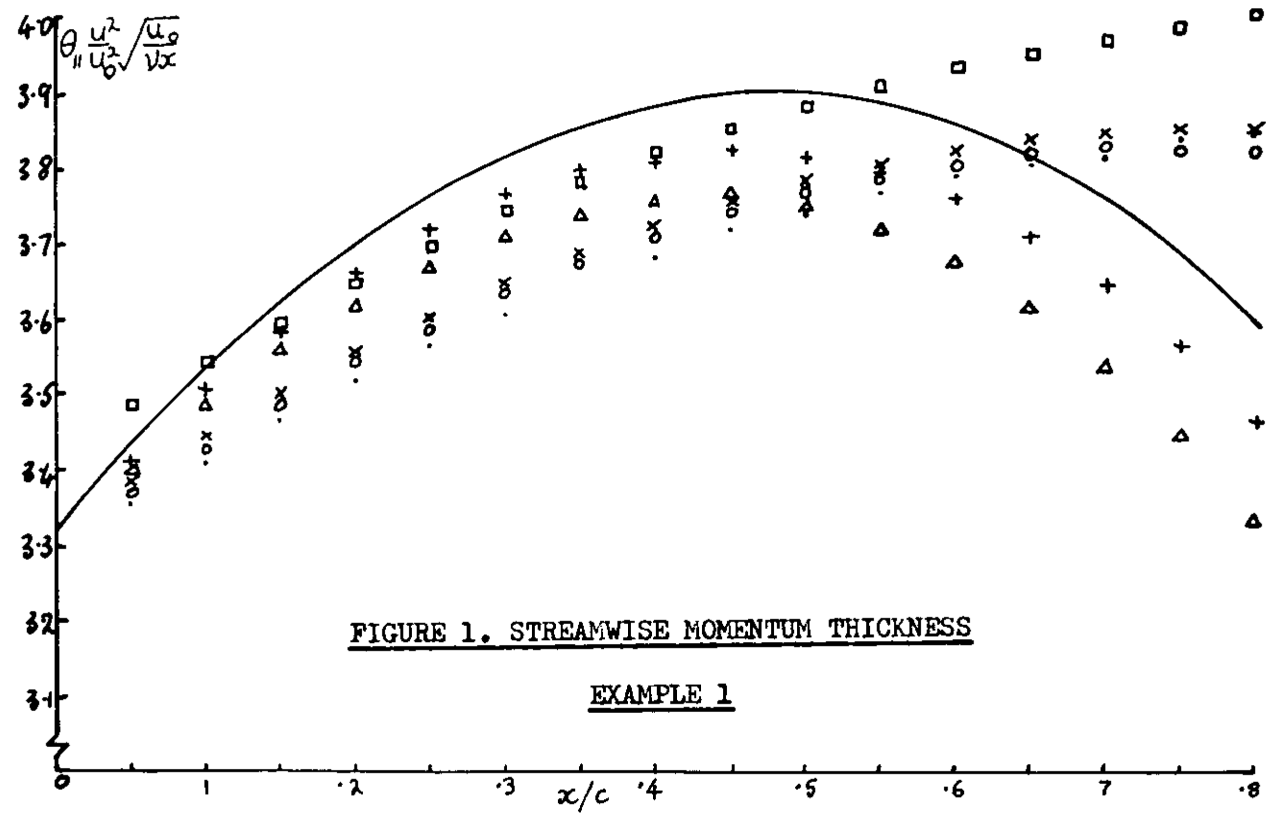

FIGURE 1. STREAMWISE MOMENTUM THICKNESS

EXAMPLE 1

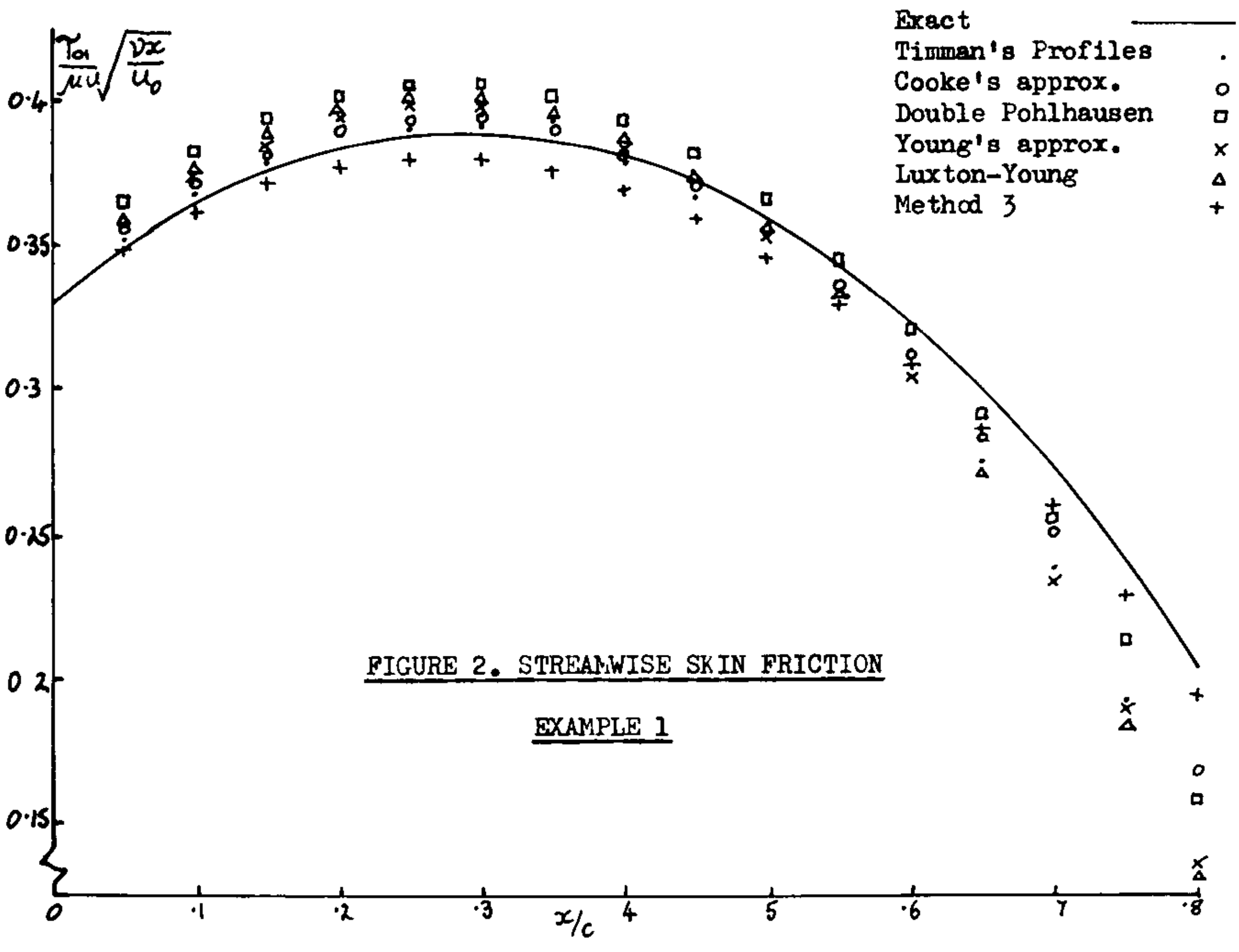

FIGURE 2. STREAMWISE SKIN FRICTION

EXAMPLE 1

