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Supersonic Laminar Boundary Layers on Cones

by

J. C. Cooke

Aerodynamics Dept., R.A.E., Farnborough

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SUPERSONIC LAMINAR BOUNDARY LAYERS ON CONES

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SUMMARY

The boundary layer flow over a cone inclined at a small angle to a supersonic stream, and over a type of caret (Maikapar, Nonweiler) surface, as generalized by Townend, is calculated by an implicit finite difference method. Prandtl number is arbitrary but viscosity must follow the Chapman-Rubesin law. Any (conical) distribution of wall temperature or heat flux can be covered; the effects of suction or blowing can only be included if the normal velocity along a ray varies inversely as distance from the apex.

Some sample calculations are made. The method begins to break down as separation is approached, but it is not difficult to find the separation line by extrapolation.

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CONTENTS

1 INTRODUCTION

We shall consider the flow over two types of conical surfaces. The first is an inclined cone in supersonic flow and the analysis is an extension of earlier work[1] which dealt with incompressible flow over such a cone when the flow is conical. In that case the real flow is approximately conical only near to the vertex but not everywhere, but the main intention then was to gain experience of a certain method of computation with a view to the present extension, and to see if one could arrive at the separation line satisfactorily. In supersonic flow we may expect the external flow to be really conical, and we find that the equations of motion can be reduced to forms rather similar to the earlier ones, except that the temperature now comes in and there is indeed an extra equation to determine the temperature. We also require to make the Chapman and Rubesin[2] assumption that viscosity is proportional to temperature. We find that the method can be used with arbitrary Prandtl number or wall temperature, and also if necessary with wall suction, provided that the prescribed temperature is conical (constant along generators) and that the suction amount on a ray is proportional to the inverse square root of distance from the apex.

The second application is to the Townend surface studied in Refs.3 and 4. In this case the apex of the cone is no longer the point furthest upstream, but is at the side (at the point O, see Fig.1) and we are considering the flow on the underside of this body. We shall give more details of this surface in Section 3.2.

No particular difficulty was found in extending the method of Ref.1 to these cases, and the results of some sample calculations will be given.

2 THE EQUATIONS OF MOTION

We may write the boundary layer equations for flow over a cone in the form[5]

$$\rho\left(U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \theta} + W\frac{\partial U}{\partial \zeta} - \frac{V^2}{r}\right) = -\frac{\partial p}{\partial r} + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial U}{\partial \zeta}\right) \qquad (1)$$

$$\rho\left(U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \theta} + W\frac{\partial V}{\partial \zeta} + \frac{UV}{r}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \frac{\partial}{\partial \zeta}\left(\mu\frac{\partial V}{\partial \zeta}\right) \qquad (2)$$

$$\rho\left(U\frac{\partial I}{\partial r} + \frac{V}{r}\frac{\partial I}{\partial \theta} + W\frac{\partial I}{\partial \zeta}\right) = \mu\left\{\left(\frac{\partial U}{\partial \zeta}\right)^2 + \left(\frac{\partial V}{\partial \zeta}\right)^2\right\} + U\frac{\partial p}{\partial r} + \frac{V}{r}\frac{\partial p}{\partial \theta} + \frac{1}{Pr}\frac{\partial}{\partial \zeta}\left(\mu\frac{\partial I}{\partial \zeta}\right) \qquad \ldots(3)$$

$$\frac{\partial}{\partial r}(\rho r\, U) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho r\, V) + \frac{\partial}{\partial \zeta}(\rho r\, W) = 0 \quad . \tag{4}$$

In these equations r denotes distance from the apex, θ the angle between any generator and a fixed generator (after developing the cone into a plane), ζ is distance normal to the surface, and U, V, W are velocity components in the r, θ and ζ directions. I is the enthalpy, ρ the density, μ the coefficient of viscosity and Pr the Prandtl number. According to the usual boundary layer approximation p is constant across the boundary layer. In conical flow we have $\partial p/\partial r = 0$ and we now make the further transformations

$$Z = \left(\frac{1}{C\mu_o \rho_o}\right)^{\frac{1}{2}} \left(\frac{p}{p_o}\right)^{-\frac{1}{2}} \frac{1}{r^{\frac{1}{2}}} \int_0^{\zeta} \rho\, d\zeta \tag{5}$$

$$U = U_e u, \quad V = V_e v, \quad z = U_e^{\frac{1}{2}} Z \tag{6}$$

$$Q_Z = -\frac{3}{2}U - V_\theta - V\frac{p_\theta}{2p} \tag{7}$$

$$Q = U_e^{\frac{1}{2}}\left(w - \frac{1}{2}K'^2\, vz\right), \quad I = I_e T \tag{8}$$

where

$$K' = \frac{V_e}{U_e} = \frac{1}{U_e}\frac{\partial U_e}{\partial \theta}, \quad M' = \frac{1}{V_e}\frac{\partial V_e}{\partial \theta}, \quad N' = \frac{1}{I_e}\frac{\partial I_e}{\partial \theta} \tag{9}$$

and C is given by the relation

$$\frac{\mu}{\mu_o} = C\frac{T}{T_o} \; ; \tag{10}$$

the subscript e denotes values in the external flow, whilst the subscript o denotes values at some reference station. The details are given in Appendix A.

The equations of motion become

$$u_{zz} - w\, u_z - K'^2\, u\, v - K'\, v\, u_\theta + K'^2\, v^2 = 0 \tag{11}$$

$$v_{zz} - w\, v_z - v\,(M'\, v + K'\, v_\theta + u) = -(1 + M')\, T \tag{12}$$

$$T_{zz} - w\,T_z - K'\,v\,T_\theta \;=\; -\frac{a_s^2}{I_e}\left\{\left(\frac{U_e\,u_z}{a_s}\right)^2 + \left(\frac{V_e\,v_z}{a_s}\right)^2\right\} \tag{13}$$

$$w_z - \frac{1}{2}K'^2\,v + \frac{3}{2}\,u + M'\,v + K'\,v_\theta \;=\; -\frac{\gamma}{2(\gamma-1)}\,N'\,K'\,v \tag{14}$$

where a_s denotes the "critical velocity" that is, the velocity at the place where the Mach number is unity.

We see that these equations are closely similar to those of Ref.1, with an extra equation and other small changes due to the effect of compressibility.

3 THE EXTERNAL FLOW

3.1 The inclined cone

There is here the usual difficulty in that the external flow passes through a shock and there is an "entropy layer". However one could argue that the invisoid flow nearest to the cone surface (which originates at the apex) must be isentropic. The condition for this in conical flow reduces to

$$V_e = \frac{\partial U_e}{\partial \theta}$$

and so in Appendix A we put $K' = L'$. We must indeed use this relation if the boundary conditions at infinity, namely $u = v = 1$, $u_{zz} = u_z = v_z = u_\theta = 0$ are to hold in equation (11). If the external flow is rotational then it is not true that $u_z = 0$ at the edge of the boundary layer and then $K' \neq L'$. We have avoided the difficulty by asserting that the flow at the edge of the boundary layer is irrotational so that we have not followed the tables[6] in determining V_e. However, taking it as accepted that the tables give the pressure correctly (whatever one's view about the entropy near to the surface), we have used them to find U_e (which will give the pressure correctly to the first order in the incidence α) and we have then found V_e simply by putting it equal to $\partial U_e/\partial\theta$.

According to the tables[6] (taking $\theta = 0$ along the windward generator)

$$\frac{U_e}{a_s} \;=\; M_1^* - \alpha\,M_2^*\cos\left(\frac{\theta}{\sin\theta_o}\right) \tag{15}$$

where θ_o is the semi-angle of the cone, α the incidence, M_1^* and M_2^* are given in the tables.

By differentiation we have

$$\frac{V_e}{a_s} = \frac{\alpha M_2^*}{\sin\theta_c} \sin\left(\frac{\theta}{\sin\theta_o}\right) \tag{16}$$

whilst the tables give

$$\frac{V_e}{a_s} = \alpha\, M_{w,2}^* \sin\left(\frac{\theta}{\sin\theta_o}\right) . \tag{17}$$

If the semi-angle of the cone is small the difference between the two values of V_e will not be very great. Thus for $M_\infty = 3$, $\theta_o = 7.5^\circ$ the value of $M_2^*/\sin\theta_o$ is 3.040 whilst $M_{w,2}^*$ from the tables is 3.042. The corresponding values for $M_\infty = 6$ are 2.863 and 2.625. However, each of these values is to be multiplied by α, so the difference between the values is still small.

The factor on the right hand side of (13) is given by

$$\frac{a_s^2}{I_e} = \frac{2a_s^2}{c^2 - q_e^2} = \frac{2}{(c/a_s)^2 - (q_e/a_s)^2} = \frac{2}{\frac{\gamma+1}{\gamma-1} - M^{*2}} = \frac{2}{6 - M^{*2}} \tag{18}$$

for $\gamma = 1.4$, where c is the velocity of efflux into a vacuum and M^* is the Mach number referred to a_s, q_e being the resultant velocity in the external flow.

The computation is done in two parts. First we must determine details of the flow on the attachment line (the most windward generator). Here we have $K' = 0$ and so all derivatives with respect to θ disappear from the equations which are solved to find u, v and T on the attachment line. These serve as starting values for the subsequent calculation in steps of amount $\delta\theta$ from each generator of the cone to the next.

3.2 The Townend surface

The shape of this surface is shown in Fig.1, and it is described in Refs.3 and 4. The flow first passes through a plane shock and the surface is at first a "caret" or Nonweiler[7] or Maikapar[8] surface, followed by an isentropic compression which is reversed Prandtl-Meyer flow in vertical planes parallel to the flow at infinity. If the Mach number after the shock is M_1, the external flow to begin with will be uniform and the development will be identical to that over a flat plate. We shall suppose that there is no heat transfer but shall not assume unit Prandtl number. When the fluid reaches the Townend surface proper the Mach number will still be M_1. After this the flow is compressed and the Mach number will fall, but it will be constant along the generators of the cone. If its value along any generator is M, then following Ref.4 we write

$$\alpha'^2 = M^2 d^2 + K^{2\delta}, \quad \beta'^2 = M^2 - 1, \quad \Delta^2 = d^2 + K^{2\delta} \tag{19}$$

where

$$K = \left(\frac{a_s}{a_e}\right)^2 = \frac{M^2 + 2\varepsilon}{2\delta}, \quad \delta = \frac{\gamma+1}{2(\gamma-1)}, \quad \varepsilon = \frac{1}{\gamma-1}, \tag{20}$$

and d is a fixed number depending on the geometry of the surface. We then have

$$d\theta = -\frac{M K^{\delta-1} \alpha'}{\Delta^2 \beta'} dM \tag{21}$$

$$\frac{U_e}{a_s} = -\frac{K^{\varepsilon} \beta}{\Delta}, \quad \frac{V_e}{a_s} = \frac{\alpha'}{K^{\frac{1}{2}} \Delta} \; ; \tag{22}$$

we also have in equation (13)

$$\frac{a_s^2}{I_e} = (\gamma-1) \left(\frac{a_s}{a_e}\right)^2 = (\gamma-1) K . \tag{23}$$

Hence

$$K' = \frac{V_e}{U_e} = -\frac{\alpha'}{K^{\delta} \beta'}, \quad K' \frac{\partial}{\partial\theta} = \frac{\Delta^2}{M K^{2\varepsilon}} \frac{\partial}{\partial M}$$

$$M' = \frac{1}{U_e} \frac{\partial V_e}{\partial\theta} = \frac{V_e}{U_e} \frac{\partial}{\partial\theta} (\log V_e) = \frac{\Delta^2 d^2}{K^{\varepsilon} \alpha'^2} - \frac{\Delta^2}{2\delta K^{2\delta}} + \frac{\Delta^2}{\alpha'^2} - 1$$

$$N' = \frac{\partial}{\partial\theta} (\log I_e) = -\frac{\partial}{\partial\theta} (\log K) = \frac{\beta' \Delta^2}{\delta\alpha' K^{\delta}}$$

so that

$$N' K' = -\frac{\Delta^2}{\delta K^{2\delta}} .$$

The solution here falls into two parts. Firstly there is parallel flow over the initial Nonweiler surface and the flow will be of Blasius type. (We ignore the effect of the corner itself, which will only affect the flow very near to the corner.) The fluid then arrives at the Townend surface and the computation proper starts. It was here found more convenient to replace the independent variable θ by M, which can be done without difficulty since M has the same value at all points on a generator. Further details of the initial flow are given in Appendix B.

4 THE SOLUTION OF THE EQUATIONS

The general method of solution has been described in Ref.1 and will not be given in any detail. We proceed step by step in the θ direction, that is from one generator to the next, using the Crank-Nicholson method, which is an implicit method requiring the inversion of a tridiagonal matrix for each unknown u, v and T at each step in θ. Inversion of such a matrix is quite simple. It is also necessary to iterate at each step.

