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The Effect of an Initial Boundary Layer on the Development of a Turbulent Free Shear Layer

By

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LONDON: HER MAJESTY'S STATIONERY OFFICE

1963

SIX SHILLINGS NET

C.P. No.682

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J. F. Nash

June, 1962

SUMMARY

The problem of predicting the mean velocity on streamlines through the pre-asymptotic turbulent free shear layer in two-dimensional incompressible flow is resolved into two parts. The linearized momentum equation in terms of a generalized axial co-ordinate ζ_o is solved in the usual way. A relation between ζ_o and the distance from the separation point is then established analytically in contrast to the previous use of empirical expressions.

It is shown that except in the region close to separation the velocity on the streamlines can be predicted by the simple approximation proposed by Kirk.

CONTENTS

Replaces N.P.L. Aero. Report No.1019 - A.R.C.23,847.

List of Symbols

x, y		space co-ordinates
u, v		velocity components
u_o		velocity in initial boundary layer
u_e		velocity at edge of shear layer
u^*	=	u/u_e
ρ		density
ψ		stream function
τ		shear stress
ε		virtual kinematic viscosity
δ		boundary-layer thickness at separation
θ		momentum thickness of boundary layer
Θ		momentum thickness of shear layer
σ		factor related to rate of spread of asymptotic mixing layer
b		thickness of shear layer
Δ		length appearing in equation (24)
m, λ		coefficients in expressions for u_o
K_1, K_2, ε_o, n		coefficients in expressions for ε
ξ	=	$\int_o^x \frac{\varepsilon}{u_R} dx$
ζ	=	$y/2\sqrt{\xi}$
ζ_o	=	$\delta/2\sqrt{\xi}$

Subscripts

R	reference value
M, C, D	values on certain streamlines
h	value outside shear layer

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1. Introduction

Attempts to derive a mathematical model of the supersonic flow past a backward-facing step or in the wake of a blunt trailing-edge wing moving at supersonic speeds have drawn attention to the problem of predicting the velocity of fluid along specified streamlines in the free shear layer forming the boundary between an external quasi-inviscid flow and a region of semi-dead air. For the purposes of analysis an idealized system is examined in which the pressure is everywhere constant and, since there are assumed to be no solid boundaries downstream of the origin of mixing along which external forces could be allowed to act, the total momentum of the flow is conserved. Within the conservation of total momentum however there is a continuous exchange of momentum due to the mixing process between the layers of fluid moving at different velocities. Air initially at rest in the dead-air region is entrained and gathers momentum at the expense of the retardation of air originating in the free stream. As the flow proceeds downstream more and more fluid comes under the influence of the mixing phenomenon and the width of the shear layer increases steadily; but the width remains small compared with the distance from the origin making the shear layer a boundary-layer type problem with large transverse gradients.

In general mixing is initiated when the stream separates from a solid boundary and a boundary layer will have developed upstream of the separation point. Thus at its origin the shear layer has a non-zero thickness and the velocity profile of the initial boundary layer, and in the turbulent case the distribution of turbulent shear stress also, represents an important boundary condition placed on the subsequent development of the layer. The flow in the early part of the mixing layer, (see Fig.8), is dominated by the transition from velocity profiles of a boundary layer type to those corresponding to fully-developed mixing further downstream, and in this region the thickness of the initial boundary layer presents a reference dimension which determines the scale of succeeding velocity profiles.

As the flow proceeds downstream the influence of the initial disturbances in the shear layer decays and at very large distances from separation the characteristic dimension loses its significance with the velocity profiles approaching similarity. The existence of this asymptotic form has led to the formulation of the usual simplified flow model in which the thickness of the layer is zero at its origin and similarity of the velocity profiles is assumed throughout. This model implies a higher order singularity at the separation point insofar as there must be a discontinuity in velocity as distinct from one in velocity gradient, but any difficulties in the solution are purely local as is the case of the flow in the boundary layer near the leading edge of a plate.

In the laminar case D. R. Chapman[1,2] has derived exact solutions of the momentum and energy equations to describe the velocity and temperature distributions through the asymptotic mixing layer for Mach numbers in the subsonic and supersonic ranges. In the turbulent case solutions

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of the mean velocity field in incompressible flow have been obtained by Tollmien[3] using Prandtl's mixing length hypothesis, and by Görtler[4] who made use of the virtual viscosity concept, good agreement with experiment being achieved[5,6]. Abramovich[7] has developed Taylor's vorticity transport hypothesis to examine the effect of heat transfer through the asymptotic turbulent shear layer for Mach numbers up to 1.0. Experiments at supersonic speeds[8,9,10] have shown that the mean velocity profiles differ little in form from those at low speeds and the only effect of increasing Mach number appears to be a decrease in the rate of spread of the shear layer. Korst et al[11] have used the error-function velocity profile (Görtler's first approximation) as the basis of their calculations at supersonic speeds.

A laminar free shear layer growing from zero thickness at separation has been approached experimentally only with difficulty[20] and it is not easy to conceive of a set of conditions which would promote a fully-developed turbulent shear layer with zero initial thickness. In most practical cases a boundary layer of finite thickness has already developed upstream of separation and the velocity profiles in the separated layer undergo a transition from the initial form to their asymptotic similar form far downstream. The effects of the initial boundary layer on the laminar shear layer have been discussed briefly in Ref.12 and the corresponding problem in turbulent flow has been considered by Korst et al[11] and Kirk[13]. Korst's method, which is treated in more detail by A. J. Chapman and Korst[14], is based on the solution of a linearised momentum equation and leads to the derivation of velocity profiles in the initial part of the shear layer in terms of a generalized streamwise co-ordinate. No attempt was made to predict theoretically an expression connecting this co-ordinate and the distance from the separation point but an empirical relation of a form suggested by Pai[15] was found to agree well with low speed measurements.

The present exercise sets out to establish, on an analytical basis, a relation between the generalized position parameter and the physical space co-ordinate, which is in agreement with the observations of A. J. Chapman and Korst near the separation point and which is valid to infinity downstream. This information is then used together with the existing solution of the momentum equation to compute the velocities along typical streamlines in the shear layer.

A simple method of predicting the characteristics of the pre-asymptotic shear layer was proposed by Kirk[13]. It was suggested that the real mixing layer could be replaced by an equivalent layer growing over a greater distance from zero thickness. In this way reference could be made to all the results for the asymptotic shear layer and the effects of the finite initial thickness translated into nothing more than a linear shift of the origin. It will be shown that except in the region close to the separation point Kirk's method yields values of velocity which are in close agreement with calculations using the present analysis.

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The present work represents an attempt to investigate certain basic principles of the turbulent free shear layer developing from an initial boundary layer at separation. The analysis considers an incompressible flow model for simplicity but there appear to be no grounds for assuming that the onset of compressibility factors has any large effect on the turbulent mixing process and it would seem that there is some prospect that the general conclusions of the work would find an application in the treatment of the turbulent shear layer at supersonic speeds.

2. The Momentum Equation

The momentum differential equation for the incompressible free shear layer at constant pressure has the same form as that of the attached boundary layer and can be written

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial \tau}{\partial y} . \qquad \ldots(1)$$

It has been shown[12] that the equation may be reduced without serious loss of accuracy to the form

$$\rho u_R \frac{\partial u}{\partial x} = \frac{\partial \tau}{\partial y} , \qquad \ldots(2)$$

where u_R is a fixed reference velocity usually taking some mean value between the velocities at each edge of the shear layer. In laminar flow the shear stress τ is known at once in terms of the local velocity gradient and the coefficient of viscosity μ which is a property of the fluid. For the turbulent case we make use of Prandtl's virtual kinematic viscosity hypothesis

$$\tau = \rho \varepsilon \frac{\partial u}{\partial y} , \qquad \ldots(3)$$

where ε is not a property of the fluid but is some function of x. The assumption that ε does not vary with y is known to lead to satisfactory phenomenological descriptions of the flow field but it must not be forgotten that it is an entirely artificial concept and has no physical backing[6].