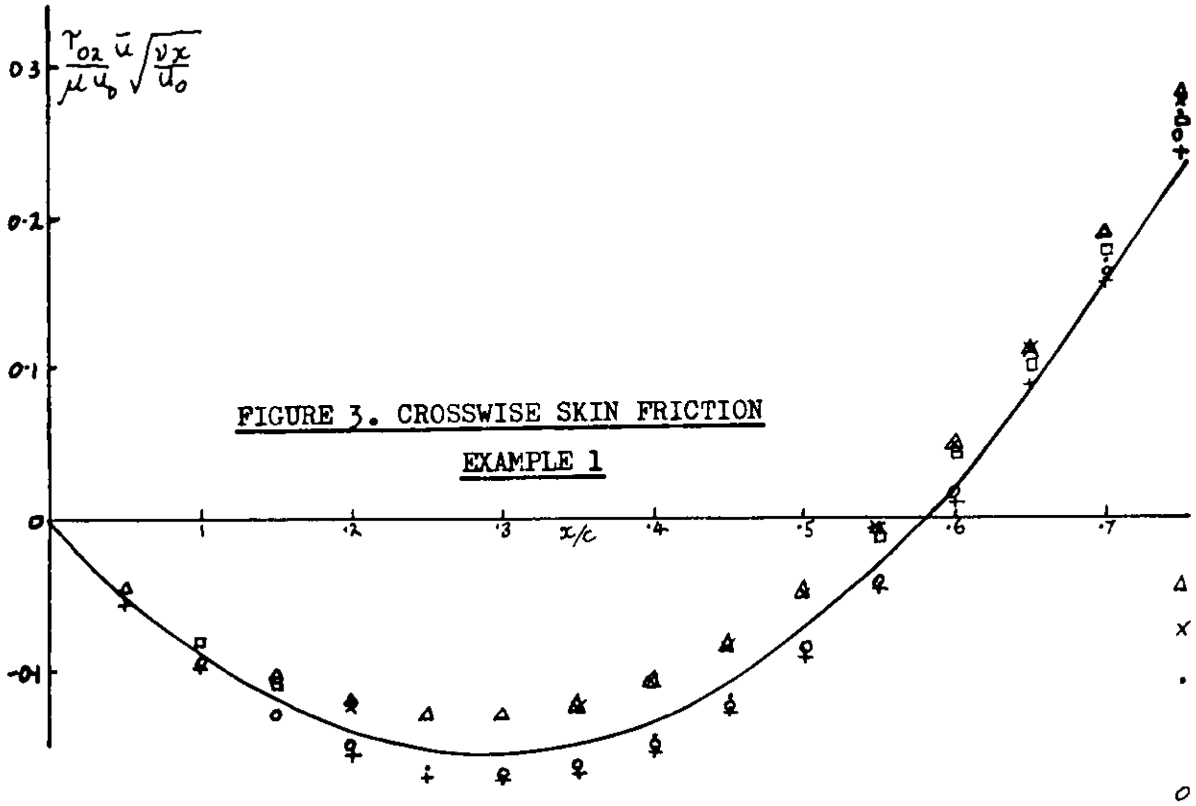

FIGURE 3. CROSSWISE SKIN FRICTION

EXAMPLE 1

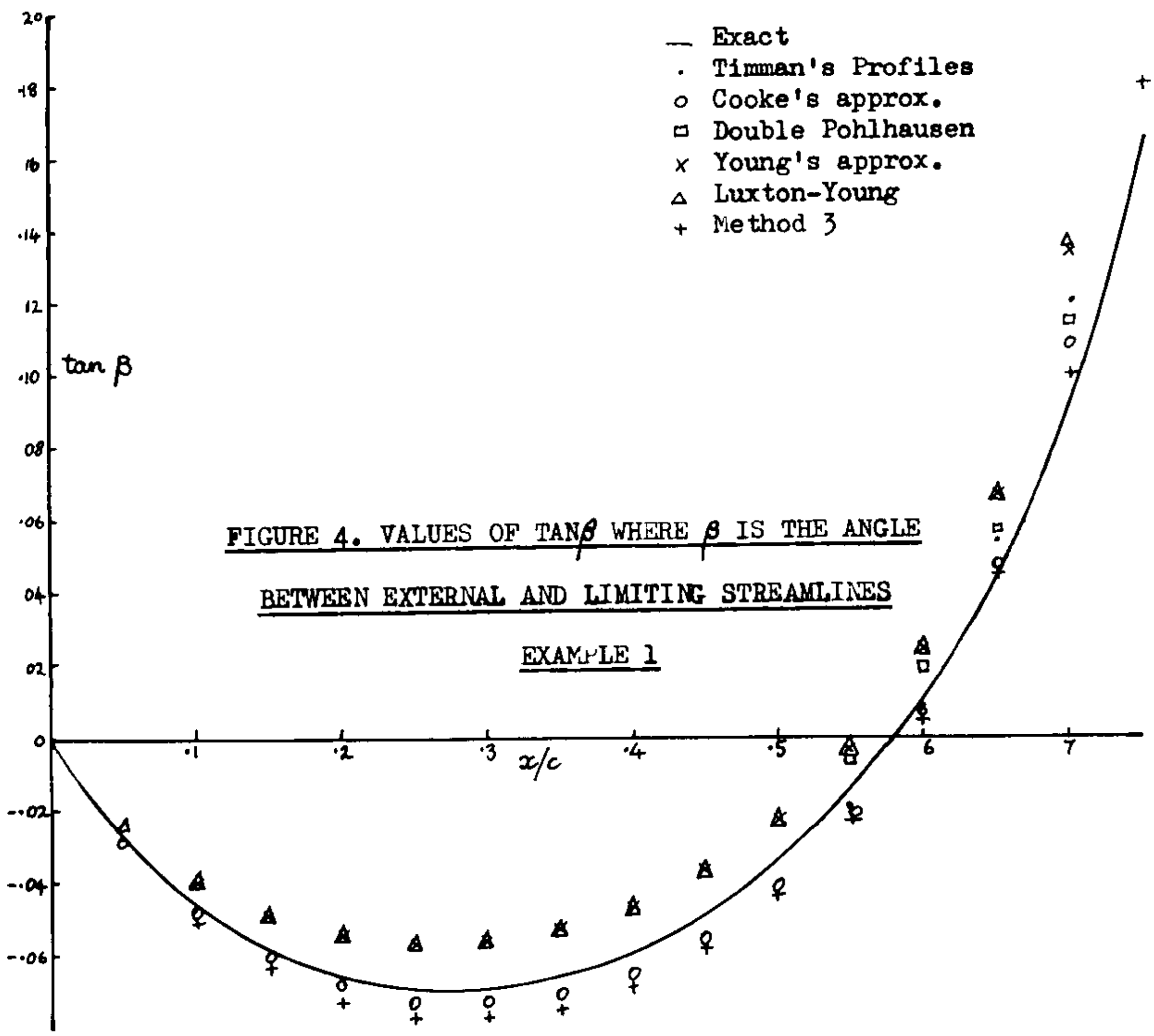

FIGURE 4. VALUES OF TANβ WHERE β IS THE ANGLE BETWEEN EXTERNAL AND LIMITING STREAMLINES