If $u_{m+1,n}$ denotes the values of u at the point $(m+1)\,\delta\theta,\ n\delta z$, it is found after suitable linearization of the equations that the u's are given by a set of linear equations

$$a_n\, u_{m+1,n+1} + b_n\, u_{m+1,n} + c_n\, u_{m+1,n-1} = d_n \qquad (1 \leqslant n \leqslant N) \qquad (24)$$

where a_n, b_n, c_n and d_n are dependent on values of u, v and T at the previous station $m\delta\theta$ and on values at station $(m+1)\,\delta\theta$ obtained from the previous iteration. The boundary conditions are $u_{m+1,0} = 0$, $u_{m+1,N+1} = 1$ where N is taken large enough to reach the outer boundary to a sufficient approximation. There are similar equations for v and T with different coefficients. We shall discuss later the boundary conditions for T. w is found from the finite difference form of equation (14) using where necessary values of u, v and T obtained from the previous iteration. w is taken to be zero at the wall, or to have a prescribed value if there is suction or blowing.

Iterations over all four equations in succession are required until there is negligible change from one iteration to the next.

The outer boundary condition for T is $T_{m+1,N+1} = 1$ but that at the wall must be further considered. We may either fix the wall temperature by putting $T = T_w$ (say) giving T_w a known value at every station, or we may assume a prescribed heat transfer. In the former case we simply write

$$T_{m+1,0} = T_w \ .$$

In the case of zero heat transfer for which $\partial T/\partial Z = 0$ at the wall, more than one method of solution was tried. Finally it was found that it was best to go one step "into" the wall and make use of $T_{m+1,-1}$. For zero wall derivative we have in finite central difference terms

$$T_{m+1,-1} = T_{m+1,1} \quad .$$

Dropping the subscript m + 1 for clarity we write the T equation corresponding to n = 0 in (24) as

$$a_o T_1 + b_o T_o + c_o T_{-1} = d_o$$

that is, for zero heat transfer, for which $T_{-1} = T_1$,

$$(a_o + c_o) T_1 + b_o T_o = d_o \quad .$$

It is found that a knowledge of u_{-1} and v_{-1} will now also be required. Taking the u equation for n = 0 we have

$$a_o u_1 + b_o u_o + c_o u_{-1} = d_o \quad .$$

Now $u_o = 0$ and so

$$u_{-1} = \frac{d_o - a_o u_1}{c_o}$$

and u_{-1} can be found from the knowledge of u_1. v_{-1} is found in a similar way.

Certain details in the method of linearization have been found to be very important. Thus in equation (12), considered as an equation for v, we have a term v^2. If v_o is the value found at one iteration and v_1 is the value to be found in the next one, then the term is v_1^2 and the simplest way of linearizing it is to write it as $v_o v_1$, with error of order $v_1 - v_o$, and this is often satisfactory. It is better, however, to write

$$\begin{aligned} v_1^2 &= v_o v_1 + v_o (v_1 - v_o) + (v_1 - v_o)^2 \\ &\simeq 2 v_o v_1 - v_o^2 \end{aligned} \tag{25}$$

with error of order $(v_1 - v_o)^2$.

Again in equation (12) there is a term vv_θ. A straightforward way of linearizing this is to write it as $v_o\ v_{1\theta}$ but it is better to write

$$v_1\ v_{1\theta} = v_o\ v_{1\theta} + v_{o\theta}\ (v_1 - v_o) + (v_1 - v_o)\ (v_{1\theta} - v_{o\theta})$$

$$\approx v_o\ v_{1\theta} + v_1\ v_{o\theta} - v_o\ v_{o\theta} \quad , \qquad (26)$$

ignoring the last term. Provided the iterations converge the method of linearization makes no difference to the final answer, but it may make a considerable difference to the rate of convergence, or may even turn a non-convergent sequence into a convergent one. This method of linearization has been called "Newtonian quasilinearization".

A main interest is the position of separation. As this line was approached it was found that more iterations per step were required. When the number of these became excessive the interval $\delta\theta$ was halved and later halved again and so on as required. By this means the separation point could be found with an accuracy of 3 to 4 significant figures. This mode of approach took considerable machine time; however it was found that the same point could be arrived at (as in Ref.1) by stopping the computation earlier and plotting $(\tan\beta)^2$ against θ, where β is the angle between limiting streamline and generators of the cone. Near to separation the points so plotted fell quite closely on a straight line which could be continued on to the point where $\beta = 0$ thus determining the value of θ at separation. This mode of procedure was suggested by the work of Brown[10] who investigated the nature of the singularity at separation in the incompressible case. Further details are given in Ref.1. An illustration of the results of this approach is given in Fig.3.

5 RESULTS

5.1 Inclined cone

Numerical calculations were carried out for a cone of semi-angle θ_o of $7\frac{1}{2}^\circ$ with $M_\infty = 3$ and 6 and λ ($= \alpha/\sin\theta_o$) having values 1 and 2 and with either $T_w = 1$ (a highly cooled wall) or zero heat transfer (z.h.t.). One computation with suction was also carried out. It was found in all cases that separation could be estimated to three or four significant figures in θ. The values of θ at separation are shown in Table 1.

Table 1

M_∞	λ	Wall conditions	θ_{sep}
3	1	z.h.t.	0.326
3	1	cooled	0.336
6	1	z.h.t.	0.353
6	1	cooled	0.361
3	2	z.h.t.	0.269
6	2	z.h.t.	0.269
3	2	z.h.t. and suction	0.273

For purposes of easier comparison we will rewrite Table 1 in a group of sub-tables. Thus to estimate the effect of wall cooling we have

M_∞	λ	Wall conditions	θ_{sep}
3	1	z.h.t.	0.326
3	1	cooled	0.336
6	1	z.h.t.	0.353
6	1	cooled	0.361

and we see that cooling the wall delays separation but not as much as might have been expected.

To consider the effect of Mach number we have

M_∞	λ	Wall conditions	θ_{sep}
3	1	z.h.t.	0.326
6	1	z.h.t.	0.353
3	2	z.h.t.	0.269
6	2	z.h.t.	0.269

and so we find that doubling the Mach number delays separation in one case and does not change it in another case.

These results are contrary to those given by Stewartson[9] for a flat plate with a continual adverse pressure gradient with zero heat transfer when the separation point is earlier for the higher Mach numbers. However in the case we are now considering there is an initial pressure gradient in such a direction as to develop a cross-flow which is larger for the higher Mach numbers and this counteracts the tendency to early separation since a larger cross-flow has to be destroyed before separation takes place. The effect of increasing Mach number was shown by Stewartson[9] in two dimensions and Cooke[11] in three dimensions to amplify the effective pressure gradient by a factor of approximate amount $1 + \frac{1}{2}(\gamma-1)\,M^2$, but it does this in the earlier part of the flow as well as the later part and the effects compensate one another. We can see this by examining Fig.4 which shows the surface flow angles for $M_\infty = 3$ and $M_\infty = 6$ when $\lambda = 2$, with zero heat transfer.

There is a similar compensating feature where we calculate the effects of wall cooling. Fig.5 shows the surface flow angle for $M = 6$, $\lambda = 1$, with and without wall cooling.

The compensating tendency also shows itself when suction is applied. Only one such case was considered, in which w was given the value -0.4 at the wall, which implies that

$$(W)_w = -0.4\,\frac{1}{\rho w}\sqrt{C\,\mu_o\,\rho_o}\left(\frac{p}{p_o}\right)^{\frac{1}{2}} U_e^{\frac{1}{2}}\,\frac{1}{r^{\frac{1}{2}}} \quad . \tag{27}$$

where the subscript w denotes values at the wall. Fig.6 shows the surface flow angles for $M_\infty = 3$, $\lambda = 2$ with zero heat transfer with or without suction.

The corresponding table for separation is

M_∞	λ	Wall	θ_{sep}
3	2	z.h.t.	0.269
3	2	z.h.t. and suction	0.273

and once more there is no great change in θ_{sep}.

For the skin friction components τ_u and τ_v we have

$$\tau_u = \mu_w \left(\frac{\partial U}{\partial \zeta}\right)_w$$

$$= \frac{A}{r^{\frac{1}{2}}} \left[(1 + 0.2\ M_e^2)^{-7/4} \left(\frac{U_e}{a_s}\right)^{3/2} \left(\frac{\partial u}{\partial z}\right)_w \right] \tag{28}$$

where

$$A = (C\mu_o \rho_o)^{\frac{1}{2}}\ a_s^{3/2}\ (1 + 0.2\ M_o^2)^{7/4} \quad , \tag{29}$$

and

$$\tau_v = \frac{A}{r^{\frac{1}{2}}} \left[(1 + 0.2\ M_e^2)^{-7/4} \left(\frac{U_e}{a_s}\right)^{\frac{1}{2}} \left(\frac{V_e}{a_s}\right) \left(\frac{\partial u}{\partial z}\right)_w \right] . \tag{30}$$

For the displacement thickness component δ_u^* we have

$$\delta_u^* = \int_0^\infty \left(1 - \frac{\rho U}{\rho_e U_e}\right) d\zeta$$

$$= B\ r^{\frac{1}{2}} \left[(1 + 0.2\ M_e^2)^{3/4} \left(\frac{U_e}{a_s}\right)^{-\frac{1}{2}} \Delta u \right] \tag{31}$$

where

$$B = (1 + 0.2\ M_o^2)^{3/4} \left(\frac{C\ \nu_o}{a_s}\right)^{\frac{1}{2}} \tag{32}$$

$$\Delta u = \int_0^\infty (1-u)\ dz + \int_0^\infty (T-1)\ dz \quad . \tag{33}$$

δ_v^* is defined in a similar way, with v instead of u.

For the heat transfer $\bar{Q}$ we have

$$\bar{Q} = \frac{k_w}{c_p} \left(\frac{\partial I}{\partial \zeta}\right)_w$$

$$= D r^{-\frac{1}{2}} \left[(1 + 0.2\ M_e^2)^{-11/4} \left(\frac{U_e}{a_s}\right)^{\frac{1}{2}} \left(\frac{\partial T}{\partial z}\right)_w \right] \tag{34}$$

where

$$D = \left(\frac{I_o k_o}{c_p}\right)(1 + 0.2\, M_o^2)^{11/4}\left(\frac{C\, a_s}{\nu_o}\right)^{\frac{1}{2}} \quad , \tag{35}$$

and k is the heat conductivity.

The results for $M_\infty = 6$, $\lambda = 1$, have been plotted in detail in Figs.7 and 8. In Figs.7(a) and 8(a) are shown the values of $(u_z)_w$, $(v_z)_w$, $(T_z)_w$, δ_u, δ_v and δ_T where

$$\delta_u = \int_0^\infty (1-u)\, dz \; , \quad \delta_v = \int_0^\infty (1-v)\, dz \; , \quad \delta_T = \int_0^\infty (T-1)\, dz \; . \quad \ldots (36)$$

In Figs.7(b) and 8(b) are shown the skin friction components, heat transfer and displacement thicknesses, as given by equations (28) to (35). Owing to the transformation used Figs.7(a) and 8(a) give somewhat deceptive results. For instance, in Fig.8(a) examining $(T_z)_w$ suggests that the heat transfer increases at first, whilst Fig.8(b) shows that in fact it is a maximum at the start. Again Fig.7(a) suggests that the v component of displacement thickness is negative. This is due to the large overshoot, but actually, when properly defined, it is positive everywhere as Fig.7(b) shows.

A few sample profiles are shown in Figs.9, 10 and 11 which give profiles of u, v and T at $\theta = 0.2$ (approximately one quarter of the way round the cone) for $M_\infty = 6$, $\lambda = 1$ for zero heat transfer and cooled wall. The overshoot for v can be seen in Fig.10.

It can be illuminating to show cross-flow profiles, that is of velocity normal to the external streamlines, and we show these for $\lambda = 2$, $M_\infty = 3$ and $M_\infty = 6$ at $\theta = 0.2$ in Fig.12, and at points fairly near to separation in Fig.13, the latter illustrating profiles of "cross-over" type.

It was considered of some interest to calculate the recovery factor r in the case of zero heat transfer. As expected, r was near to $(Pr)^{\frac{1}{2}}$ in all cases and we show in Fig.14 a few values for $M_\infty = 3$, $\lambda = 1$, zero heat transfer.

It is helpful to show some limiting streamlines or skin friction lines. These are the curves which one sees in examining oil-flow patterns. We imagine the cone to be developed on to a plane and the result is shown in Fig.15 for the case $M_\infty = 6$, $\lambda = 2$ with zero heat transfer.

5.2 Townend surface

The results here were disappointing in that the adverse pressure gradient suddenly imposed along BO (Fig.1) on a boundary layer that is already growing between AO and BO leads almost immediately to separation.

In each case examined the geometry was such that the free stream had a Mach number of 6.8, which was reduced to 4.6174 at the shock. Various values of the angles ε and δ (see Fig.2) were tried and the reversed Prandtl Meyer compression led to separation at a Mach number only very slightly lower than the initial value. The results are given in Table 2.

Table 2

Separation on Townend surface

$M_\infty = 6.8, \qquad M_1 = 4.6174$

ε	δ	γ	M_{sep}
23°.95	40°.18	1.4	4.602
23°.95	40°.18	1.3	4.596
45°.00	62°.28	1.4	4.520
45°.00	62°.28	1.3	4.484
85°.00	87°.30	1.3	4.405

A typical plot of the surface flow angle is shown in Fig.16. It was found necessary to take steps in Mach number which were very small indeed. There is an initial rapid fall in surface flow angle, which flattens out a little and then again falls rapidly. The same technique of plotting $\tan^2\beta$ against Mach number served to determine the separation line. It was expected that a value of γ less than 1.4 might delay separation and for $\gamma = 1.3$ a slightly later position was indeed found.