From (2) and (3) we obtain

$$u_R \frac{\partial u}{\partial x} = \varepsilon \frac{\partial^2 u}{\partial y^2} , \qquad \ldots(4)$$

and/

and making the transformation

$$\xi = \int_0^x \frac{\varepsilon}{u_R} \, dx \qquad \ldots(5)$$

we have finally

$$\frac{\partial u}{\partial \xi} = \frac{\partial^2 u}{\partial y^2} . \qquad \ldots(6)$$

This equation will be recognised as the classical one-dimensional diffusion equation for which standard solutions are available[17]. The problem thus resolves itself into two parts, the first being to solve equation (6) together with the appropriate boundary conditions and the second to interpret the function ε with a view to deriving some plausible relation between ξ and x.

3. Solution of the Equation

Equation (6) is parabolic and the velocity field throughout the free mixing layer will be defined once we specify the velocity profile at the separation point. At $\xi = 0$ let

$$u = u_o(y) \quad \text{for} \quad 0 < y < \infty$$

$$u = 0 \quad \text{for} \quad -\infty < y < 0.$$

The solution is then given by

$$u(\xi,y) = \frac{1}{\sqrt{\pi}} \int_0^\infty u_o(\alpha) \, e^{-(\alpha-\zeta)^2} \, d\alpha , \qquad \ldots(7)$$

where

$$\zeta = \frac{y}{2\sqrt{\xi}} . \qquad \ldots(7a)$$

In order to obtain a convenient reference length for the solution we may define the thickness δ of the initial boundary layer, and the boundary conditions can then be written

$$u_o = u_o(y) \quad \text{for} \quad \xi = 0, \; 0 < y < \delta$$

$$u_o = u_e \quad \text{for} \quad \xi = 0, \; \delta < y < \infty$$

where u_e is the free stream velocity.

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The solution then becomes

$$u(\xi,y) = \frac{u_e}{2}[1 + \mathrm{erf}(\zeta - \zeta_o)] + \frac{1}{\sqrt{\pi}}\int_o^{\zeta_o} u_o(\alpha)\, e^{-(\alpha-\zeta)^2}\, d\alpha, \qquad ...(8)$$

where

$$\zeta_o = \frac{\delta}{2\sqrt{\xi}}. \qquad ...(8a)$$

It remains to specify the velocity profile, $u_o(y)$, in the initial boundary layer. For the turbulent case the $\frac{1}{m}$ th power law profiles

$$u = u_e\left(\frac{y}{\delta}\right)^{\frac{1}{m}}$$

are known to be a good empirical fit to measurements, with m taking values around 7. Substituting this information into equation (8) we have

$$u^* = \frac{u}{u_e} = \tfrac{1}{2}[1 + \mathrm{erf}(\zeta - \zeta_o)] + \frac{1}{\sqrt{\pi}\,\zeta_o^{\frac{1}{m}}}\int_o^{\zeta_o} \alpha^{\frac{1}{m}}\, e^{-(\alpha-\zeta)^2}\, d\alpha. \qquad ...(9)$$

Thus the solution of the velocity field is determined in terms of generalized co-ordinates ζ_o and ζ, where $\zeta/\zeta_o = y/\delta$ and the relation between ζ_o and x remains to be found.

The integral in equation (9) is difficult to evaluate in closed form for large values of m but A. J. Chapman and Korst[14] have carried out graphical integrations to compute the velocity profiles for values of ζ_o between 1 and 5. As $\zeta_o \rightarrow 0$, i.e., at infinity downstream, the velocity profiles approach similarity in the single parameter ζ:-

$$u^* = \tfrac{1}{2}(1 + \mathrm{erf}\ \zeta). \qquad ...(10)$$

This equation is identical to Görtler's 1st approximation. This fact may be used to give some assessment of the accuracy of the present method. Görtler's 1st approximation is known to be in good agreement with the measured mean velocity profile in the asymptotic mixing layer at least near the centre of the layer but with some deviation towards the edges. The present method is considered to be of sufficient accuracy to establish overall trends but is not intended to yield quantitative data which is accurate to better than a few percent.

The/

The computation of velocity profiles in the shear layer can be simplified if an approximation is made to the initial boundary-layer profile to enable a closed solution to be obtained. It is known that linear profiles of the form

$$u_o^* = 1 - \lambda\left(1 - \frac{y}{\delta'}\right) \qquad \ldots(11)$$

with $0 < \lambda < 1$, can be chosen to resemble the $\frac{1}{m}$th power curves fairly closely, and for $m = 7$ the value $\lambda = \frac{1}{4}$ appears reasonable (see Fig.1). It is not necessary to retain the same thickness for the two profiles and in the present analysis we allow the thickness δ' of the linear profile to vary such that the same momentum thickness is preserved. For $m = 7$ and $\lambda = \frac{1}{4}$ we obtain

$$\delta' = 0.933\,\delta$$

which also specifies that

$$\zeta_o' = 0.933\,\zeta_o. \qquad \ldots(12)$$

By substituting equation (11) into equation (8), we determine the velocity field in the mixing layer in the form

$$u^* = \tfrac{1}{2}(1 + \operatorname{erf}\zeta) - \frac{\lambda}{2}\left[\left(1 - \frac{\zeta}{\zeta_o'}\right)\{\operatorname{erf}\zeta - \operatorname{erf}(\zeta - \zeta_o')\} - \frac{1}{\sqrt{\pi}\,\zeta_o'}\left\{e^{-\zeta^2} - e^{-(\zeta-\zeta_o')^2}\right\}\right]. \qquad \ldots(13)$$

We see from Fig.2 that for values of ζ_o smaller than about 2 it is not possible to distinguish between the velocity profiles computed from equation (13) and those obtained from graphical solutions of equation (9). Near the separation point then we must retain the correct initial conditions but further downstream we see that it is possible to use an approximate set of boundary conditions without significant loss of accuracy. Fig.2 also serves to demonstrate that the calculated velocity profiles are in good agreement with the measurements of Ref.14.

4. The Function ε

We now face the problem of specifying the variation of ε with x in order to determine the relation between the generalized streamwise parameter ζ_o and the physical co-ordinate x.

Let/

Let us first recall Prandtl's discussion of the virtual kinematic viscosity concept (see Ref.18, e.g.). The mixing length hypothesis of turbulence had led to expressions for the virtual kinematic viscosity in the form

$$\varepsilon = \ell^2 \left| \frac{\partial u}{\partial y} \right| \qquad \ldots(14)$$

where ℓ was the mixing length. For free turbulent flows Prandtl suggested that the size of the fluid elements which are caused to move in a lateral direction by the turbulent mixing process is of a comparable order to the width b of the shear layer, and further that the fluid elements experience an overall velocity gradient which is proportional to the maximum velocity difference across the layer divided by b. Hence in the present context equation (14) could be modified to

$$\varepsilon = K_1 b u_e \qquad \ldots(15)$$

where K_1 is some constant to be found.

Recognizing that the asymptotic turbulent mixing layer grows linearly with the distance from its origin, Görtler was able to write

$$\varepsilon = \frac{1}{4\sigma^2} u_e x \qquad \ldots(16)$$

and from equations (5) and (7a), taking $u_R = \frac{1}{2}u_e$,*

$$\zeta = \frac{\sigma y}{x} \qquad \ldots(17)$$

where/

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*Korst[19] takes $u_R = u_e$ but scales the value of ε to correct for this so that equation (16) becomes

$$\varepsilon = \frac{1}{2\sigma^2} u_e x .$$

This interpretation of u_R is retained by A. J. Chapman and Korst[14] who find an empirical relation for ε in the pre-asymptotic shear layer. For this reason their quoted values of the constant ε_0 appearing in that relation must be halved to be consistent with the more general convention.

Chapman and Korst are not correct in writing $u_R = u_e$ for laminar flow where of course the kinematic viscosity values cannot be scaled accordingly (see Ref.12, e.g.).

where σ is a constant related to the rate of spread of the shear layer, and which has a value at low speeds of approximately 12. (Ref.6.)