EXAMPLE 1

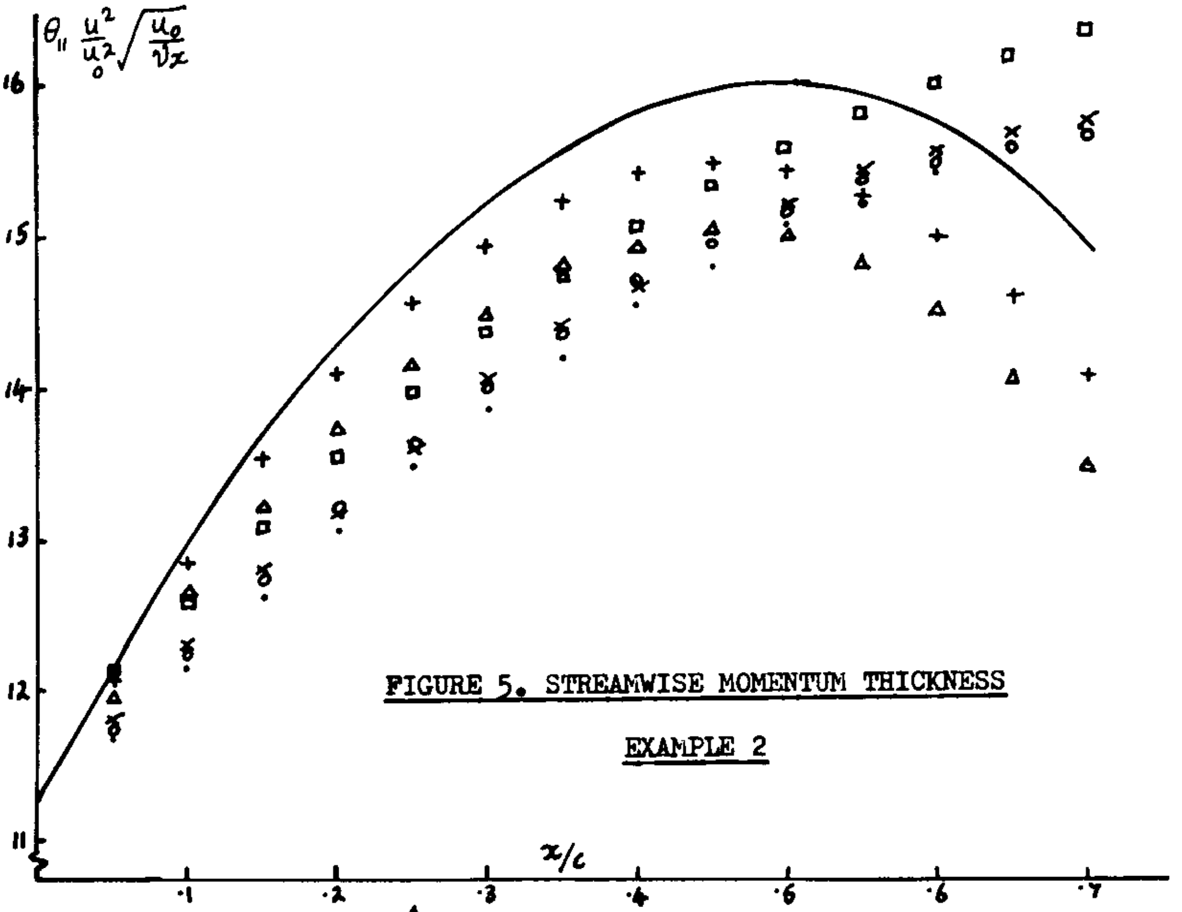

FIGURE 5. STREAMWISE MOMENTUM THICKNESS

EXAMPLE 2

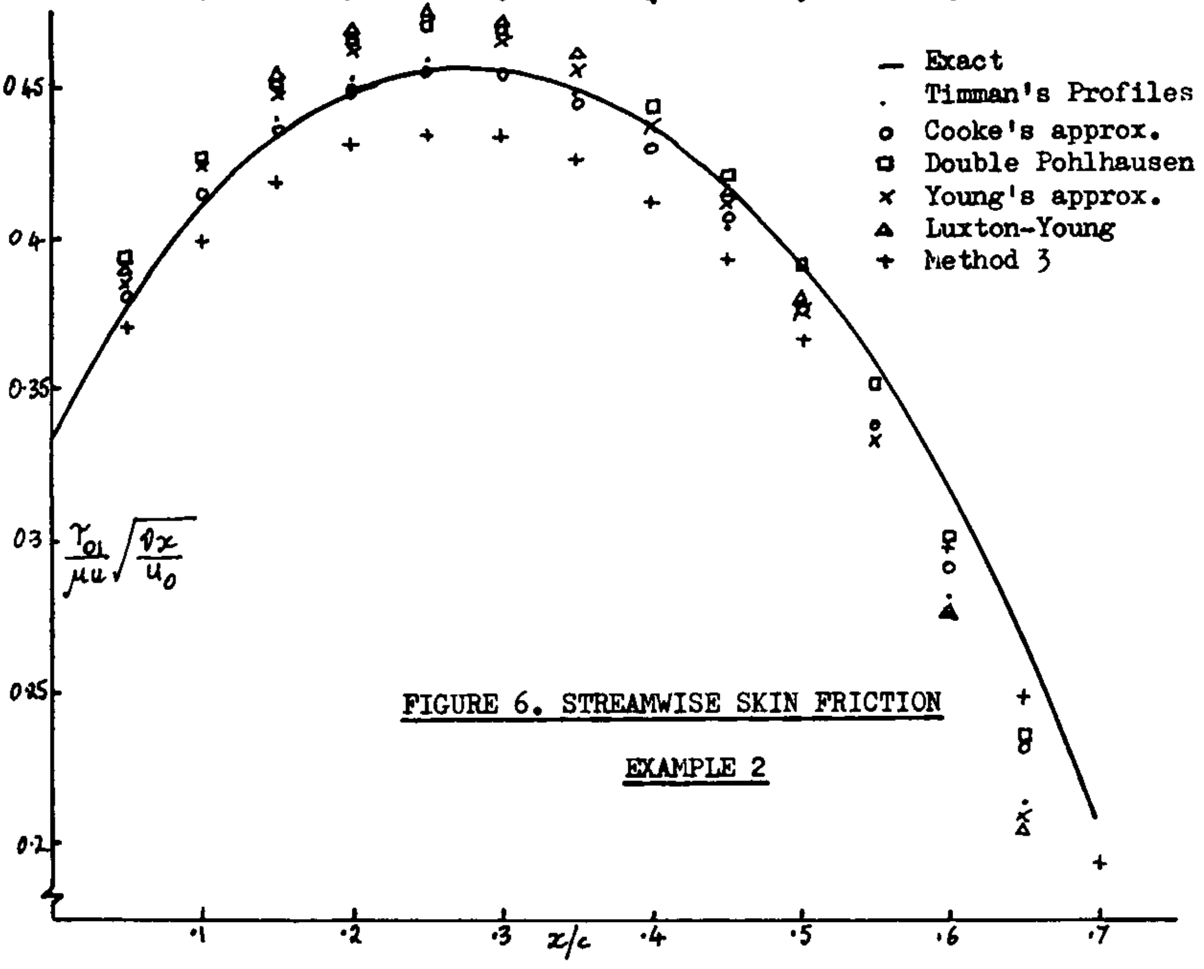

FIGURE 6. STREAMWISE SKIN FRICTION

