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Qualitative Solution of the Stability Equation for a Boundary Layer in Contact with Various Forms of Flexible Surface

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SUMMARY

An appropriate form of the boundary layer stability equation is developed for the condition where the fluid is in contact with an isotropic and homogeneous elastic medium, and various approximate analytical solutions obtained for certain types of surface, so as to reveal at least qualitatively the origin and characteristics of neutral oscillations. In the worked solutions the elastic medium is treated as non-dissipative, and the interior boundary is supposed either fixed, or free of stress, or exposed to fluid: the boundary layer, also, is treated as that over a flat-plate in an incompressible fluid.

The results obtained show that the presence of such a resiliant surface introduces the possibility of a number of other modes of oscillation (over the complete Reynolds number range) apart from those of Tollmien-Schlichting waves. Most of these modes have speeds of propagation determined largely by the properties of the elastic material, and their presence may well be effectively a matter of 'non-viscous' flow stability - a subject not treated here. The Tollmien-Schlichting mode has its minimum Reynolds number increased by the presence of the surface, but if the interior boundary is free there may be an upper limit as well. Indeed, a sufficiently thin free surface, or one of low rigidity, apparently eliminates neutral oscillations of this mode altogether, only at the expense, however, of the introduction of a mode of flexural waves.

List of Contents — Page

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Page

1. Introduction

The theoretical study of the stability of the laminar boundary layer in contact with a flexible surface is a problem of current interest, but is one beset by considerable difficulties. Even in the absence of surface elasticity, there are still perhaps certain aspects of the standard treatment (described, for example, in Ref. 1) which are not completely satisfactory, and numerical solutions involve a great deal of laborious calculation. It is not to be expected, therefore, that a theoretical understanding of the same problem complicated further by the presence of surface elasticity will be readily obtained. Enthusiasm for the task is also moderated by the suspicion that the experimental evidence concerning the effect of a flexible surface in reducing skin friction, whilst of great significance, need not necessarily be due to enhanced stabilisation of the laminar boundary layer: it could be that turbulence is present, but that the surface modifies it to such an extent that its usual consequences are substantially alleviated. We shall not remark upon the merits of this argument; we draw attention to it because, although it helps to present the need for theoretical work. it also justifies endeavours such as that of Ref. 2, and that which we attempt here, to examine the qualitative - rather than the quantitative - nature of the solution, before embarking on long calculations which may not be relevant.

Another reason for eschewing at this stage too detailed a survey is the vast number of variables which may be of some significance. With an inflexible surface, one has to consider the Reynolds number and some parameter indicating the velocity profile of the boundary layer as variables, - at least, if one excludes compressibility effects, and restricts the consideration to two-dimensional flow, and postulates only the existence of neutral oscillations. But even with an isotropic, homogeneous and non-dissipating elastic surface, there are two additional independent parameters denoting the elastic behaviour of the solid surface together with a third, denoting the magnitude of elastic stresses compared with aerodynamic stresses - say, the ratio of the density of the fluid to that of the surface material (σ). If one were to include damping, a simple treatment might require the use of two more independent variables, and if one were to consider non-isotropic materials one might be forced to employ as many as 36 independent elastic constants for a single material! As well as this, one is faced with a variety of possible conditions to be applied at the boundary between the elastic skin and the effectively inelastic structure to which it is attached. Yet it may well be, in practical applications, that inhomogeneity, anisotropy, dissipation, and the precise form of the interior boundary condition all play an important rôle.

It need hardly be pointed out, therefore, that the enormity of the problem forces one to simplify it - perhaps to the extent that it loses its practical significance, though one can only hope not. What we try to do here, therefore, is to examine qualitatively the existence of neutral oscillations in the two-dimensional flat plate boundary layer in contact with a homogeneous and isotropic, non-dissipating surface, and in doing so we allow the density ratio between surface and fluid, and the two elastic constants of the surface material, to vary over the complete physical range. These elastic constants are most conveniently described by the ratios to the free-stream speed (u_δ) of the speeds of compression waves (c_1) and shear waves (c_2) in the material; there are occasions where it is more convenient to use the speed of propagation of 'Rayleigh' surface waves (c_3), or the speed of longitudinal waves on an extensive thin plate (c_4), but these speeds, it will be appreciated, can be related to c_1 and c_2. We also hypothesise four simple interior boundary conditions, one the obvious concept of an elastic sheet of uniform thickness rigidly attached to a fixed structure, and another the less practical but 'opposite' notion of a surface of uniform thickness free of stress at its interior surface; the other two relate to a surface in contact with fluid at its inner boundary. We are then presented with four similar problems, involving 3 non-dimensional physical parameters which affect the

functional/

functional dependence on Reynolds number (R_δ) of the speed of propagation (c), and the wavelength (proportional to $1/\delta$), of neutral oscillations.

The preliminary paragraphs (2, 3 and 4) are concerned with the establishment of a suitably modified form of 'stability equation', like that for an inflexible surface, but taking account of the feed-back between the elastic deformation of the surface, caused by fluid stresses, and the structure of the fluid oscillation. This stability equation is merely a convenient form of statement of the Eigen-value problem which allows us to find discrete values of c and δ for the hypothesised neutral oscillation at any chosen Reynolds number, and its form for an inflexible surface is familiar. In deriving it, since we are interested in how the inflexible surface solution is changed, we must make similar simplifications to those employed for the inflexible surface - which are based on the assumptions that $1/R_\delta$, c/u_δ and δ, or some combination of them, are small quantities. However, one of the less satisfying, but perhaps more provocative, of our deductions is that our analysis yields results implying the existence of neutral oscillations whose speed is, for instance, close to a natural wave speed of the elastic medium, even where this implies large values of c/u_δ. Although one should not attach credence to such deductions, because values of c/u_δ around unity - let alone higher values - are quite outside the scope of our approximations, nonetheless one is tempted to wonder whether such fast oscillations may exist, and the only way of finding this out would be to start afresh the formulation of a stability equation with approximations introduced to suit such a possibility. An exact form of the stability equation cannot, of course, be obtained for an inflexible surface, so any hopes of an all-embracing form for the flexible surface problem are perhaps too ambitious.

Having established this equation in a general form - suitable for boundary layer velocity distributions other than flat plates, for amplified or attenuated oscillations and for dissipating surface materials - we particularise it to our problem, and then proceed in paragraph 5 to introduce further and coarser simplifications (which would seem adequate for a merely qualitative interpretation) so as to evaluate the terms of this equation depending on the boundary-layer characteristics. Again in paragraphs 6, 7 and 8 we introduce certain simplifications to deal with the terms of the equation involving the elastic constants - as they are in general too complicated to display a qualitative solution; paragraph 6 deals with the application of the boundary condition of rigidity at the interior surface, and the next paragraph reproduces the work for a freely mounted surface. They are both detailed discussions of the algebraic solution of the stability equations for a number of extreme instances in which the expression for the elastic constants simplifies - for very thick or very thin layers of the surface material, and for oscillations propagated slowly or rapidly compared with the speed c_2. From the form of the algebraic equations thus produced, it is usually possible to deduce the qualitative behaviour of the solution for neutral oscillations as it appears, for example, on diagrams of (c/u_δ) or δ versus R_δ.

Paragraph 8 makes a brief reference to two other particular instances in which the inner boundary condition has a simple form, supposing this boundary to be exposed to a fluid which has extreme (large or small) values of density and viscosity. A detailed summary of these solutions is presented in paragraph 9 together with a key to the few quantitative results which thrust themselves out of the analysis. However, it is not intended that this discussion should be a substitute for numerical results: rather it is hoped that it may be an aid in showing the areas in which solutions may exist and have some particular interest. Numerical work, if attempted, would be built on the results of paragraph 4: only the introduction to the sections 6, 7 and 8 - where unsimplified expressions for the elastic constants are stated - would be of relevance.

2. Equations for the Displacement of the Solid

We suppose that the material of the solid is isotropic and subjected to small steady stresses superposed on which are oscillatory stresses also small compared with the modulus of the material. We may thus consider the latter as independent of the former in their effect on the material's displacement. In the absence of the oscillatory stresses we suppose that the surface of the solid is exposed to a fluid at the plane $y = 0$, the x-axis being taken in the downstream direction and y being measured positive into the fluid. If X and Y are complex functions of x and y whose real parts denote the displacement parallel to the x- and y-axes of the material particle at (x, y) resulting from the oscillatory stresses applied by the fluid at the surface, then we can write the boundary condition at this surface as

$$\left.\left(\frac{\partial X}{\partial y}+\frac{\partial Y}{\partial x}\right)\right|_{y=0} = \tau e^{i\alpha(x-ct)}$$
$$\left.\left[\frac{\lambda}{G}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)+2\frac{\partial Y}{\partial y}\right]\right|_{y=0} = \tilde{\omega} e^{i\alpha(x-ct)} \qquad \ldots (2.1)$$

where G is the modulus of rigidity, λ is the modulus of elongation, and

$$G\tau e^{i\alpha(x-ct)}, \quad G\tilde{\omega} e^{i\alpha(x-ct)}$$

are complex functions whose real parts are the oscillatory shear and normal stresses on the surface in the directions of the x- and y-axes respectively. Here α is a real constant denoting the wave number, and c is a complex number denoting in its real part the speed of the disturbance along the surface, and its imaginary part determining the degree of amplification or attenuation with time t. Throughout the present work we shall, however, treat c as real, which is equivalent to restricting our considerations to the existence of neutral oscillations.

Supposing that the material is bounded on the interior at the plane $y = -d/\alpha$, say, there will be two additional equations to denote conditions at this interior boundary: these conditions will be later stipulated to suit the problem considered. A solution of the equations of small non-dissipative elastic displacements whose form allows these to be satisfied together with the two equations of (2.1) is given by

$$\alpha X e^{-i\alpha(x-ct)} = (A_1/r_1)\sinh r_1\eta + A_2 \cosh r_2\eta + A_3 \cosh r_1\eta + r_2 A_4 \sinh r_2\eta$$
$$i\alpha Y e^{-i\alpha(x-ct)} = A_1 \cosh r_1\eta + (A_2/r_2) \sinh r_2\eta + r_1 A_3 \sinh r_1\eta + A_4 \cosh r_2\eta \qquad \ldots (2.2)$$

where
$$r_1^2 = 1 - (c/c_1)^2, \quad r_2^2 = 1 - (c/c_2)^2, \quad \eta = \alpha y$$
$$c_1^2 = (\lambda + 2G)/\rho_S \quad c_2^2 = G/\rho_S \qquad \ldots (2.3)$$

and where ρ_S is the density of the material. Thus substituting from (2.2) in (2.1) we find that

$$2A_1 + (1 + r_2^2)A_4 = \tau$$
$$2A_2 + (1 + r_2^2)A_3 = i\tilde{\omega} \qquad \ldots (2.4)$$

and/

and application of the interior boundary conditions will generally yield two other equations enabling the A-coefficients of (2.2) to be determined in terms of the surface stresses.

If the material exerts some dissipative effect, then the equation (2.2) may still be applied if we suppose that this effect can be represented as for a 'Voigt-Solid'[5] by expressing the stress tensor in the form

$$\Pi_{nm} = \left[\lambda + (\mu_1 - 2\mu_2)\frac{\partial}{\partial t}\right]\frac{\partial X_j}{\partial x_j}\delta_{nm} + \left(G + \mu_2\frac{\partial}{\partial t}\right)\left(\frac{\partial X_n}{\partial x_m} + \frac{\partial X_m}{\partial x_n}\right) \quad \ldots (2.5)$$

with μ_1 and μ_2 suitably chosen constants. In this expression X_n, x_n are respectively the displacements and the co-ordinates, and δ_{nm} is zero unless $n = m$, when it is unity. Evidently μ_1 and μ_2 are 'coefficients of viscosity' for the material, and further $\mu_1 = 4\mu_2/3$ if the mean normal stress is independent of the rate of change of dilatation. If μ_1 and μ_2 are not both zero, the equation (2.3) needs some modification in order that (2.2) should satisfy the equations of displacement: we find in fact that

$$c_1^2 = (\lambda + 2G - i\alpha\mu_1 c)/\rho_s, \quad c_2^2 = (G - i\alpha\mu_2 c)/\rho_s . \quad \ldots (2.6)$$

This is the only modification needed, and equation (2.4) remains unaltered.

3. Expression of Surface Stresses in terms of Fluid Stream Function

We suppose that the fluid velocity components parallel to the axes are (u, v) and if the fluid is incompressible these can be represented by a stream function ψ where

$$u = \mathcal{R}\left\{\frac{\partial\psi}{\partial y}\right\}, \quad v = \mathcal{R}\left\{-\frac{\partial\psi}{\partial x}\right\}. \quad \ldots (3.1)$$

As is usual in such problems, we separate ψ into two parts denoting the steady flow and a superposed oscillation by writing:

$$(\alpha/c)\psi = \int^{\eta} U(\eta)d\eta + \phi(\eta)e^{i\alpha(x-ct)} \quad \ldots (3.2)$$

where $U(\eta)$ is real, and $\phi(\eta)$ a complex function. Supposing that $|\phi|$ is sufficiently small, the equations of fluid motion then show that the static pressure of the fluid is p, say, where

$$[p - p_1(x)]/\rho c^2 = \mathcal{R}\left\{\left[U'\phi + (1-U)\phi' + \frac{i}{R}(\phi' - \phi''')\right]e^{i\alpha(x-ct)}\right\} \quad \ldots (3.3)$$

where ρ is the fluid density and

$$R = \rho c/\alpha\mu . \quad \ldots (3.4)$$

It is also to be noted that in accordance with the boundary-layer approximation to the viscous flow over a plate:

$$\frac{1}{\rho c^2}\frac{dp_1}{dx} = \frac{\alpha}{R}U''(0) . \quad \ldots (3.5)$$

Now, denoting by a subscript ϕ the oscillatory parts of p, u and v, the oscillatory stresses at the deformed surface where $y = \mathcal{R}\{Y\}|_{(x_0,0)} = Y_0$, say, are given by

G$\mathcal{R}$/

$$\left.\begin{aligned} G\mathcal{R}\{\varpi e^{i\alpha(x-ct)}\}\Big|_{x=x_0} &= \left(-p_\phi + 2\mu \frac{\partial u_\phi}{\partial X}\right)\Bigg|_{(x_0+X_0,Y_0)} \\ G\mathcal{R}\{\tau e^{i\alpha(x-ct)}\}\Big|_{x=x_0} &= \mu\left(\frac{\partial u_\phi}{\partial y} + \frac{\partial v_\phi}{\partial x}\right)\Bigg|_{(x_0+X_0,Y_0)} \end{aligned}\right\} \quad . \quad \ldots (3.6)$$

Remembering that we are dealing with small fluid oscillations, causing small displacements at the surface, we retain only first-order quantities in X_0 and Y_0, or ϕ, and find from (3.1), (3.2), (3.3) and (3.5) in (3.6) that

$$(G\varpi/\rho c^2)\Big|_{x=x_0} = -\frac{1}{R}U''(0)\alpha X_0 e^{-i\alpha(x_0-ct)} - U'(0)\phi(0) - \phi'(0) + \frac{i}{R}[\phi'''(0)+\phi'(0)]$$

$$(G\tau/\rho c^2)\Big|_{x=x_0} = \frac{1}{R}U''(0)\alpha Y_0 e^{-i\alpha(x_0-ct)} + \frac{1}{R}[\phi''(0)+\phi(0)]\ .$$

Substituting for X_0 and Y_0 from (2.2) by noting that these are the surface displacements at $\eta = 0$, and substituting for τ and ϖ from (2.4) we find after some algebra that

$$\left.\begin{aligned} &\left[2+\frac{i}{R}\sigma U''(0)(1-r_2^2)\right]A_1 + \left[(1+r_2^2)+\frac{i}{R}\sigma U''(0)(1-r_2^2)\right]A_4 = \frac{\sigma}{R}(1-r_2^2)[\phi''(0)+\phi(0)] \\ &\left[2+\frac{i}{R}\sigma U''(0)(1-r_2^2)\right]A_2 + \left[(1+r_2^2)+\frac{i}{R}\sigma U''(0)(1-r_2^2)\right]A_3 \\ &\qquad = -\sigma(1-r_2^2)\left\{[U'(0)\phi(0)+\phi'(0)]i + \frac{1}{R}[\phi'''(0)+\phi'(0)]\right\} \end{aligned}\right\}$$

$$\ldots (3.7)$$

Here we have written

$$\sigma = \rho/\rho_s$$

which can be called the relative density: if σ is large the surface material is evidently light, but if σ is small it is heavy, relative to the fluid.

4. Boundary Condition for the Fluid at the Surface

The boundary conditions at the surface within the fluid are the kinematic condition that

$$v\Big|_{(x_0+X_0,Y_0)} = \mathcal{R}\left\{\frac{\partial Y}{\partial t}\right\}\Bigg|_{(x_0,0)} \qquad \ldots (4.1)$$

and the assumed 'no-slip' condition

$$u\Big|_{(x_0+X_0,Y_0)} = \mathcal{R}\left\{\frac{\partial X}{\partial t}\right\}\Bigg|_{(x_0,0)}\ . \qquad \ldots (4.2)$$

Remembering/

Remembering, again, that we are concerned only with first-order effects of a small oscillation, we can interpret (4.2) from (3.1) and (3.2) as

$$U'(0)Y_0\alpha + \phi'(0)e^{i\alpha(x_0-ct)} = \frac{1}{c}\frac{\partial X}{\partial t}\bigg|_{(x_0,0)}$$

and so on using (2.2) we have

$$A_2 + A_3 = i\phi'(0) + U'(0)(A_1 + A_4) . \qquad \ldots (4.3)$$

Again from (4.1) we have similarly

$$A_1 + A_4 = i\phi(0) \qquad \ldots (4.4)$$

and evidently (4.3) can be written as

$$A_2 + A_3 = i[\phi'(0) + U'(0)\phi(0)] . \qquad \ldots (4.5)$$

Together with the two as yet undetermined boundary conditions to be applied at the interior solid surface at $y = -d/\alpha$, the pair of equations (3.7) and equations (4.4) and (4.5) are together six linear equations in A_1, A_2, A_3, A_4 and, say,

$$A_5 = i\phi(0) , \quad A_6 = i[\phi'(0) + U'(0)\phi(0)] , \qquad \ldots (4.6)$$

relating these to $\phi''(0)$ and $\phi'''(0)$. We may conveniently summarise them in the form

$$\sum_{n=1}^{6} a_{mn}A_n = a_m \qquad (m = 1,2, \ldots 6) \qquad \ldots (4.7)$$

where the equations for $m = 1,2$ are taken to relate to the interior surface boundary condition, and the a_{mn} and a_m are given in Table 1 below for $m = 3, 4, 5$ and 6, read from (3.7), (4.4) and (4.5), but simplified by the omission of the terms of order σ/R, where these occur compared with unity, Evidently A_1, A_2, A_3 and A_4 can be eliminated to provide the boundary condition at the surface within the fluid, which will be a connection between A_5 and A_6 given by (4.6) and $\phi''(0)$ and $\phi'''(0)$.