In the practical case when the shear layer grows not from zero thickness but from an initial boundary layer at separation, the linear variation of ε with x can no longer be expected to apply and we seek some more representative expression. Pai[15] suggested that the linear relation could be generalized by some power law of the form

$$\varepsilon = \varepsilon_o\left(\frac{x}{L}\right)^n \qquad \ldots(18)$$

where L is some convenient characteristic length (e.g., the boundary-layer thickness at separation). A. J. Chapman and Korst were able to show that for short distances downstream of separation equation (18) was a good fit to their measurements (Fig.3) if n took the value 0.7 and ε_o was a constant whose value depended on the shape of the initial boundary-layer profile. A relation of this form suffers however from the disadvantage that it can be valid only for a limited distance downstream and Pai recognised that for any value of n different from unity the condition at infinity could not be satisfied. In the present investigation we require an expression for ε which is consistent with the observations of Chapman and Korst near the separation point and which is valid to infinity downstream.

By definition

$$\xi = \int_o^x \frac{\varepsilon}{u_R}\,dx$$

and

$$\zeta_o = \frac{\delta}{2\sqrt{\xi}}\,.$$

Hence eliminating ξ we have

$$\frac{1}{\zeta_o^2} = \frac{4}{\delta^2}\int_o^x \frac{\varepsilon}{u_R}\,dx. \qquad \ldots(19)$$

Now introducing an expression for ε in the form of equation (15) and taking $u_R = \frac{1}{2}u_e$, we can write

$$\frac{1}{\zeta_o^2} = \frac{8K_1}{\delta^2}\int_o^x b\,dx \qquad \ldots(20)$$

and solving this equation formally for x

$\frac{x}{\delta}\Big|$

$$\frac{x}{\delta} = \frac{\delta}{4K_1}\int_0^x \frac{d\left(\frac{1}{\zeta_0}\right)}{\zeta_0 b} . \qquad \ldots(21)$$

The constant K_1 may be determined by reference to Görtler's problem. Consider a turbulent mixing layer starting from zero thickness at separation and developing to a thickness b' in a distance $\sigma\delta/\zeta_0$. From equations (15) and (16) the virtual kinematic viscosity is given by

$$\varepsilon = K_1 b' u_e = \frac{1}{4\sigma^2} u_e\left(\frac{\sigma\delta}{\zeta_0}\right) \qquad \ldots(22)$$

from which we have

$$K_1 = \frac{\delta}{4\sigma\zeta_0 b'} . \qquad \ldots(23)$$

Substituting this expression for K into equation (21) and remembering that $\zeta_0 b'$ is a constant we obtain

$$\frac{x}{\delta} = \sigma\int_0^{\frac{1}{\zeta_0}} \frac{b'}{b}\, d\left(\frac{1}{\zeta_0}\right)$$

$$= \frac{\sigma}{\zeta_0} - \frac{\Delta}{\delta}, \qquad \ldots(24)$$

where

$$\frac{\Delta}{\delta} = \sigma\int_0^{\frac{1}{\zeta_0}}\left(1 - \frac{b'}{b}\right) d\left(\frac{1}{\zeta_0}\right). \qquad \ldots(24a)$$

The problem is thus reduced to one of finding the ratio of the thickness b of the real shear layer to the thickness b' of a shear layer developing over the same distance $\sigma\delta/\zeta_0$ from zero initial thickness according to equation (10). In the asymptotic case considered by Görtler the question of the definition of the mixing layer thickness did not arise since the velocity profiles were similar, and the constant K was in any case related to a second constant σ which could be measured directly. In the present context however the velocity profiles are essentially dissimilar and it is possible to define the thickness in a number of ways; it will be shown that the various definitions of b lead to different forms of the relation between ζ_0 and x.

(a)/

(a) The first attempt interprets b as the physical thickness of the shear layer measured between the stations where the velocity is 0.01 and 0.99 of the free-stream velocity. The variation of this thickness b with ζ_o can be found from the velocity profiles computed in the previous section:-

$$\frac{b}{\delta} = \frac{1}{\zeta_o}(\zeta_{u^*=0.99} - \zeta_{u^*=0.01}) \qquad \text{...(25)}$$

and

$$\frac{b'}{\delta} = \frac{3.29}{\zeta_o}. \qquad \text{...(25a)}$$

Fig.3* shows that the values of x found by this method, curve "a", are everywhere too low, the term Δ in equation (24) being too large, by almost a factor of 2, to agree with the experimental points. Further calculations considering the thickness of the shear layer between stations where the velocity is 0.1 and 0.9 of the free-stream velocity yielded values of x which were only marginally better.

(b) The second attempt was to interpret b as the momentum thickness of the mixing layer, θ, defined by

$$\theta = \int_{-\infty}^{\infty} u^*(1 - u^*)dy. \qquad \text{...(26)}$$

In this case the ratio b/b' is given by

$$\frac{b}{b'} = \frac{\theta}{\theta_{asympt}} = 2.5\int_{-\infty}^{\infty} u^*(1 - u^*)d\zeta. \qquad \text{...(27)}$$

The variation of x with ζ_o predicted by this method is illustrated by curve "b" in Fig.3, and again it is seen that the values of Δ are larger than is required to agree with the measurements.

(c) It is apparent from these two experiments that the identification of b with either the physical thickness or the momentum thickness of the shear layer does not lead to an adequate

description/

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*The distances are plotted as ratios of θ instead of δ for consistency with the subsequent analysis; for the boundary-layer profile $u^* = (y/\delta)^{\frac{1}{7}}$, $\theta = \frac{7}{72}\delta$.

description of the variation of the virtual kinematic viscosity with x. Now in the original arguments from which equation (15) was derived the thickness b was introduced as a measure of both the mixing length and the overall velocity gradient across the layer. It would seem plausible then to retain a velocity gradient in the expression for ε in place of a thickness factor and we posit a relation of the form

$$\varepsilon = \frac{K_2 u_e}{\left(\frac{\partial u^*}{\partial y}\right)_R} . \qquad \ldots(28)$$

In the present analysis we compute the representative velocity gradient on the curve $\zeta = 0$ where it is close to its maximum value. As in the previous cases the constant K_2 can be expressed in terms of the conditions in the turbulent shear layer growing from zero thickness and we obtain the following equation for Δ:-

$$\frac{\Delta}{\delta} = \sigma \int_0^{\frac{1}{\zeta_0}} \left\{1 - \sqrt{\pi}\left(\frac{\partial u^*}{\partial \zeta}\right)_{\zeta=0}\right\} d\left(\frac{1}{\zeta_0}\right) . \qquad \ldots(29)$$

From equation (8) we have

$$\sqrt{\pi}\left(\frac{\partial u^*}{\partial \zeta}\right)_{\zeta=0} = e^{-\zeta_0^2} + \frac{2}{\zeta_0^{\frac{1}{m}}} \int_0^{\zeta_0} \alpha^{\frac{m+1}{m}} e^{-\alpha^2} d\alpha . \qquad \ldots(30)$$

Values of x computed using equations (24), (29) and (30) are shown as curve "c" in Fig.3 and it is seen that this curve is in good agreement with the experimental points. If we can now show that the present relation between x and ζ_0 satisfies the conditions far downstream we shall feel some confidence in using it for our later calculations.

At large distances from separation, i.e., for small values of ζ_0, equation (30) can be expanded as a power series

$$\sqrt{\pi}\left(\frac{\partial u^*}{\partial \zeta}\right)_{\zeta=0} = 1 - \frac{\zeta_0^2}{15} + \frac{\zeta_0^4}{29} - \ldots \qquad \ldots(31)$$

and

$$\int_{\frac{1}{\zeta_0}}^{\infty} \left\{1 - \sqrt{\pi}\left(\frac{\partial u^*}{\partial \zeta}\right)_{\zeta=0}\right\} d\left(\frac{1}{\zeta_0}\right) = \frac{\zeta_0}{15} - \frac{\zeta_0^3}{87} + \ldots \qquad \ldots(32)$$

This/

This equation approaches zero, and hence the quantity Δ tends to a definite limit, as $\zeta_o \to 0$. Our calculations indicate that for small values of ζ_o

$$\frac{\Delta}{\delta} = \sigma\left(0.222 - \frac{\zeta_o}{15} + \frac{\zeta_o^3}{87} - \dots\right)$$

and therefore

$$\frac{x}{\delta} = \frac{\sigma}{\zeta_o}\left(1 - 0.222\,\zeta_o + \frac{\zeta_o^2}{15} - \frac{\zeta_o^4}{87} + \dots\right) \text{(Fig.4)} \qquad \dots(33)$$

Thus in the limit at infinity downstream

$$\frac{x}{\delta} = \frac{\sigma}{\zeta_o} \qquad \dots(34)$$

which is the correct asymptotic value.