EXAMPLE 2

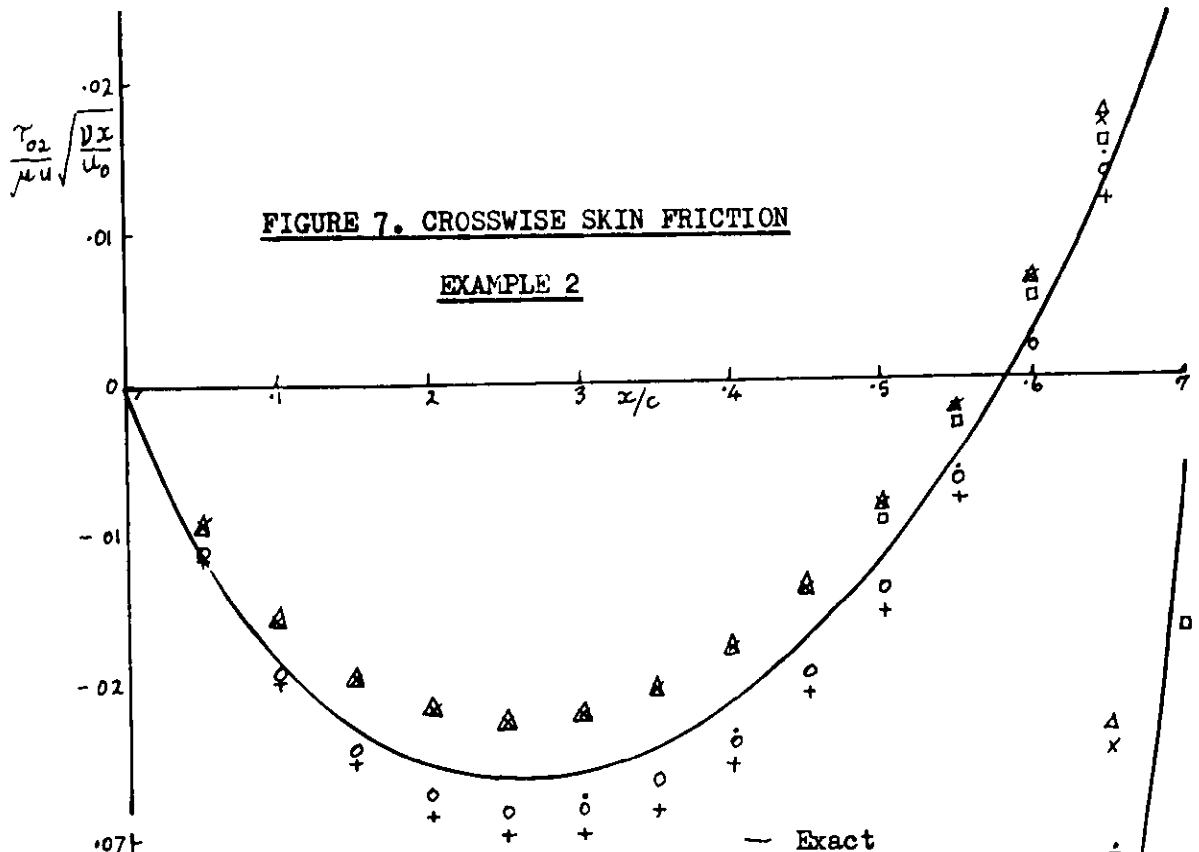

FIGURE 7. CROSSWISE SKIN FRICTION
EXAMPLE 2
$\frac{\tau_{02}}{\mu u}\sqrt{\frac{\nu x}{u_0}}$
·02
·01
0
-·01
-·02
·1
·2
·3
·4
·5
·6
·7
x/c