One is forced to the conclusion that this type of surface is not suitable for obtaining useful compression unless the flow is turbulent or the boundary layer is bled away. In view of this no further analysis of results was carried out for this surface.

6 CONCLUSIONS

There appears to be no difficulty in extending to three dimensions the solution of compressible boundary layer equations by implicit finite difference methods, at least in the case where there are only two independent variables (θ and z here), even though there are in effect three dependent variables. Such problems are only quasi three-dimensional, and the addition of one more independent variable increases the work enormously and brings other complications in its wake.

Solutions for the inclined circular cone have been found and can be carried far enough to estimate the position of separation. This position for a given inclination is not changed very much either by cooling the wall or by applying suction, or even by changing the Mach number. This unexpected result is due to the fact that increasing (say) the Mach number leads to an effective increase in pressure gradient and this increases the cross-flow to begin with, so that a larger cross-flow has later to be destroyed, thus counteracting the expected tendency to earlier separation.

In the case of the Townend surface all of the cases tried gave very early separation.

Appendix A

TRANSFORMATION OF THE EQUATIONS OF MOTION

We take $\partial p/\partial r = 0$ and make the transformation (5). In addition we follow Moore[5] in writing

$$\rho r\, U = \psi_\zeta , \qquad \rho r\, V = \phi_\zeta , \qquad \rho r\, W = -\psi_r - \frac{1}{r}\,\phi_\theta \tag{37}$$

which satisfies (4), and then we put

$$\psi = \left(\frac{p}{p_o}\right)^{\frac{1}{2}} \sqrt{C\,\mu_o\,\rho_o}\; r^{3/2}\, f(Z,\theta)$$

$$\phi = \left(\frac{p}{p_o}\right)^{\frac{1}{2}} \sqrt{C\,\mu_o\,\rho_o}\; r^{3/2}\, g(Z,\theta)$$

so that

$$U = f_Z , \qquad V = g_Z .$$

We write

$$\frac{\mu}{\mu_o} = C\,\frac{I}{I_o}$$

and we find that r will drop out from the equations of motion and they reduce to

$$f_{ZZZ} + \left(\frac{3}{2}\,f + g_\theta + g\,\frac{p_\theta}{2p}\right) f_{ZZ} - g_Z\, f_{Z\theta} + g_Z^2 = 0$$

$$g_{ZZZ} + \left(\frac{3}{2}\,f + g_\theta + g\,\frac{p_\theta}{2p}\right) g_{ZZ} - g_Z\, g_{Z\theta} - f_Z\, g_Z = \frac{1}{\rho}\, p_\theta$$

$$\frac{1}{Pr}\, I_{ZZ} + \left(\frac{3}{2}\,f + g_\theta + g\,\frac{p_\theta}{2p}\right) I_Z - g_Z\, I_\theta = -\frac{1}{\rho}\, g_Z\, p_\theta - f_{ZZ}^2 - g_{ZZ}^2 .$$

These equations were in effect first obtained by Moore[5]. The equations may also be written

$$U_{ZZ} - QU_Z - VU_\theta + V^2 = 0$$

$$V_{ZZ} - QV_Z - VV_\theta - UV = \frac{1}{\rho} p_\theta$$

$$\frac{1}{Pr} I_{ZZ} - QI_Z - VI_\theta = -\frac{V}{\rho} p_\theta - \left\{(U_Z)^2 + (V_Z)^2\right\}$$

$$Q_Z + \frac{3}{2} U + V_\theta + V \frac{p_\theta}{2p} = 0 \quad .$$

Next we write

$$U = U_e u , \quad V = V_e v , \quad Z = U_e^{\frac{1}{2}} z , \quad Q = U_e^{\frac{1}{2}}\left(w - \frac{1}{2} K' L' vz\right)$$

$$I = I_e T , \quad L' = \frac{1}{U_e}\frac{\partial U_e}{\partial \theta} , \quad K' = \frac{V_e}{U_e}$$

and we have

$$u_{zz} - w u_z - K' L' uv - K' vu_\theta + K'^2 v^2 = 0 \tag{38}$$

$$v_{zz} - w u_z - v (M' v + K' v_\theta + u) = \frac{1}{\rho U_e V_e} p_\theta \tag{39}$$

$$\frac{1}{Pr} T_{zz} - w T_z - K' N' v T - K' v T_\theta = -\frac{K' v}{\rho I_e} p_\theta - \frac{1}{I_e}\left\{(U_e u_z)^2 + (V_e v_z)^2\right\} \quad \ldots (40)$$

$$w_z - \frac{1}{2} K' L' v + \frac{3}{2} u + M' v + K' v_\theta = -\frac{K' v p_\theta}{2p} \tag{41}$$

where

$$M' = \frac{1}{U_e}\frac{\partial V_e}{\partial \theta} , \quad N' = \frac{1}{I_e}\frac{\partial I_e}{\partial \theta} \quad .$$

At infinity we have $u = v = T = 1$, and all derivatives of u, v and T with respect to z and θ are zero. From equation (38) we see that this implies that $K' = L'$ which is the condition for irrotational external flow. This we shall suppose to be the case; we have already discussed this in Section 3.

If we apply the boundary conditions at infinity in equations (39) and (40) we find

$$\frac{p_\theta}{\rho_e U_e V_e} = -(1 + M') , \qquad -\frac{K'}{\rho_e I_e} p_\theta = -K' N' .$$

These results can also be obtained directly from the external isentropic flow.

To find the right hand side of equation (41) we use the fact that in isentropic flow

$$\frac{p}{p_o} = \left(\frac{I_e}{I_o}\right)^{\frac{\gamma}{\gamma-1}}$$

and so

$$\frac{1}{p} p_\theta = \frac{\gamma}{\gamma-1} \frac{1}{I_e} \frac{\partial I_e}{\partial \theta} = \frac{\gamma}{\gamma-1} N' .$$

We also have from the equation of state and the fact that the pressure is constant through the boundary layer

$$\frac{\rho_e}{\rho} = \frac{I}{I_e} = T .$$

Hence the equations of motion beoome finally

$$u_{zz} - w\, u_z - K'^2\, uv - K'\, vu_\theta + K'^2\, v^2 = 0$$

$$v_{zz} - w\, v_z - v\, (M'\, v + K'\, v_\theta + u) = -(1 + M')\, T$$

$$\frac{1}{Pr} T_{zz} - w\, T_z - K'\, v\, T_\theta = -\frac{1}{I_e}\left\{(U_e\, u_z)^2 + (V_e\, v_z)^2\right\}$$

$$w_z - \frac{1}{2} K'^2\, v + \frac{3}{2} u + M'\, v + K'\, v_\theta = -\frac{\gamma}{2(\gamma-1)} N'\, K'\, v .$$

Appendix B

STARTING CONDITIONS ON THE TOWNEND SURFACE

The equations of motion we have given will apply everywhere on the surface but in the initial (Nonweiler) part they can be simplified, since we have parallel flow here. Nevertheless it may be useful to derive the simplified equations directly from the main equations.

We open out the surface on to a plane as in Fig.2. If U_1 is the external velocity after passing through the shock and M_1 is the Mach number we have

$$U_e = -U_1 \cos\theta , \quad V_e = U_1 \sin\theta . \tag{42}$$

Also

$$\frac{U_1}{a_s} = \frac{a_1 M_1}{a_s} = \frac{M_1}{K_1^{\frac{1}{2}}}$$

where K_1 is given by equation (20) with $M = M_1$. Hence

$$\frac{U_e}{a_s} = -\frac{M_1}{K_1^{\frac{1}{2}}} \cos\theta , \quad \frac{V_e}{a_s} = \frac{M_1}{K_1^{\frac{1}{2}}} \sin\theta .$$

Now the flow direction everywhere will be parallel to the line AB, and if the magnitude is q we have $U = -q\cos\theta$, $V = q\sin\theta$ and so from equations (6) we have $u = v$. Also we have

$$K' = -\tan\theta , \quad M' = -1 , \quad N' = 0$$

and the equations (11) and (12) reduce to one and the same equation. The resulting equations are (using (23))

$$u_{zz} - w\,u_z - \tan\theta\; u\,u_\theta = 0$$

$$\frac{1}{Pr} T_{zz} - w\,T_z - \tan\theta\; u\,T_\theta = -(\gamma-1) M_1^2 u_z^2$$

$$w_z - \frac{1}{2}(1 - \tan^2\theta)\,u + \tan\theta\; u_\theta = 0 .$$

If we write

$$\eta = \frac{z}{f}, \qquad w = f \tan\theta\, \eta\, u \frac{\partial}{\partial\theta}(\log f) + \frac{g}{f}$$

$$f = \left\{2 \sin\theta \cos\theta \,(\cot\varepsilon - \cot\theta)\right\}^{\frac{1}{2}}$$

the equations reduce to

$$u'' - g u' = 0$$

$$\frac{1}{Pr} T'' - g T' = -(\gamma-1) M_1^2 u'^2$$

$$u + g' = 0$$

where primes denote differentiation with respect to η. These are the standard Blasius equations for compressible flow over a flat plate. Also initially when $\theta \simeq \varepsilon$ we have $f = 0$ as in the standard Blasius transformation.

We then use any convenient method of solving these equations in terms of η. When we come to the later flow where z is to be used its value will be found from the relation $z = f_1 \eta$ where f_1 is the value of f when $\theta = \theta_1$, (the value of θ where the Townend surface proper begins), and so the two solutions can be joined.

In the case of the Townend surface there are difficulties in the change-over from the initial flat plate computation to the subsequent one since the shape of the surface is such that there is a suddenly imposed pressure gradient at the change-over. Consequently the initial flow used does not satisfy the new differential equations. The effect is to cause oscillations in the subsequent steps. The method of dealing with this difficulty is to take a few very small steps, with $\delta M = -0.0005$. The oscillations are then very much smaller and they damp out after a few steps (4 steps were found to be adequate). Consequently, after proceeding a very small distance downstream we have arrived at a solution which does satisfy the differential equations and then we can proceed by larger steps and there is no further difficulty. An alternative approach might be to smooth the coefficients of the differential equations over a short distance so as to ensure that there are no discontinuities and then proceed by a few very small steps over the region where the changes are large. However the first method described above proved adequate.

SYMBOLS

A	defined by (29)
a_n, b_n, c_n, d_n	coefficients in (24)
a_s	critical velocity of sound
B	defined by (32)
C	Chapman-Rubesin constant
D	defined by (35)
I	enthalpy
k	heat conductivity
K	defined by (20)
K', L', M', N'	defined by (9)
M	Mach number
M^*	q_e/a_s
M_1^*, M_2^*	values from tables used in (15) and (16)
m	defined by $m\delta\theta = \theta$
n	defined by $n\delta z = z$
N	number less one of intervals δz
p	pressure
Pr	Prandtl number
q	resultant velocity
Q	defined by (8)
$\bar{Q}$	rate of heat transfer
r	distance from apex of cone
T	I/I_e
U, V, W	velocity components
V_+	cross-flow velocity
u, v, w	defined by (6) and (8)
z	defined by (6)
Z	defined by (5)
α	incidence of cone
α'	defined by (19)
β	surface flow angle
β'	$(M^2 - 1)^{\frac{1}{2}}$
γ	ratio of specific heats
δ_u, δ_v	defined by (36)
δ_u^*, δ_v^*	components of displacement thickness

SYMBOLS (Contd)

Δ	defined by (19)
Δu	defined by (28)
δ	see Figs.1 and 2
ε	see Figs.1 and 2
θ	angle between generators of the cone when developed into a plane
θ_o	semi-angle of cone
ζ	distance measured normal to the surface
λ	$\alpha/\sin\theta_o$
μ	viscosity
ρ	density
τ_u, τ_v	components of skin friction
ϕ, ψ	defined by (37)
Subscripts	
e	denotes values in the main stream
w	denotes values at the wall
o	denotes values at some reference condition
∞	denotes values at infinity upstream
sep	denotes values at separation
1	denotes values just after the Nonweiler surface shock

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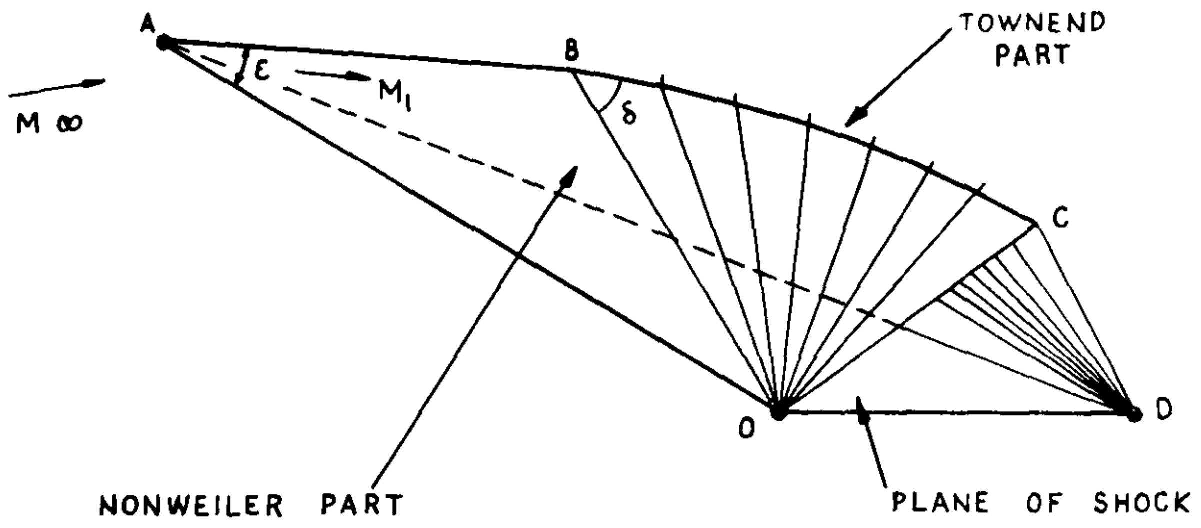