We have then the complete solution of the mean velocity field which is in agreement with measurements near the separation point and which converges to the correct form at infinity.

Korst et al[11] indicate a method of finding the displacement effect of the shear layer with a view to determining a y-scale relative to the streamlines in the external flow. Our present analysis is mainly concerned with the computation of the velocities along streamlines in the layer and the precise location of a particular streamline is of no immediate concern. It must be pointed out then that the illustration of the flow field in Fig.8 shows the shear layer relative to intrinsic co-ordinates and the absolute y-scale can be derived only by specifying boundary conditions in the free stream.

5. The Velocity Along Certain Streamlines

We are now in a position to use the foregoing results to compute the velocity along streamlines through the shear layer. In the present analysis it will be convenient to define a stream function ψ such that

$$u = \frac{1}{\rho}\frac{\partial\psi}{\partial y}, \quad v = \frac{1}{\rho}\frac{\partial\psi}{\partial x}. \qquad \dots(35)$$

Consider first the streamline ψ_M defined by

$$\psi_h - \psi_M = \int_{-\infty}^{\psi_h} u^* d\psi \qquad \dots(36)$$

where ψ_h is some streamline in the free stream outside the shear

layer./

layer. [ψ_M is a streamline since the integral on the right-hand side of equation (36) is a constant for all x from considerations of the conservation of momentum.]

It can be shown that the velocity u^*_M on this "median" streamline ψ_M is a function only of the shape of the velocity profile. In the asymptotic mixing layer the velocity profiles are similar, hence the velocity u^*_M is constant (equal to u^*_0, say) and ψ_M is then termed the constant-velocity streamline. The value of u^*_0 depends fairly critically on the profile shape; calculated from the profile given by Tollmien's method the value is 0.58, whereas Görtler's 1st approximation leads to a value of 0.62.

Tracing the streamline ψ_M back into the pre-asymptotic region of the shear layer, the velocity u^*_M departs from its constant value u^*_0 on account of the dissimilarity of the velocity profiles. At the separation point u^*_M is a function of the initial boundary-layer profile, and for profiles of the form $u^* = (y/\delta)^{\frac{1}{m}}$ the value of u^*_M is given by

$$u^*_M = \left(\frac{1}{m+2}\right)^{\frac{1}{m+1}} \qquad \ldots(37)$$

and for $m = 7$, $u^*_M = 0.76$.

The fall of u^*_M through the pre-asymptotic mixing layer from this value at separation can be computed from numerical integrations of the velocity profiles. From equations (35) and (36), for any station x

$$\int_{y_M}^{h} u^* dy = \int_{-\infty}^{h} u^{*2} dy \qquad \ldots(38)$$

where the ordinate h is outside the layer. With y_M determined from equation (38) the velocity u^*_M is known at once from the velocity profile. The variation of the velocity along the median streamline is illustrated in Fig.5.

The median streamline ψ_M can be used as a convenient datum from which to measure the stream function and we may define a non-dimensional variable

$$\psi^* = \frac{\psi - \psi_M}{\rho u_e \theta}, \qquad \ldots(39)$$

where θ is the momentum thickness of the initial boundary layer. Consider the form of equation (36) at the separation point, if ψ_D is the dividing streamline

ψ_h/

$$\psi_h - \psi_M = \int_{\psi_D}^{\psi_h} u^* d\psi \qquad ...(40)$$

and this may be written

$$\psi_M - \psi_D = \int_{\psi_D}^{\psi_h} (1 - u^*) d\psi$$

$$= \rho u_e \theta ; \qquad ...(41)$$

or

$$\psi_D^* = -1. \qquad ...(42)$$

Thus we obtain the well-known result that the mass flow between the median streamline, as we have termed it, and the dividing streamline is proportional to the momentum thickness of the initial boundary layer.

The variation of the velocity, u_D^*, along the dividing streamline is shown in Fig.5, and this diagram illustrates the important significance of the two streamlines $\psi^* = 0$ and $\psi^* = -1$. The velocity on the median streamline approaches its asymptotic value closely at distances from separation equal to 40 or 50 times the momentum thickness (4 to 5 times the boundary-layer thickness), while the velocity on the dividing streamline has by then reached no more than 80% of its asymptotic value. Thus even when the velocity profiles closely resemble the similar asymptotic form the velocity u_D^* may still be distinct from its final value.

The velocity profiles at several streamwise stations in the shear layer, in terms of the non-dimensional stream function ψ^* are shown in Fig.6. [The square of the velocity, rather than the absolute value, is plotted to preserve the linearity of the profiles near $u^* = \frac{1}{2}$.] The effect of the initial boundary layer on the velocity on a number of streamlines is shown in Fig.7. It is observed that at a *fixed distance from the separation point increase of the initial* boundary-layer momentum thickness causes a rapid change in the velocity on all the streamlines except those close to the median streamline $\psi^* = 0$.

6. The Method of Kirk

Kirk[13] suggested that some distance, x, downstream of separation the turbulent shear layer behaves in much the same way as an equivalent layer developing from zero thickness over a greater distance $x+x'$. It was assumed implicitly that over the distance x' the equivalent mixing layer attains a momentum thickness θ equal to the momentum thickness θ of the real boundary layer. Thus the origin of mixing of the equivalent layer is at a distance x' upstream of the

real/

real separation point where x' is proportional to θ. To Görtler's first approximation the momentum thickness of an asymptotic turbulent shear layer increases as $\frac{1}{30}$th of the distance from the origin and hence Kirk's arguments lead to

$$x' = 30\theta. \qquad \ldots(43)$$

The velocity profiles in the equivalent mixing layer are similar and the $u^* \sim \psi^*$ curves can be obtained from equation (10) by simple quadrature:-

$$\psi^* = \frac{x + x'}{\sigma\theta} \int_{\zeta_0}^{\zeta} u^* d\zeta. \qquad \ldots(44)$$

Some data computed by Kirk's method are presented in Fig.7, and it is seen that for values of x/θ greater than 80 (8 boundary-layer thicknesses) the velocity along the dividing streamline is in good agreement with the values predicted by our analysis. For the streamlines in the lower-velocity part of the shear layer, for $\psi^* < -1$, the agreement is good much nearer to separation.

7. Conclusions

The mean velocity field in the two-dimensional turbulent free shear layer developing from an initial boundary layer at separation has been solved using the linearized momentum equation. Reference has been made to the work of Korst and others in which solutions of the equation were found in terms of a generalized axial co-ordinate ζ_0 which was related to the distance from the separation point x by an integral equation involving the virtual kinematic viscosity ε.

In contrast to the previous work in which an empirical expression was found to describe the variation of ε with x, and hence of ζ_0 with x, the relation between ζ_0 and x has been established as part of the solution. This relation is in agreement with the measurements of A. J. Chapman and Korst near the separation point and also satisfies the conditions at infinity downstream.

The present analysis has been used to compute the velocity of fluid along certain streamlines through the shear layer. It is found that except in the region close to the separation point the velocities can be predicted satisfactorily by the simple approximation proposed by Kirk.

Acknowledgement/

Acknowledgement

The author wishes to acknowledge the assistance of Miss F. M. Worsley with the calculations.