Table 1

Values of a_{mn} and a_m in equation (4.7)

m \ n	1	2	3	4	5	6	a_m
3	1	0	0	$\frac{1}{2}(1+r_2^2)$	0	0	$\frac{\sigma}{2R}(1-r_2^2)\,\phi''(0)$
4	0	1	$\frac{1}{2}(1+r_2^2)$	0	0	$\frac{\sigma}{2}(1-r_2^2)$	$\frac{-\sigma}{2R}(1-r_2^2)\,\phi'''(0)$
5	0	1	1	0	0	-1	0
6	1	0	0	1	-1	0	0

In the classical theory of oscillatory disturbances in viscous flow, the equations of fluid motion are solved in such a way that ϕ is represented as the sum of two functions, namely,

ϕ/

$$\phi = C_1\Phi(\eta) + C_2 f(\zeta) \qquad \ldots (4.8)$$

say, where

$$\zeta = (\eta-\eta_1)(RU_1')^{1/3}, \quad U(\eta_1) = 1, \quad U_1' \equiv U'(\eta_1) \qquad \ldots (4.9)$$

and where, in general, Φ is independent of R. If c is real, the function $f(\zeta)$ is taken to satisfy the condition $f(\zeta) \to 0$ as $\zeta \to +\infty$, together with the equation

$$\left.\begin{aligned} if^{iv}(\zeta) + \zeta f''(\zeta) &= 0 \\ \text{or} \qquad if'''(\zeta) + \zeta f'(\zeta) &= f(\zeta) \end{aligned}\right\} . \qquad \ldots (4.10)$$

It follows from (4.8), (4.9) and (4.10) that, for $R \to \infty$,

$$\frac{1}{R}\phi'''(0) = -iU_1'[f(\zeta_1)-\zeta_1 f'(\zeta_1)]C_2 + \mathcal{O}(1/R) \qquad \ldots (4.11)$$

where

$$\zeta_1 = -\eta_1(RU_1')^{1/3} . \qquad \ldots (4.12)$$

Again,

$$\frac{1}{R}\phi''(0) = iU_1'\eta_1\zeta_1 f'(\zeta_1)G(\zeta_1)C_2 + \mathcal{O}(1/R) \qquad \ldots (4.13)$$

where for convenience we have placed

$$G(\zeta_1) = \left[2\int_\infty^{\zeta_1} f(\zeta)d\zeta - \zeta_1 f(\zeta_1)\right] \Big/ [\zeta_1^2 f'(\zeta_1)] \qquad \ldots (4.14)$$

as we may deduce using equation (4.10). Further from (4.8) and (4.9),

$$\left.\begin{aligned} \phi(0) &= C_1\Phi(0) + C_2 f(\zeta_1) \\ \text{and} \qquad \phi'(0) &= C_1\Phi'(0) - C_2[\zeta_1 f'(\zeta_1)/\eta_1] \end{aligned}\right\} . \qquad \ldots (4.15)$$

Substituting in (4.7) from (4.11), (4.13) and (4.15), we find we now have a set of six linear equations which can be represented as

$$\left.\begin{aligned} \sum_{n=1}^{6} b_{mn}B_n &= 0 \qquad (m = 1, 2, \ldots . 6) \\ \text{where} \qquad B_n &= A_n \quad (n = 1, 2, 3, 4) \\ B_5 &= C_1 , \quad B_6 = C_2 \end{aligned}\right\} \qquad \ldots (4.16)$$

$$\left.\begin{aligned} \text{and where} \quad b_{mn} &= a_{mn} \quad (n = 1, 2, 3, 4) \\ b_{m5} &= ia_{m5}\,\Phi(0) + ia_{m6}\,[U'(0)\Phi(0)+\Phi'(0)] \\ b_{m6} &= ia_{m5}\,f(\zeta_0)+ia_{m6}\,U'(0)\{f(\zeta_1)-[\zeta_1 f'(\zeta_1)/\eta_1 U_1']\}-a_m/C_2 \end{aligned}\right\} (4.17)$$

it being here assumed that $a_1 = a_2 = 0$, as we shall indeed later verify. The boundary condition now becomes the condition that the set of equations (4.17) is compatible; that is, that the determinant

$$|b_{mn}| = 0 . \qquad \ldots (4.18)$$

Or/

Or if we place $\Delta = |a_{mn}|$, and denote by Δ_{mn} the first minor of Δ corresponding to the element a_{mn} then, from (4.17), the condition (4.18) becomes

$$\Delta\{[E/(1+\lambda)]-F\} = \{[iE/U'(0)]\sum_{m=1}^{6}\Delta_{m6}\, a_m - i(1+E)\sum_{m=1}^{6}\Delta_{m5}\, a_m\}/[C_2\zeta_1 f'(\zeta_1)]$$

where

$$\left.\begin{aligned} E &= U'(0)\Phi(0)/\Phi'(0), \quad F = -f(\zeta_1)/[\zeta_1 f'(\zeta_1)] \equiv F(\zeta_1) \\ \text{and} \quad \eta_1 U_1' &= (1+\lambda). \end{aligned}\right\} \quad \ldots (4.19)$$

Thus on expansion and rearrangement, we find from Table 1, (4.11) and (4.13) that

$$E = \frac{(1+\lambda)\Delta F + D_1'(1+F) + D_2' G}{\Delta + D_3'(1+F) + D_4' G} \quad \ldots (4.20)$$

where

$$\left.\begin{aligned} D_1' = &= -\tfrac{1}{2}\sigma(1-r_2^2)\eta_1 U_1'^2 \Delta_{45} \\ D_2' &= \tfrac{1}{2}\sigma(1-r_2^2)\eta_1^2 U_1'^2 \Delta_{35} \\ D_3' &= \tfrac{1}{2}\sigma(1-r_2^2)\eta_1 U_1'^2 [\Delta_{45}U'(0) - \Delta_{46}]/U'(0) \\ D_4' &= \tfrac{1}{2}\sigma(1-r_2^2)\eta_1^2 U_1'^2 [\Delta_{36} - \Delta_{35}U'(0)]/U'(0) \end{aligned}\right\} \quad \ldots (4.21)$$

Equation (4.20) is rather too complicated for a general review such as we intend here, and we shall simplify it on the basis that both $|F|$ and $1/U_1'$ (or $1/U'(0)$) are generally small quantities. As is well-known such assumptions, or others similar to them, figure prominently in the theory of hydrodynamic stability (in the derivation, for example, of equation (4.10)). To be precise, we shall ignore terms of order

$$F^2/U_1', \quad F/U_1'^2 \quad \ldots (4.22)$$

compared with unity in both the denominator and numerator of the right-hand side of (4.20), allowing that, in general, σ and the Δ's are of unit order. Then, noting that we may demonstrate, on expanding G in terms of F, for $F \to 0$ (i.e., $\zeta_1 \to \infty$) that

$$G = F[1 - 2F + \mathcal{O}(F^2)] \quad \ldots (4.23)$$

we can simplify the right-hand side of (4.20) to read

$$E = \frac{[(1+\lambda)\Delta + D_1' + D_2']F + D'_1}{(\Delta + D_3') + (D_3' - D_2')F}. \quad \ldots (4.24)$$

Now, if we adopt the notation

$$\left.\begin{aligned} w &= 1/(1+E) \\ \mathcal{F}(\zeta_1) &= 1/(1+F) \end{aligned}\right\} \quad \ldots (4.25)$$

and note furthermore that, from Table 1,

$$\Delta = \tfrac{1}{2}\sigma(1-r_2^2)\Delta_{46} + \Delta_0, \quad \Delta_0 = \Delta_{56} \quad \ldots (4.26)$$

then (4.24) becomes

$\mathcal{F}(\zeta_1)/$

$$\mathcal{F}(\zeta_1) \;=\; \frac{[(1+\lambda)\Delta_0+D_1]w+D_2}{(\Delta_0+D_3)+\lambda\Delta w} \qquad \ldots (4.27)$$

where

$$\left.\begin{aligned} D_1 &= \tfrac{1}{2}\sigma(1-r_2^2)\eta_1 U_1'\Delta_{46}[U'(0)-U_1']/U'(0) \\ D_1-D_3+D_2 &= -\tfrac{1}{2}\sigma(1-r_2^2)\eta_1 U_1'^2\Delta_{45} \;=\; U_1'D_2^*, \text{ say,} \\ D_3 &= \tfrac{1}{2}\sigma(1-r_2^2)\eta_1 U_1'[\Delta_{46}+(1+\lambda)\Delta_{35}] \end{aligned}\right\} \qquad \ldots (4.28)$$

For the boundary-layer flow over a flat plate for which

$$U''(0) \;=\; U'''(0) \;=\; 0$$

the assumptions that

$$\lambda \;=\; 0, \qquad U_1' \;=\; U'(0) \qquad \ldots (4.29)$$

have a special justification, and then (4.27) simplifies further to

$$(\Delta_0+D_3)\mathcal{F}(\zeta_1) \;=\; \Delta_0 w+D_2 \qquad \ldots (4.30)$$

which is the form of the so-called "stability equation" we shall use in what follows. It will be seen that for an inflexible surface (4.20), (4.24) and (4.27) all reduce to the well-known form

$$\mathcal{F}(\zeta_1) \;=\; w(1+\lambda)/(1+\lambda w) \qquad \ldots (4.31)$$

(see for example, Lin[1], equation (3.6.9), p.40) whilst (4.30) is the simpler form

$$\mathcal{F}(\zeta_1) \;=\; w \qquad \ldots (4.32)$$

used usually as a first approximation to (4.31) in an iterative process. The form of stability equation proposed by Brooke-Benjamin[2] can be written as

$$\mathcal{F}(\zeta_1) \;=\; w+U_1'(D_2^*/\Delta_0) \qquad \ldots (4.33)$$

which can be regarded as a particular form of (4.30) for the case where $D_3 \ll \Delta_0$, that is in general for

$$\tfrac{1}{2}\sigma(1-r_2^2) \;=\; \tfrac{1}{2}\rho c^2/G \ll 1 .$$

We shall call this the 'rigid surface' approximation, on the grounds that it implies large G, but we see that it is inevitably seriously in error in the neighbourhood of zeros of Δ_0, which indeed play an important rôle in the stability theory. We shall reserve the term 'inflexible surface' to imply the more stringent condition that $G = \infty$, for which the mode of oscillation is given by the well-known Tollmien-Schlichting solution. A 'heavy surface' is one for which σ is small compared with unity, and a 'light surface' conversely one with large σ; in aerodynamic applications of engineering interest, it is likely that all surfaces would be 'heavy'.

5. Expressions for w

The function $\mathcal{F}(\zeta_1)$ of equation (4.27) and (4.30) is a well-known complex function of the variable ζ_1, and is, for example, tabulated by Lin (Ref.1, Table 1, p.42). Its behaviour is shown for real negative ζ_1 in Fig.1. The function w is more complicated to deal with, and most numerical assessments of it are founded upon the assumption that

$$\delta \ll 1 \qquad \ldots (5.1)$$

where/

where δ represents the ratio of boundary-layer thickness to the wavelength $(1/\alpha)$. This, for an inflexible surface at least, is in any case a justifiable assumption if $U'(0)$ is assumed large as in (4.22). Numerical estimates are often founded upon an expansion in a power series of δ^2, and the leading terms are given (according to Lin, Ref.1, equation (5.5.8), p.86) by:

$$w = 1 + U'(0)\left\{\int_0^\delta (U-1)^{-2}d\eta + [U(\delta)-1]^{-1}\right\}\{1 + O(\delta^2)\} \quad \ldots (5.2)$$

where the path of integration of the line integral is indented below the singularity at $\eta = \eta_1$, and where $U(\delta) = u_\delta/c$, u_δ being the speed of flow outside the boundary layer. Thus placing

$$w = u + iv \quad \ldots (5.3)$$

where u and v are real, we find that

$$v = -\pi[U''(\eta_1)U'(0)/U_1'^3]\{1 + O(\delta^2) + O[U(\delta)^{-3}]\} \quad \ldots (5.4)$$

$$u = U'(0)\{⨍_0^\delta(U-1)^{-2}d\eta + [U(\delta) - 1]^{-2}\}\{1 + O(\delta^2)\}\,. \quad \ldots (5.5)$$

We note that, in this approximation, v is independent of δ. In particular if $U(\delta)$ is large, we find that, for the Blasius flat plate velocity distribution

$$U''(\eta_1) = \tfrac{1}{2}\{U^{iv}(0)/[U'(0)]^2\}\{1 + O[U(\delta)^{-3}]\}$$

so that in (5.4), we find from the Blasius solution and (4.29) that, correct to terms of order $[U(\delta)]^{-3}$ and δ^2 compared with unity,

$$U(\delta) \simeq 1.92\, v^{-1/3}\,. \quad \ldots (5.6)$$

Again in (5.5), we find that

$$u = U'(0)[U(\delta)]^{-2} - \frac{1}{\pi}v\ell n[U(\delta)] + O[U(\delta)^{-1}] \quad \ldots (5.7)$$

from which we observe that, as u is in general finite, δ is of order $1/U(\delta)$. Accordingly, for a flat plate velocity distribution, from (5.6) and (5.7)

$$U'(0) \simeq [U(\delta)]^2 u \simeq 3.7\, u/v^{2/3} \quad \ldots (5.8)$$

correct to terms of order δ compared with unity.

In the arguments that follow, we shall be considering the modifications to the boundary-layer stability due to a flexible surface on a flat plate, and (5.6) and (5.8) have a simplicity and qualitative significance which renders them particularly suitable for our descriptive treatment. However, the latter in particular would not usually be judged sufficiently accurate for quantitative assessments, and we note that our approximations are severely in error wherever δ or c/u_δ is large.

We shall frequently refer to the Reynolds number R_δ based on the velocity u_δ and the boundary-layer thickness: this is evidently

$$R_\delta = R\delta U(\delta) \quad \ldots (5.9)$$

and for a flat plate velocity distribution, supposing that δ is the displacement thickness

$$R_\delta = 0.58\, R[U(\delta)]^2/U'(0)\,. \quad \ldots (5.10)$$

Further,/

Further, substituting for R from (4.12) in terms of ζ_1, and using (4.29)

$$\left.\begin{aligned} R_\delta &= 0.58\, U'(0)[U(\delta)]^2(-\zeta_1)^3 \\ &= 0.58\, u(-\zeta_1)^3/v^{4/3} \end{aligned}\right\} . \qquad \ldots (5.11)$$

Another useful relation is

$$\delta R_\delta = \tfrac{1}{3}\,[U(\delta)]^3(-\zeta_1)^3 = 2.38\,(-\zeta_1)^3/v\,. \qquad \ldots (5.12)$$

It will of course be recalled that in our treatment of neutral oscillations, ζ_1 is real (and in fact negative in the range of interest, as shown in Fig. 1).

6. Non-Dissipating Material Fixed at its Interior Surface

If the material of the solid is fixed on its surface at $y = -d/\alpha$ then here both the real parts of X and Y vanish. Thus from (2.2)

$$\left.\begin{aligned} -(A_1/r_1)\sinh r_1 d + A_2\cosh r_2 d + A_3\cosh r_1 d - r_2 A_4 \sinh r_2 d &= 0 \\ A_1\cosh r_1 d - (A_2/r_2)\sinh r_2 d - r_1 A_3\sinh r_1 d + A_4\cosh r_2 d &= 0 \end{aligned}\right\} \ldots (6.1)$$

If we identify these with the equations $m = 1$ and 2 of (4.7), then we can deduce from Table 1 that

$$\begin{aligned} \Delta_0 = {} & \tfrac{1}{4}(1+r_2^2)^2[(1/r_1 r_2)\sinh r_1 d - \sinh r_2 d - \cosh r_1 d\cosh r_2 d) \\ & + 1 + r_2^2 + r_1 r_2 \sinh r_1 d\sinh r_2 d - \cosh r_1 d\cosh r_2 d \qquad \ldots (6.2) \end{aligned}$$

$$\Delta_{45} = \tfrac{1}{2}(1-r_2^2)[r_1\sinh r_1 d\cosh r_2 d - (1/r_2)\sinh r_2 d\cosh r_1 d] \quad \ldots (6.3)$$

$$\begin{aligned} \Delta_{35}+\Delta_{46} = {} & (1+r_2^2)[(1/r_1 r_2)\sinh r_1 d\sinh r_2 d - \cosh r_1 d\cosh r_2 d + 1] \\ & + 2[r_1 r_2 \sinh r_1 d\sinh r_2 d - \cosh r_1 d\cosh r_2 d + 1] \\ & + \tfrac{1}{2}\sigma(1-r_2^2)[(r_1 r_2 + r_1^{-1} r_2^{-1})\sinh r_1 d\sinh r_2 d - \cosh r_1 d\cosh r_2 d + 1)] \end{aligned}$$
$$\ldots (6.4)$$

6.1 Very thick surface

If the surface is very thick, so that $r_1 d$ and $r_2 d$ are much greater than unity, and we suppose that both r_1 and r_2 are real and positive (i.e., $c < c_2$), then from (6.2), (6.3) and (6.4)

$$\Delta_0 = [r_1 r_2 - \tfrac{1}{4}(1+r_2^2)^2][(r_1 r_2 - 1)/4r_1 r_2]\exp[(r_1+r_2)d] \qquad \ldots (6.2A)$$

$$\Delta_{45} = \tfrac{1}{2} r_1(1-r_2^2)[(r_1 r_2 - 1)/4 r_1 r_2]\exp[(r_1+r_2)d] \qquad \ldots (6.3A)$$

$$\Delta_{35}+\Delta_{46} = [2r_1 r_2 - 1 - r_2^2 + \tfrac{1}{2}\sigma(1-r_2^2)(r_1 r_2 - 1)][(r_1 r_2 - 1)/4r_1 r_2]\exp[(r_1+r_2)d]. \quad (6.4A)$$

6.11 Surface material 'rigid'

From (4.33), (4.28), (6.2A) and (6.3A), the stability equation is

$$\mathcal{Y}(\zeta_1) = w - \tfrac{1}{4}(\rho c^2/G) r_1(1-r_2^2)U'(0)/[r_1 r_2 - \tfrac{1}{4}(1+r_2^2)^2] \quad \ldots (6.5)$$

We see that the identity between v and the imaginary part of $\mathcal{Y}(\zeta_1)$ is unaltered by the modification introduced by the flexible surface, which is

seen/

seen to affect only the real part of w. Thus from (5.5) and (5.11), if $\overline{U'(0)}$ is the solution of the Eigen-value equation (4.30) for an inflexible surface corresponding to a certain $U(\delta)$ and ζ_1, there will exist a solution for a flexible surface for which $U(\delta)$ and ζ_1 are the same, but δ is changed so that

$$\left.\begin{aligned}(\delta/\bar{\delta}) = [\overline{U'(0)}/U'(0)] = (\bar{R}_\delta/R_\delta) &= 1 - \tfrac{1}{4}(\rho u_\delta^2/G)(1-r_2^2)r_1/[r_1r_2 - \tfrac{1}{4}(1+r_2^2)^2] \\ &= 1 - \chi\,, \quad \text{say}\end{aligned}\right\} \quad (6.6)$$

the barred quantities referring to the values for an inflexible surface. For small c/c_2 we find that

$$\tfrac{1}{2}(1-r_2^2)r_1/[r_1r_2 - \tfrac{1}{4}(1+r_2^2)^2] \rightarrow [1-(c_2/c_1)^2]^{-1} = 4(c_2/c_4)^2$$

whilst for some $c = c_3 < c_2$, the expression in braces in (6.6) vanishes, this speed corresponding to that of the well-known Rayleigh surface waves in a thick homogeneous and isotropic material. Thus, for $c < c_3$, we see from (6.6) that a neutral oscillation propagated at this speed will have a larger wavelength (i.e., δ will be smaller) and occur at a larger R_δ than when the fluid is in contact with an inflexible surface. Indeed for some values of c close to c_3, but less than it, no solution exists and apparently no neutral oscillation can exist. However, for $c > c_3$ we see that the expression in braces in (6.6) changes sign, and δ is increased and R_δ decreased compared with inflexible surface values. For speeds close to, but greater than, c_3 we see in fact that R_δ would be very small, suggesting that if any neutral oscillation exists on an inflexible surface for which $c > c_3$, then neutral oscillations can exist on a flexible surface at indefinitely small R_δ. However, plainly our solution is then not strictly valid as the wavelength is also indefinitely small, so that δ is large. On the other hand, provided $c_3 > \bar{c}_{max}$, where $\bar{c}_{max}$ is the maximum speed for which neutral oscillations exist for an inflexible surface ($= 0.42\, u_\delta$ for a flat plate), it would appear that a flexible surface is stabilising, insofar as the minimum R_δ for neutral oscillations is increased. These results have all been pointed out by Brooke-Benjamin[2], but we shall now discover that they are considerably modified (even if G is large) by the effect of the term of (4.29) neglected in (4.33).

6.12 Surface materials of smaller rigidity

It will be seen that if D_3 is not neglected in the expression for $\Delta_0 + D_3$ of (4.30), then we have from (6.4A) and (4.25),

$$\left.\begin{aligned}\Delta_0 + D_3 &= \{r_1r_2 - \tfrac{1}{4}(1+r_2^2)^2 + \sigma(1-r_2^2)[r_1r_2 - \tfrac{1}{2}(1+r_2^2)] - \tfrac{1}{4}\sigma^2(1-r_2^2)^2(1-r_1r_2)\} \\ &\qquad \times \frac{r_1r_2-1}{4r_1r_2}\exp[(r_1+r_2)\,d] \\ &= \tfrac{1}{4}(1-r_2^2)^2\{\sigma-[1-(r_1r_2)^{\frac{1}{2}}]^{-1}+2(1-r_2^2)^{-1}\}\{\sigma-[1+(r_1r_2)^{\frac{1}{2}}]^{-1}+2(1-r_2^2)^{-1}\} \\ &\qquad \times \frac{(r_1r_2-1)^2}{4r_1r_2}\exp[(r_1+r_2)\,d]\end{aligned}\right\} \quad \ldots (6.7)$$

and we find that this expression has the same sign as Δ_0 for small c, but vanishes for $c = c_3(\sigma)$, say, where $c_3(\sigma) < c_3(0)$, which, of course, is the speed c_3 mentioned in the previous paragraph as the Rayleigh

surface/

surface wave speed. The values of $c_3(\sigma)$ are shown in Fig. 3. Thus (Δ_0+D_3) maintains the same sign for all c in the range between $c_3(\sigma)$ and c_2, and so has the opposite sign to Δ_0 in $c_3 > c > c_3(\sigma)$, but the <u>same</u> sign for $c_2 > c > c_3$. The interior boundary condition is ambiguous if $d \to \infty$ with $c > c_2$, since then Δ_0 and the other determinants become periodic functions of d, implying disturbances which do not attenuate on penetrating the depth of the material. Thus we exclude for the present the consideration of oscillations propagated at such high speeds.

From (5.3) and (4.30), we find that

$$v = (\Delta_0+D_3)I/\Delta_0 \qquad \ldots (6.8)$$

where

$$I \equiv \mathcal{I}\{\mathcal{F}(\zeta_1)\} .$$

From (5.6), v should be small for our solution to be valid, and this generally implies small I. However, we shall apply the solution to finite I as it may, nonetheless, preserve some qualitative significance. In relating u to $H \equiv \mathcal{R}\{\mathcal{F}(\zeta_1)\}$ we observe that in (4.30) we must use the expression for D_2 which from (4.28) and (4.29) is

$$D_2 = D_2^* U'(0)+D_3 = \tfrac{1}{2}\sigma(c/c_2)^2[-\Delta_{45}U'(0)+(\Delta_{46}+\Delta_{35})] .$$

Accordingly, from (4.30) and (5.7), we can form the following relations:

$$\{U'(0)/[U(\delta)]^2+(D_2^*/\Delta_0)U'(0)+(D_3/\Delta_0)+\mathcal{O}[U(\delta)^{-1}] = (\Delta_0+D_3)H/\Delta_0 \ldots (6.9a)$$

$$U'(0)/[U(\delta)]^2 = (\Delta_0+D_3)H/[(1-\chi_1)\Delta_0]$$

where

$$\chi_1 = -(D_2^*/\Delta_0)[U(\delta)]^2\{1+[D_3/D_2^*U'(0)]\}+\mathcal{O}(\delta) . \qquad \ldots (6.9b)$$

$$\left.\begin{aligned} u &= (\Delta_0+D_3)H^*/[(1-\chi)\Delta_0] \\ \text{where} \quad H^* &= H-D_3/(\Delta_0+D_3) \\ \text{and} \quad \chi &= -(D_2^*/\Delta_0)[U(\delta)]^2 = \tfrac{1}{2}\sigma(u\delta/c_2)^2(\Delta_{45}/\Delta_0) \end{aligned}\right\} . \qquad \ldots (6.9c)$$

In view of the crudity of the approximation involved in the expression (5.8) for u, some simplification of these relations need not substantially detract from their accuracy or, what is more important, their qualitative significance. Thus, from (6.9b), if we can assert that (D_3/D_2^*) is bounded, then the replacement of χ by χ_1 in this equation involves in general no more crude an assumption than the use of (5.8) in place of (5.7). However, an exception arises in the neighbourhood of $\chi = 1$ where H must be replaced by H^* as in (6.9c). Now $\chi = 1$ must correspond either with $\delta = 0$, $H^* = 0$ or with $U(\delta) = 0$, since (Δ_0+D_3) would not in general be zero for $\chi = 1$ nor Δ_0 infinite. The condition $U(\delta) = 0$ implying $c = \infty$ would, in any case, be outside the range of validity of our analysis, and if $\delta = 0$ we see from (6.9b) and (6.9c) that $\chi_1 = \chi$. Thus the replacement of χ_1 by χ in (6.9b) only results in additional and significant errors where $\chi \to 1$ as $H \to 0$; the correct behaviour is then represented by $H^* \to 0$.