FIG. 1 THE TOWNEND SURFACE

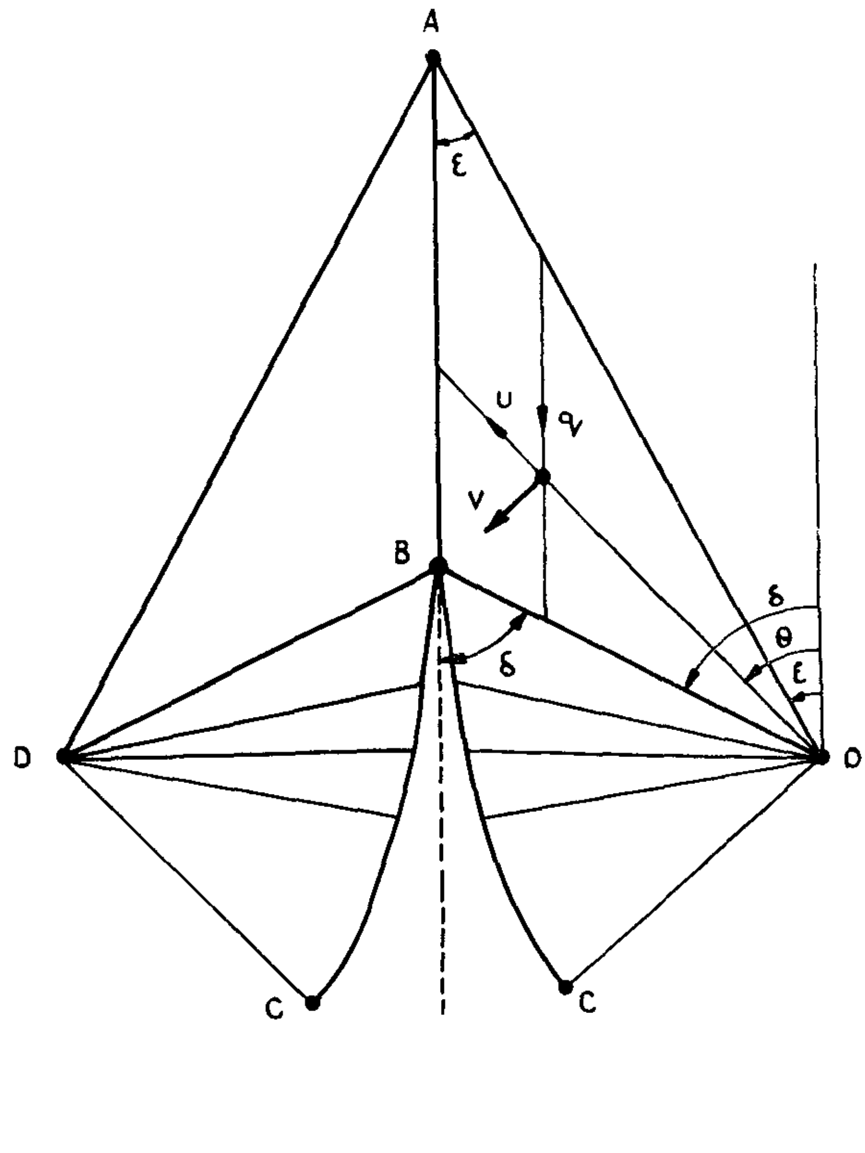


FIG. 2 THE SURFACE DEVELOPED ON A PLANE

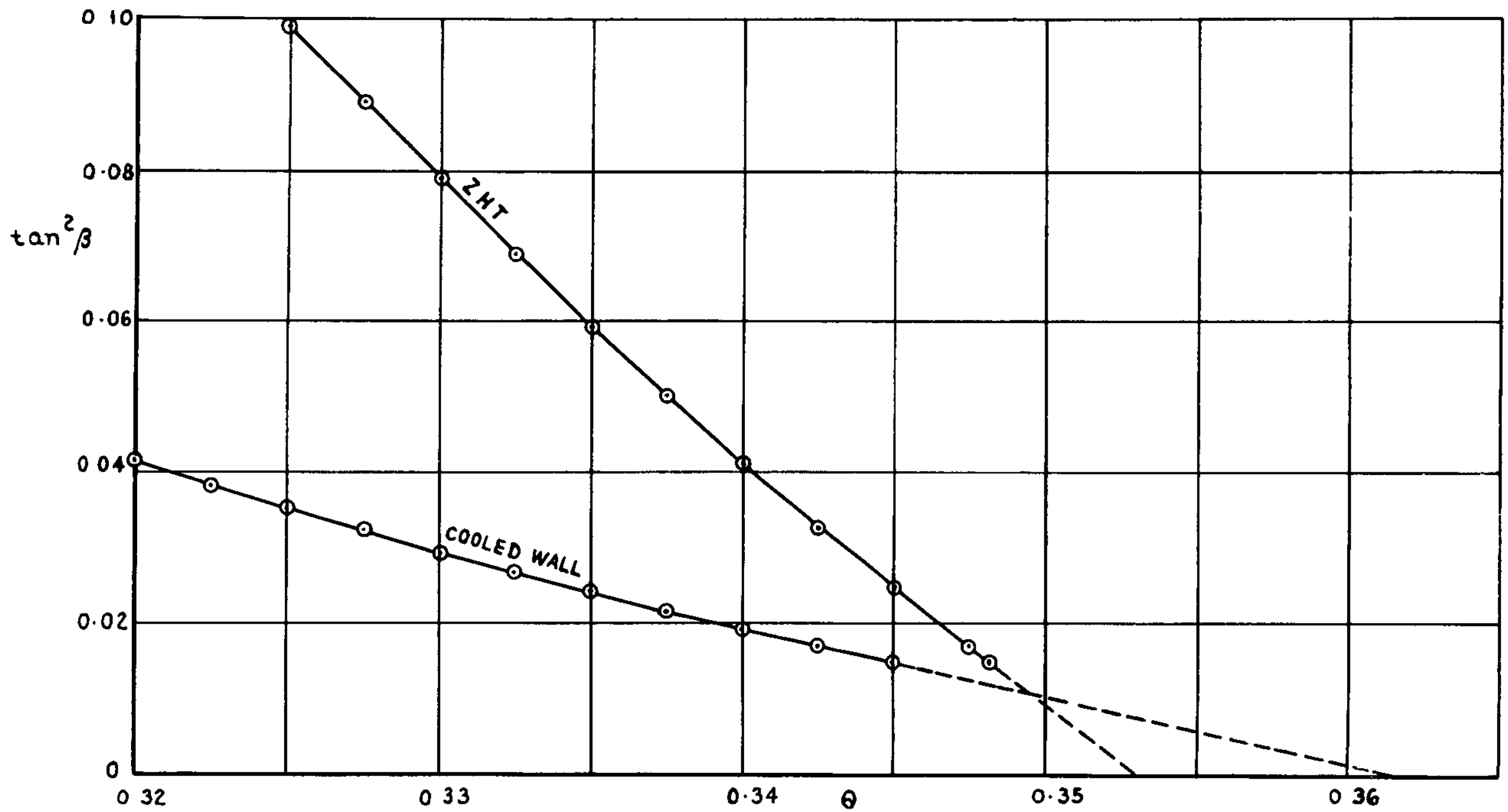


FIG. 3 DETERMINATION OF THE SEPARATION POINT $M_{\infty}=6$, $\lambda=1$

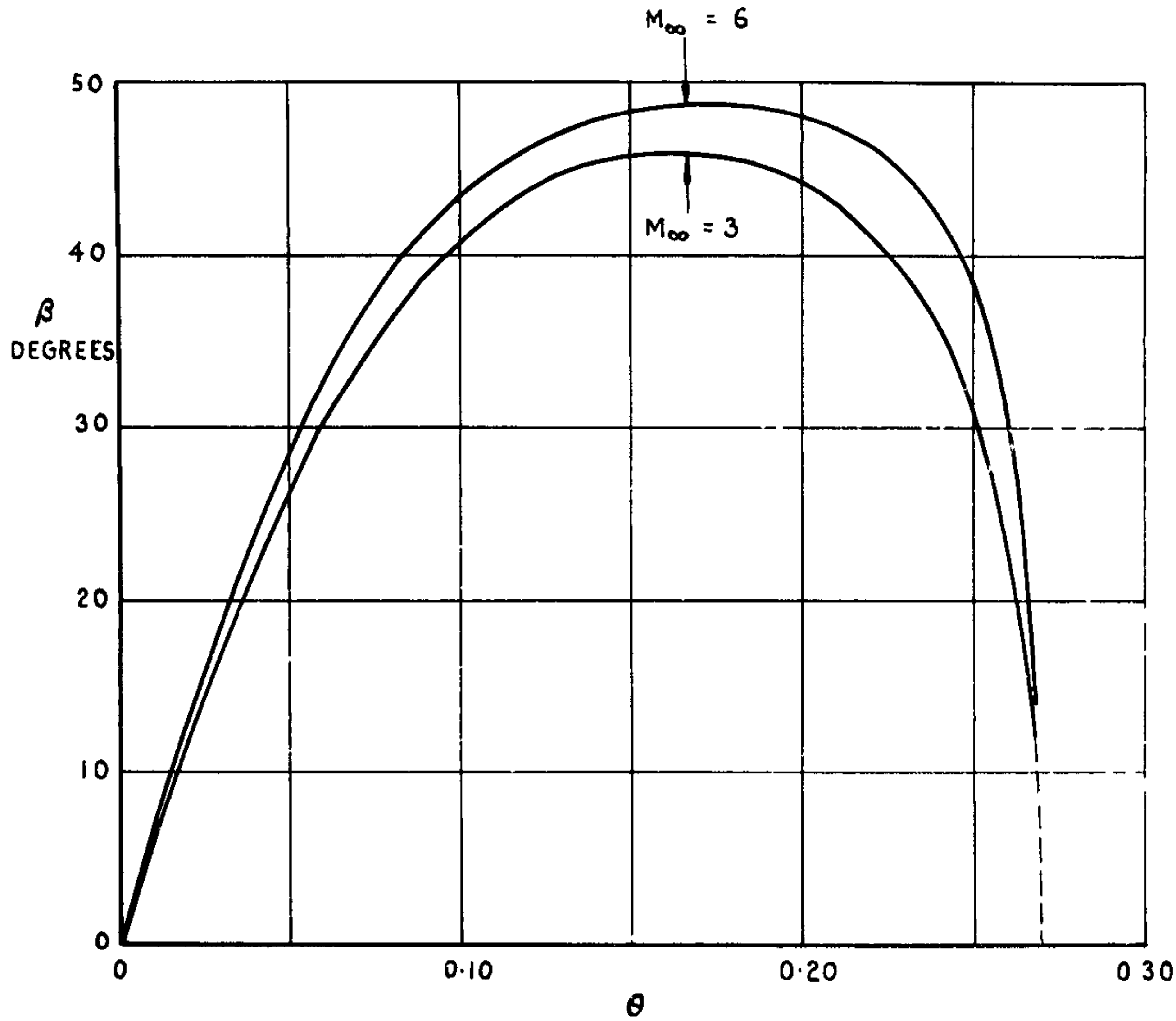


FIG. 4 SURFACE FLOW ANGLE WITH λ = 2, ZERO HEAT TRANSFER

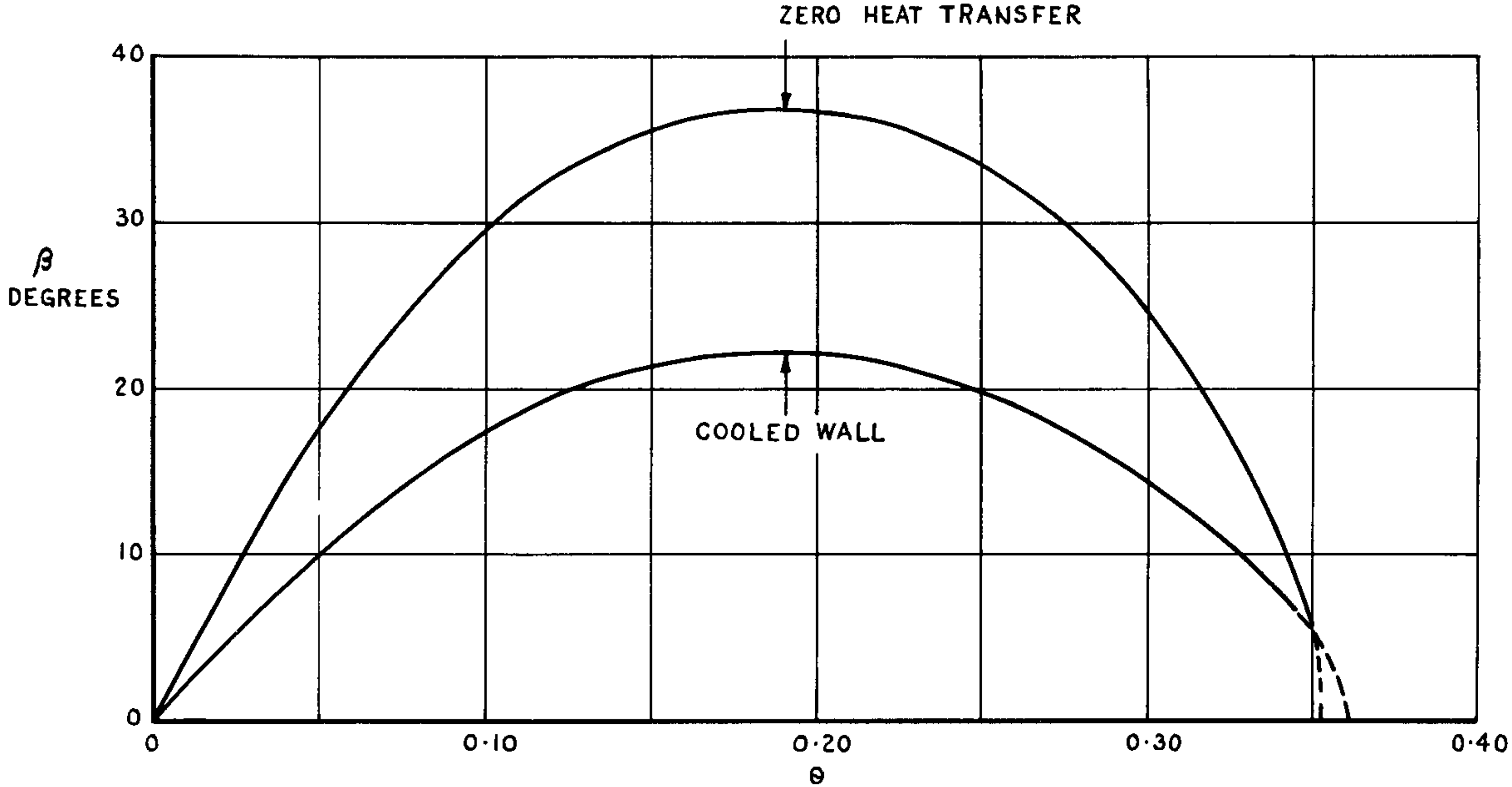


FIG. 5 SURFACE FLOW ANGLE WITH $M_\infty = 6$, $\lambda = 1 \cdot 0$