References

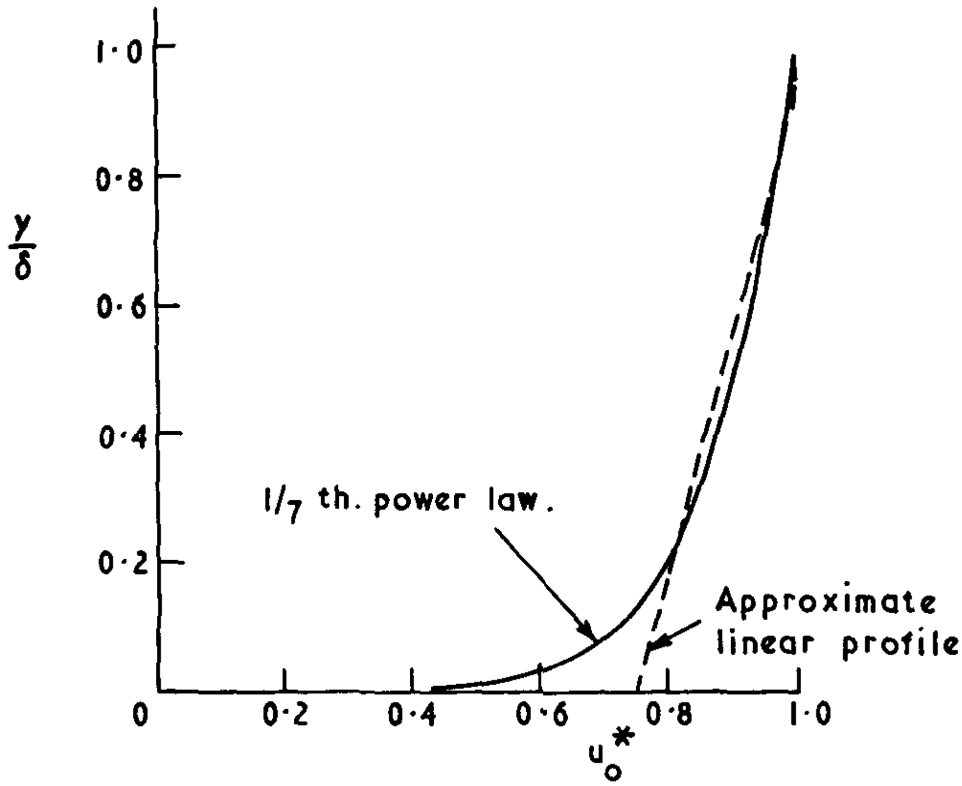
No.	Author(s)	Title, etc.
1	D. R. Chapman	Laminar mixing of a compressible fluid. N.A.C.A. Rep.958, 1950.
2	D. R. Chapman	A theoretical analysis of heat transfer in regions of separated flow. N.A.C.A. TN 3792, October, 1956.
3	W. Tollmien	Berechnung turbulenter. Ausbreitungsvorgänge. ZAMM, Vol.6, p.468, 1926. (Translated as N.A.C.A. TM 1085, 1945)
4	H. Görtler	Berechnung von Aufgaben der freien Turbelenz auf Grund eines neuen Näherungsansatzes. ZAMM, Vol.22, p.244, 1942.
5	H. Reichardt	Gesetzmässigkeiten der freien Turbulenz. VDI-Forschungsheft, No.414, p.1, May, 1942. (Translated as Ministry of Aircraft Production, R.T.P. Trans No.1752)
6	H. W. Liepmann and J. Laufer	Investigation of free turbulent mixing. N.A.C.A. TN 1257, August, 1947.
7	G. N. Abramovich	The theory of a free jet of a compressible gas. Central Aero-Hydrodynamical Institute, Moscow. Rep.377, 1939. (Translated as N.A.C.A. TM 1058, March, 1944)
8	P. B. Gooderum, G. P. Wood and M. J. Brevoort	Investigation with an interferometer of the turbulent mixing of a free supersonic jet. N.A.C.A. Rep.963, 1950.

9/

No.	Author(s)	Title, etc.
9	N. H. Johannesen	The mixing of free axially symmetrical jets of Mach number 1.40. A.R.C. R. & M.3291. January, 1957.
10	A. F. Charwat and J. K. Yakura	An investigation of two-dimensional supersonic base pressures. J.A.S., Vol.25, p.122, February, 1958.
11	H. H. Korst, R. H. Page and M. E. Childs	Compressible two-dimensional jet mixing at constant pressure. University of Illinois, Dept. of Mechanical Engineering, M.E.-TN-392-1, 1954.
12	J. F. Nash	Laminar mixing of a non-uniform stream with a fluid at rest. A.R.C. C.P.613. September, 1960.
13	F. N. Kirk	An approximate theory of base pressure in two-dimensional flow at supersonic speeds. R.A.E. Tech. Note Aero.2377, 1954. (Published 1959)
14	A. J. Chapman and H. H. Korst	Free jet boundary with consideration of initial boundary layer. Proc. 2nd U.S. National Congress of Applied Mechanics, Michigan, June, 1954.
15	S. I. Pai	Two-dimensional jet mixing of a compressible fluid. J.A.S., Vol.16, p.463, 1949.
16	H. E. Bailey and A. N. Kuethe	Supersonic mixing of jets and turbulent boundary layers. WADC Tech. Report 57-402 ASTIA Document No. AD150992 University of Michigan, June, 1957.
17	J. Crank	The Mathematics of Diffusion. Oxford University Press, 1957.
18	H. Schlichting	Boundary layer theory. Pergamon Press, 1955.

No.	Author(s)	Title, etc.
19	H. H. Korst	A theory for base pressures in transonic and supersonic flow. J.App.Mech., Vol.23, No.4, p.593, December, 1956.
20	D. R. Chapman, D. M. Kuehn and H. K. Larson	Investigation of separated flows in supersonic and subsonic streams with emphasis on the effect of transition. N.A.C.A. Report 1356, 1958.

FIG. 1



Velocity profile of the initial boundary layer

The approximate profile is defined by

$$u_o^* = 1 - \lambda \left(1 - \frac{y}{\delta'}\right),$$

with $\lambda = 1/4$

$\delta' = 0{\cdot}933\ \delta$.