We can sum up by stating that

$$\left.\begin{aligned} u = U'(0)/[U(\delta)]^2 &= (\Delta_0+D_3)H/[(1-\chi)\Delta_0] \\ \text{for} \quad (D_3/D_2^*) = \mathcal{O}[U(\delta)], &\quad \text{and excluding } 1-\chi = \mathcal{O}(H) \text{ as } H \to 0 \end{aligned}\right\} . \qquad (6.10)$$

The restriction on (D_3/D_2^*) usually, as in the present context, is in the form of an upper bound on σ. Thus from (6.3A) and (6.4A), for $c < c_2$,

$|D_3/D^*_2|/$

$$|D_3/D_2^*| = |(\Delta_{46}+\Delta_{35})/\Delta_{45}| = O\,[\sigma(c/c_2)^2] \quad \text{as } \sigma\to\infty$$
$$= O\,(1) \quad \text{as } \sigma\to 0$$

and (6.10) is applicable, except possibly near $H = 0$, without significant error if

$$\sigma(u_\delta/c_2)^2 = \rho u_\delta^2/G = O\,[U(\delta)^3]\,.$$

In other words, our analysis would be inapplicable to indefinitely large values of $\rho u_\delta^2/G$.

We shall now consider possible solutions to the Eigen-value problem posed by equations (6.8), (5.6) and (6.10), assuming in turn values of c in the ranges $(0,\ c_3(\sigma))$, $(c_3(\sigma),\ c_3)$ and $(c_3,\ c_2)$.

(i) $\underline{c < c_3(\sigma)}$

From (6.8) and (5.6) we see that we have a connection between ζ_1 and $U(\delta)$ given in broad essentials by a relation of the form

$$I = Pc_3^2(\sigma)(c/u_\delta)^3/[c_3^2(\sigma)-c^2]$$

where P is positive. We see that this gives a value of c for each and every value of ζ_1 for which $I > 0$; and $c\to 0$ as $I\to 0$, which corresponds to $\zeta_1\to -\infty$ and $\zeta_1\to -2.3$ (approximately). As P can be shown to be a monotonic function of c, it follows that c will be maximum for $I = I_{max}$, and the smaller the value of $c_3(\sigma)/u_\delta$ the closer will this maximum of c be to $c_3(\sigma)$; but note that $c_{max} < c_3(\sigma)$, and there is no solution (and therefore no neutral oscillation) in the range $c_3(\sigma) > c > c_{max}$.

Supposing that the value of χ for $c = c_{max}$ is less than unity, it follows from (6.10) that there will also be a positive value for $U'(0)/[U(\delta)]^2$ - and so for δ - for each and every ζ_1 in the range of positive I, and it follows from (5.11) that on a $c - R_\delta$ diagram neutral oscillations would lie on a looped curve with two arms stretching to $R_\delta = \infty$ (as $\zeta_1\to-\infty$ or $\zeta_1\to -2.3$) and with its lobe bounded above by $c = c_{max} < c_3(\sigma)$. Using equations (5.11), (6.8) and (6.10) we see that

$$R_\delta = 0.58(-\zeta_1)^3(H/I^{4/3})[\Delta_0/(\Delta_0+D_3)]^{1/3}(1-\chi)^{-1} \qquad \ldots (6.11)$$

where χ is given as before by (6.6), and consequently as χ is positive in the range considered,

$$R_{min} > \bar{R}_{min} = 0.58(-\zeta_1)^3(H/I^{4/3})_{min}$$

where $\bar{R}_{min}$ is the minimum value of R_δ for which neutral $U'(0)$ remains finite and bounded, but since u and v_3 are bounded in this limit, (5.11) shows that $R_\delta\to\infty$, in proportion to $(-\zeta_1^3)$. In this range therefore, we find on a c-R_δ or δ-R_δ diagram, as in Fig.4b, a continuous extension of the incomplete curve of neutral oscillations propagated at speeds between $c_3(\sigma)$ and c_3, extending it now to $R_\delta = \infty$, with the c-R_δ curve asymptotic to $c_3(0)$ from above, and the δ-R_δ curve likewise asymptotic to a finite limit, $\delta = \delta_\infty$ (say), where

$$\delta_\infty = 0.58(u_\delta/c_3)[1-(c_3/c_1)^2]^{\frac{1}{2}}/\{[2-(c_3/c_2)^2][1+\sigma(c_3/c_2)^2-\tfrac{1}{4}\sigma(c_3/c_2)^4)]\}$$
$$= 0.54(u_\delta/c_3)/(1+0.7\sigma)\,, \quad \text{for } c_2 \ll c_1\,.$$

At/

At the finite Reynolds number where $c = c_3$, (i.1., where $I \to 0$ for finite ζ_1) the value of $\delta = 0.435\ \delta_\infty$, since H then has the approximate value 2.3; thus δ tends to δ_∞ from below: this Reynolds number is from (5.11)

$$R_\delta = 9.4\ (u_\delta/c_3)^3/\delta_\infty .$$

We are asserting a mode of neutral oscillation, with finite speed ($c = c_3$) and wavelength, to exist for $R_\delta = \infty$; however, it will be observed that, as we have assumed in our solution that both c/u_δ and δ are small, this cannot be confidently asserted in general for very small u_δ/c_2 (which would imply that c/u_δ is large), nor for large u_δ/c_2 (as then δ would be large). Indeed, our solution is only strictly valid for large σ and for a limited range of values of u_δ/c_2. Nonetheless, $U'(0)$ will be observed to be generally unaffected by the magnitude of u_δ/c_2, so that in the limit $|\zeta_1| \to \infty$, which corresponds to $R_\delta \to \infty$, the quantities of (4.22) remain small; thus our error results merely in the adoption of the simplified forms of the expressions (5.6) and (5.8) for u and v, and it seems probably that the solution will indeed have a wider range of applicability than we have just suggested.

Note that in the range of speeds here considered, we find (as we did for $c < c_3(\sigma)$) that it is possible to derive solutions as a transformation of the values for neutral oscillations on an inflexible surface. In fact, oscillations exist over a flat inflexible surface. Further we see from (5.12) that

$$R_\delta \delta = -2.38\ (\zeta_1^3/I)[\Delta_o/(\Delta_o + D_3)]$$

so that on a δ-R_δ diagram, not only will the looped curve be shifted to higher R_δ, but δR_δ will also be increased, compared with the values for the same ζ_1 for the Tollmien-Schlichting oscillation. Apparently the value of δ is decreased by the effect of the χ term, and decreased by the effect of the D_3 term: the nett effect would generally be in doubt.

If χ increases to unity for some $c = c^* < c_{max}$, (where values of c^* are shown in Fig. 2), then δ will tend to vanish as $\chi \to 1$, $U'(0)$ will tend to infinity, though $U(\delta)$ will remain bounded, and also from (6.11) we see that $R_\delta \to \infty$. There will then be two loops on a c-R_δ diagram, (Fig. 4(a)) as $\chi = 1$ will have two roots if regarded as an equation for ζ_1, corresponding to the same value of $c = c^*$. These two loops would each have the same two asymptotes $c = 0$ and $c = c^*$, and both curves lie everywhere in the range $c < c^*$. Likewise the δ-R_δ diagram would have two lobes, each of the four arms asymptotic to $\delta = 0$ at $R_\delta = \infty$. As before, of course, $R_{min} > \bar{R}_{min}$ for both lobes.

The condition $\chi = 1$ for $c < c_{max}$ is inevitably satisfied for sufficiently large values of $u_\delta/c_3(\sigma)$, or for sufficiently large $\rho u_\delta^2/G$. Indeed, it would appear from (6.6) that χ is least for $c \to 0$, and

$$\chi|_{c=0} = \tfrac{1}{2}\rho u_\delta^2/\{G[1-(c_2/c_1)^2]\} = 2\sigma(u_\delta/c_4)^2 \qquad \ldots (6.12)$$

so that if $\sigma(u_\delta/c_4)^2 > \frac{1}{2}$ we see that $\chi > 1$ for all non-zero c; and so no neutral oscillation would exist in the range of speed of propagation considered above.

(ii) $c_3(\sigma) < c < c_3$

We observe from (6.8) that $I < 0$, and as negative values of I are bounded (by $I = I_{min} < 0$, say) there will be no solution for $c_3(\sigma) < c < c_{min}$ say, where evidently c_{min} is the root of an equation of the form

$1/(-I_{min})/$

$$1/(-I_{min}) = P(u_\delta/c)^3[c^2-c_3^2(\sigma)]/(c_3^2-c^2)$$

where P is positive. We first suppose that $c_{min} > c^*$; indeed, we have already suggested that, under certain circumstances, $c^* < c_3(\sigma)$, and then plainly this supposition would be true. We see that (6.8) will have a solution corresponding to all negative I, c increasing as I decreases; in particular as $I \to 0$ through negative values, we observe that $c \to c_3$. In regard to the corresponding values of u, we note that from (6.10) it will be positive and finite, - and so also $U'(0)\ \delta$ and R_δ will be finite, - at the speed $c = c_{min}$ (provided that $c_{min} < c^*$). Corresponding to smaller values of $|\zeta_1|$, both I and H will decrease, the latter becoming zero whilst I is still negative. (As shown in Fig.1, this happens for $|\zeta_1|$ a little less than 0.9.) As $c < c_3$ for $I < 0$, and $U(\delta)$ is finite, it follows from (6.10) that $U'(0)$ will tend to zero with H, and likewise from (6.11), so will R_δ. (Plainly this also implies $\delta \to \infty$, and this magnitude, and that of R_δ, are both quite outside the range of validity of our theory, so this result can only be treated as displaying a trend.) On the other hand, as $|\zeta_1|$ increases towards 2.3, (-I) will decrease until in the limit of $I \to 0$ we have seen that $\Delta_0 \to 0$ and $c \to c_3$.

In this range, therefore, neutral oscillations will lie in an incomplete curve on a c-R_δ or δ-R_δ diagram, extending from indefinitely small R_δ to some finite value, with c increasing from c_{min} to c_3, and δ decreasing from some indefinitely large value. Equation (5.12) shows in particular that because v decreases whilst $|\zeta_1|$ increases (being inversely proportional, by (5.6), to $[U(\delta)]^3$), therefore δR_δ increases over a bounded interval as R_δ increases from zero.

The condition that $c^* > c_{min}$ is unlikely to lie within the range of validity of our theory, and in any case it hardly corresponds to physical conditions of much interest. It requires a very large value of σ, but yet a relatively small value of $\frac{1}{2}\rho u_\delta^2/G$, and so the value of c_2/u_δ is inevitably large. Even for indefinitely large σ, - such that $c_3(\sigma)/c_2$ is small - the condition that $c^* = c_3(\sigma)$ can be shown to imply that

$$(c/u_\delta)^2 = 1/[1+(c_2/c_1)^2]$$

and since c_2/c_1 is usually small compared with unity, and in any case always bounded, this is a ratio not very much less than unity. If we were to consider speeds of propagation $c = c^* > c_3(\sigma)$, plainly not only is c increased, but u_δ must be lower, for the same σ. Consequently (c/u_δ) is inevitably large. For what it is worth, we may however deduce that if $c^* > c_{min}$, then the neutral oscillations may appear on a c-R_δ diagram asymptotic from above to $c = c^*$ at infinite R_δ (and δ is zero at $R_\delta = \infty$). At $c = c_3$ the values of δ and R_δ are - as deduced above - both finite.

(iii) $\underline{c_3 < c < c_2}$

In this range $(\Delta_0+D_3)/\Delta_0$ is positive and likewise so also is $(\Delta_0+D_3)/[\Delta_0(1-\chi)]$ which in particular is bounded. Thus there will in general be a solution for each and every ζ_1 for which, from (6.8), $I > 0$: and plainly for $c \to c_3+0$, there is a solution which is continuous with that for $c \to c_3-0$ from below. As $|\zeta_1|$ increases above 2.3 (corresponding, approximately, to $I = 0$), (6.8) shows that c will rise, reaching a maximum, and will then decrease to $c = c_3$ as $|\zeta_1| \to \infty$ and $I \to 0$ again. However, it should be noted that if

$[c_2^2c_4/u_\delta^3(1+\sigma)^2]/$

$$[c_2^2 c_4/u_\delta^3(1+\sigma)^2] < (I_{max}/3.57) \simeq 0.04$$

then some values of I close to I_{max} would give no solution, because $c \simeq c_2$ for $I < I_{max}$. As $|\zeta_1|$ increases to infinity, (6.10) shows that in particular, if $\bar{c}$ is the speed of propagation of such an oscillation corresponding to a certain ζ_1, then (5.6) and (6.8) show that $c > \bar{c}$. Likewise $R_\delta < \bar{R}_\delta$ for identical ζ_1.

6.13 The 'heavy surface' approximation

It is instructive to compare the 'heavy surface' solution obtained from paragraph 6.12 by allowing $\sigma \to 0$ with the 'rigid surface' approximation of Brooke Benjamin[2], outlined above in paragraph 6.11. If $c_3 > \bar{c}_{max}$, the maximum speed at which Tollmien-Schlichting oscillations are propagated (over an inflexible surface), then in the notation of paragraph 6.12 (i) $c^* > c_{max}$ and there is essentially no significant difference between the single-lobe c-R_δ and δ-R_δ diagrams discussed in paragraph 6.12 (i), and the equivalent diagrams deduced from paragraph 6.11. However, if $c_3 < \bar{c}_{max}$ paragraph 6.11 shows that these diagrams become double lobed in the region $c < c_3$, both c-R_δ curves being asymptotic to c_3 from below; whereas, by comparison, it is possible to show from the arguments of paragraph 6.12, and equations (6.6) and (6.7) that c^* is only less than c_{max} for $c_3 < 0.95\,\bar{c}_{max}$ (if σ is small), and provided this condition is met the curves become double-lobed, the c-R_δ curves being asymptotic to a value which is only slightly less than c_3 if σ is small.

Again, the 'rigid surface' approximation shows no other mode of neutral oscillation if $c_3 > \bar{c}_{max}$, but an additional lobe with $c > c_3$ if $c_3 < \bar{c}_{max}$, both arms of the c-R_δ being asymptotic to $c = c_3$ from above at $R_\delta = 0$, and both arms of the δ-R_δ curve tending to infinity at $R_\delta = 0$. However, the analysis of paragraph 6.12 shows a neutral oscillation to exist at all Reynolds numbers, irrespective of the value of σ, - the value of c being everywhere close to c_3 if σ is small, and the c-R_δ curve being asymptotic to c_3 from above at $R_\delta = \infty$ only. The δ-R_δ diagram curve shows that δ is infinite at $R_\delta = 0$, but finite for $\delta \to \infty$. It is in this range of c that the differences between the solutions are most accentuated, because a 'heavy surface' impresses its own neutral oscillations on the boundary layer.

For very small c_3/u_δ, the difference is even more marked: in particular we have seen that for $(c_4/u_\delta)^2 < 2\sigma$, the lobes in the region $c < c_3$ disappear altogether according to the analysis of paragraph 6.12, though of course such an inequality does not arise if $\sigma \to 0$.

6.2 Thin surface

We defined a 'thick' surface as one having indefinitely large thickness, and if we likewise define a 'thin' surface as one having indefinitely small thickness and evaluate equations (6.2), (6.3) and (6.4) in the limiting condition $d \to 0$, we find

$$\Delta_0 = -c^4/4c_2^4[1+\mathcal{O}(r_2^2 d^2)] \quad \ldots (6.2B)$$

$$\Delta_{45} = -c^4 d/2c_1^2 c_2^2[1+\mathcal{O}(r_2^2 d^2)] \quad \ldots (6.3B)$$

$$\Delta_{35}+\Delta_{46} = \sigma c^6 d^2/2c_1^2 c_2^4[1+\mathcal{O}(r_2^2 d^2)+\mathcal{O}(c_2^2/\sigma c^2)] \quad \ldots (6.4B)$$

In the limit we see from (4.28) that D_2 and D_3 will be negligible compared with Δ_0, and (4.30) becomes identical with (4.32) - the stability equation for an inflexible surface. This, of course, is quite

understandable/

understandable, but as we are interested in conditions in which the surface elasticity has some appreciable effect upon the stability, it is evident that this limit is not of itself of interest.

An examination of the orders of magnitude involved shows that the term D_2 has a finite effect if $|r_2 d|$ (or $|r_1 d|$) is small, provided that

$$c_1^2/\sigma u_\delta^2 = \mathcal{O}(d)$$

whilst D_3 has a finite effect if

$$c_1 c_2/\sigma c^2 = O(d) .$$

We suppose accordingly that, in our solution which is only valid if $U(\delta)$ is large,

$$\left. \begin{aligned} c/c_1 &= \mathcal{O}[U(\delta)^n] , \quad 1 > n > 0 \\ d &= \mathcal{O}[U(\delta)^{-1-n}] \end{aligned} \right\} \quad \dots (6.13)$$

and then plainly $|r_1 d|$ and $|r_2 d|$ are of order $1/U(\delta)$, and so small within the range of validity of our solution. The effects of the terms D_2 (and D_3 if $n = 1$) on the solution are then evidently appreciable within the limitation (6.13), provided σ is not small, but the error in equations (6.2B) and (6.3B) will be seen to be of order $1/U(\delta)^2$ and so negligible within our approximation, whilst that in (6.4B) is also of the same magnitude when the effect of the term D_3 is appreciable.

In practice it is convenient to relate the thickness of the surface to that of the boundary layer, by writing, say

$$d = \theta U(\delta)/U'(0) \quad \dots (6.14)$$

where θ is proportional (and of the same magnitude as) the ratio of the surface to the boundary-layer thickness. Then from (5.8), if u is in general of order unity, we can re-interpret (6.13) as

$$c/c_2 = \mathcal{O}(\theta^{-1}) , \quad \theta = \mathcal{O}[U(\delta)^{-n}] \quad \dots (6.15)$$

so that the surface thickness is, in certain conditions, assumed small compared with that of the boundary layer; nonetheless, we have asserted - and will indeed show - that its elasticity can have a significant effect, provided at least σ is not small.

The restriction on c/c_1 in (6.13) is, of course, artificial - as plainly to cover all eventualities we ought to allow that c is bounded only by the magnitude of u_δ. It will be recalled that it was necessary to assume $c < c_2$ in the discussion of the thick surface, and the present assumption is of the same category, though by no means so restrictive. It is merely a device to eliminate from consideration the 'periodic' solutions which will be the subject of the next paragraph.

Using (6.2B), (6.3B), (6.4B) we find from (6.8) and (6.10) that

$$\left. \begin{aligned} v &= (\Delta_0 + D_3)I/\Delta_0 = [1-(\sigma c^2 d/c_1 c_2)^2]I \\ \text{and} \quad u &= (\Delta_0 + D_3)H/[\Delta_0(1-\chi)] \quad \text{where} \quad \chi = \sigma u_\delta^2 d/c_1^2 \end{aligned} \right\} \quad \dots (6.16)$$

the expression for u being applicable (except possibly for $H \to 0$) if (6.15) is satisfied. Using the interpretations of u and v in (5.6) and (5.8), and noting the definition of θ in (6.14), we can recast these equations as

y/

$$\left.\begin{aligned} y &= I_o^{2/3}\, x(1-x^2)^{2/3} \\ x &= (I_o/H)\beta(1-\gamma y) \\ \text{where } I_o = (I/7.14)\,, \quad x &= \beta/[U(\delta)U'(0)]\,, \quad y = \beta U(\delta)/U'(0), \\ \beta &= \sigma u_\delta^2\theta/c_1 c_2\,, \quad , \quad \gamma = c_2/c_1 \end{aligned}\right\} \quad (6.17)$$

We note that y is proportional to $\beta\delta$, and further:

$$\left.\begin{aligned} c/u_\delta &= (x/y)^{\frac{1}{2}} \\ R_\delta &= 0.58\beta(y/x^3)^{\frac{1}{2}}(-\zeta_1)^3 \end{aligned}\right\} . \qquad \ldots (6.18)$$

Evidently, x satisfies an equation of the type

$$\left.\begin{aligned} x^{-1}-A(1-x^2)^{-2/3} &= \bar{x}^{-1} \\ \text{where} \qquad A = \gamma I_o^{-2/3} \quad , &\quad \bar{x} = \beta I_o/H \end{aligned}\right\} . \qquad \ldots (6.19)$$

As A is essentially positive, this has one, and only one, solution for x in $(0,1)$ for any value of $\bar{x}$. If $\bar{x} < 0$, there is also a single solution with $x > 1$; and if $\bar{x} > 1$, there are possibly two solutions with $x > 1$ though these only exist if A is sufficiently small.