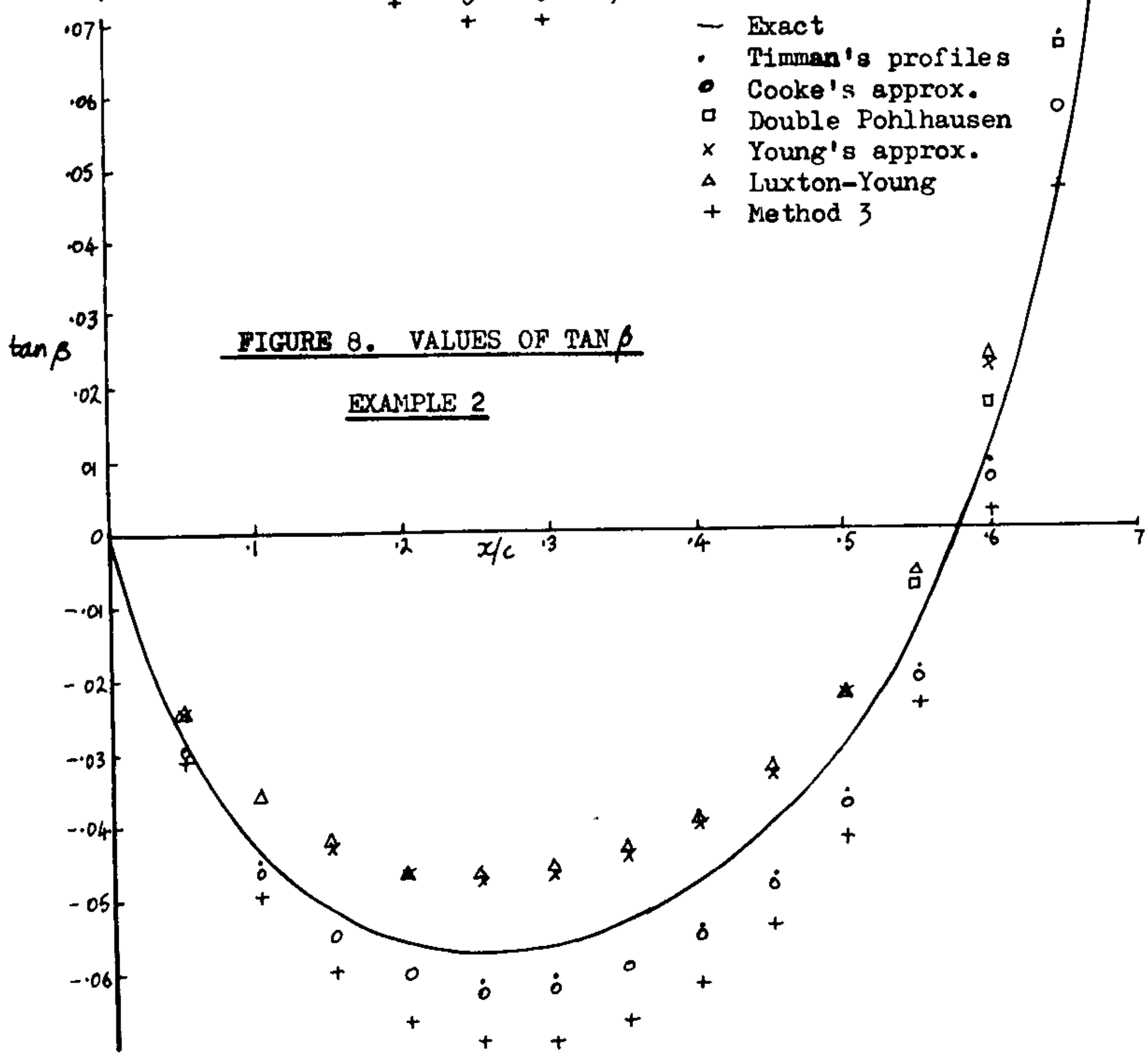

FIGURE 8. VALUES OF TAN β
EXAMPLE 2
$\tan\beta$
·07
·06
·05
·04
·03
·02
·01
0
-·01
-·02
-·03
-·04
-·05
-·06
·1
·2
·3
·4
·5
·6
·7
x/c
— Exact
• Timman's profiles
o Cooke's approx.
□ Double Pohlhausen
× Young's approx.
△ Luxton-Young
+ Method 3