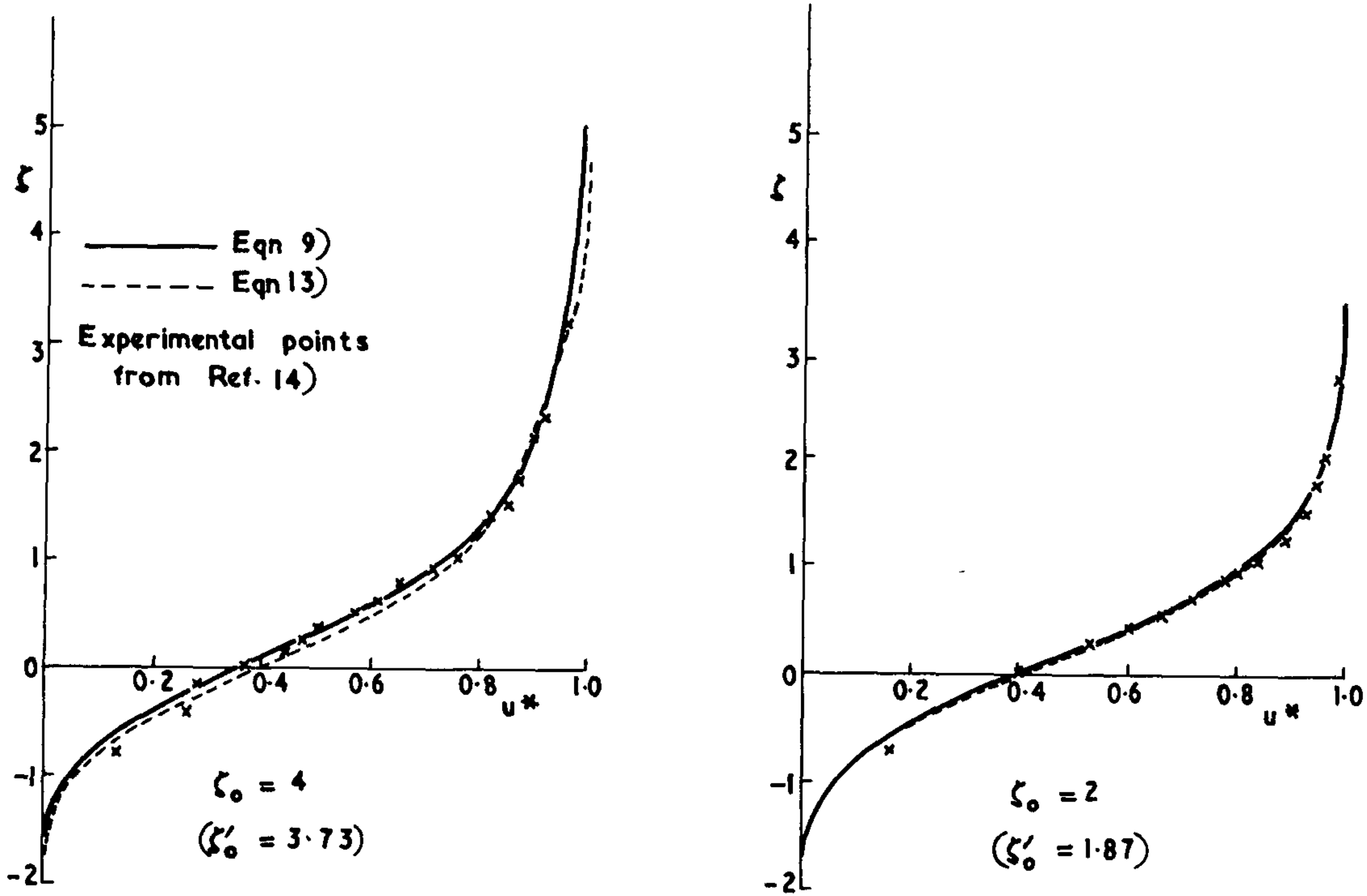
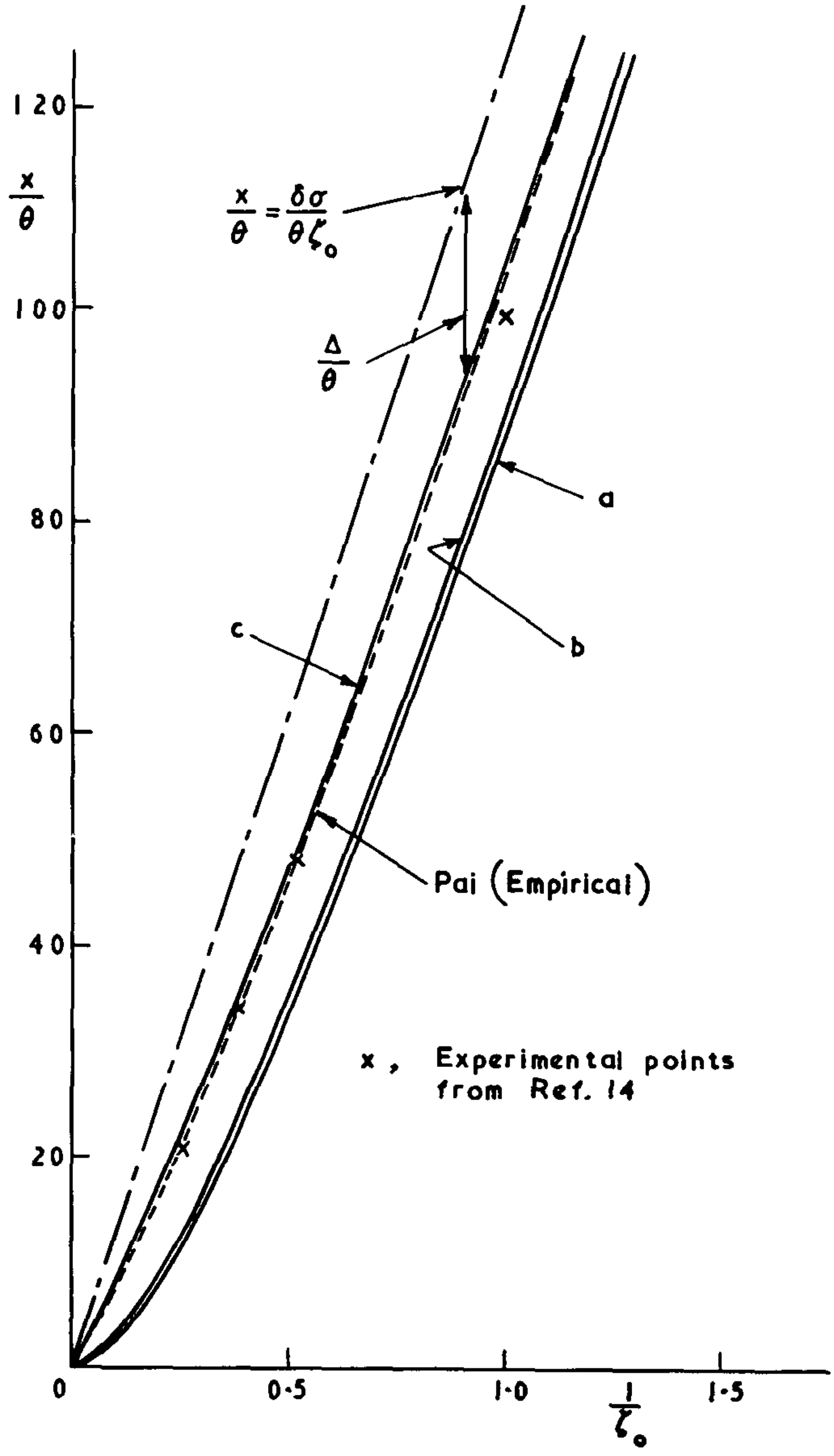


FIG. 2.

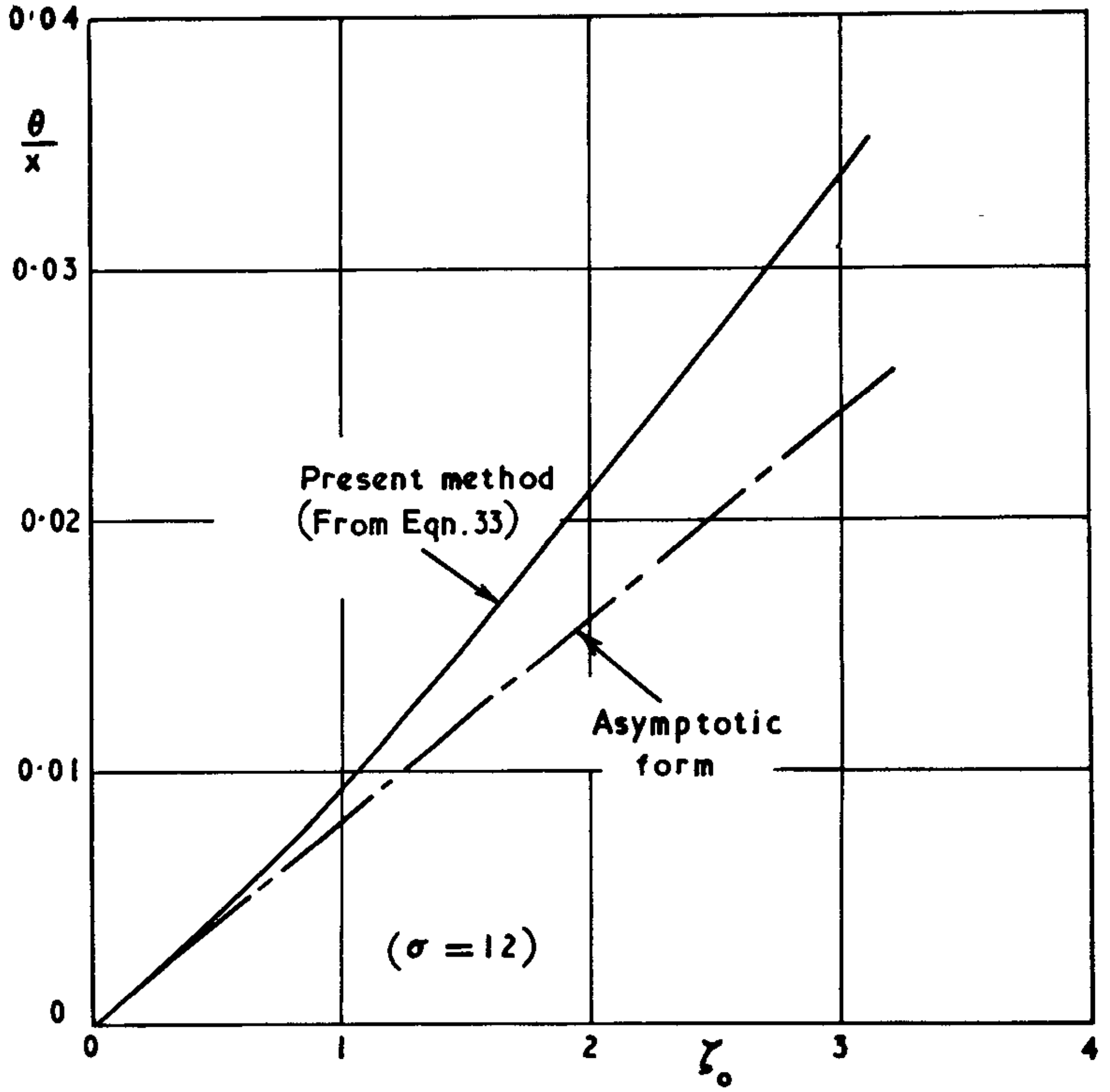
Comparison of shear-layer velocity profiles using the two initial profiles in Fig. 1.