Following the solution for varying ζ_1, we see that for $\zeta_1 \to -\infty$ where $I \to 0$ and $H \to 1$, the solution becomes identical with that for an inflexible surface $(\beta = 0)$, for which

$$x/\beta = \bar{x}/\beta = I_o/H \;; \quad y/\beta = I_o^{1/3}/H\,.$$

Thus, δ, c and $1/R_\delta$ will all tend to zero in this limit. A solution exists for all ζ_1 for which I remains positive, in which $0 < y < 1/\gamma$ and so $\chi < 1$. From (6.17) and (6.18)

$$\begin{aligned} c/u_\delta &= [I_o(1-x^2)]^{1/3} \\ R_\delta &= 0.58(-\zeta_1)^3 H I_o^{-4/3}(1-x^2)^{-2/3}(1-\gamma y)^{-1} \end{aligned}$$

and since x and y are both positive for $\beta \neq 0$, it follows that c/u_δ is decreased, and R_δ is increased, compared with the Tollmien-Schlichting inflexible surface solution for the same ζ_1. As ζ_1 tends to -2.3, for which $I \to 0+$, the difference once again becomes negligible, and δ, c and $1/R_\delta$ once more tend to zero. Thus neutral oscillations would be plotted again as lobes on c-R_δ and δ-R_δ diagrams, but c_{max} would be smaller, and $R_{\delta\,min}$ larger, than for the Tollmien-Schlichting solution; the larger the value of β, the greater is the modification to these bounds.

As $|\zeta_1|$ further decreases, I becomes increasingly negative. We note that as $I \to 0$ from below, then $x \to 1$ and $y \to \infty$; this corresponds to $c/u_\delta \to 0$, and δ and $R_\delta \to \infty$. This indicates a breakdown in the necessary conditions (6.15) or (6.13) governing the 'thin surface' approximation, and further investigation of the equations shows that there is no 'thin surface' continuation of the solution for $I < 0$ in the neighbourhood of $I = 0$. This is not meant to imply that no solution exists in this region for finite θ, but merely that this part of the solution is inevitably associated with large values of $\delta\theta$. On the other hand, for $\delta\theta = \infty$, we have seen in paragraph 6.12 (ii) that the solution for $I \to 0-$ involves $c \to c_3$ and δ finite, which is incompatible with large $\delta\theta$ if θ is bounded as here; thus if the

'thin surface'/

'thin surface' solution is to link up with the 'thick surface' solution, it must do so either at $R_\delta = 0$ (corresponding to $H = 0$) or at $R_\delta = \infty$ (corresponding to $|\zeta_1| \to \infty$) where $\delta = \infty$. Use of the full equations would seem necessary to establish the precise form of the link, but some better indication is obtained by including the term of order $c_2^2/\sigma c^2$ in (6.4B). If this is done, it appears that $R_\delta \to 0$ and $\delta \to \infty$ for some $I < 0$ where, supposing that $\delta\theta$ is small and $c_2^2/c_1^2 < 1/3$

$$(c/c_1)^2 \to [1-3(c_2/c_1)^2]/\sigma.$$

No more credence can be attached to this limit than that obtained from (6.19) as clearly $\delta\theta$ is not small, but it shows that the inclusion of this term could account for the existence of some upper bound of R_δ in the solution, as would be indicated by the possible limit $R_\delta = 0$ at $H = 0$ for $\delta\theta = \infty$. Moreover, it confirms the trend already observed in the 'thick surface' solution for the speed of propagation to decrease with σ at $R_\delta = 0$: we found in paragraph 6.12 for instance, that $c_3(\sigma)$ varied inversely with $1/\sigma$.

In the range of $I < 0$ away from $I = 0$, where (6.19) has more accuracy, (insofar as the values of x and y may be finite) we find that there are two solutions corresponding to $x > 1$ and $x < 1$, and of these only the former is relevant as clearly, from (6.16), x must be greater than unity where $I < 0$ if v is to be positive. A solution exists for all $I < 0$ except in the neighbourhood of $\zeta_1 = 0$: here $|\zeta_1|$ reaches a minimum and then increases again whilst x continues to increase monotonically; x reaches an indefinitely large value as $H \to 0-$ through negative values, varying in proportion to $(-H)^{-3/4}$. Correspondingly we find that $y \to 1/\gamma$, so that this limit implies $R_\delta \to 0$ and $c \to \infty$, but $\delta = \delta_0$ where

$$\theta\delta_0 = 0.58\, c_1^2/\sigma u_\delta^2 .$$

The behaviour $c \to \infty$ for finite δ is however once again incompatible with (6.13), besides, of course, being incompatible with large values of $U(\delta)$ and we shall suggest, in the next paragraph 6.3, a better approximation to this particular limit.

Except, therefore, possibly at the extremes of the Reynolds number range - that is, from (6.18), except for large or small values of $R_\delta c_1 c_2/\sigma\theta u_\delta^2$, - we find that c decreases with increasing R_δ, its general magnitude being indicated by the value at $H = 0+$ (or rather at $H^* = 0+$, though from (6.9c) it can be shown that this distinction is unimportant), where

$$(c/u_\delta) = (c_2/c_1)^{1/2} x^{1/2}, \quad \text{where} \quad (x^2-1)^{2/3}/x = 2.1\,\gamma$$

and evidently x is close to unity $y\,(c_2/c_1)$ is small. This value is reached at a Reynolds number

$$R_\delta = 0.38\sigma u_\delta^2\theta/(c_1 c_2^3 x^3)^{1/2}$$

and corresponds to a wavelength such that $\delta = \delta_0$. For higher R_δ we have seen that δ increases, but for lower R_δ, δ tends to return once more to the value δ_0. We note that for small values of $\sigma u_\delta^2\theta/c_2^2$ the Reynolds number scale of the solution becomes vanishingly small, as also does the wavelength of the disturbance; since $\theta \to 0$ or $c_2 \to \infty$ corresponds to an inflexible surface, it is only to be expected that modes of oscillation such as this must cease to exist: this is additional evidence that there must be an upper bound to the Reynolds number range for which the solution exists.

6.3 Fast propagated oscillations

In both paragraphs 6.1 and 6.2, we have limited our solution to bounded values of c: for a thick surface we supposed that $c < c_2$, and

for/

for a thin surface, we took $c/c_2 = O(\sigma^{-1})$. We shall now assume that c/c_1 is large, and more specifically that $\theta c/c_1$ is large. For then we see that if we place $cd/c_2 = \psi$, it follows from (6.14) that

$$U'(0) = u_\delta \theta/\psi c_2$$

and

$$U(\delta)U'(0) = c_2\psi/\theta c \qquad \dots (6.20)$$

and consequently δ, which is of the same magnitude as $U(\delta)/U'(0)$ is small for finite ψ. The appropriate form of equations (6.2), (6.3) and (6.4) is now

$$\Delta_0 = -\tfrac{1}{4}(c/c_2)^4 \cos\gamma\psi \cos\psi + O(c^2/c_2^2) \qquad \dots (6.2C)$$

$$\Delta_{45} = -\tfrac{1}{2}(c^3/c_1c_2^2)\sin\gamma\psi\cos\psi + O(c/c_2) \qquad \dots (6.3C)$$

$$\Delta_{35}+\Delta_{46} = \tfrac{1}{2}\sigma(c^4/c_2^3c_1)\sin\gamma\psi\sin\psi + O(c^2/c_2^2) + O(\sigma c^2/c_2^2) \qquad \dots (6.4C)$$

where, as before, $\gamma = c_2/c_1$. We should more properly put $|r_2 d|$ in place of cd/c_2 in the expression for ψ to avoid complications in the order of magnitude of the error of the above equations where ψ or $\gamma\psi$ are, either of them, integral multiples of $\pi/2$, but this distinction will be ignored.

Then in (6.8), from (5.6)

$$\left.\begin{aligned} v &= 7.14/[U(\delta)]^3 = (\Delta_0+D_3)I/\Delta_0 \\ &= \left[1 - \frac{\sigma^2c^2\sin\gamma\psi\sin\psi + O(\sigma^2c_1^2) + O(\sigma c_1^2)}{c_1c_2\cos\gamma\psi\cos\psi + O(c_1^4/c^2)}\right] \end{aligned}\right\}. \qquad \dots (6.21)$$

Strictly, the error terms included should be included as they are evidently not negligible in the neighbourhood of $\psi = N\pi/2\gamma$ or $N\pi/2$, where N is any non-negative integer. However, their effect is merely to shift the zeros or poles of the right-hand side of (6.21) to values close to, but not equal to, $\psi = N\pi/2$ or $N\pi/2\gamma$, and provided we exclude $\psi = 0$, only a small difference results from their inclusion. Even at $\psi = 0$, there is no significant loss in accuracy, as we note that the expressions (6.2C), (6.3C) and (6.4C) above reduce to (6.2B), (6.3B) and (6.4B) if ψ is small, the error terms remaining negligible. Thus we can rewrite (6.21) approximately as

$$\left.\begin{aligned} 1/I_0 &= \left\{[U(\delta)]^2 - \frac{1}{\gamma}s^2\tan\gamma\psi\tan\psi\right\}U(\delta) \\ \text{where}\quad s &= \sigma u_\delta/c_1 \end{aligned}\right\} \qquad \dots (6.21a)$$

In framing the equation for u, we note that, in (6.10), D_3/D^*_2 does not satisfy the necessary bounds, and in general we must use (6.9a) instead, so that from (6.20), putting $\tau = \theta u_\delta/c_2$, we find that

$$\gamma\tau[1-sU(\delta)\tan\gamma\psi] = -(H-1)s^2\psi\tan\gamma\psi\tan\psi + \gamma H\psi[U(\delta)]^2 \qquad \dots (6.22)$$

Re-arranging (6.22) and (6.21a), we then find after some algebra that

$$\left.\begin{aligned} \gamma\psi H[U(\delta)]^2+(\gamma\tau s\tan\gamma\psi)[U(\delta)]+[(s^2\psi\tan\gamma\psi\tan\psi-\gamma\tau)Hs^2\psi\tan\gamma\psi\tan\psi] &= 0 \\ (\gamma\tau s\tan\gamma\psi)[U(\delta)]^2+(\psi s^2\tan\gamma\psi\tan\psi-\gamma\tau)[U(\delta)]+(H/I_0)\gamma\psi &= 0 \end{aligned}\right\}$$

$$\dots (6.23)$$

and since both these equations, regarded as quadratics in $U(\delta)$, must have a common solution, we find the following relation between their coefficients:

|[(H-1)/

$$\begin{vmatrix} [(H-1)s^2\psi\tan\gamma\psi\tan\psi+\tau\gamma] & \gamma\tau s\tan\gamma\psi \\ -(H/I_0)\gamma\psi & (s^2\psi\tan\gamma\psi\tan\psi-\tau\gamma) \end{vmatrix} \times \begin{vmatrix} H\gamma\psi & \gamma\tau s\tan\gamma\psi \\ \gamma\tau s\tan\gamma\psi & (s^2\psi\tan\gamma\psi\tan\psi-\tau\gamma) \end{vmatrix}$$

$$= \left\{\begin{vmatrix} H\gamma\psi & [(H-1)s^2\psi\tan\gamma\psi\tan\psi+\gamma\tau] \\ \gamma\tau s\tan\gamma\psi & -(H/I_0)\gamma\psi \end{vmatrix}\right\}^2 \qquad \ldots (6.24)$$

This, it need hardly be pointed out, is an awkward equation to analyse. However, it can be shown that for small ψ, it reduces - as would be expected - to a form compatible with (6.17), ψ being proportional to $(xy)^{\frac{1}{2}}$ in the notation of that equation. Thus for large $(xy)^{\frac{1}{2}}$ the equations (6.23) furnish a more accurate extension of (6.7). Whereas in paragraph 6.2, we suggested that $I \to 0$ from below corresponded to $x \to 1$ and $y \to \infty$ (which would indicate $\psi \to \infty$) we now see that the limit may correspond in fact to $\psi = \pi/2$ with

$$\tan\psi \sim [H\gamma\cot(\gamma\pi/2)]/[(-I_0)^{2/3}(H-1)^{1/3}s^2]\ , \quad U(\delta) \sim [(1-H)/I_0]^{1/3}$$

but this plainly implies $c \to 0$ (since $H > 1$ at this limit). As it is necessary for c to be larger than c_2 in order that the periodic character of solutions of the type we are considering here to exist at all, we may reasonably suppose that this limiting solution is invalid. In fact, in the original approximation of paragraph 6.2, it also appeared that $c \to 0$, so that it would have been surprising to find an extension of this solution, continuous with it in the limit, in the range of imaginary r_2.

However, there is another solution of paragraph 6.2 implying large (xy), and this appeared as the limit $x \to \infty$, $y \to 1/\gamma$, where $H \to 0$ through negative values. This did indeed imply $c \to \infty$, and we see now that it must correspond again with $\psi = \pi/2$, and another possible solution in the neighbourhood of $\psi = \pi/2$ is represented by

$$(H-1) \sim -(2\tau\gamma/\pi s^2)\cot(\pi\gamma/2)\cot\psi\ , \quad U(\delta) \sim \pi(H-1)/(2\tau I_0).$$

Clearly this corresponds to $H \to 1$ from below, as $\psi \to \pi/2$ from below, and implies in this limit $c \to \infty$, and so $\delta \to 0$ and $R_\delta \to 0$. In the treatment of paragraph 6.2 we found in fact that the limit corresponded to $c \to \infty$ and $R_\delta \to 0$, but the value of δ was found to be finite.

We can restrict our discussion of the solutions of (6.24) to those which correspond to large roots $U(\delta)$ of (6.23), as plainly they are the only ones which are justified by our assumptions. Indeed, of course, our solution is only strictly applicable to vanishingly small δ as well as vanishingly small c/u_δ, and we see from (6.20) that $c = 0$ is bound to imply $\delta = \infty$; thus we shall restrict our search to solutions which in general yield small, but non-zero, values of c/u_δ. For this reason we omit solutions in the region of $I = 0$, and then (6.24) shows that, for all values of $|H/I|$ in a bounded interval, large values of $U(\delta)$ must correspond to values of ψ such that approximately

$$\left.\begin{aligned} (\psi/\gamma\tau)\tan\gamma\psi\tan\psi &= s^{-2}-\gamma^{-1/2}\tan^{3/2}\gamma\psi\tan^{1/2}\psi \\ \text{and} \qquad \tau s^3\tan^2\gamma\psi\tan\psi &>> \gamma\psi \end{aligned}\right\} . \qquad \ldots (6.24a)$$

The corresponding values of $U(\delta)$ and δ are given by

$$c/c_1 = (\gamma\cot\gamma\psi\cot\psi)^{1/2}/\sigma\ , \quad \theta U(\delta)/U'(0) = \psi(\gamma\tan\gamma\psi\tan\psi)^{1/2}$$

and it may be observed that the implied value of $U(\delta)$ is such as to render zero the expression in curly brackets in (6.21a), which, of course, is a result of supposing that $U(\delta)$ is so large that $1/U(\delta)$ can be neglected.

Properly,/

Properly, of course, we should equate the curly bracket of (6.21) to zero to represent this solution, but the difference appears unimportant. A solution exists of the equation of (6.24a) in which (c_1/c), (c/u_δ) and δ are all vanishingly small for any bounded (H/I), provided that (u_δ/c_1) is indefinitely large, that σ is neither too large nor too small compared with unity, and provided that θ is not too small. This solution is given by:

$$\left.\begin{array}{l} c/c_1 = [\gamma u_\delta^2 \cot\psi/\sigma c_1^2]^{1/3}, \quad \theta U(\delta)/U'(0) = \psi[c_1^2 \tan\psi/\sigma^2 u_\delta^2]^{1/3} \\ \tan\gamma\psi \simeq (c_1/\sigma u_\delta)(c_2 \cot\psi/\sigma u_\delta)^{1/3} \end{array}\right\} \quad \ldots (6.25)$$

Since $\tan\gamma\psi$ is small, we can replace ψ by $(N\pi/\gamma)$ with good approximation, and so observe that the solution is only real if $\tan N\pi/\gamma$ is positive. If θ or σ is small, we see that δ is large, whilst if σ is large, we see that (c/c_1) is small - as opposed to the large value we have supposed it to have. From (5.11) the corresponding value of R_δ is

$$R_\delta = 0.58\,(\sigma\theta u_\delta/\psi c_2)(\sigma u_\delta \tan\psi/c_1)^{2/3}(-\zeta_1)^3 \quad \ldots (6.26)$$

but it must be recalled that the solution does not apply in the neighbourhood of those values of ζ_1 for which $I = 0$, and in particular it does not apply for $\zeta_1 \to -\infty$. (Our basic solution also breaks down near $\zeta_1 \to -\infty$ as the terms of (4.22) are then no longer small.)

Owing to the periodic character of the solution, there are evidently an indefinite number of such solutions, with δ increasing, and R_δ decreasing, as ψ takes on higher values. Plainly our solution is not valid applied to very large values of ψ, but it is probably a correct deduction that, as (θ/ψ) occurs as a parameter rather than just θ, the number of possible solutions of this type increases with θ.

The implication of (6.25) that c is independent of ζ_1 is of course only an approximation to the truth: equation (6.21a) shows that $U(\delta)$ will be increased or decreased according to whether I is positive or negative; thus as (6.25) applies both for positive and negative I, there will in fact be two separate modes of oscillation, with the values of both c and δ displaced one way or the other from those quoted in (6.25). Equation (6.26) then shows that they exist in different ranges of R_δ. We have already mentioned that our approximation is unjustified in the region of $I = 0$, but it is obscure what the appropriate form of the solution will be in this region, since so much would seem to depend on the relative positions of zeros or poles of $\tan\psi$ relative to those of $\tan\gamma\psi$, - for which there is of course no general rule. It is easily deduced, however, that there could be no continuous solution for all ψ, so that each successive value of $\psi = N\pi/\gamma$, as N increases, will indicate a different mode of oscillation.

The solution which we have demonstrated is certainly not the only one which exists, and which is within the range of validity of our analysis under certain circumstances: there are, for instance, solutions in the neighbourhood of $\psi = N\pi/2$, at least if σ is very small. In particular, if γ is very small - a possibility of some practical relevance as it corresponds to a material of large bulk modulus but small rigidity, like rubber - it is possible to show the existence of modes of oscillation with speed c where $c_2 \ll c \ll c_1$. For then, from (6.24a)

$$\psi^2 \tan\psi \simeq (\theta c_2/\sigma^2 u_\delta)(c_1/c_2)^2$$

provided that $(c_2\psi/c_1) \ll \pi/2$, and so we discover that, except for values of I in the neighbourhood of $I = 0$,

$$(c/c_2) = (\psi u_\delta/\theta c_2)^{1/2}, \qquad U(\delta)/U'(0) = \sigma^{-1}(\psi c_2/\theta u_\delta)^{1/2}.$$

Such/

Such a solution is compatible with the assumptions provided σ is not too small, and provided (ψ/θ) is bounded above and below - in other words, provided the surface is of the same order of thickness as the boundary layer.

6.4 Slowly propagated oscillations

We now assume that c/c_2 is small. This evidently covers all the possible speeds of propagation if u_δ/c_2 is small, but it also includes, of course, a possible type of oscillation even if u_δ/c_2 is not small. Expanding the right-hand-side of equations (6.2), (6.3) and (6.4) in a series of ascending powers of c/c_2, we find that

$$\Delta_0 = -\tfrac{1}{4}(c/c_2)^4[1+(1-\gamma^2)^2d^2+(1-\gamma^4)\sinh^2 d][1+O(c^2/c_2^2)] \quad \ldots (6.2D)$$

$$\Delta_{45} = -\tfrac{1}{4}(c/c_2)^4[(1+\gamma^2)\sinh d \cosh d-(1-\gamma^2)d][1+O(c^2/c_2^2)] \;. \quad (6.3D)$$

$$\Delta_{35}+\Delta_{46} = -\tfrac{1}{2}(c/c_2)^4\{(1-\gamma^2)^2d^2-\gamma^2(1+\gamma^2)\sinh^2 d+\tfrac{1}{4}\sigma(c/c_2^2)^2 \times[(1-\gamma^2)d^2-(1+\gamma^2)^2\sinh^2 d]\}[1+O(c^2/c_2^2)]. \quad \ldots (6.4D)$$

We see that these are a more general form of equations (6.2B), (6.3B) and (6.4B) to which they reduce if d is taken as small.

In equations (6.8) and (6.10) we find that D_3/Δ_0 is small unless σ is large - and this possibility will not be envisaged here, as the effect would be to produce a more complicated form of the equations (6.17) which would be difficult to analyse in general terms. Thus the stability equation reduces to Brooke-Benjamin's form (4.33) for a 'rigid' surface: namely,

$$v = I\,, \quad u = H/(1-\chi)\,, \quad \chi \equiv \chi(d) = \tfrac{1}{2}\sigma(u_\delta/c_2)^2\frac{(1+\gamma^2)\sinh d \cosh d-(1-\gamma^2)d}{1+(1-\gamma^2)^2d^2+(1-\gamma^4)\sinh^2 d} \quad \ldots (6.27)$$

This implies, as we saw in paragraph 6.11, that the only solutions are those obtained by transforming the value of δ for the Tollmien-Schlichting solution at a certain $U(\delta) = \overline{U(\delta)}$ and ζ_1 in the ratio of $(1-\chi)$ to 1, as in equation (6.6). However, this now yields an implicit equation for δ, rather than an explicit one as in (6.6); for from (6.14) we see that, if θ is known, then

$$d = \theta[\overline{U(\delta)}/\overline{U'(0)}](\delta/\bar{\delta}). \quad \ldots (6.28)$$

Thus from (6.27) we find an equation for d as follows:

$$d/[1-\chi(d)] = I_0^{1/3}\,\theta/H = [\overline{U(\delta)}/\overline{U'(0)}]\theta. \quad \ldots (6.29)$$

Now $\chi(d)$ is a monotonic, increasing function of d, and so therefore is $d/(1-\chi)$, unless $\chi(\infty)$ exceeds unity, when $d/(1-\chi)$ is negative for some large d. Thus one solution to (6.28), and one only, is always to be found for positive d. As θ is reduced, so are the values of $\chi(d)$ and d; accordingly, $\delta/\bar{\delta}$ as well therefore as $\bar{R}_\delta/R_\delta$ are increased. On the other hand as $\theta \to \infty$, we see that $d \to \infty$ if $\chi(\infty) < 1$, and so $\delta/\bar{\delta} \to (1-\chi)$. However, if $\chi(\infty) > 1$, d tends to some finite limit (as $\theta \to \infty$) corresponding to $\chi = 1$, and so $\delta \to 0$. This, of course, is compatible with the absence of a solution for $\chi(\infty) > 1$ for infinitely thick surfaces as we found in paragraph 6.1; there the value of $\chi(\infty)$ appeared as $\chi|_{c=0}$, and was quoted in equation (6.12)

7. Non-Dissipating Material Free at its Interior Surface

Although it could never correspond precisely with a real physical condition, the assumption that the interior surface is free of any stresses provides an interesting comparison with that of the material fixed at its

interior./

interior. It obviously may lead to a solution of some relevance to the problem of a skin fixed at only discrete points, which are wide apart compared with the wavelength of the oscillation, though in most such practical applications of interest the skin may be under considerable tension. If the tension is comparable with the product of G and the square of skin thickness, the linearised equations for the displacement due to stress which we have assumed will not be applicable.