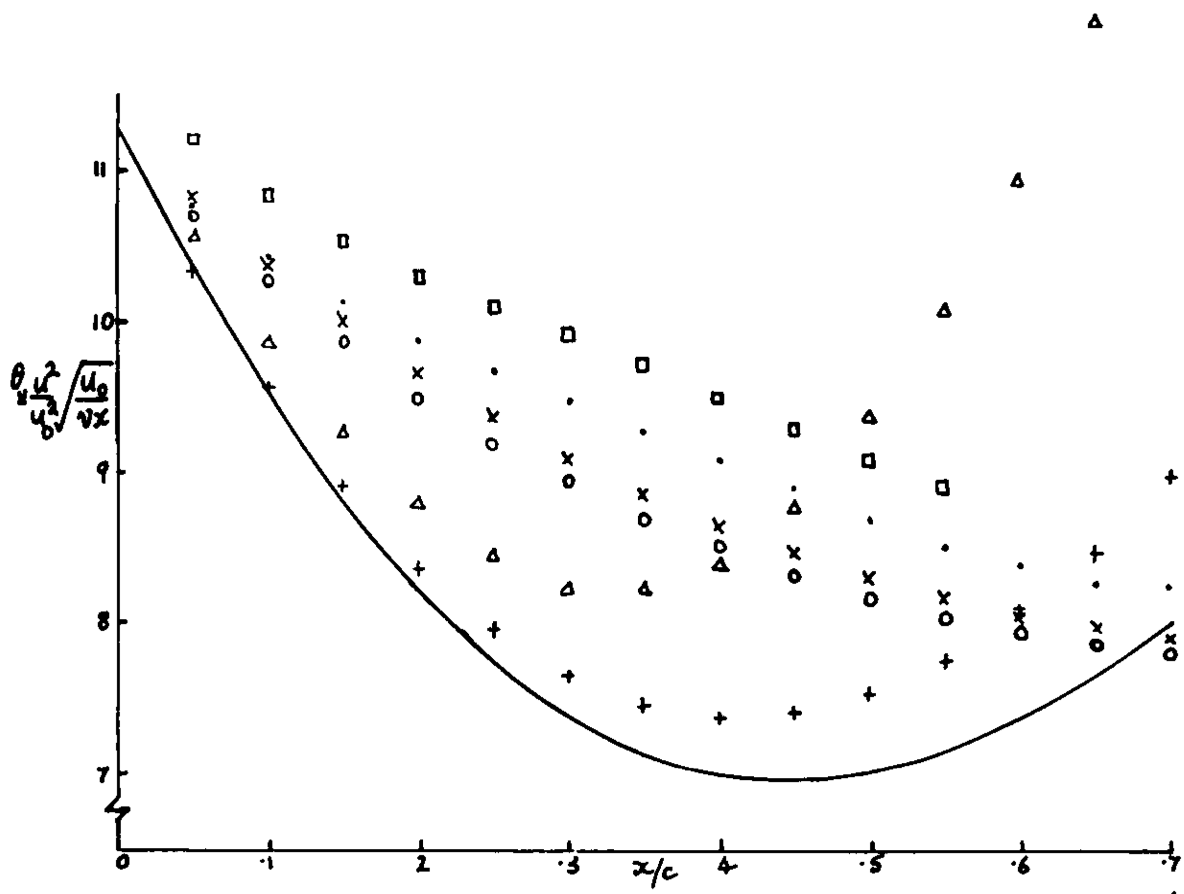

FIGURE 9. STREAMWISE MOMENTUM THICKNESS. EXAMPLE 3

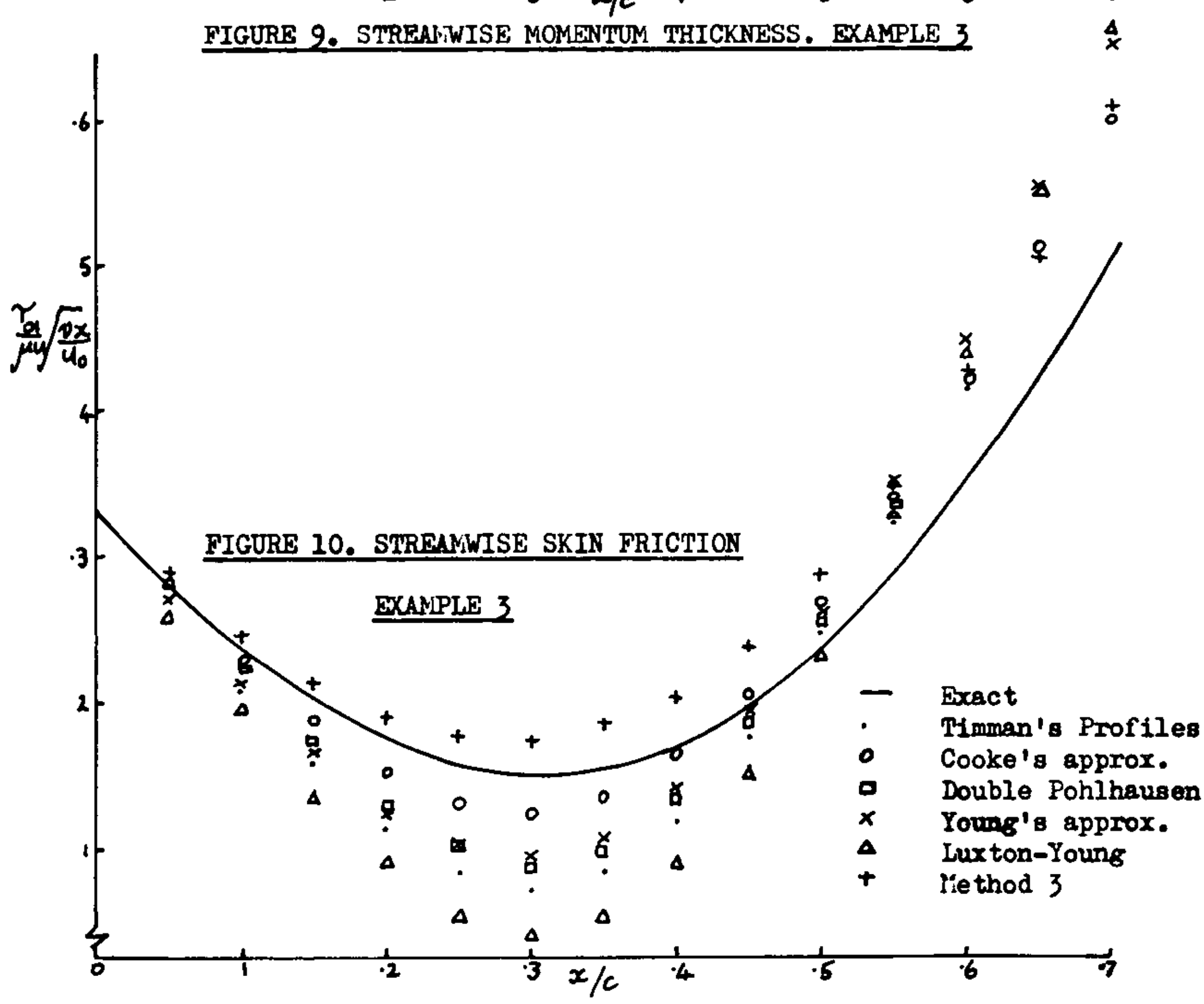

FIGURE 10. STREAMWISE SKIN FRICTION