FIG. 3

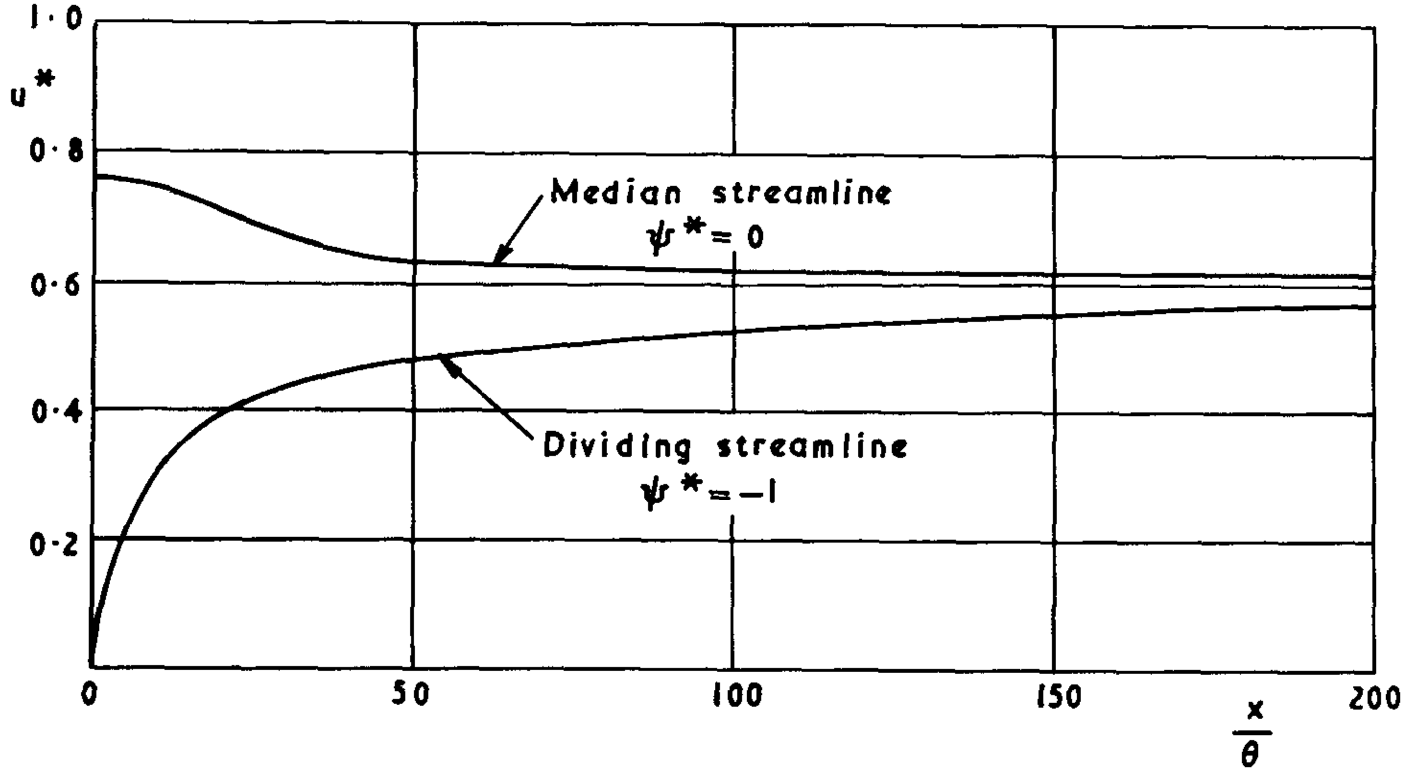


The relation between x and ζ_o by various methods

FIG. 4



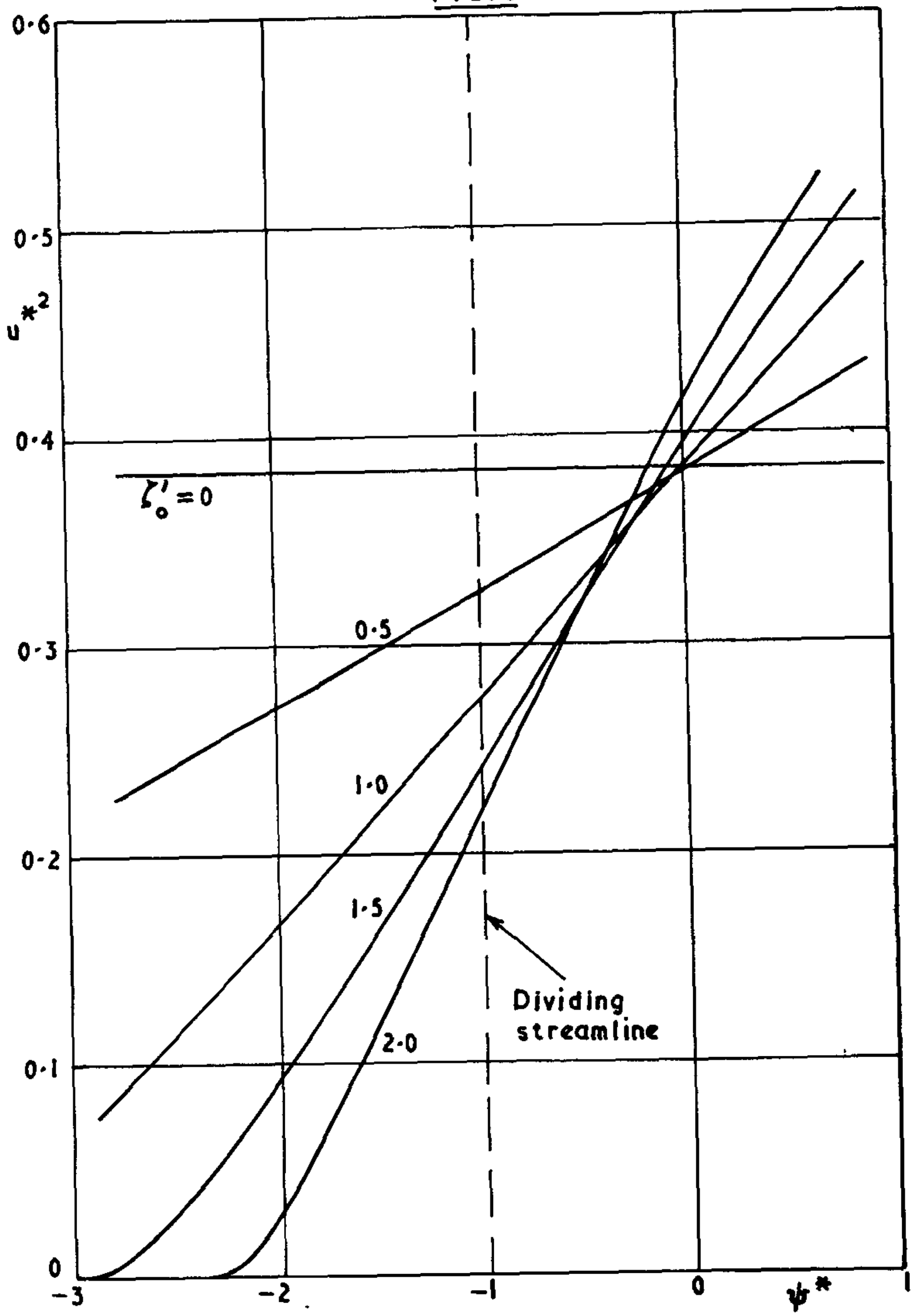
The relation between x and ζ_0 far downstream.