The boundary condition to be applied is obtained from an expression like (2.1), but relating to displacement derivatives at $\eta = -d$, and with no applied forces, both ϖ and τ would be zero. Applying (2.2) we find accordingly that

$$\left.\begin{aligned} -\tfrac{1}{2}(1+r_2^2)(A_1/r_1)\sinh r_1 d + A_2\cosh r_2 d + \tfrac{1}{2}(1+r_2^2)A_3\cosh r_1 d - r_2 A_4 \sinh r_2 d &= 0 \\ A_1\cosh r_1 d - \tfrac{1}{2}(1+r_2^2)(A_2/r_2)\sinh r_2 d - r_1 A_3 \sinh r_1 d + \tfrac{1}{2}(1+r_2^2)A_4\cosh r_2 d &= 0 \end{aligned}\right\} (7.1)$$

If we identify these with the equations $m = 1$ and 2 of (4.7), then we can deduce from Table 1 that

$$\Delta_0 = \tfrac{1}{2}(1+r_2^2)^2(1-\cosh r_1 d\cosh r_2 d) + [r_1 r_2 + (1/16 r_1 r_2)(1+r_2^2)^4]\sinh r_1 d\sinh r_2 d \quad (7.2)$$

$$\Delta_{45} = \tfrac{1}{2}(1-r_2^2)[r_1 r_2 \sinh r_1 d\cosh r_2 d - \tfrac{1}{4}(1+r_2^2)^2\sinh r_2 d\cosh r_1 d]/r_2 \quad \ldots (7.3)$$

$$\begin{aligned}\Delta_{35}+\Delta_{46} = {} & \tfrac{1}{2}(1+r_2^2)(3+r_2^2)(1-\cosh r_1 d\cosh r_2 d) + [2r_1 r_2 + (1/4 r_1 r_2)(1+r_2^2)^3] \\ & \times \sinh r_1 d\sinh r_2 d - \tfrac{1}{2}\sigma(1-r_2^2)\{[1+\tfrac{1}{4}(1+r_2^2)^2]\cosh r_1 d\cosh r_2 d - (1+r_2^2) \\ & - [r_1 r_2 + (1/4 r_1 r_2)(1+r_2^2)^2]\sinh r_1 d\sinh r_2 d\}. \quad \ldots (7.4)\end{aligned}$$

We shall now expand these expressions for $d = \infty$, small d, and for large and small c/c_1 as we did for the fixed surface relations. We have no need to write down the expressions for $d = \infty$ since, understandably, they are identical to those for a surface fixed at its inner boundary, except for a factor common to all determinants, which is immaterial. For a 'thin' boundary, however, we find that

$$\Delta_0 = -\tfrac{1}{4}(c/c_2)^6 d^2[1-\gamma^2-\tfrac{1}{4}(c/c_2)^2][1+\mathcal{O}(r_2^2 d^2)+\mathcal{O}(c_2^2 d^2/c^2)] \quad \ldots (7.2B)$$

$$\Delta_{45} = \tfrac{1}{2}(c/c_2)^4 d[1-\gamma^2-\tfrac{1}{4}(c/c_2)^2][1+\mathcal{O}(r_2^2 d^2)] \quad \ldots (7.3B)$$

$$\Delta_{35}+\Delta_{46} = -(\sigma c^6/8c_2^6)[1+\mathcal{O}(d^2 c_2^2/\sigma c^2)+\mathcal{O}(r_2^2 d^2)]. \quad \ldots (7.4B)$$

For large c/c_1, we have on placing $\psi = cd/c_2$,

$$\Delta_0 = (c_1 c^6/16 c_2^7)\sin\gamma\psi\sin\psi + \mathcal{O}(c^4/c_1^4) \quad \ldots (7.2C)$$

$$\Delta_{45} = (c^5/8c_2^5)\cos\gamma\psi\sin\psi + \mathcal{O}(c^3/c_1^3) \quad \ldots (7.3C)$$

$$\Delta_{35}+\Delta_{46} = -(\sigma c^6/8c_2^6)\cos\gamma\psi\cos\psi + \mathcal{O}(c^2/c_1^2)+\mathcal{O}(c^2/\sigma c_1^2). \quad \ldots (7.4C)$$

These it will be seen reduce to the equations B if ψ is small. Finally, for small c/c_2, we find that

$$\Delta_0 = \tfrac{1}{4}(c/c_2)^4(1-\gamma^2)^2(\sinh^2 d - d^2)[1+\mathcal{O}(c^2/c_2^2)+\mathcal{O}(c^2/d^2c_2^2)] \quad \ldots (7.2D)$$

$$\Delta_{45} = \tfrac{1}{4}(c/c_2)^4(1-\gamma^2)(\sinh d\cosh d + d)[1+\mathcal{O}(c^2/c_2^2)] \quad \ldots (7.3D)$$

$$\begin{aligned}\Delta_{35}+\Delta_{46} = {} & -\tfrac{1}{2}(c/c_2)^4\{(1-\gamma^2)\gamma^2\sinh^2 d + [1+\tfrac{1}{4}\sigma(c/c_2)^2](1-\gamma^2)^2 d^2 \\ & + \tfrac{1}{4}\sigma(c/c_2)^2(\cosh^2 d - \gamma^4\sinh^2 d)\}[1+\mathcal{O}(c^2/c_2^2)]. \quad \ldots (7.4D)\end{aligned}$$

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We shall generally deal much more briefly with these equations than with the equivalent equations of paragraph 6, because the principles involved in the analysis are in most cases the same, or similar. The thick surface solution being identical with that of the fixed surface, we start with a discussion of the effect of reducing thickness on slowly propagated oscillations.

7.1 Slowly propagated oscillations

Assuming that $\dot{\sigma}$ is bounded, using (7.2D) and (7.3D) and proceeding as before (in paragraph 6.4), we arrive at an expression like (6.24), namely,

$$v = I\,, \quad u = H/(1-\chi)\,, \quad \chi \equiv \chi(d) = 2\sigma(u_\delta/c_4)^2[\sinh d\cosh d+d]/(\sinh^2 d-d^2) \quad (7.5)$$

which is valid provided that $(c/c_2)^2$ is small, and moreover provided that

$$d \gg \sigma^{1/2}\,(c/c_4) \qquad \ldots\ (7.6a)$$

$$d \gg (c/c_4) \qquad \ldots\ (7.6b)$$

conditions which will be established in paragraph 7.2. Whereas $\chi(d)$ was a monotonic function increasing with d for a fixed surface, it will be seen that it is monotonic and decreasing (with increase of d) for a free surface. (In fact $\chi \to \infty$ as $d \to 0$.) The equation for d, supposing that θ is known, is that stated before, namely in (6.28), and further as in (6.29).

$$(\delta/\bar{\delta}) \;=\; (\bar{R}_\delta/R_\delta) \;=\; 1-\chi(d)\,.$$

Now, if $\chi(\infty) > 1$ then $\chi > 1$ for all positive d, and there is no solution - at least if u_δ is not small compared with c_2, then none for small c/c_2. On the other hand, if $\chi(\infty) < 1$, then there may be a solution which would correspond to those values of ζ_1 in an interval about that for which δ is largest: in fact as $d/(1-\chi)$ has a minimum value (being infinite at $d = \infty$ and at some finite value), there would be two solutions corresponding to each ζ_1 (and c/u_δ) if any exist at all. For some $\delta = \delta_{min} < \bar{\delta}_{max}$ these two solutions would merge into one, and for smaller values of $\bar{\delta}$, no solution would exist. Thus as $|\zeta_1|$ decreases from infinity and $\bar{\delta}$ and $c/u_\delta = 1/\bar{U}(\bar{\delta})$ both increase, there will be a region for which no solution exists: when $\bar{\delta}$ rises above $\bar{\delta}_{min}$ a solution will exist (with $\delta < \bar{\delta}$) the two values of δ gradually diverging; ultimately, as $|\zeta_1|$ decreases further and $\bar{\delta}$ starts to fall again so the two solutions will once again converge. The smaller the value of $\chi(\infty)$ may be, the greater is the difference between the two solutions, and the smaller the value of $\bar{\delta}_{min}$. An increase in the value of θ brings about the same effects, whilst increase of $\chi(\infty)$ or decrease of θ have the opposite effects. If, in particular, $1/\theta$ and $\chi(\infty)$ are sufficiently large that

$$(1/\theta)[d/(1-\chi)]_{min} > [\bar{U}(\bar{\delta})/\bar{U}'(0)]_{max} \;=\; 0.366$$

then no solution exists. Even, however, when it does exist, it will be observed that δ, (c/u_δ) and R_δ are always finite.

The maximum value of θ satisfying (7.7) is shown in Fig.5. For small $\chi(\infty) = 2\sigma(u_\delta/c_4)^2$,

$$[d/(1-\chi)]_{min} \sim 8[\chi(\infty)/9]^{1/3}$$

occurring for $d = 2[3\chi(\infty)]^{1/3}$. This value is compatible with (7.6a), and with (7.6b) as well provided that, if θ is supposed small,

$$u_\delta/c_2 \;=\; O(\theta)\,.$$

Solutions/

Solutions in this range of small θ thus excluded can be treated by the methods of paragraph 7.2 below. With this restriction, therefore, we find that the maximum skin thickness for which the mode of oscillation is absent is given by

$$\theta_{max} \sim 10.5\, \chi^{1/3}(\infty)\ , \quad \text{for} \quad \chi(\infty) \to 0\ , \qquad \dots (7.8)$$

Whereas, of course, for $\chi(\infty) \to 1$, we have that $\theta_{max} \to \infty$.

Where a solution exists, then plotted on a c-R_δ or δ-R_δ diagram, the neutral oscillations will be found (as sketched in Fig.4(e)) to lie on a closed curve, bounded below by $R_\delta > R_{\delta_{min}}$, and above by a finite value of R_δ. Likewise the values of δ are bounded above by a value less than $\overline{\delta}_{max}$, and below by a finite value; the values of c are also bounded above and below by finite values, and generally $c_{max} = \bar{c}_{max}$.

7.2 Thin surface

7.21 Low speeds of propagation

In order to connect the solution of 7.1 with that for a thin surface it is necessary to relax (7.6) and consider d and c/c_2 to be similar small magnitudes. Then from equations (7.2D) and (7.2B) we see that

$$\Delta_0 = \tfrac{1}{4}(1-\gamma^2)d^2(c/c_2)^4[(1-\gamma^2)(d^2/3)-(c/c_2)^2][1+O(c^2/c_2^2)+O(d^2)]\ . \quad (7.9)$$

and similarly from (7.4B) and (7.4D),

$$\Delta_{35}+\Delta_{46} = -\tfrac{1}{2}(c/c_2)^4[(1-\gamma^2)d^2+\tfrac{1}{4}\sigma(c/c_2)^2][1+O(c^2/c_2^2)+O(d^2)]\ . \quad \dots (7.10)$$

As in (6.8) and (6.10) we can find from (4.28), (4.30), (5.6), (7.9) and (7.10) that

$$\left.\begin{aligned} v &= (\Delta_0+D_3)I/\Delta_0 = \tfrac{3}{4}\sigma^2 I(\xi-\xi_1)(\xi+\xi_2)/(3\xi-1) = 7.14/[U(\delta)]^3 \\ u &= (\Delta_0+D_3)H/I[(1-\chi)\Delta_0] = U'(0)/[U(\delta)]^2, \quad \chi = -(2c_4/I_0\sigma c_4)[\xi^{3/2}/(\xi-\xi_1)(\xi+\xi_2)] \end{aligned}\right\} \quad \dots (7.11)$$

where we have placed

$$\xi = 4c^2/d^2c_4^2,\quad \frac{\sigma\xi_1}{2} = \left[\left(\frac{1+\sigma}{\sigma}\right)^2+\frac{1}{3}\right]^{1/3} - \left(\frac{1+\sigma}{\sigma}\right),\quad \frac{\sigma\xi_2}{2} = \left[\left(\frac{1+\sigma}{\sigma}\right)^2+\frac{1}{3}\right]^{1/3} + \left(\frac{1+\sigma}{\sigma}\right)\ . \quad \dots (7.12)$$

If σ is small, ξ_1 is close to, but less than $1/3$; if σ is large, then ξ_1 is small but non-zero. Now from (5.6), (5.8) and (6.14)

$$\xi = (4u_\delta^2/\theta^2 c_4^2)[U'(0)]^2/[U(\delta)]^4 = 4u^2u_\delta^2/\theta^2 c_4^2 \qquad \dots (7.13)$$

and so eliminating $U'(0)$ and $U(\delta)$ between equations (7.11) and (7.13) we arrive at an equation:

$$A(3\xi-1)(\xi-\xi_1)(\xi+\xi_2)\xi^{1/2} + \xi^2(3\xi-1) = B(\xi-\xi_1)^2(\xi+\xi_2)^2$$

where $\quad A = \tfrac{1}{2}\sigma(u_\delta/c_4)I_0\ , \quad B = \tfrac{3}{4}(\sigma^3/\theta)(u_\delta/c_4)^2 HI_0\ .$

This appears a complicated equation to solve, and perhaps the easiest method would be to plot the curve of A versus B on a diagram, using A and B

as/

as cartesian co-ordinates, the curve being generated by allowing ζ_1 (and so H and I_0) to vary. Then the straight lines ξ = constant can be superimposed, and the intersections would be solutions: however, only those for that semi-infinite range of A for which the first term on the left-hand side of (7.14) is positive would be valid solutions, because this is the condition, from (7.11), for positive $U(\delta)$.

If this is done it will be found that there are possibly solutions for a limited range of values above (and including) $\xi = 0$, but none in an interval about either side of $\xi = \xi_1$. These small values of ξ correspond, it will be appreciated, with the solutions - if any - of paragraph 7.1, modified for small d. In fact, for small ξ, (7.14) can be expanded as

$$(1-\chi)\xi^{1/2} = (B/A)\xi_1\xi_2(1+a_1\xi + \dots\dots)$$

and it can be shown that $a_1 = -3\sigma$. The expression for ξ in (7.13) shows that this is equivalent to

$$(1-\chi)u = H(1+a_1\xi + \dots\dots)$$

and the series is evidently merely that for (Δ_0+D_3/Δ_0); using the definition of ξ in (7.12), we see that D_3/Δ_0 is small if - as asserted earlier - the condition of (7.6a) is satisfied. The condition (7.6b) states alternatively that $\xi << 1/3$, which justifies the neglect of (D_3/Δ_0), as in (7.5), in the condition $\sigma \to 0$.

For some ξ in the range $\xi_1 < \xi \leqslant 1/3$, a solution exists corresponding to negative I and positive H. Moreover, if $\chi > 1$ at $H = 0$ there will also be solutions corresponding to still smaller values of $|\zeta_1|$, where H and I are both negative, though a range about $\zeta_1 = 0$ gives no solution. (If it did, it would, of course, be of doubtful validity.) Now $\chi < 1$ at $H = 0$ implies, from (7.12) and (7.14), that

$$(u_\delta/c_4) < [2\sqrt{3}/(12+\sigma)]/I_0|_{H=0} = 21/(12+\sigma) \; . \qquad \dots (7.15)$$

If this condition is satisfied, then as $|\zeta_1|$ increases from the value corresponding to $H = 0$, ξ will decrease from 1/3, reaching a minimum, and then it would increase to the value 1/3 again as $I \to 0$ with further increase of $|\zeta_1|$. Now, as $H \to 0$, we see from (7.11) that $U(\delta) \to 0$, and by (7.13), $\delta \to \infty$ (and, of course, $U'(0) \to 0$). Equation (5.11) shows that $R_\delta \to 0$, but plainly the large values of δ and c invalidate the solution in this limit. On the other hand, as $I \to 0$, δ and $U(\delta)$ remain finite, as also will R_δ; thus plotted on an c-R_δ or δ-R_δ diagram, neutral oscillations of this mode would lie on an incomplete curve, extending from infinite c and δ at $R_\delta = 0$.

If (7.15) is not satisfied, the condition $H \to 0+$ corresponds to a value of $\xi \neq 1/3$, and $\chi \to 1$; but $U(\delta)$ and $U'(0)$ will be non-zero. The solution will be continuous here with that for the values of $|\zeta_1|$ for which H and I are negative. Ultimately as $\xi \to 1/3$, then $H \to 0$ through negative values, and this corresponds to the same limiting behaviour as noted before, - namely, c/u_δ and δ tending to infinity, and R_δ tending to zero.

The solution for $I \to 0$ through negative values (and correspondingly $\xi \to 1/3$ from below) is in fact continuous with that existing in the range $\xi > 1/3$, which is satisfied by positive H and I. If u_δ/c_2 or σ is sufficiently small, this continuation will be such that a single value of ξ will in general correspond to two values of I, and the solution only exists for a finite range of ξ above 1/3. As $|\zeta_1| \to \infty$, and $I \to 0$ from positive values, ξ will decrease to 1/3 again, implying that $U(\delta)$ has the same finite value as for the limit of $I \to 0$ from below, but that $U'(0)$ is decreased and so δ increased. However, since $|\zeta_1|$ is unbounded in the limit, so is R_δ; consequently on the

c-R_δ/

$c\text{-}R_\delta$ and $\delta\text{-}R_\delta$ diagrams (Fig.4(g)) the aforementioned incomplete curves are thus continued to $R_\delta = \infty$, being asymptotic to the values

$$(c/c_4)^3 \;=\; u_\delta\theta/[(12+\sigma)c_4 H]\;, \qquad \delta \propto U(\delta)/U'(0) \;=\; 2(c/c_4\theta)\xi^{-1/2} \quad \ldots (7.16)$$

where evidently $H = 1$ and $\xi = 1/3$. Since (c/u_δ) and δ must both be small we see that this solution is strictly valid only for large σ. However, if (u_δ/c_4) is sufficiently small, $U'(0)$ will be large, and it will then be only the approximations to u and v which are at fault.

There is another solution yielded by (7.14) for small u_δ/c_2 or σ which involves a range of values of ξ with a lower bound greater than 1/3 but no bounded upper limit. This is evidently properly dealt with by relaxing the assumption that c/c_2 and d are comparable magnitudes, as this implies $\xi = O(1)$. This we do in paragraph 7.22.

Likewise if u_δ/c_2 or σ is large there are again two solutions, one being continuous with that already found for $\xi \leqslant 1/3$, and for all $\xi > 1/3$ yielding values of $I < I_{max}$ corresponding to bounded values of $|\zeta_1|$. The other corresponds to $\zeta_1 = -\infty$ at $\xi = 1/3$, and is compatible with $|\zeta_1|$ decreasing to a value for which $I < I_{max}$ and then increasing again as $\xi \to \infty$. The solution $|\zeta_1| \to \infty$ as $\xi \to 1/3$ implies the asymptotes given in (7.16) above, with $H = 1$.

7.22 Solutions with larger speeds of propagation

We now turn our attention to those solutions in which θ, and so d, may be vanishingly small, but c/c_2 is finite. We should expect such solutions to be continuous with those discussed above which apparently exist for $\xi \to \infty$. The relevant equations for Δ_0, etc., may now be taken as (7.2B), (7.3B) and (7.4B) and we see that from (4.28), (6.9b) and (6.14),

$$\left.\begin{aligned} (\Delta_0 + D_3)/\Delta_0 &= [c_4^2-(1-\sigma^2)c^2]/(c_4^2-c^2) \\ \chi_1 &= -(\sigma/\theta)U'(0)\{(u_\delta/c)+[\sigma c^2/\theta(c_4^2-c^2)]\} \end{aligned}\right\} \quad \ldots (7.17)$$

and from (5.6), (5.8) and (6.8)

$$I_0(u_\delta/c)^3 \;=\; (c_4^2-c^2)/[c_4^2-(1-\sigma^2)c^2] \qquad \ldots (7.18)$$

so that in (6.9b), ignoring the undetermined small term of order δ compared with unity,

$$(u_\delta/c)U'(0)\{1+(\sigma/\theta)U'(0)[(u_\delta/c)+\sigma c^2\theta^{-1}/(c_4^2-c^2)]\} \;=\; (H/I_0). \quad \ldots (7.19)$$

We shall subdivide our discussion of these equations according to the value of c relative to $c_4(\sigma) = c_4/(1-\sigma^2)^{1/2}$ and $c_4(0) = c_4$.

(i) $c < c_4$

Here solutions must lie within the range of $I > 0$. We note that the expression in (7.18):

$$(c_4^2-c^2)c^3/\{[c_4^2-(1-\sigma^2)c^2]c_4^3\}$$

has a maximum at $c = c_{max}$ where

$$(c_{max}/c_4)^2 \;=\; [6-\sigma^2-\sigma(24+\sigma^2)^{1/2}]/[6(1-\sigma^2)]$$

and where its value is unity for small σ, decreasing to $0.24\sqrt{0.6} = 0.186$ at $\sigma = 1$, and finally to zero as $\sigma \to \infty$. Plainly if

$$(u_\delta/c_4)^3(I_0)_{max} \;=\; 0.08\,(u_\delta/c_4)^2$$

is/

is greater than this maximum, solutions do not exist in the neighbourhood of $I = I_{max}$, as we found when discussing the pair of 'large u_δ/c_4 or σ' solutions involving large ξ at the end of the previous section. On the other hand, if σ or u_δ/c_4 is sufficiently small, continuous solutions will exist for all $I > 0$, though in both instances we see that, for each I and ζ_1 for which a solution exists, there are two corresponding values of c/c_4 on either side of the maximum. For $I \to 0$, the more slowly propagated oscillation - which according to our present approximation would have vanishing speed - will in fact obey the then more exact relations found in the previous paragraph for solutions in the neighbourhood of $\xi = 1/3$. Thus we see how we may "marry" the solutions with those just discussed in paragraph 7.21.