EXAMPLE 3

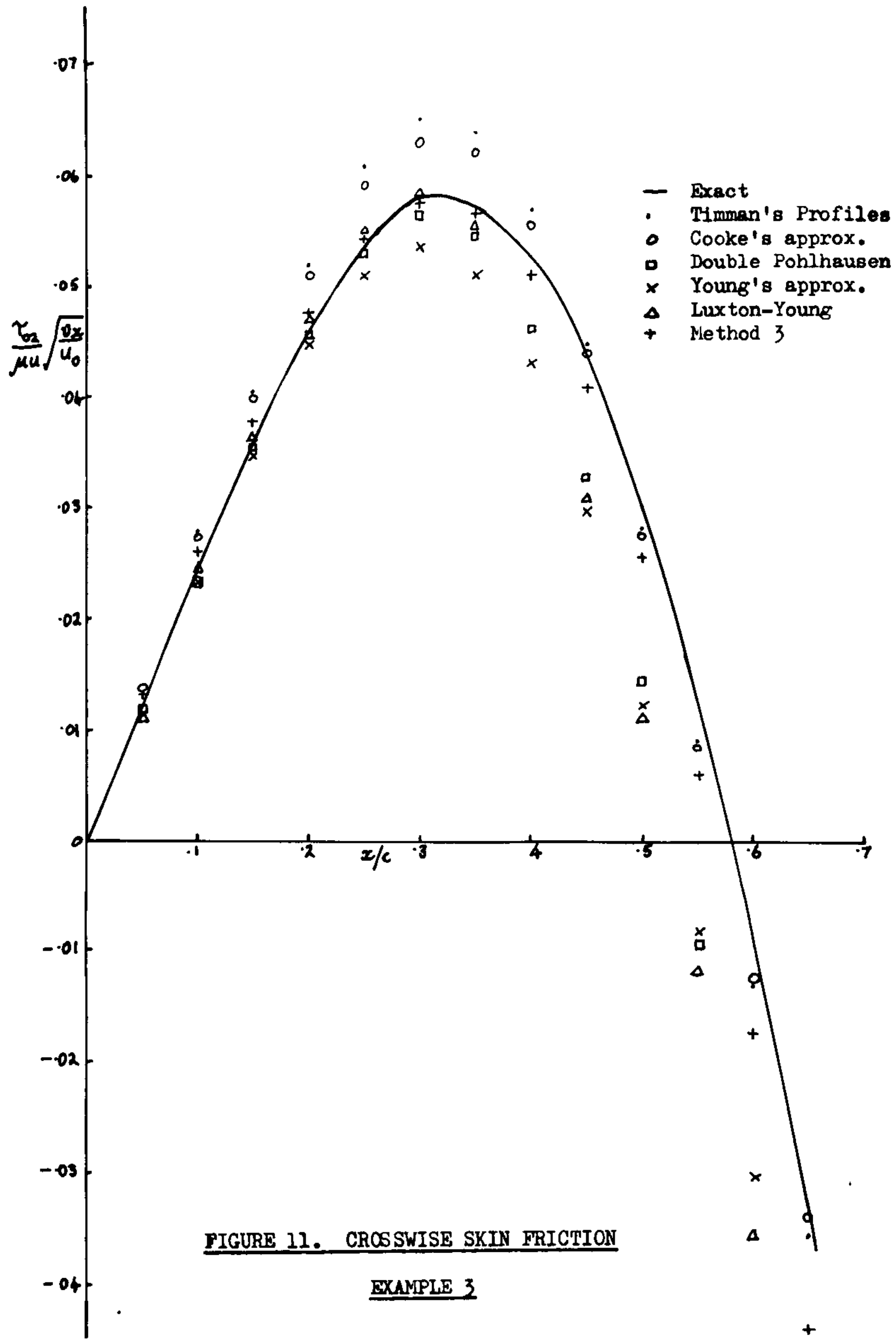

FIGURE 11. CROSSWISE SKIN FRICTION

EXAMPLE 3

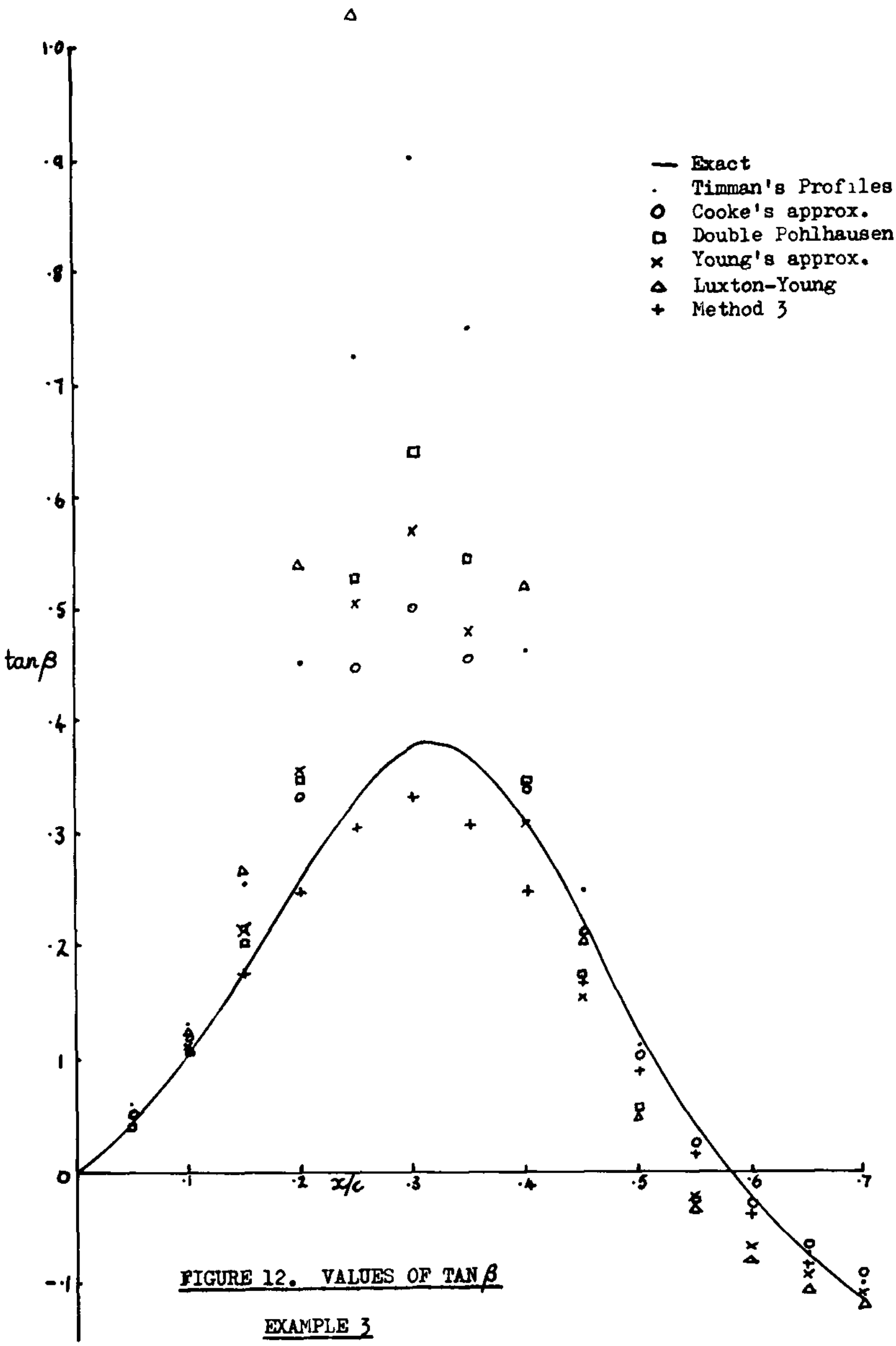

FIGURE 12. VALUES OF TAN β

EXAMPLE 3

DETACHABLE ABSTRACT CARDS

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September 1967

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