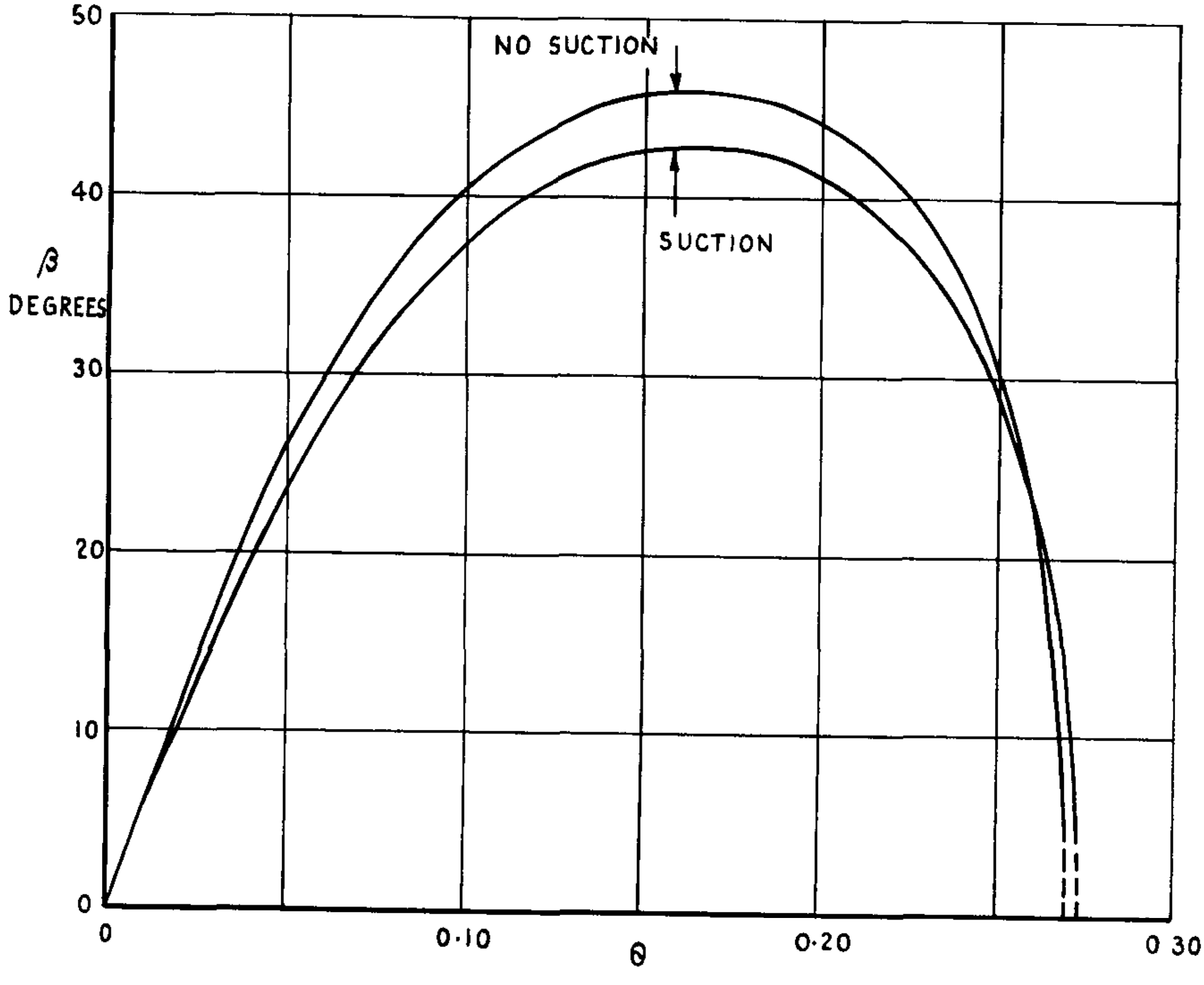
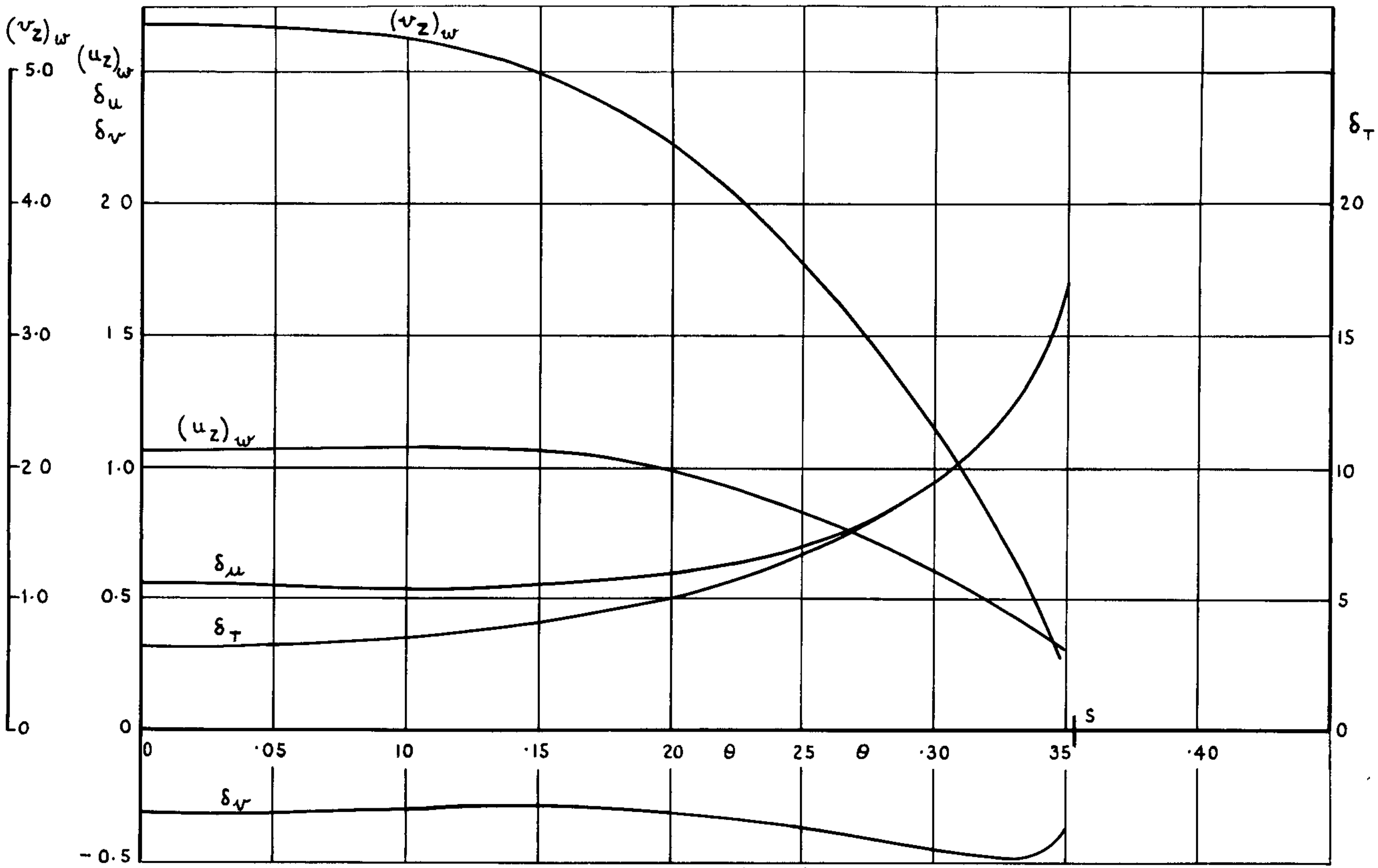


FIG. 6 SURFACE FLOW ANGLE WITH $M_{\infty} = 3$, $\lambda = 2$, ZERO HEAT TRANSFER



COMPRESSIBLE CONE FLOW (Z.H.T.) ATLAS RESULTS

FIG.7 a VALUES OF $(u_z)_w$, $(v_z)_w$, δ_u, δ_v FOR $M_\infty = 6$, $\lambda = 1$, ZERO HEAT TRANSFER

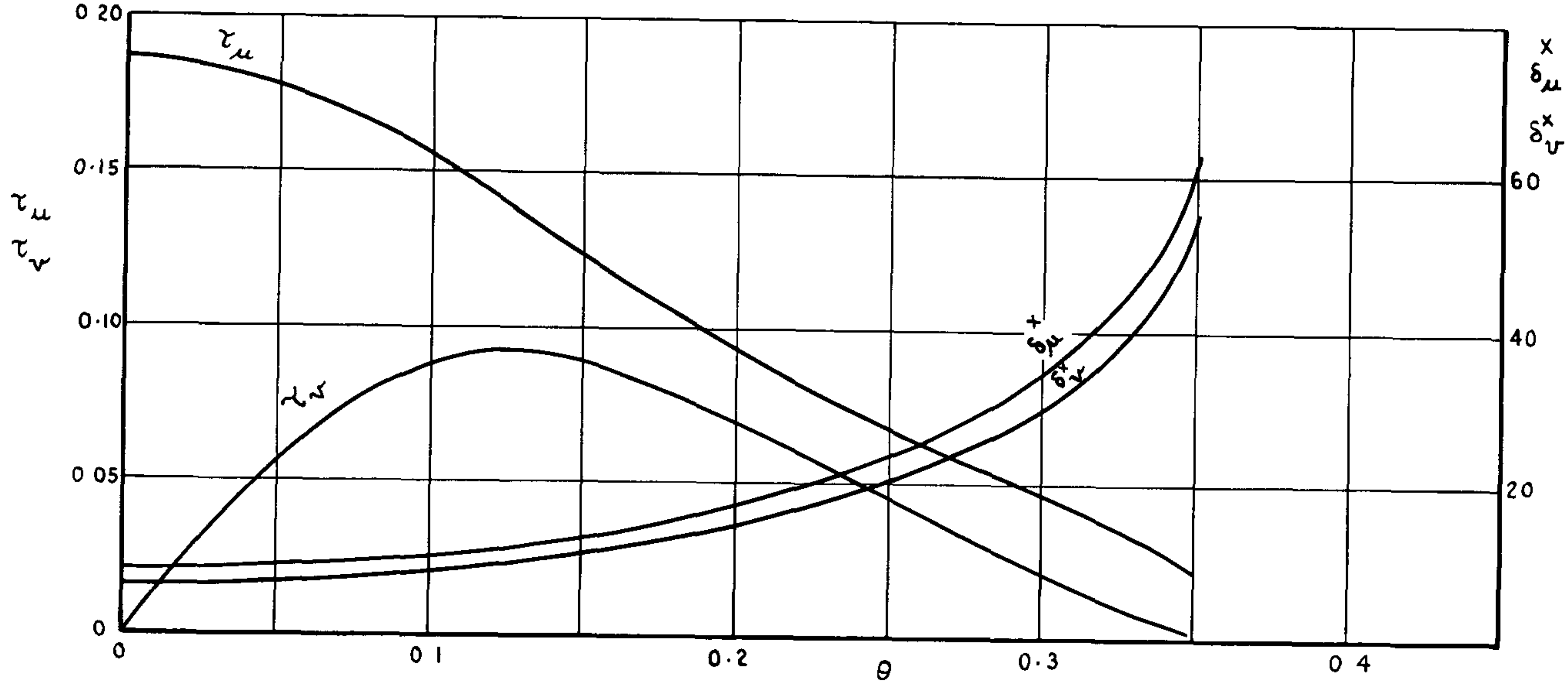


FIG. 7 b VALUES OF τ_u , τ_v , δ^x_u , δ^x_v FOR $M_\infty = 6$, $\lambda = 1$, ZERO HEAT TRANSFER