Considering first the form of the solution for sufficiently small u_δ/c_4 (such that solutions exist for all I), and ignoring for the moment the particular solution corresponding to small c, we see that the other solution will correspond to $c = c_4$ at the limits $I \to 0+$ (for $\zeta_1 \to -\infty$, and $\zeta_1 \to -2.3$) with c reaching some minimum (greater than c_{max}) at $I = I_{max}$. If u_δ/c_4 is smaller than unity, then of course such solutions would presumably not exist. However, since $(I_o)_{max} = 0.081$, it follows that, if σ is small, this form of solution applies for $u_\delta/c_4 < 2.3$, and close to this limit, the value of (c/u_δ) can be numerically quite small compared with unity.

We see that we can express equation (7.19) as

$$(u_\delta/c)I_oU'(0)[1+\sigma u_\delta U'(0)/(\theta c)] = H-[\sigma c^2U'(0)/\theta u_\delta]^2/[c_4^2-(1-\sigma^2)c^2] \qquad (7.19a)$$

in a form evidently analogous with (6.9c), and from this form it is perhaps more apparent that as $I \to 0$ (and $c \to c_4$) then

$$\delta = 0.58\, U(\delta)/U'(0) = 0.58/(\theta H^{1/2}). \qquad \ldots (7.20)$$

In other words $\theta\delta$, - the ratio of skin thickness to wavelength - is 0.58 at the one limit (corresponding to $|\zeta_1| \to \infty$) and 0.38 at the other ($\zeta_1 = -2.3$). Since δ and c are finite, then from (5.12) the limit $|\zeta_1| \to \infty$ also corresponds to $R_\delta \to \infty$, but the other limit corresponds to a finite value of Reynolds number given by

$$R_\delta = 8.15\, \theta(u_\delta/c_4)^3 \qquad \ldots (7.21)$$

and it can be anticipated that the solution will here be continuous with another existing for smaller values of R_δ and larger c.

The value of δ from (7.20) at these two limits corresponds to values of $\xi = H$, and this is compatible with the prediction of paragraph 7.21 above that the solution will originate from some value of $\xi > 1/3$: however, since d is not negligible compared with c/c_4 at these limits, the present solution will not be accurate except in a qualitative sense.

We note that for $u_\delta/c_4 < 2.3$ and $\sigma \to 0$, the value of c tends to c_4 for all $I > 0$ in this mode of oscillation, so that from (7.19a)

$$(c_4/u_\delta)^3(U'(0)/\theta^2) = I_o\{[1+(4Hc_4^4/I_o^2\theta^2u_\delta^4)]^{1/2}-1\}$$

and the value of $\delta = 0.58\, u_\delta/U'(0)c_4$ is everywhere of the magnitude of $1/\theta$, if θ is small, reaching a maximum at the two end limits, where $I \to 0$, given by (7.20) above.

The other solution of (7.18) and (7.19) corresponding to that found earlier to exist in a bounded range above $\xi = 1/3$ tends, as $\sigma \to 0$, to give $c = \bar{c}$, where $\bar{c}$ is the speed of propagation for the Tollmien-Schlichting

oscillation/

oscillation corresponding to ζ_1; further (7.14) shows that, as $(\sigma/\theta) \to 0$,

$$(\sigma/\theta)U'(0) = \{[1+(4R/I_0)(\sigma/\theta)]^{1/2} - 1\}I_0^{1/3} . \qquad \ldots (7.22)$$

Thus in general δ will also tend to coincide with $\bar{\delta}$, the inflexible surface value, except near the limits $I \to 0$, where $\delta \to \infty$. The value of ξ corresponding to $c = \bar{c}$ and $\delta = \bar{\delta}$ is

$$\bar{\xi} = (u_\delta H/c_4\theta)^2 \qquad \ldots (7.23)$$

and this can clearly be large - and so our solution valid - if

$$\theta << (u_\delta/c_4) .$$

However close to the limits $I \to 0$, we find from (7.22) that ξ tends to zero as $I^{1/2}$, (and $R_\delta \to \infty$) and this is compatible with the anticipated result that this solution is another approximation to that found earlier to exist in a bounded range above $\xi = 1/3$, and indeed (7.14) yields the solution (7.23) for $I \neq 0$ and $\sigma \to 0$. This same limiting behaviour of $\xi \to 0$ as $I \to 0$ is characteristic of the approximation to this mode of oscillation even where σ is not small, and the correspondence between the present solution and that of (7.14) is always present. Equation (7.22) shows that $U'(0)$ is then always less than the inflexible surface value, and so generally δ will be larger, and R_δ smaller.

If (u_δ/c_4) exceeds the critical value given by

$$(u_\delta/c_4)^3_{crit} = (c_4^2-c^2)c^3/\{c_4^3[c_4^2-(1-\sigma^2)c^2]\}_{max}/(I_0)_{max} \qquad \ldots (7.24)$$

then $c = c_{max}$ for some $I < I_{max}$. The solution which originates from $\xi = 1/3$ in the neighbourhood of $I = 0+$ for finite $|\zeta_1|$ will correspond to $c = c_{max}$ for some larger $|\zeta_1|$, and then yield larger values of c as $|\zeta_1|$ decreases once again, until as I returns to zero, $c \to c_4$, and δ and R_δ have limit values ascribed in (7.20) and (7.21).

The other solution originates from the condition $\xi = 1/3$ at $\zeta_1 = -\infty$ given in the discussion of paragraph 7.21 relevant to small values of (c/c_4). This condition corresponds to $R_\delta = \infty$, with the c and δ values as given by equation (7.16). As $|\zeta_1|$ decreases, c will increase to c_{max} and continue to increase as $|\zeta_1|$ returns once more to infinity: at this limit, $R_\delta \to \infty$, $c \to c_4$ and - according to (7.20) - the value of δ tends to $(0.58/\theta)$.

(ii) <u>$c_4 < c < c_4(\sigma)$</u>

Here we find a solution, in the range of values of ζ_1 for which $I < 0$, which is continuous with one of the solutions in the region $I > 0$; as I and $|\zeta_1|$ decrease, we find that c increases to some value less than $c_4(\sigma)$, and then decreases again as $\zeta_1 \to 0$. There is however no solution for δ in the neighbourhood of small ζ_1, and a continuous solution is found with $|\zeta_1|$ reaching a non-zero minimum, and then decreasing once again, until in the limit as $H \to 0$ through negative values, $\delta \to \infty$ and $R_\delta \to 0$.

However, if for some values of c in the range of the solution

$$c^3/[u_\delta(c^2-c_4^2)] \leqslant (\theta/\sigma) \qquad \ldots (7.25)$$

then $\delta \to 0$ and $R_\delta \to \infty$ for that value of $I < 0$ for which (7.23) is satisfied as an equality (with $H > 0$). A second solution then exists

involving/

involving the other limit of $\delta \to \infty$ and $R_\delta \to 0$ which gives two values of c and δ over a range of values of ζ_1.

It is pertinent to note that if u_δ/c_4 and σ are small, (7.23) implies from (7.18) that

$$\theta \geqslant (c_4/\sigma u_\delta)[1+2.5\, c_4^3/u_\delta^3]$$

but if either (u_δ/c_4) or σ is sufficiently large that

$$(2+3\sigma^2)u_\delta^3/c_4^3 \geqslant 15\sqrt{3}$$

(7.23) requires the less stringent condition that

$$\theta \geqslant \tfrac{1}{2}(c_4/u_\delta)\sigma\sqrt{3}.$$

Tracing these various modes of oscillation for all $c < c_4(0)$ on c-R_δ and δ-R_δ diagrams, we see that if (u_δ/c_4) is less than the critical value of (7.24), one mode exists (involving large ξ) which starts from $\delta = 0.58/\theta$ and $c = c_4$ at $R_\delta = \infty$, decreases at first in its values of c and δ as R_δ decreases (Fig.4(g)), but reverses this trend, with δ tending to indefinitely large values as $R_\delta \to 0$. If θ is sufficiently large that (7.25) is satisfied there is a branch in the δ-R_δ and c-R_δ curves (Fig.4(h)), returning to $R_\delta = \infty$ with $c > c_4$, and with δ reaching vanishingly small values in the limit.

If u_δ/c_4 is larger than the critical value, there are two modes involving large ξ, one originating from the asymptotes given by equation (7.16) at $R_\delta = \infty$ (which is a valid expression provided the given value of c is small compared with c_4) reaching a minimum R_δ and returning to $R_\delta = \infty$, with c asymptotic to c_4 from below, and δ tending to $0.58/\theta$ (Fig.4(f)). The second mode is an extension of the solution for small ξ, extending to infinite c and δ at $R_\delta = 0$, which reaches some small value of c/c_4 and δ at finite R_δ (corresponding to $\xi = 1/3$), then increases its values of c through c_4, and δ through $0.38/\theta$ at some finite R_δ, finally returning to $R_\delta = 0$ with $c > c_4$ and indefinitely large δ. Once again if θ is sufficiently large, these latter curves branch to $R_\delta = \infty$, with $c > c_4$ and with δ vanishing in the limit.

(iii) $\underline{c_4 > c_4(\sigma)}$

This speed range only exists of course if $\sigma < 1$, and if σ is close to unity it will involve speeds considerably larger than c_4. Here, once again, we may expect to find solutions of (7.18) only in the range for which $I > 0$, and since

$$(c^2-c_4^2)c^3/[(1-\sigma^2)c^2-c_4^2]c_4^3$$

has a minimum at $c = c_{min}$, say, where

$$(c_{min}/c_4)^2 \;=\; [6-\sigma^2+\sigma(24+\sigma^2)^{1/2}]/6(1-\sigma^2)$$

and at the minimum its value is close to unity for small σ, but becomes indefinitely large if $\sigma \to 1$, we therefore find that such solutions exist only for sufficiently large u_δ/c_4, and then only for a finite range of positive I about I_{max}. The value of $U'(0)$ is only positive if, from (7.14),

$$c^3/[u_\delta(c^2-c_4^2)] < (\theta/\sigma)$$

and placing $c = c_{min}$, we see that this implies

$$(u_\delta\theta/c_4) > (c_{min}/c_4)^3\{6(1-\sigma^2)/[5\sigma+(24+\sigma^2)^{1/2}]\}.$$

The/

The right-hand-side is equal to $\sqrt{1.5}$ if σ is small, but increases to infinity as $\sigma \to 1$. Supposing that θ is sufficiently large to justify this inequality, then a solution exists which will lie on a closed curve on both c-R_δ and δ-R_δ diagrams, existing for a range of finite R_δ.

7.3 Fast propagated oscillations

Comparing (7.2C), (7.3C) and (7.4C) with the equations (6.2C) and (6.3C) and (6.4C), we find that the only difference - apart from a constant of proportion - is that 'cos' and 'sin' are everywhere interchanged. Thus to each and every solution of paragraph 6.3 for the fixed surface there will be, in general, an analogous solution with the value of ψ changed by $\pi/2$ or $\pi/2\gamma$. This will, of course, affect the numerical values of the solution, but not its qualitative interpretation. The solutions mentioned in paragraph 6.3 relevant to the region about $\psi = \pi/2$ will here be seen to be relevant to those for small ψ, which, of course, have already been remarked upon. There is likewise a singular behaviour of the solution in the region of $\psi = \pi$, but it is difficult to determine whether this connects with any of the solutions already discussed.

8. Surfaces Exposed to Fluid at the Inner Boundary

Where a fluid exists at the inner boundary, its depth, compressibility, viscosity and density are all parameters of importance. In general, the structure of the equations for the boundary condition would need to take account of most - if not all - of these parameters, but in a few extreme examples its presence can be simulated quite simply. Thus if one is willing to assume that the fluid is incompressible, and has very large density and viscosity, then the boundary condition is effectively that for a rigidly mounted surface: in practice there will be some movement at the boundary but this could be small compared with the movement at the outer surface exposed to the moving fluid boundary layer. Again, if the inner fluid is very light and assumed inviscid, then the 'free' boundary condition is appropriate, though clearly the inner boundary will not be completely free of stress - but rather the stress there will be much less than at the external boundary.

Two other extremes of this type will now be considered and their effects on the stability problem briefly outlined. The first is the case of an indefinitely heavy, inviscid incompressible fluid, and the other the case of a light, very viscous fluid; we shall refer to these respectively as HI, and LV, fluids for brevity. For the HI fluid, the appropriate boundary condition at the interior is that shear stress is absent, and there is no normal displacement: for the LV fluid, it is that normal stress is absent, and there is no tangential displacement. Interpreting this algebraically, and considering first the HI fluid, we find the boundary condition is formed by one from each of the pairs of boundary conditions (6.1) and (7.1). These become, after some manipulation:

$$\left.\begin{aligned} A_1 - A_3 r_1 \tanh r_1 d &= 0 \\ -(A_2/r_2)\tanh r_2 d + A_4 &= 0 \end{aligned}\right\} \qquad \ldots (8.1)$$

whilst, for the LV fluid we have similarly

$$\left.\begin{aligned} -(A_1/r_1)\tanh r_1 d + A_3 &= 0 \\ A_2 - A_4 r_2 \tanh r_2 d &= 0 \end{aligned}\right\} \qquad \ldots (8.2)$$

These are both simple conditions, and can be identified as the equations for $m = 1$ and 2 of (4.7). We can deduce, from Table 1, that the determinants of interest in our problem have the values shown in Table 2, where we also include the appropriate limiting forms for small c/c_2 and small d. For large d, the equations are identical with those for the other boundary conditions, and for large c/c_1 they can be reduced to forms analogous with those of paragraph 6.3 by a suitable change in ψ.

Table 2

Values of Determinants

Fluid	Determinant	General Form*	Limit $(c/c_2) \ll 1$	Limit $d \ll 1$
HI	Δ_0	$-\frac{1}{2}(t_2/r_2)(1+r_2^2)^2+r_1 t_1$	$\frac{1}{2}(1-\gamma^2)(c^2/c_2^2)$ $\times(\tanh d+d\ \mathrm{sech}^2 d)$	$c^2(c_4^2-c^2)d/4c_2^4$
	Δ_{45}	$\frac{1}{2}(1-r_2^2)(t_1 t_2/r_1 r_2)$	$\frac{1}{2}(c^2/c_2^2)\ \tanh^2 d$	$c^2 d^2/2c_2^2$
	$\Delta_{35}+\Delta_{46}$	$-(1+r_2^2)(t_2/r_2)+2r_1 t_1$ $+\frac{1}{2}\sigma(1-r_2^2)(r_1 r_2 t_1-t_2)/r_2$	-	$(1-2\gamma^2)c^2 d/c_2^2$ $-\frac{1}{2}\sigma c^4 d/(c_1 c_2)^2$
LV	Δ_0	$-\frac{1}{2}(t_1/r_1)(1+r_2^2)^2+r_2 t_2$	$\frac{1}{2}(1-\gamma^2)(c^2/c_2^2)$ $\times(\tanh d-d\ \mathrm{sech}^2 d)$	$-c^4 d/4c_2^4$
	Δ_{45}	$c^2/2c_2^2$	$c^2/2c_2^2$	$c^2/2c_2^2$
	$\Delta_{35}+\Delta_{46}$	$-(1+r_2^2)(t_1/r_1)+2r_2 t_2$ $+\frac{1}{2}\sigma(1-r_2^2)(r_1 r_2 t_2-t_1)/r_1$	-	$-c^2 d/c_2^2$ $-\frac{1}{2}\sigma c^4 d/c_2^4$

*Note: $t_1 = \tanh r_1 d$, $t_2 = \tanh r_2 d$

8.1 Slowly propagated oscillations

Here, as in paragraphs 6, 4 and 7.1, the behaviour depends on the value of χ, which is given by

$$\chi = 2\sigma(u_\delta^2/c_4^2)[\sinh^2 d/(\sinh d \cosh d+d)] \quad \text{for HI fluid}$$

$$= 2\sigma(u_\delta^2/c_4^2)[\cosh^2 d/(\sinh d \cosh d-d)] \quad \text{for LV fluid} .$$

For the HI fluid, χ is a monotonic increasing function of d, and the inflexible surface mode of oscillation will be modified in the manner described in paragraph 6.4; for LV fluid, χ is monotonic and decreasing with increase of d, which as in paragraph 7.1 indicates the disappearance of this mode for

$$0.366\ \theta < [d/(1-\chi)]_{min} .$$

For small values of $\sigma(u_\delta/c_4)^2$, this implies a value for θ_{max} which is $2^{-1/3}$ times that found for a free surface in (7.8) - that is, a reduction of about 20% in the permissible thickness for the elimination of the Tollmien-Schlichting oscillations.

8.2 Thin surfaces

8.21 Heavy incompressible fluid

From (6.8) we have

$$7.14/[U(\delta)]^3 = v = (\Delta_0+D_3)I/\Delta_0$$
$$= I\{c_1^2c_4^2-[1-2\sigma(1-2\gamma^2)]c_1^2c^2-\sigma^2c^4\}/[c_1^2(c_4^2-c^2)] \quad .. \ (8.3)$$

whilst (6.9a) gives

$$U'(0)/U(\delta) = [(H-1)c^2/I_0u_\delta^2]+(u_\delta/c)+[\sigma u_\delta^2\theta/(c_4^2-c^2)] . \quad ... \ (8.4)$$

We can simplify (8.4) by writing

$$\left.\begin{aligned} c_5^2 &= \tfrac{1}{2}(c_1/\sigma)^2\{[(1-2\sigma)^2+8\sigma\gamma^2]^{1/2}-[(1-2\sigma)+4\sigma\gamma^2]\} \\ c_6^2 &= \tfrac{1}{2}(c_1/\sigma)^2\{[(1-2\sigma)^2+8\sigma\gamma^2]^{1/2}+[(1-2\sigma)+4\sigma\gamma^2]\} \end{aligned}\right\} \quad ... \ (8.5)$$

For small σ, c_6/c_1 is large, and c_5 is slightly in excess of c_4; as σ increases, c_6 decreases and c_5 reaches a maximum and then decreases, being equal to c_4 again where

$$\sigma = (1-2\gamma^2)/[2\gamma^2(1-\gamma^2)] .$$

As σ becomes indefinitely large both c_5 and c_6 tend to zero inversely with σ.

With this notation, (8.3) becomes

$$c_4^5(c_5^2-c^2)(c_6^2-c^2)/[c_5^2c_6^2(c_4^2-c^2)] = (c_4/u_\delta)^3/I_0 \quad ... \ (8.6)$$

and (8.4) can be rewritten as

$$1/\delta \propto U'(0)/U(\delta) = (c/u_\delta)^2\{H-1+\theta\sigma(u_\delta/c_4)[1-(c/c_5)^2]^{-1}[1+(c/c_6)^2]^{-1}\}I_0^{-1}+(u_\delta/c) .$$
$$... \ (8.7)$$

We shall very briefly discuss the results obtained from these expressions by distinguishing whether c_5 is greater than, or less than, c_4.

(i) $\sigma < (1-2\gamma^2)/[2\gamma^2(1-\gamma^2)]$

Here we see that $c_5 > c_4$, and there are two solutions for $c < c_4$ which, - as in paragraph 7.22, depending on the magnitude of u_δ/c_4, - take the form of either a pair of solutions starting and terminating at $c = 0$ and $c = c_4$, each limit corresponding to $I \to 0$ through positive values (at $H = 1$ for one solution and for $H = 2.3$ for the other); or else to a pair of solutions, one increasing from $c = 0$, reaching a maximum and returning to $c = 0$, and the other decreasing from $c = c_4$, reaching a minimum, and returning to $c = c_4$, as - in each case, - $|\zeta_1|$ increases from 2.3 (where $I = 0$) to infinity. As in paragraph 7.22, the latter corresponds to small (u_δ/c_4), and the former to large (u_δ/c_4). In each instance, where $c \to 0$ or $c \to c_4$ as $I \to 0$, we find that $\delta \to 0$ and $R_\delta \to \infty$; in fact $\delta \propto 1/R_\delta$ as $c \to c_4$, unless $|\zeta_1| \to \infty$ when $\delta \propto R_\delta^{-1/3}$. In the other limit where $c \to 0$, both c and δ vary as $R_\delta^{-1/4}$, unless $|\zeta_1| \to \infty$ when c and δ vary as $R_\delta^{-1/10}$.

For small σ, the solution approximates, as in paragraph 7.22, to that for the Tollmien-Schlichting mode.

For $c_4 < c < c_5$, I must be negative and any solutions which exist must be in a range bounded by some $I < 0$ and positive H: such a

solution/

solution would start at $\delta = \infty$ and $R_\delta = 0$ with a value $c > c_4$, and would either return to this condition or, if θ is sufficiently small, finish at $|\zeta_1| \to 0$, corresponding to $c = c_4$ and $\delta = 0$ at $R_\delta = 0$. There may also be a closed solution in the range $c > c_5$ if c_2/u_δ is sufficiently small.

(ii) $\underline{\sigma > (1-2\gamma^2)/[2\gamma^2(1-\gamma^2)]}$

Here a solution exists for all $I > 0$ in the range $c < c_5 < c_4$, the value of c reaching a maximum with I, and tending to zero as $I \to 0$.

For $c_5 < c < c_4$, there is a solution starting, as we mentioned above, at $\delta = \infty$ and $R_\delta = 0$ with $c > c_4$, and finishing at $|\zeta_1| \to \infty$ with $c = c_4$ and $\delta = R_\delta = 0$; again there may be a closed solution (for a range of I on either side of I_{max}) for $c > c_4$.