Variation of velocity along median and dividing streamlines

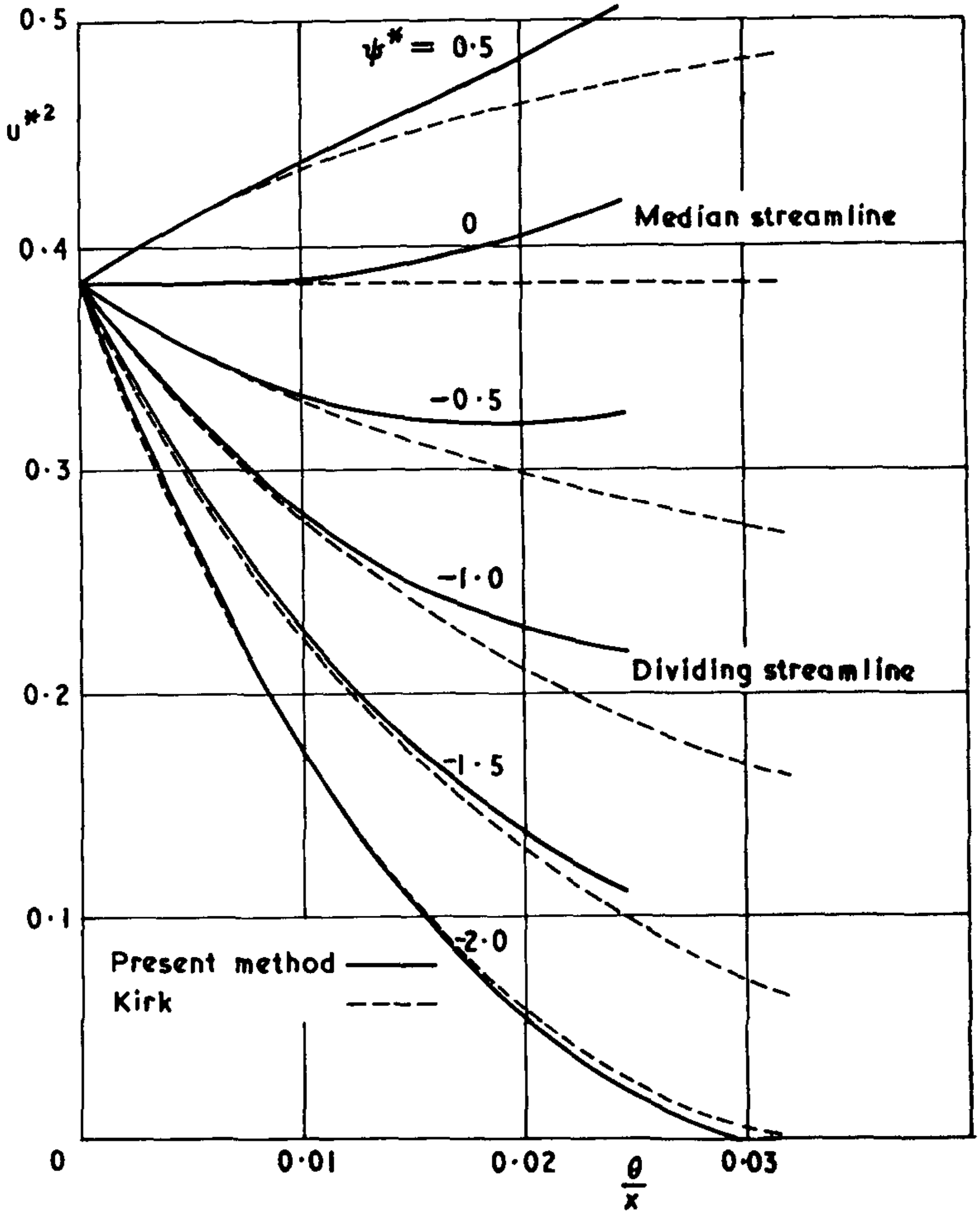
FIG. 5

FIG. 6



Velocity profiles in terms of stream-function for several streamwise stations.

FIG. 7



Velocity along streamlines, comparison with Kirk's method

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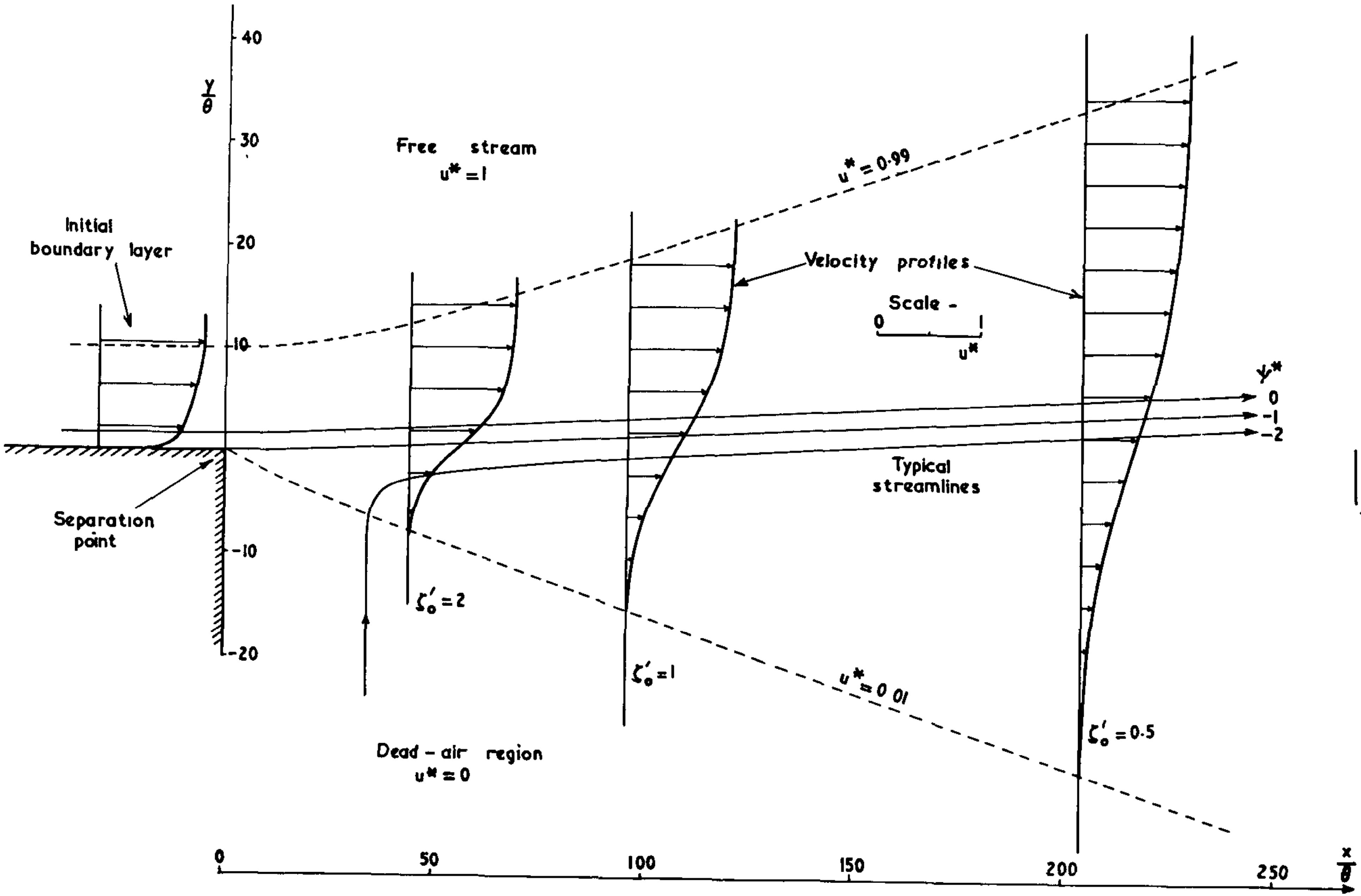

The pre-asymptotic mixing layer.

FIG 8.

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