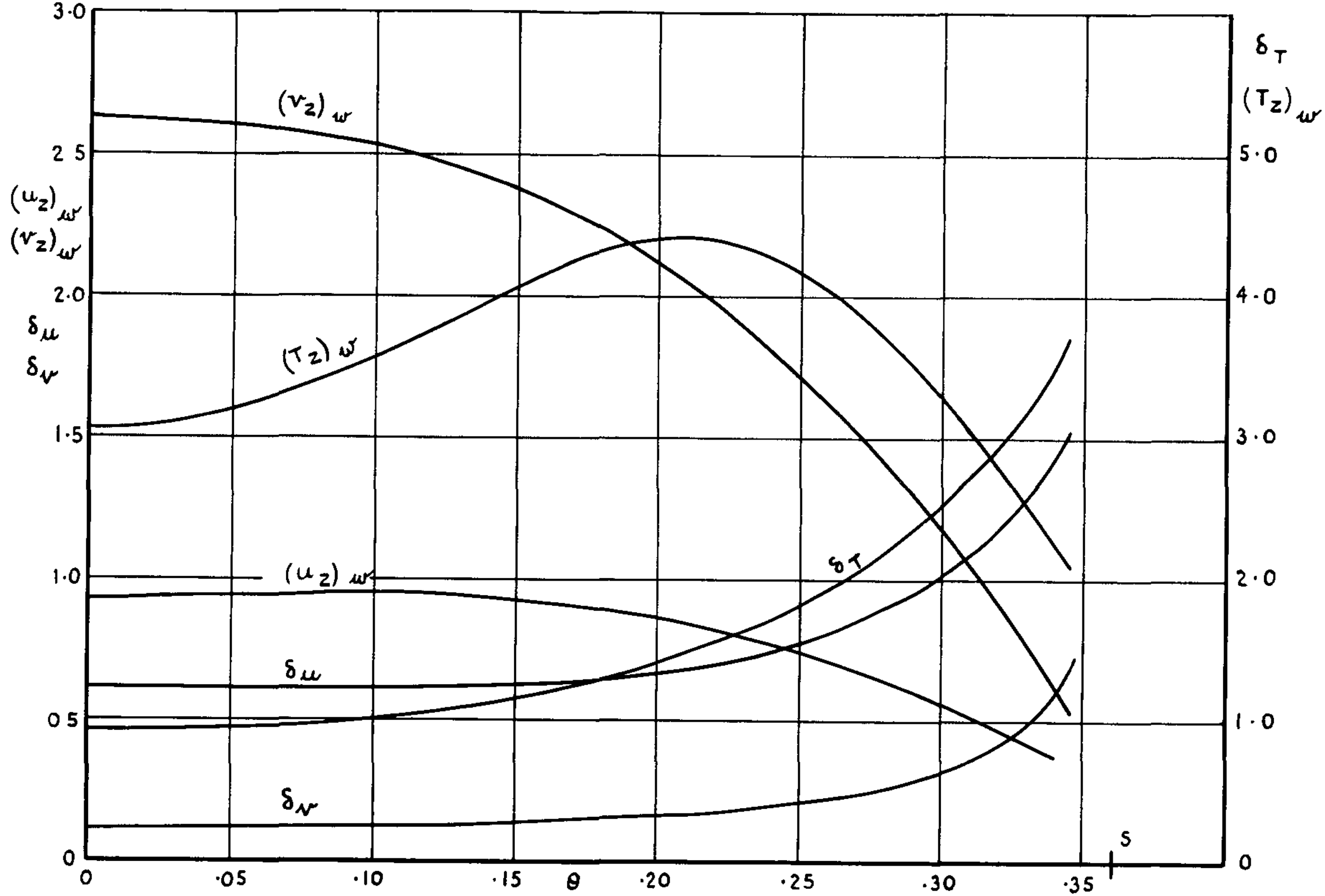


FIG. 8 a VALUES OF $(u_z)_w, (v_z)_w, (T_z)_w, \delta_u$ AND δ_v FOR $M_\infty = 6, \lambda = 1$, COOLED WALL

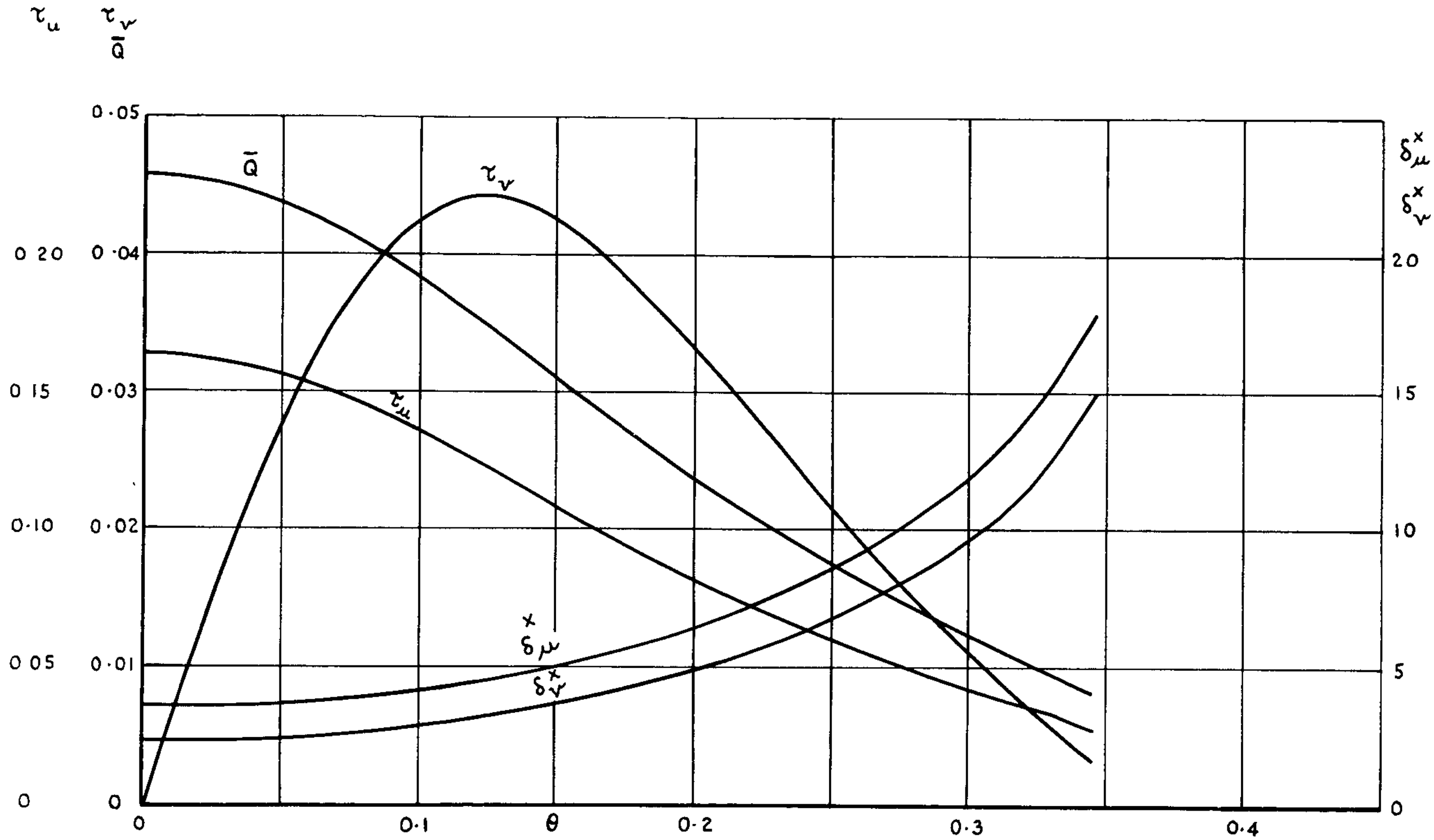


FIG. 8 b VALUES OF τ_u, τ_v, $\bar{Q}$, δ_u^x, δ_v^x FOR $M_\infty=6$, $\lambda=1$, COOLED WALL