8.22 Light, viscous fluid

From (6.8) we have, on taking account of the fact (as in paragraph 7.21) that d may be of order c/c_2:

$$\left.\begin{aligned} 7.14/[U(\delta)]^3 &= 3(1+2\sigma)I[(\xi_0-\xi)/(1-3\xi)] \\ \text{where} \qquad \xi &= c^2/c_4^2d^2, \quad \xi_0 = 1/[3(1+2\sigma)] \end{aligned}\right\} \quad \ldots (8.8)$$

whilst from (6.10), which suffices since $(\Delta_{35}+\Delta_{46})$ is small compared with Δ_{45}, we have

$$\left.\begin{aligned} U'(0)/[U(\delta)]^2 &= [3(1+2\sigma)H/(1-\chi)][(\xi_0-\xi)/(1-3\xi)] \\ \text{where} \qquad \chi &= (\sigma c_4/I_0u_\delta)\xi^{3/2}/(\xi_0-\xi) \end{aligned}\right\} \quad \ldots (8.9)$$

and since $\qquad \xi = (u_\delta u/c_4\theta)^2$

we find, as in (7.14), an equation for ξ of the form

$$\left.\begin{aligned} A(3\xi-1)(\xi-\xi_0)\xi^{1/2} + \xi^2(3\xi-1) &= B(\xi-\xi_0)^2 \\ \text{where} \quad A = (1+2\sigma)u_\delta I_0/(\sigma c_4), \quad B &= 3(1+2\sigma)^2u_\delta^2HI_0/(\sigma\theta c_4^2) \end{aligned}\right\} \quad \ldots (8.10)$$

and this remains a good approximation even if $c/c_2 \gg d$ (i.e., even if ξ is large) except for some modification to the meaning of ξ_0.

As in paragraph 7.21, the solution for $\xi < \xi_0$ represents the modified form of the inflexible surface solution, if it exists (as of course it will do if, for instance, $\sigma \to 0$). Likewise, the solution for $\xi > \xi_0$ follows the same principles as those discussed in that section: with ξ tending to 1/3 as $HI \to 0$, except that now, at this limit, instead of (7.16) we have

$$\left.\begin{aligned} (c/c_4)^3 &= (u_\delta\theta/6c_4H), \quad \delta \propto U(\delta)/U'(0) = (c/c_4\theta)\xi^{-1/2} \\ R_\delta &= 10.9\,(-\zeta_1)^3H^{4/3}(u_\delta/c_4)^{5/3}\theta^{-1/3} \end{aligned}\right\} \ldots (8.11)$$

In the present instance, however, there is no solution in the range of unbounded ξ, and the solution starts at $H = 0$, with $c = \delta = \infty$ and $R_\delta = 0$, and proceeds through $I = 0$ to $|\zeta_1| \to \infty$, where ξ returns once more to the value 1/3; at this limit c and δ are finite, but R_δ is infinite. If σ is small, the solution for $I > 0$ follows closely the Tollmien-Schlichting solution, except at the limits.

9. Summary and Discussion of Solutions

We envisage the two-dimensional boundary layer to exist over a plane surface, which is composed of isotropic elastic material with a uniform thickness, supposed to be θ times the boundary-layer thickness defined by $u_\delta / \frac{\partial u}{\partial y}$, where $\frac{\partial u}{\partial y}$ is the surface velocity gradient in the boundary layer. (This thickness for a flat plate boundary layer is 1.7 times the displacement thickness.) The interior surface of the material is supposed either rigidly fixed, exposed to fluid of special properties (paragraph 9.7), or else free (that is, strictly, exposed to vacuum) - the latter being an unrealistic condition but intended to simulate the condition where an unstretched elastic covering is bonded only at discrete points to a rigid structure.

In framing the stability equation, which is in effect the statement of the Eigen-value problem concerning the evaluation of the speed of propagation (c) of the assumed neutral oscillations and of their downstream wavelength, we find it desirable to introduce some special simplifications, as the form of the equation derived by making only the usual simplifications (equation (4.20)) appears unduly complicated for a general analysis such as we attempt. These special simplifications (that the terms of (4.22) are small) would be justified if $(c\delta/u_\delta\zeta_0^3)$ and $(c^2\delta^2/u_\delta^2\zeta_0^{3/2})$ are small compared with unity, where δ is the boundary-layer displacement thickness divided by the oscillation wavelength and ζ_0 is described below. Then, for a flat plate boundary layer velocity distribution, which is closely linearly near the surface, we deduce (in equation (4.30)) that the stability equation can be represented by

$$\mathcal{F}(\zeta_0) = (\Delta_0 w + D_2)/(\Delta_0 + D_3) .$$

Here $\mathcal{F}(\zeta_0)$ is a complex function (as shown in Fig.1) of ζ_0, which is a parameter involving the Reynolds number R_δ based on the boundary-layer displacement thickness, as well as involving the terms c/u_δ and δ; w is a complex function of c/u_δ and δ, whose form depends on the assumed velocity distribution. For an inflexible surface D_2 and D_3 are zero, and the equation reduces to the familiar form $\mathcal{F}(\zeta_0) = w$, where w is usually separated into its real and imaginary parts u and v. In contact with an elastic surface, Δ_0, D_2 and D_3 are functions of c/u_δ and δ, and of θ, u_δ/c_2, c_2/c_1 and σ as well, where σ is the ratio of fluid density to that of the elastic material, and c_1 and c_2 are respectively the speeds of propagation of compression waves and of shear waves within the material. The form of the functional relationships for Δ_0, D_2 and D_3 depends on the interior boundary condition, - whether it is fixed or free, or exposed to a fluid. It may frequently be justifiable to ignore D_3 compared with Δ_0 in which event the form of the stability equation reduces to that already proposed by Brooke-Benjamin[2]. However, since Δ_0 may vanish under certain conditions corresponding to a mode of oscillation of the surface *in vacuo*, (that is, with $\sigma = 0$), the inclusion of D_3 leads to some important differences in interpretation.

In an endeavour to understand the qualitative aspects of the various possible solutions of the stability equation arising from variation in the four parameters θ, u_δ/c_2, c_2/c_1 and σ, some specially simple forms of expression for w in terms of c/u_δ and δ are adopted (equations (5.6) and (5.8)) in which in particular the real part (u) is crudely approximated in a manner which is only justifiable if δ is small compared with unity. This is merely a matter of expedience, and in any quantitative solution relatively little more labour would be involved by taking more precise expressions - though even so iterative methods, which are convergent, only if δ is small would commonly be adopted. We certainly consider solutions which in practice would not justify these simplifications, and it is pertinent to enquire what significance may be attached to these. Where the frequency parameter $c\delta/u_\delta$ remains small

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or R_δ is large, it is merely our approximations to u and v which are at fault; large values of δ would indicate a modification to our results - and a difficulty in deriving the numerical values - whilst large values of c/u_δ would imply in fact values close to unity (since in our approximation this implies by equation (5.6) large v, and the more precise form of this relation - equation (5.4) - shows that v takes all positive values for $c/u_\delta \leqslant 1$). Where the frequency parameter is implied to be large, and $1/R_\delta$ small, we can infer nothing more than that our theory is inapplicable and the reality of the solutions obtained must remain in doubt.

However, the inclusion of the elastic constants introduces another aspect to these approximations. We shall, for example, consider modes of oscillation propagated at speeds close to the Rayleigh surface-wave speed (c_3), at those close to the speed of waves on an extensive thin plate (c_4), and various 'resonance' solutions involving speeds of propagation larger than that of compression waves (c_1) or shear waves (c_2) within the material. Provided, of course, that such speeds are small compared with u_δ - as we are at liberty to assume - then such solutions have reality, but they clearly would not be found to exist by an exact analysis if the relevant speed of propagation were greater than u_δ. The possibility of the existence of oscillations propagated faster than u_δ is not of course envisaged in the basic framework of our theory - let alone in the approximations - and it would take us too far afield to discuss this matter here. However, because our approximations take no account of the fact that c cannot exceed u_δ, these solutions, involving discrete speeds of propagation connected with the elastic properties of the material, appear to exist in our treatment irrespective of the relative value of c/u_δ; and even if we consequently imply in our analysis that c/u_δ is greater than unity, this cannot of itself be taken to imply that they are in reality absent, as our analysis gives an over-estimate of c/u_δ under such conditions. In fact, in an exact treatment these modes of oscillation will be found to cease to exist for some value of $c < u_\delta$, and it may be that the condition $c = u_\delta$ in our approximation is no bad guide to their disappearance. Whether this disappearance implies their absence is effectively a problem of 'inviscid flow' stability, and possibly a question susceptible to analysis.

We investigate solutions for large and small values of u_δ/c_2, with the purpose of definitely including or excluding such modes of oscillation, for both large and small values of θ. In this way we hope to be able to interpolate broadly the effect of these parameters over their complete range of variation, though with only the edges of the jig-saw complete, it is obviously diffiult to try to predict all that lies in the interior, - for general values of θ and u_δ/c_2. The parameter c_2/c_1 is rarely of crucial importance, and in any case for common materials it has only a limited variation between a small (positive) value typical of non-rigid but relatively incompressible materials like rubber, - to about 1/3, - a figure typical of most metals and alloys.

9.1 Thick surfaces - large θ

We first limit our considerations to speeds of propagation $c < c_2$, which would in fact cover all conditions we could legitimately consider if u_δ/c_2 is not very large. The oscillations within the elastic material attenuate exponentially away from the surface into the depth of the material, and the solution for large θ is indistinguishable from that for infinite θ. Our solution being only qualitative we have little basis for any numerical indication of what we mean by 'large θ', but provided u_δ/c_2 is not large, a surface thickness of, say, 4 times the wavelength could be interpreted as large, and the 'wavelength' mentioned here being that of the neutral oscillation, its value could be taken for this purpose to be that for an inflexible surface at the relevant Reynolds number. (Further amplification of this point will be found in paragraph 9.3 below.)

9.11 Modification to Tollmien-Schlichting mode

For small values of σ and u_δ/c_2, the lobe diagrams of c-R_δ and δ-R_δ, familiar in the study of the Tollmien-Schlichting waves of inflexible surface stability, are modified only slightly, and what modification there is, depends primarily on the value of $\sigma u_\delta^2/c_4^2$ (or of $\rho u_\delta^2/G$, where G is the modulus of rigidity). The effect of the surface elasticity, as Brooke-Benjamin[2] has pointed out, is to decrease δ below the inflexible surface values and to increase R_δ, thus increasing the minimum Reynolds number for which neutral oscillations exist. (There is as well a slight decrease in c/u_δ.) If u_δ/c_2 is small, the proportionate reduction in $(1/R_{\delta_{min}})$ amounts to

$$2\sigma u_\delta^2/c_4^2 = \frac{\sigma u_\delta^2}{2c_2^2}\bigg/\left(1-\frac{c_2^2}{c_1^2}\right) = \left[\tfrac{1}{2}\rho u_\delta^2/G\left(1-\frac{c_2^2}{c_1^2}\right)\right].$$

As a better approximation for larger values of u_δ/c_2, - though generally rather an overestimate - if in Fig.2 the value of $c^\times$ is taken as the speed of propagation of Tollmien-Schlichting waves at minimum Reynolds number (i.e., $0.43\ u_\delta$), then the proportionate reduction can be taken as the ratio of $\sigma u_\delta^2/c_4^2$ to that value read from the figure.

As σ and u_δ/c_2 further increase, the above mentioned effects become more and more pronounced - the decrease in speed of propagation being particularly noticeable if σ is large (i.e., if the surface is very light in density compared with the fluid). Indeed, the maximum value of c is always less than the value of $c_3(\sigma)$ shown in Fig.3 as a function of σ. Ultimately, the lobes of the diagrams lose their characteristic shape when $c^\times$ (given in Fig.2) is less than the maximum speed of propagation of neutral disturbances. For then, at the higher speeds of propagation of neutral oscillations, the Reynolds number tends to infinity and the wavelength to infinity, producing a double lobe as indicated in Fig.4(a)*. With further reduction the lobes tend to shrink towards zero speed and infinite R_δ, until for

$$\sigma u_\delta^2/c_4^2 > \tfrac{1}{2} \quad \text{i.e.,} \quad \tfrac{1}{2}\rho u_\delta^2 > \left(1-\frac{c_2^2}{c_1^2}\right)G$$

they are altogether absent.

9.12 The Rayleigh wave mode

However, before this happens, a new mode of neutral oscillation shows its presence. If this value of $\sigma u_\delta^2/c_4^2$ is obtained by relatively large σ, but small values of u_δ/c_4, this mode only appears as a possibility with such a high speed of propagation, and such a small wavelength and Reynolds number, that our analysis is hardly applicable. However, if u_δ/c_3 has a greater value than unity, the mode is recognised on a c-R_δ diagram as existing at all R_δ, having a value rather greater than $c_3(\sigma)$ at low Reynolds number, increasing through the value $c_3 = c_3(0)$ at some finite R_δ, and tending to decrease to it again as $R_\delta \to \infty$ (Fig.4(b)).

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*None of the diagrams of Figs.4(a) - (j) is intended to be anything more than a free-hand sketch, composed on the basis of a study of the various formal, and approximate, solutions of paragraphs 6, 7 and 8. Generally the information available consists only of a knowledge that in certain ranges of Reynolds number c and δ are increasing (or decreasing) with increase of R_δ, and that their asymptotic variation at the extremes of Reynolds number is of a particular form. Possibly (where indicated) numerical values at one or two particular points may be known, but otherwise the curves shown have no quantitative significance.

The value c_3 has some special significance as it is the speed of propagation of surface waves in the material in the absence of fluid stresses - the so-called Rayleigh wave speed. Thus we are asserting such a neutral oscillation also to exist even at $R_\delta = \infty$ in the presence of a fluid, though only if it has a specific wavelength. This wavelength varies inversely as u_δ/c_3 (and also decreases somewhat as σ is decreased), so that although the mode of oscillation is apparently within the range of validity of our solution for large u_δ/c_3, insofar as the speed of propagation is then sufficiently small, the wavelength is then too small to justify our approximations, at least in the simplified expressions for u and v, - the equation connecting $\mathcal{F}(\zeta_0)$ and w remaining valid.

9.13 Conditions for a 'heavy' surface

It is therefore judged that, despite the quantitative breakdown of our analysis, this mode of oscillation does truly exist for $u_\delta < c_3$ in some form for all σ, including vanishingly small values, and this seems physically plausible since neutral oscillations of this speed (c_3) can exist, as we have pointed out, in the absence of the fluid, - though then they can have arbitrary wavelength. Indeed, if $\sigma = 0$ - that is, the surface is indefinitely heavy compared with the gas - there can be no effect of the fluid on the surface, though the surface itself constrains the fluid, by virtue of the assumed kinematic and 'no-slip' boundary conditions. Thus the surface can perform its own modes of neutral oscillation and impress these on the fluid, any stabilising or destabilising effect of the latter being as nought if we assume $\sigma = 0$. Such conditions (in the limit of $\sigma = 0$) correspond to an indeterminacy in the stability equation, D_2 and D_3 being zero for $\sigma = 0$, and Δ_0 being zero for a natural surface mode of oscillation, such as that given by $c = c_3$. Neutral oscillations of <u>any</u> finite wavelength can exist at this speed if $\sigma = 0$, irrespective of R_δ.

However, if θ is very large, but finite, and the interior surface is fixed, such neutral modes exist for the surface itself (with $\sigma = 0$), again for any wavelength, but with some discrete speed of propagation $c > c_3$ depending on the wavelength (i.e., the oscillations are dispersive). The speed is c_3 for zero wavelength (i.e., infinite $\theta\delta$), and the absence of neutral oscillations at speeds other than c_3 for $\theta = \infty$ arises simply because any finite wavelength is necessarily as nought by comparison with the surface thickness.

These results, of course, would be modified in practice by the presence of frictional dissipation, because then only attenuated oscillations - rather than neutral ones - could exist at $\sigma = 0$; there is, of course, no question of the existence of amplified oscillations if $\sigma = 0$, even for a non-dissipating surface. In practice, in any case, the condition $\sigma = 0$ could not be reproduced, but this limiting condition has some importance as a guide to conditions for small σ. Thus we know that, in the three-dimensional R_δ-δ-c 'space', neutral oscillations for non-zero σ lie on twisted curves, whose projections on the c-R_δ and δ-R_δ planes are plotted on diagrams such as those of Fig.4. The description of lines of constant amplification factor will generate surfaces or 'sheets' in this space, and since for a non-dissipating material only neutral oscillations exist (at least for the mode in question) at $\sigma = 0$, it is evident that as $\sigma \to 0$, this sheet must tend to degenerate to the planes $c = c_3$, and $\delta = 0$ for $c > c_3$, if θ is infinite, or to some cylinder $\delta = f(c)/\theta$ for $c > c_3$, if θ is finite. Furthermore, the amplification factors (whether positive or negative) must tend to zero as $\sigma \to 0$. Thus for small, but non-zero σ, we can expect this cylinder to be deformed to some extent, as indicated for instance by the projections of the line of neutral oscillation in Fig.4(b), and the amplification or attenuation factors to grow particularly in regions removed from the neutral line. On the sheet this neutral line will delineate regions of amplified and attenuated disturbances, and the region of amplification (or attenuation) would only terminate at the limits of this space, or else on another neutral line. The latter possibility certainly arises when $\sigma = 0$ and $\theta = \infty$, for then $c = c_2$ is also a mode of neutral oscillation for all wavelengths and Reynolds numbers (corresponding to the propagation of shear waves throughout the material), but for θ large but finite, the two cylindrical sheets of neutral oscillation in R_δ-δ-c space are separate, meeting only on the line $\delta = 0$ and $c = \infty$; we shall return to this point in paragraph 9.2 as we shall see that there exists an indefinite number of rapidly propagated modes of oscillation with $c > c_2$.

9.14 Disappearance of the Rayleigh wave mode

Another important point concerns the magnitude of u_δ relative to c_3. We see that there is nothing to prevent the neutral oscillations existing at $\sigma = 0$, even if $c_3 > u_\delta$, though their presence would then obviously not be uncovered by an examination of boundary-layer stability by the usual methods. (We have referred earlier to this particular problem.) Supposing, as seems reasonable, that for σ small but non-zero the oscillations only exist with speeds of propagation above some certain minimum (corresponding to zero wavelength), the sheet of oscillations in the R_δ-δ-c space will only appear in the range $c \leqslant u_\delta$ if c_3/u_δ is sufficiently small; certainly, anyway, this applies to the neutral line, which as indicated in Fig. 5(a) grows from a closed curve as c_3/u_δ is reduced below unity. This closed curve indicates a finite region of either amplified or attenuated oscillation - to hazard a guess, the former would seem more plausible - which gradually expands as c_3/u_δ is reduced. If it is indeed the higher speed oscillations which are unstable[2], the shape of the sheet of oscillations for finite θ and zero σ would suggest that it is the longer wavelengths which are the unstable ones.

Where c_3/u_δ is reduced to such proportions that the c and δ values of the unmodified form of the inflexible surface mode of oscillations impinge on those representing the natural surface mode, the latter suffer some distortion as indicated in Fig. 4(c): if $\sigma \to 0$ in such circumstances, the otherwise limiting cylindrical shape of the sheet of oscillations is indented, and the neutral curves of both modes tend to coincide over part of the speed and wavelength range.

9.2 Rapidly propagated oscillations - large u_δ/c_2

The fast moving oscillations we have here in mind, are those which are propagated at speeds greater than c_2, the speed of shear waves within the material, and particularly those which travel at speeds greater than c_1, the compression wave speed. These, of course, only lie within the range of validity of our study if $c_2 < u_\delta$, or $c_1 < u_\delta$, as the case may be. The question of their existence in other circumstances, as we have already pointed out, cannot be answered except for $\sigma = 0$, when they are certainly present.

However, with this reservation on the magnitude of u_δ, we find that solutions can certainly exist which indicate a number of modes of neutral oscillation. There is considerable analytical difficulty in discovering the precise variation of speed and wavelength of all such oscillations as a function of Reynolds number, and we are content to derive what is little more than an existence theorem. In any particular case, however, it may not be unduly difficult to derive numerical solutions - but this has not been done as it is apparent that the detailed structure of c-R_δ and δ-R_δ diagrams for such oscillations is very critical to the choice of the parameters, particularly of c_2/c_1.

9.21 Compression wave resonance

We find that solutions exist for $u_\delta > c > c_1$ provided at least that σ is neither very small, nor very large, compared with unity; and that

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there are a number of such solutions, each independent of the next, the number generally increasing in rough proportion to the surface thickness, and the number of those lying within the range of validity of our analysis increasing, of course, with u_δ/c_1. They can, in a very real sense of the term, be regarded as 'harmonics' of some fundamental, the wavelength of the higher harmonics being progressively smaller. The fundamental itself would have a wavelength increasing with θ, σ and u_δ/c_1, but the speed of propagation of fundamentals and harmonics alike increases with u_δ/c_1, but decreases with increasing σ (that is, it is slower for a surface of low inertia). We know such oscillations also exist for $\sigma = 0$ for all wavelengths, but in our analysis if we allow σ to tend to zero, the wavelengths are too small for our approximations to u and v to be valid; high values of σ tend, as we have noted, to reduce c (apparently below c_2) and certainly no solutions of this type can exist for $c < c_2$.

Each 'harmonic' will - if lines of constant amplification factor are supposed to generate a sheet in the R_δ-δ-c space - correspond to a separate sheet, but the precise shape of such sheets as revealed by projections on c-R_δ and δ-R_δ diagrams has not been determined: it would indeed be a formidable task. However, it is apparent that solutions for the neutral line occur in pairs, with the value of $c\delta$ for each more or less the same and a constant, but one existing for high R_δ and tending to have a reduced speed of propagation (and so higher δ) at both high and low R_δ, whilst the other, existing for lower R_δ, tends to increase its speed of propagation at the extremes; for both there is quite a wide range of R_δ over which c (and δ) vary only a little. However, these are only general trends, as it is not clear whether the range of R_δ for which either member of the pair of solutions exists is bounded or not.