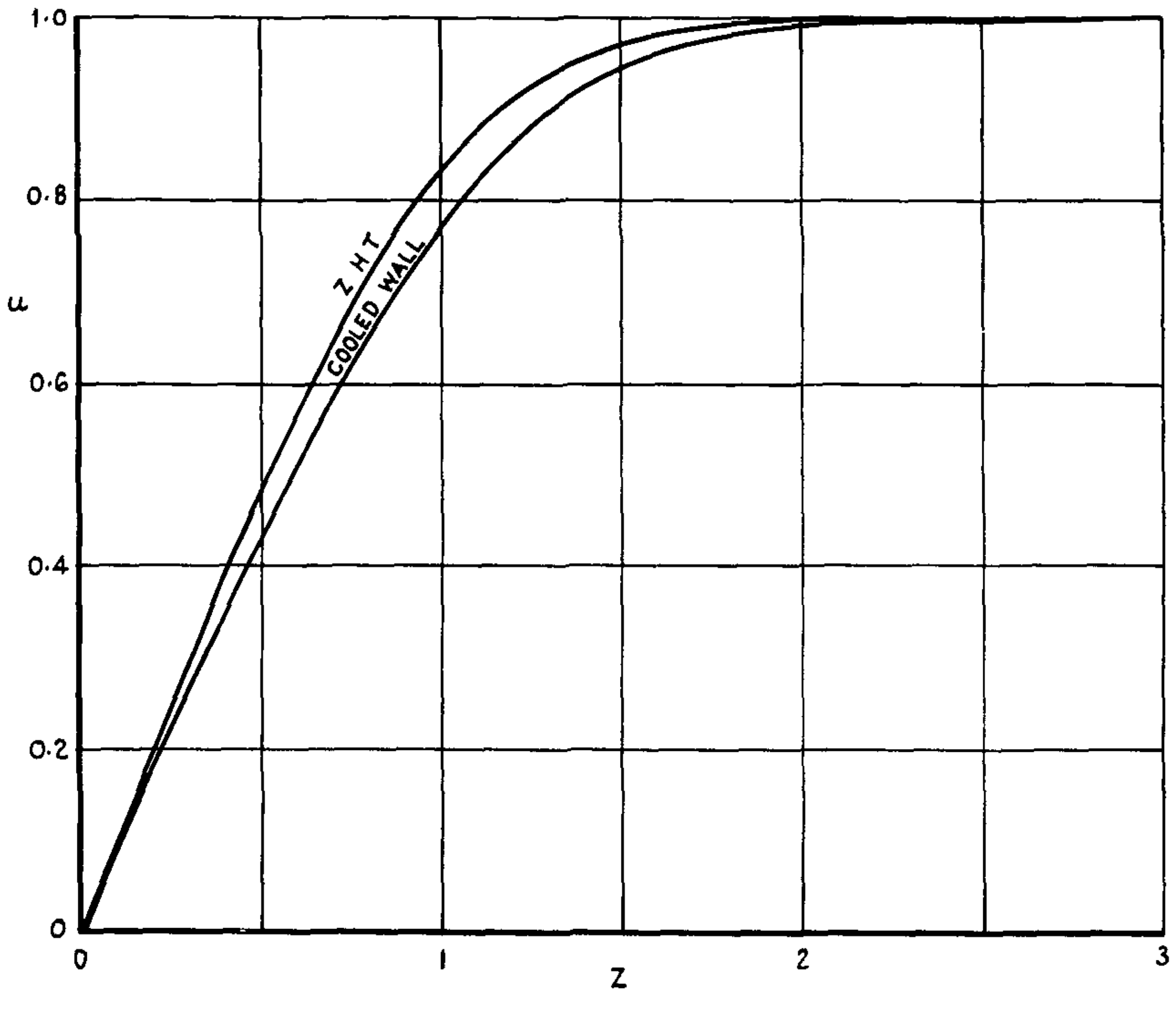


FIG 9 u PROFILES, $M_{\infty}=6$, $\lambda=1$ AT $\theta=0{\cdot}2$

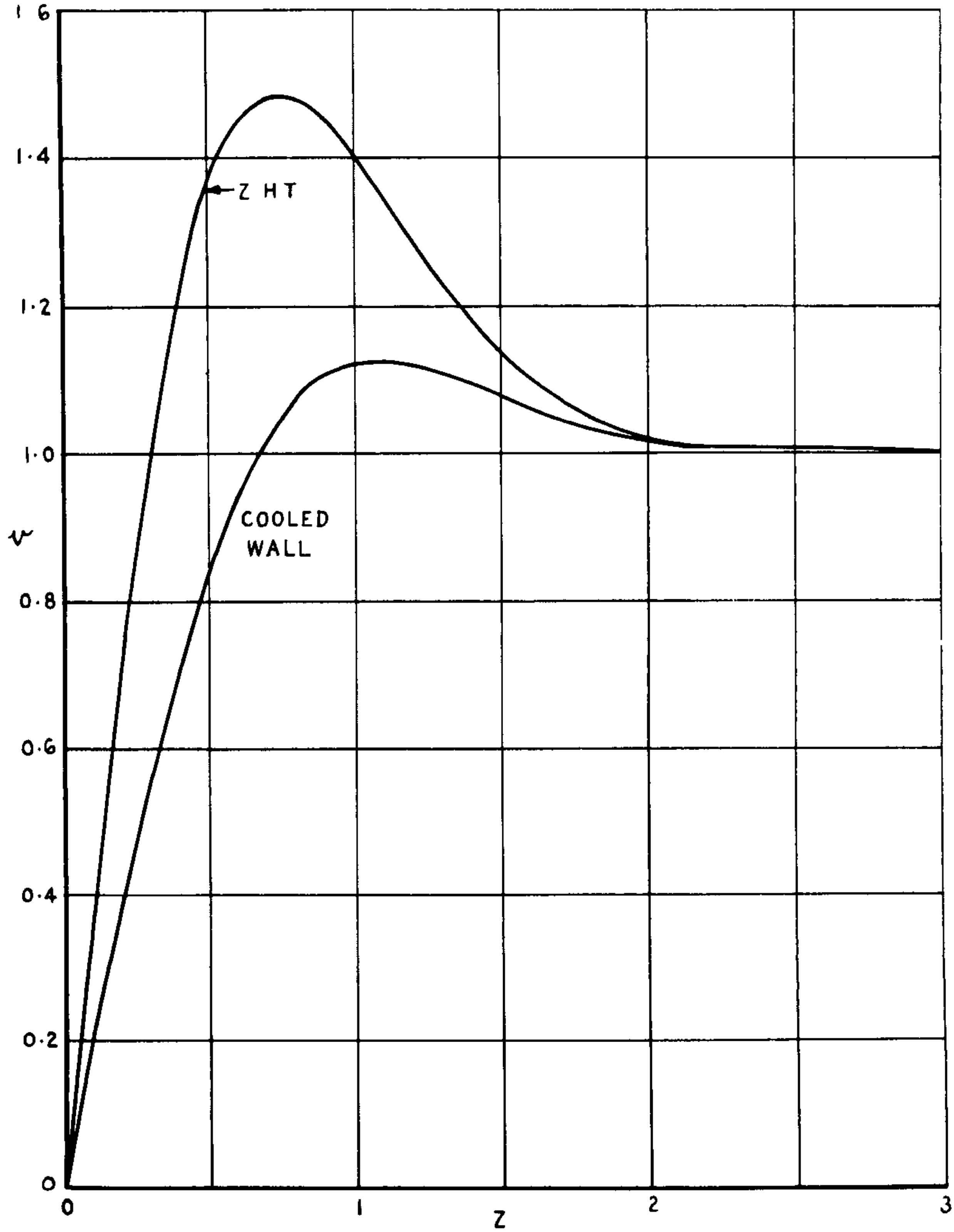


FIG 10 V PROFILES, $M_\infty = 6$, $\lambda = 1$ AT $\theta = 0{\cdot}2$

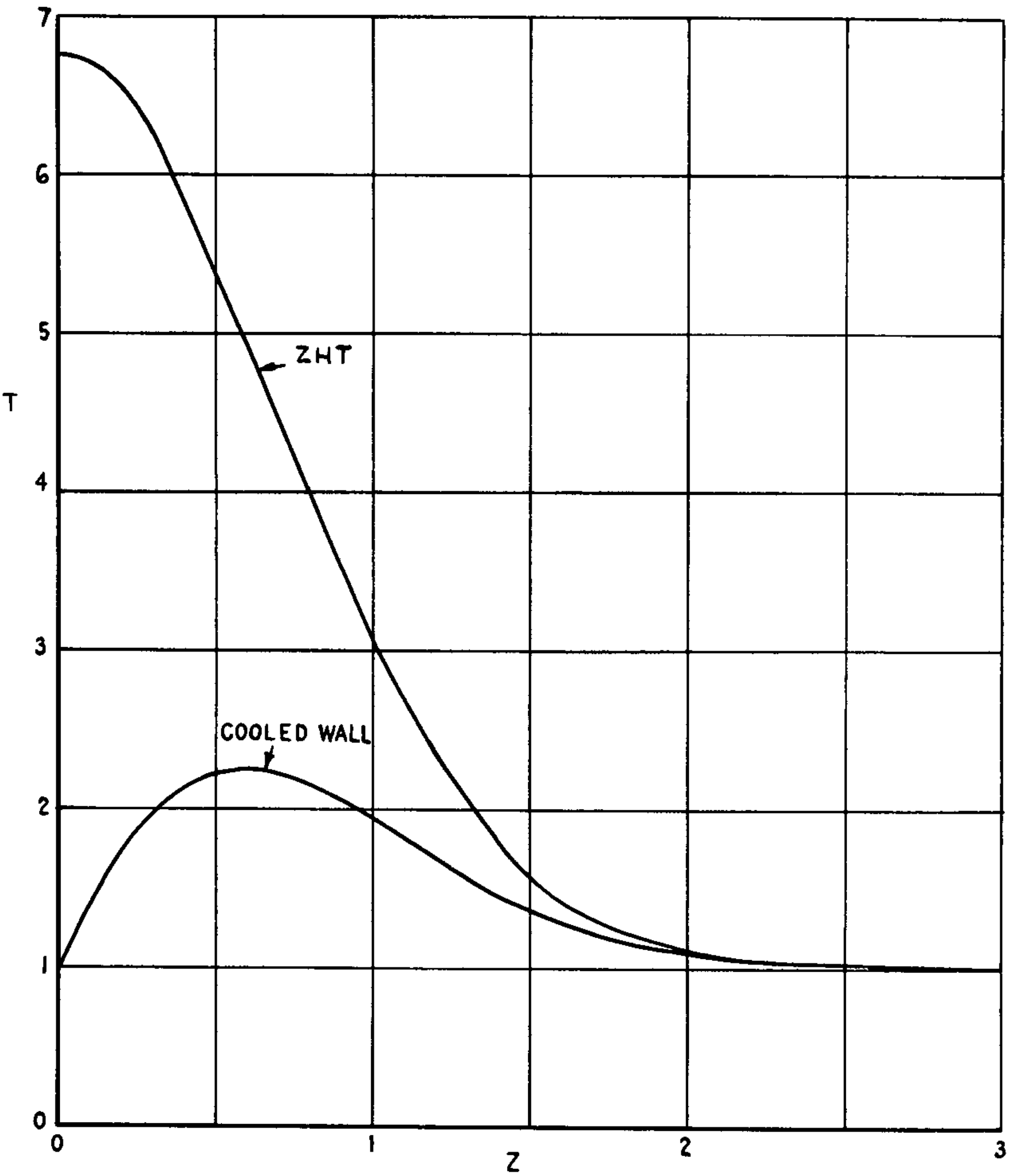


FIG. 11 T PROFILES $M_\infty = 6$, $\lambda = 1$ AT $\theta = 0{\cdot}2$

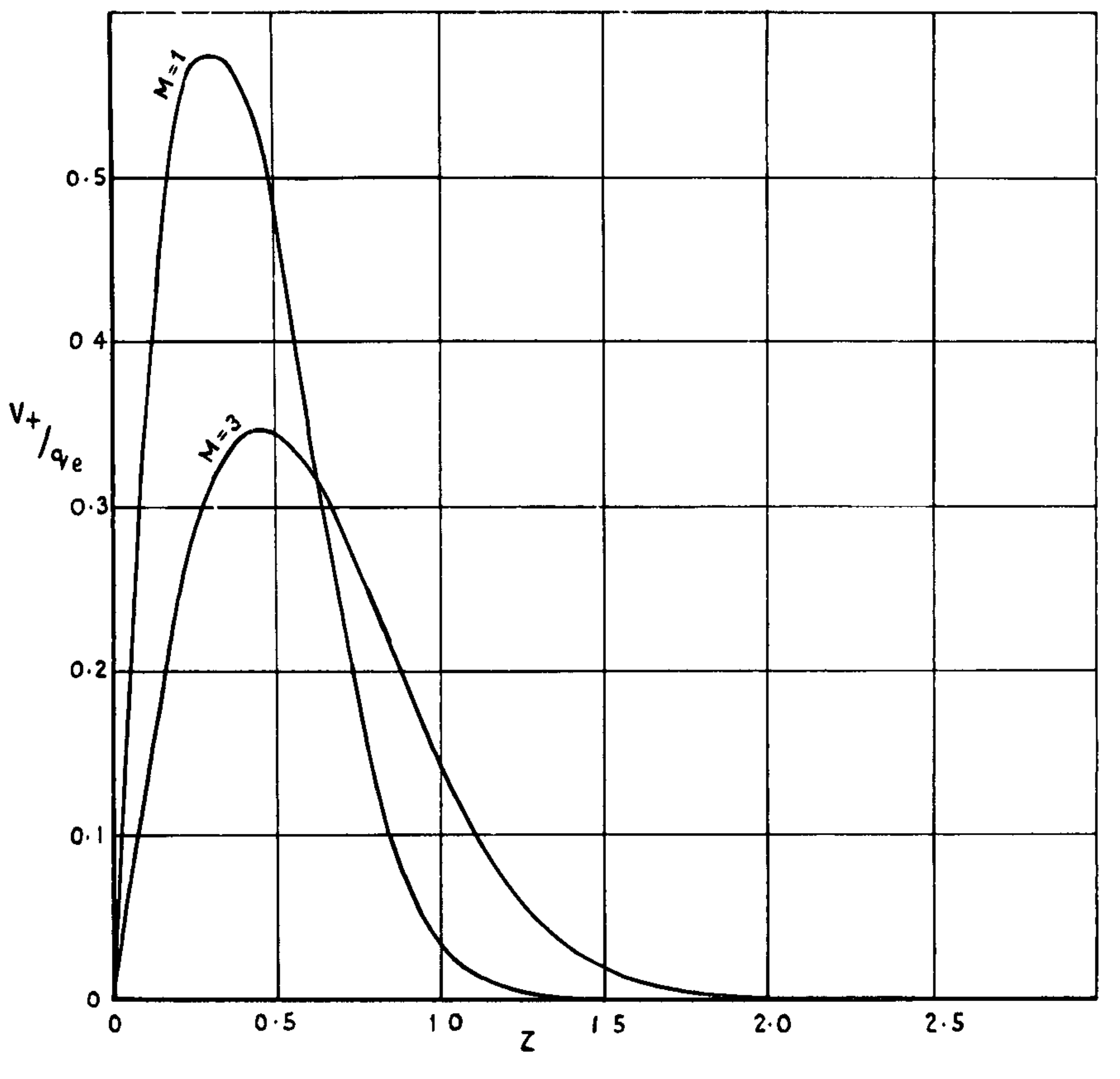


FIG.12 CROSS-FLOW PROFILES FOR λ=2, ZERO HEAT TRANSFER θ=0·2

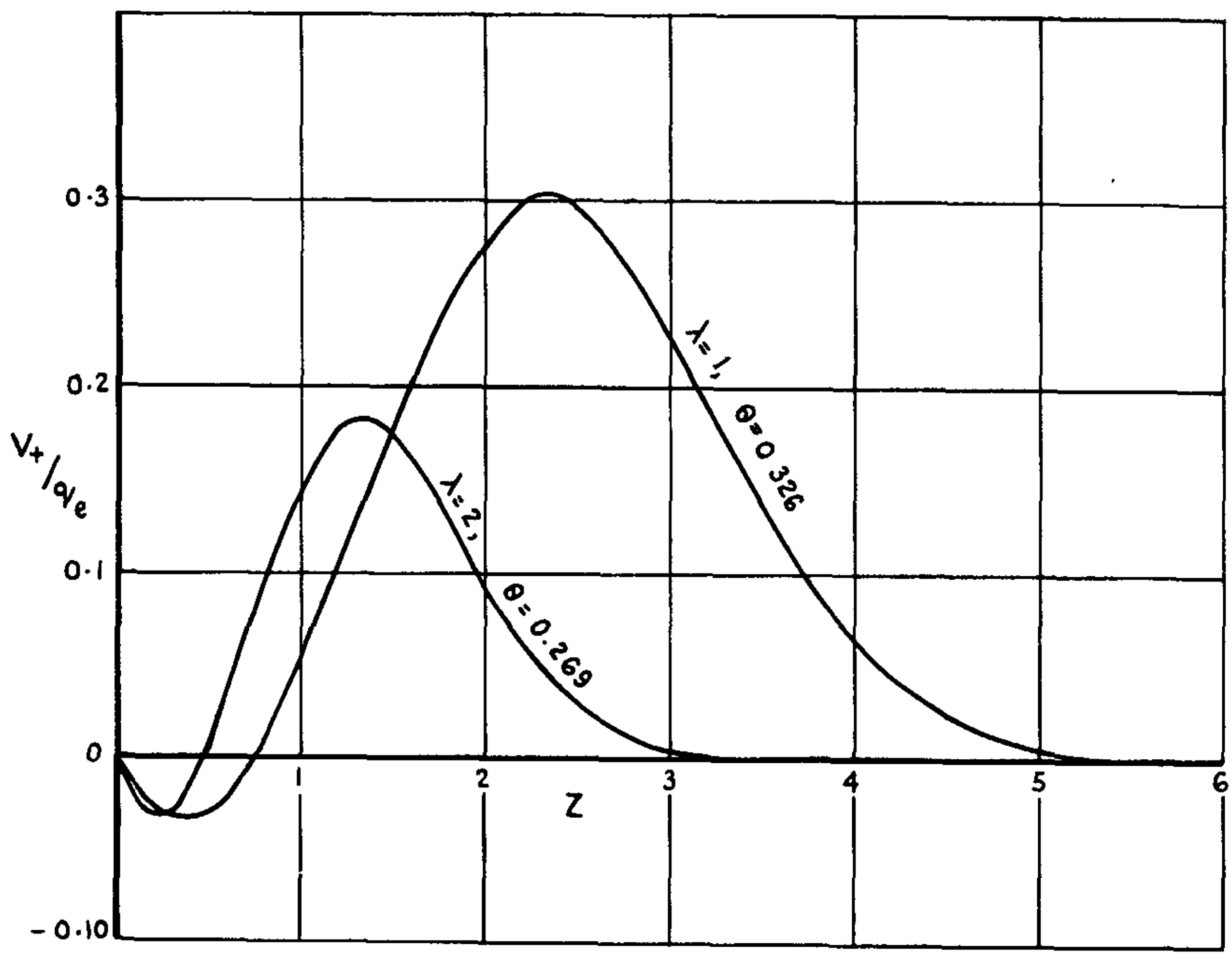


FIG. 13 CROSS-FLOW PROFILES FOR M_∞=3, ZERO HEAT TRANSFER

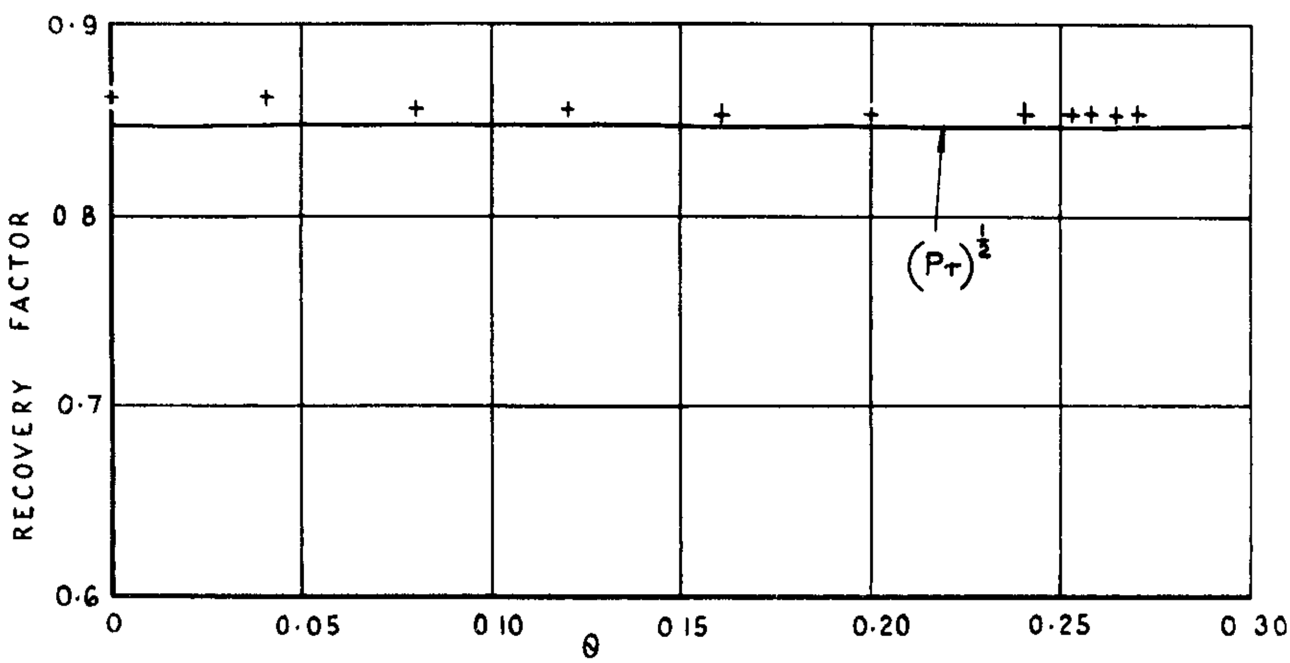


FIG. 14 RECOVERY FACTOR, $M_\infty = 3$, $\lambda = 1$, ZERO HEAT TRANSFER

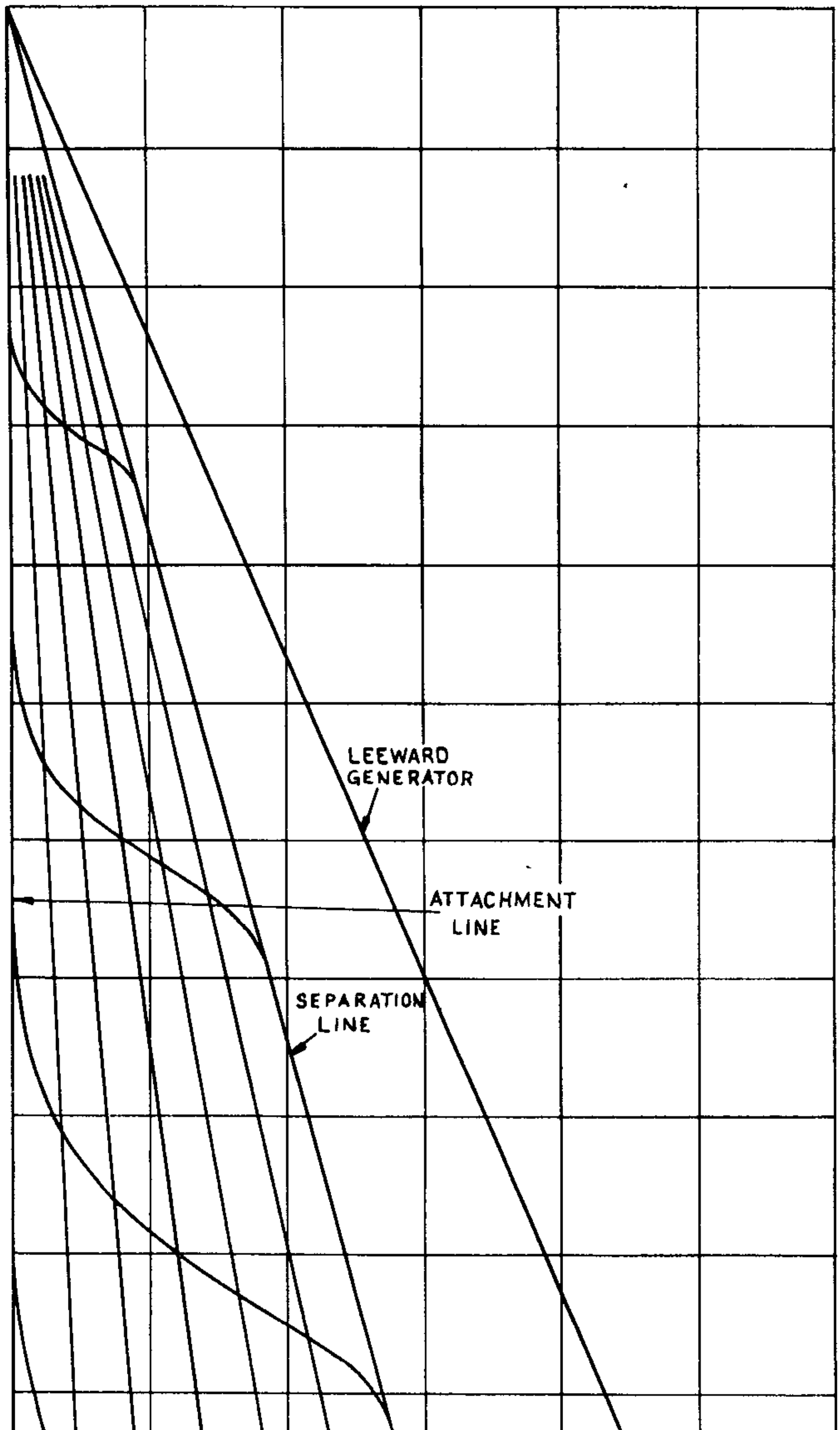


FIG. 15 LIMITING STREAMLINES $M_\infty = 6, \lambda = 2$, ZERO HEAT TRANSFER

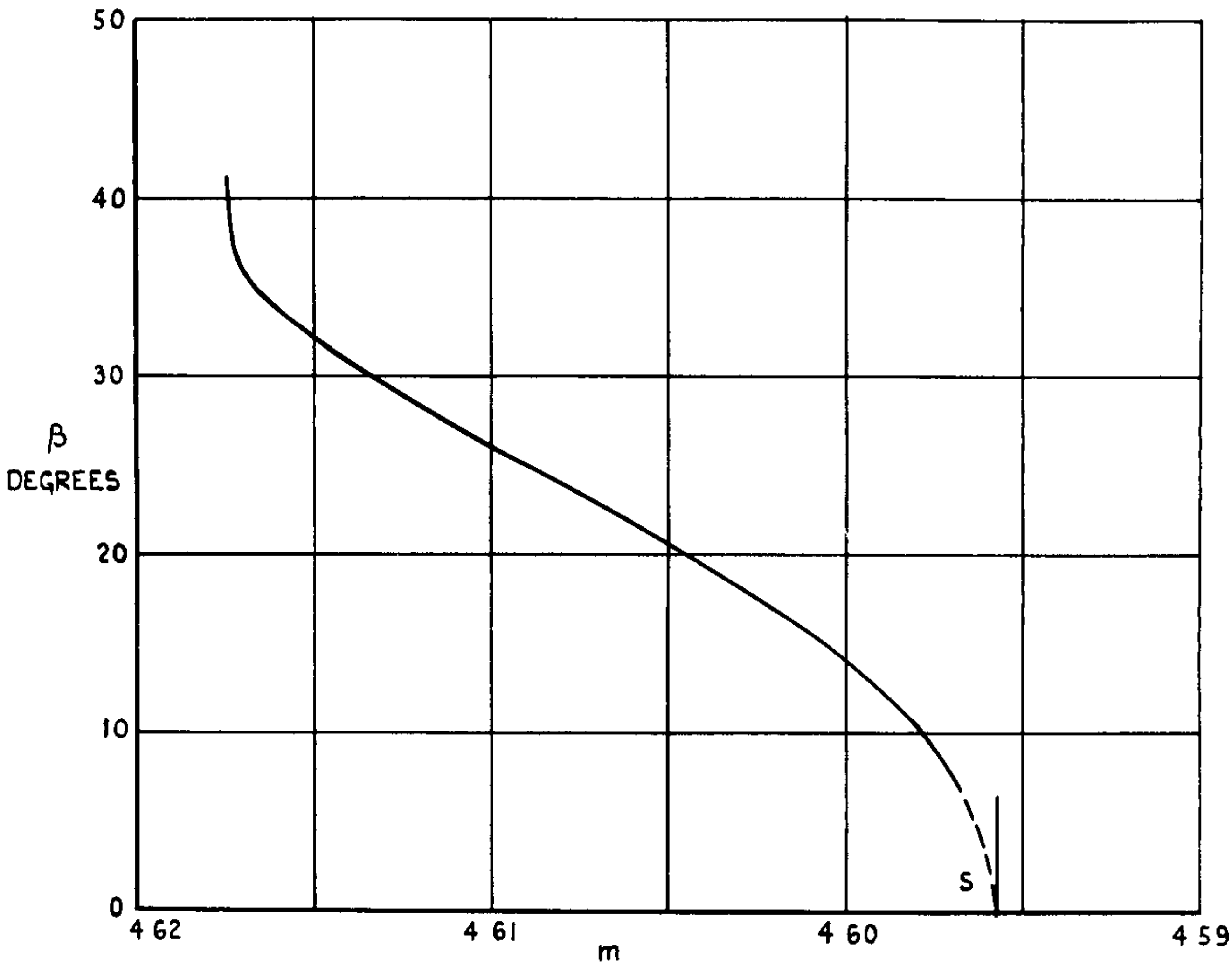


FIG. 16 SURFACE FLOW ANGLE.

TOWNEND SURFACE $\varepsilon = 23^{\circ}\!\cdot 95$, $\delta = 40^{\circ}\!\cdot 18$, $y = 1\cdot 3$

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SUPERSONIC LAMINAR BOUNDARY LAYERS ON CONES

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Some sample calculations are made. The method begins to break down as separation is approached, but it is not difficult to find the separation line by extrapolation.

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