9.22 Shear wave resonances

It is not at all clear, either, whether these solutions, on which we have based our comments, are the only ones. In general, it would appear that there were also others. Thus if $c_1 \gg c_2$, corresponding to a material of high bulk modulus but low rigidity, there are certainly solutions for $c_1 > c > c_2$, and $c < u_\delta$, which have a different character; these exist at least provided the surface has the same order of thickness as the boundary layer, and provided (as before) that σ is not small, though again it is known that such oscillations also exist for $\sigma = 0$, and in particular an indefinite number exist at $c = c_2$ for a surface of infinite thickness.

9.23 Physical origin of resonances

What is the physical nature of these oscillations? They are well-known in seismography[4], and it would appear that they are resonances of the material of the surface, tuned, as it were, by the thickness of the material, and not its surface dimensions. The general family of solutions discussed above, which exist for $c > c_1$, can be shown, for example, for a surface fixed at its interior, to correspond roughly to the condition in which compression waves reflected from the interior surface arrive back at the surface exactly in phase with the surface waves. Thus the 'fundamental' of the oscillations is such that the wave originating (say) from a peak compression is reflected to arrive back at the following peak compression; the 'harmonics' correspond to waves which have shorter wavelengths, and for which the reflected wave arrives at the surface two or more cycles later, but still in phase. The thicker the surface, and the greater the length travelled by the wave, the greater is the number of cycles delay that is possible. With a free surface, compression waves are reflected as expansions from the internal surface, and here an analogous resonance is experienced if the thickness of the material is such that the reflected waves are exactly 180° out-of-phase.

In fact, the presence of two types of waves (of shear and of compression) within the material, greatly complicates the whole physical

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picture and the analysis. A single type of wave - shear or compression - reflects or refracts as the case may be, as two distinct waves - one of each type - from any boundary. As a result, the solution is not exactly periodic as it would be with a single-wave system (unless c_2/c_1 is a rational fraction), though it is, broadly speaking, quasi-periodic.

Although we have discussed this solution in some detail, it will be evident that it may be rather an academic problem, as the sensitive nature of the resonance implies in practice that any slight non-uniformity at the elastic material's interior surface will modify it, - or maybe even eliminate it, on much the same grounds that it is easy to produce a bad note on a violin string by imprecise fingering.

9.3 Thin rigidly-attached surface

We interpret a 'thin' surface as one whose thickness is comparable with that of the boundary layer to which it is exposed, but at the same time we exclude from consideration those neutral oscillations of the quasi-periodic character studied in the previous paragraph.

9.31 Modification to Tollmien-Schlichting mode

If the surface is rigidly attached at its inner boundary, the Tollmien-Schlichting mode of oscillation found for an inflexible surface exists in a slightly modified form, the extent of the modification being appreciable only at smaller Reynolds numbers and tending to vanish for $R_\delta \to \infty$. The parameters affecting the modification are $\sigma u_\delta^2 \theta/c_2^2$ and, to a less extent, c_2/c_1 (but not the value of σ by itself), and these determine the magnitude of the increase in Reynolds number and the reduction in the speed of propagation which is noted if the solution is compared with the inflexible surface values. Plainly, for $\sigma \to \infty$, or $c_2 \to \infty$, or $\theta \to 0$, the modification vanishes, since then in each case the surface becomes virtually inflexible. On the other hand, no matter how large the value of $\sigma u_\delta^2 \theta/c_2^2$ may be, the mode of oscillation never vanishes (as we found to happen for an infinite surface).

9.32 Rayleigh wave mode

Another mode of oscillation also exists dependent on the same parameters. The behaviour of this in the range of finite $(R_\delta c_2^2/\sigma\theta u_\delta^2)$ as indicated in Fig.4(d), and it involves wavelengths of oscillation such that δ is of order $1/R_\delta$ in this range. For smaller Reynolds numbers, the mode becomes associated with the 'fundamental' of the shear wave resonances and $c \to \infty$ but $\delta \to 0$, whilst its behaviour at high Reynolds numbers is plainly associated with the Rayleigh-wave mode for thick surfaces as it involves wavelengths very small compared to the surface thickness. We have found no simple solution for this particular anomalous behaviour, though it appears that either neutral oscillations may only exist for a range of R_δ with an upper bound - the curves on the diagrams returning to $R_\delta = 0$ at finite $c \leqslant c_3$ with infinite δ, - or else c tends to c_3 at $R_\delta = \infty$, again with infinite δ. Which of these alternatives may actually appear may be dependent individually on the parameter σ, θ, u_δ^2/c_2^2 and c_2/c_1; we have adduced some evidence which points to the former alternative if $c_2^2/c_1^2 < 1/3$, but this may not be the only parameter of significance.

This association with both the shear wave resonance and the Rayleigh-wave mode is not surprising, as we know that for $\sigma = 0$, the latter mode involves speeds of propagation greater than c_2 for sufficiently small values of $\theta\delta$, and indeed from the relation connecting $\theta\delta$ and c, - which we wrote as $\delta = f(c)/\theta$ in paragraph 9.1, - we can determine that $c \to \infty$ as $\delta \to 0$, in which limit it joins the shear wave resonance. Thus for $\sigma = 0$, neutral oscillations exist in the R_δ-δ-c 'space' on the cylinder $\delta = f(c)/\theta$; and for $\sigma \to 0$ the sheet generated by solutions for the mode discussed above for both amplified and attenuated oscillations will tend to

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this shape, with the neutral line delineating stable and unstable regions. The part of the Reynolds number range of the neutral curve for which we have obtained a solution will tend towards the line $\delta = \infty$ and $R_\delta = 0$ as $\sigma \to 0$, and in the limit, at higher R_δ it must lie on this cylindrical sheet, tending to emanate from the point $c = c_3$ on the line $\delta = \infty$ and $R_\delta = 0$, and returning either to this point, or to the point $c = c_3$ on the line $\delta = R_\delta = \infty$.

For $\theta \to 0$, or $c_2 \to \infty$, the elastic surface becomes - in the limit - effectively a rigid one, and we should expect only the Tollmien-Schlichting oscillations to exist. Here again that part of the Reynolds number range for which we have found the form of the neutral line, tends to the line $\delta = \infty$ and $R_\delta = 0$ for all values of c above some minimum, and the remaining part must either tend towards the plane $\delta = \infty$ (if $\theta \to 0$) or $c = \infty$ (if $c_2 \to \infty$), thus disappearing from the range of physical interest.

9.4 Rigidly attached surface of moderate thickness - $u_\delta/c_3 < 1$

A modified form of the Tollmien-Schlichting oscillation has been observed to be present for both thick and thin surfaces and it is possible to identify it for all thicknesses of material - in other words, without the restriction that θ is about unity. It is found that, whatever the value of θ, there is the reduction of c/u_δ and δ, and the increase of R_δ, compared with the inflexible surface values which we have seen before to exist, and as for a thin surface this is most marked for small R_δ. The reason is that the effect of the surface on the oscillation depends on the ratio of surface thickness to wavelength, rather than to boundary-layer thickness: thus the significant parameter is not θ but rather $\theta\delta$, and since the values of δ for neutral oscillations decrease as R_δ increases, we see that at sufficiently high R_δ any finite value of θ would be - in effect - a 'thin' surface, and in particular at $R_\delta = \infty$ the effects of surface elasticity are quite negligible for the indefinitely large wavelengths of oscillation when these exist.

It follows that for any finite surface thickness, part at least of the usual δ-R_δ and c-R_δ curves of neutral oscillation always remains, although for an infinite surface we have seen them to be absent if $\sigma u_\delta^2/c_4^2 > 1/2$. To reconcile this apparent contradiction, it is to be noted that if indeed $\sigma u_\delta^2/c_4^2$ exceeds this limit, then as $\theta \to \infty$, both $\delta \to 0$ and $R_\delta \to \infty$ asymptotically, so that although the mode never disappears for finite θ, it at least tends to do so in the limit.

9.5 Freely mounted surface of moderate thickness

We shall now consider surfaces which are free of stress at their interior boundary, excluding from consideration the generality of resonance modes which we noted to exist in paragraph 9.2.

Plainly for a surface of infinite thickness it would not matter whether the interior surface were rigidly attached or not as, excluding these resonances, the disturbance attenuates through the material of the surface and the interior 'boundary' - at infinity - is undisturbed whether or not it is constrained. But in practice for any surface of finite thickness the interior boundary condition has a significant effect.

9.51 Modification to Tollmien-Schlichting mode

Considering the modification to the Tollmien-Schlichting oscillation then as θ reduces from indefinitely large values, we can generally represent the effect of the surface elasticity as reducing δ and increasing R_δ for disturbances of the same speeds as those for the inflexible surface. However, whereas on a rigid surface, the effect gradually diminishes as θ is decreased, the opposite is true if the surface is - as we now envisage - free.

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Indeed, there appears to be no possibility of the existence of neutral oscillations of this form which have a value of $\theta\delta$ below some minimum - that is, whose wavelength is sufficiently large compared with the surface thickness. The physical reason for this is not apparent to the author. But the effect is that even where $\sigma u_\delta^2/c_4^2$ is less than 1/2, and so not sufficiently large to cause this mode of oscillation to be absent for $\theta = \infty$, then for finite θ the neutral oscillations at large R_δ will be absent, and as θ decreases, the extent of the Reynolds number range in which neutral oscillations exist gets progressively smaller, not only by an increase in the lower Reynolds number bound, but by a decrease in the upper bound as well. The δ-R_δ and c-R_δ diagrams of neutral oscillations form closed curves (Fig.4(e)) and ultimately for a sufficiently thin surface, vanish altogether: the maximum skin thickness for which the oscillation is absent is shown in Fig.6 (which is only a good approximation if $\theta >> u_\delta/c_4$) and it will be seen that the smaller the value of $\sigma u_\delta^2/c_4^2$ the thinner must be the surface to reveal this effect. On the other hand, if $\sigma u_\delta^2/c_4^2$ exceeds that limit of 1/2 for which these oscillations are absent for infinite θ, they will remain absent for all finite θ as well.

9.52 Practical realisation of the 'free' surface

This, of course, is just the kind of stabilising effect we are looking for, and it is a pity that it arises from a concept of a boundary condition which is apparently completely unreal. However, it would be reasonable to suppose that if the surface was in fact rigidly bonded on short discrete spanwise strips, whose downstream extent was small compared with the critical Tollmien-Schlichting wavelength, and the downstream separation between each being large compared with this wavelength, then at least the mathematical model represents a fair description of the conditions between strips. But, of course, as the critical wavelengths for large Reynolds numbers tend to be very large, it would seem likely that neutral (and amplified) oscillations would in any such practical example still exist at large R_δ. Alternatively, it may be that a surface bonded at discrete studs arranged over the surface reproduces in some measure the conditions of the concept, and the development of studded rubber skins by Kramer[6] springs naturally to mind. (The discussion in paragraph 9.7 below throws some light on the action of the fluid damping also used in Kramer's experiments.) In practice, too, a question arises concerning the structural integrity of a skin of this kind. It may be desirable to use a material with a high modulus, but this drives us to thinner skins in order to obtain the effect we are looking for (see paragraph 9.67).

However, there is obviously more to the matter of reducing skin thickness than we have so far mentioned, as plainly a thin skin may well flap like a flag or any other form of flimsy surface exposed to moving fluid. In other words, it is reasonable to suppose that, if θ is small enough, there will be other modes of oscillation - even apart from the resonances already mentioned - which may bear some resemblance to the well-known Kelvin-Helmholtz type of instability. We do indeed find such modes of oscillation, though their relationship to the Kelvin-Helmholtz instability is not at all clear.

9.6 Unstretched free membrane

9.61 Longitudinal and flexural waves

Implying by the term 'membrane' a surface whose thickness is of the order of that of the boundary layer, and supposing first that (u_δ/c_4) is sufficiently large so that oscillations propagated at speed $c = c_4$ may properly be considered, we find two modes of neutral oscillation to exist. On a c-R_δ diagram (Fig.4(f)), it is seen that one mode generally exists only for R_δ less than some upper bound, the value of c/c_4 being bounded below by a small value proportional to $(u_\delta\theta/c_4)^{1/3}$, and the

maximum/

maximum Reynolds number varying as $(u_\delta/c_4)^{5/3}\,\theta^{1/3}$, if θ is small. The δ-R_δ diagram shows that both arms of the neutral curve extend towards indefinitely large δ as $R_\delta \to 0$, though the solution is not, of course, valid in such a circumstance. In fact, however, this behaviour suggests that this solution might in some circumstances, be related to the fundamental of the shear-wave 'resonance' modes, but its limiting behaviour for small R_δ is difficult to determine correctly - even if the basic theory provides an adequate approximation, which seems doubtful. The minimum value of δ in this mode is twice the minimum value of $(c/\theta c_4)$, and this is also large compared with unity if θ is small.

The other mode of oscillation exists for larger R_δ, and on a c-R_δ diagram it forms a lobe extending towards the other, and its two arms are asymptotic at infinity, the one to $c = c_4$, and the other to a value which is a little larger than the minimum value of the other mode of oscillation. Likewise, there is a lobe on the δ-R_δ diagram, one arm asymptotic to $\delta = 0.58/\theta$ (corresponding to $c = c_4$) and the other again to a minimum a little larger than that of the other mode.

It will be seen that we are implying the existence of two conditions of neutral oscillation at $R_\delta = \infty$, the one with a speed of propagation equal to c_4, which, as we have already noted, is the speed at which waves spread over an extensive, unstretched, thin membrane, and the other with a smaller speed and larger wavelength (related by $\sqrt{3}\theta\delta = 2c/c_4$), which can be recognised as a mode involving long flexural waves over a thin plate* - that is, anti-symmetric vibrations, whereas those propagated at speed $c = c_4$ are longitudinal, and symmetric, vibrations. It is natural to associate the lobe on the δ-R_δ and c-R_δ diagrams, which is formed by these asymptotes, with a region of amplified oscillation, and to suspect that at $R_\delta = \infty$ the limits of the neutral curve mark a region of Kelvin-Helmholtz instability. However, it should be noted that the wavelength of the neutral oscillation, being related to the thickness of the elastic layer, vanishes with $\theta \to 0$, which is a result which would not apply if we considered the reaction at the inner surface of the elastic material to a fluid medium (see paragraph 9.7): thus, although our result is analogous to that of the Kelvin-Helmholtz instability, its boundary condition is not designed to deal with the stability of an infinitesimal interface between two fluids.

9.62 <u>Effect of value of σ</u>

Increasing the value of σ has the effect of depressing the minimum values of c and δ of the neutral curves, but on the other hand they do not increase indefinitely if σ is decreased, the minima being inversely proportional to $(12+\sigma)$. The most noticeable effect of a change in σ is in the separation between the two lobes, this being large if σ is large; but provided $u_\delta/c_4 < 2.3$ (see Fig.7) then as σ is reduced the two lobes will join, and for smaller σ (i.e., for surfaces heavy in density compared with the fluid) the neutral curves separate into two lines in the projections of the c-R_δ and δ-R_δ diagrams (Fig.4(g)). As $\sigma \to 0$, the speed of propagation of the more rapid 'longitudinal wave' mode will tend to c_4 over virtually all the Reynolds number range, and the value of $\theta\delta$ will remain finite except as $R_\delta \to 0$. Likewise, except in this region, the speed of propagation of the 'flexural wave' mode will tend to follow the law $c=\frac{1}{2}\sqrt{3}\theta\delta c_4$, with c and δ both increasing somewhat as R_δ increases an exception also arises in this mode due to the fact that the neutral curve is indented round the outside of the Tollmien-Schlichting mode (which, of course, exists in virtually unmodified form if $\sigma \to 0$) on both diagrams, tending towards it from larger values of δ and smaller R_δ.

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*See, for example, equation (6.21) of Ref.4, which corresponds with our equation (7.16) for $\xi = 1/3$.

9.63 Solution for a 'heavy' surface

At $\sigma = 0$, the solution for elastic waves on a free surface[7] shown in Fig. 8 indicates that - viewed in a R_δ-δ-c space, - neutral oscillations exist on two cylindrical sheets given by $\delta = f_1(c)/\theta$ and $\delta = f_2(c)/\theta$, say, for all Reynolds numbers; and for small $\theta\delta$ these yield the values of $c = c_4$ and $c = \frac{1}{2}\sqrt{3}\theta\delta c_4$, respectively, consistent with the location of the neutral curves for $\sigma > 0$. Since, for both symmetric and anti-symmetric modes, $c \to c_3$ as $\theta\delta \to \infty$, it may well be that the undiscovered limiting behaviour of these neutral curves as $R_\delta \to 0$ (with $\sigma > 0$) is not associated with a shear wave resonance, but is such that they tend to $c = c_3$ and $\theta\delta = \infty$; they certainly must do so, at least, if σ is small.

For small σ, then, there are two sheets of oscillations in R_δ-δ-c space, each divided by the neutral line into a region of (weak) amplification and (weak) attenuation. As θ increases these surfaces must either coalesce or one must vanish. For the solution for $\sigma = 0$ suggests as we have already noted that if $\theta = \infty$, the oscillations are on the plane $c = c_3$ for all finite R_δ and δ, but other oscillations of the symmetric model will lie on the plane $\delta = 0$ for $c_3 < c < c_4$, and those of the antisymmetric mode in the same plane on the strip $c < c_3$; Fig. 8 applies of course for any θ and makes it clearer how the surfaces tend to this condition as θ increases. Certainly our solution suggests that both the speed of propagation of the anti-symmetric mode, and its wave-number, increase as θ increases, and although it is not permissible to treat θ as large in our solution, this is at least compatible with the trend towards coalescence. As we shall note in paragraph 9.64 below, however, there are anomalies in the behaviour of the solution, where σ is not indefinitely small.

If $\sigma \to 0$ with $u_\delta/c_4 > 2.3$, we find (see Fig. 7) that the neutral curves retain the lobed shape of Fig. 4(f) in the limit, and since c_4 is then less than the maximum speed of propagation of disturbances of the inflexible surface mode, these lobes tend to be 'wrapped around' each side of the neutral curve of this mode. This gives us a clue to the means by which the neutral curve in this configuration apparently passes from one sheet of oscillation to the other. The sheet of the flexural wave oscillation is, we know, distorted so that its neutral line tends to coincide with that on the sheet of the inflexible surface mode: possibly the two sheets are joined. Whatever the form of this deformation - and there are various possibilities - the effect will plainly be that one part of the sheet of the longitudinal wave oscillation is brought into contact with the flexural wave oscillation, as c_4 is reduced (below $u_\delta/2.3$ if σ is small). There are, as a consequence, certain deductions one could make about the orientation of the regions of amplified and attenuated oscillations on each side of the neutral lines on each sheet, in order that the sheets match where they join, but the variety of possibilities is such that only a study of attenuated, or amplified, oscillations could yield a reliable indication of the sheet geometry.

9.64 Effect of thickening the surface

For larger values of θ (indefinitely large if $\sigma \to 0$ or $u_\delta/c_4 \to 0$, but otherwise finite) we find an anomolous behaviour in the neutral line on the longitudinal wave mode, shown in the detail of Fig. 4(h), which is again almost certainly associated with the intersection of its sheet of oscillation with another - presumably that of the flexural wave mode. As applied to Fig. 4(g), such a modification could change the shape of the neutral line of the faster propagated mode to a lobe extending in both branches to $R_\delta = \infty$ in the δ-R_δ and c-R_δ diagrams: the minimum Reynolds number of this lobe would increase with θ, and so leave effectively only the single neutral line in the range of finite R_δ as $\theta \to \infty$, as of course is compatible with Fig. 4(b). As applied to Fig. 4(f),

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the modification would make the low Reynolds number neutral line S-shaped, extending it to $R_\delta = \infty$, and here as θ increases, the high Reynolds number lobe progressively retreats towards $R_\delta = \infty$, again leaving in the limit, as we would expect, the single neutral line.

9.65 Disappearance of the longitudinal wave mode

In more accurate assessments of boundary-layer stability, the problem of the relative disposition of these sheets of oscillations would be apparently simplified by the restriction $c < u_\delta$. The neutral line of the longitudinal wave mode tends to vanish - as a closed curve - as u_δ is lowered below c_4 (Fig.5(b)); or if σ is sufficiently large that, with $c_4 < u_\delta$, the stability diagram is that shown in Fig.4(f), the lowering of u_δ below c_4 joins the two sheets of oscillation along $c = u_\delta$, the neutral line retreating as shown in Fig.5(c) to the lower speeds of propagation characteristic of the flexural wave mode alone. It is doubtful whether the latter ever 'disappears', no matter how small u_δ may be: if σ is small we should expect it to move towards $c = u_\delta$ as u_δ is decreased, the associated wavelength being increased, as governed by the value of $\theta\delta$ shown in Fig.8, to correspond with this speed.

9.66 Increase of surface rigidity

Indeed as $c_4 \to \infty$, we must reproduce the effects of a rigid surface (since this also implies $c_2 \to \infty$, and an infinite rigidity), and Fig.8 shows that, even without a restriction of $c < u_\delta$, the lower speed flexural wave mode would exist only for vanishingly small δ, and the longitudinal wave mode would vanish (towards $c = \infty$). For any finite R_δ, the speed of propagation of the former mode would be virtually constant at its asymptotic value for large R_δ, only varying significantly at vanishingly small R_δ if $c_4 \to \infty$. This asymptotic value implies $c/u_\delta \sim [c_4^2\theta/u_\delta^2(12+\sigma)]^{1/3}$ - that is large values of c/u_δ in the limit $c_4 \to \infty$, - and the trend of c towards u_δ noted above to apply if $u_\delta/c_4 \to 0$ where σ is small, evidently also applies even if σ is large. The corresponding wavelengths tend to zero. Thus whatever might be said in other circumstances, for that region of small σ or small u_δ/c_4 where the neutral curves appear as in Fig.4(g), it would appear plausible that the shear wave mode is generally associated with amplified oscillations which are more rapidly propagated (and have a smaller wavelength).

9.67 A tentative 'prescription' for a thin stabilising skin

It is interesting to see the consequences of such a supposition about the amplification of the flexural mode on the 'prescription' for a free elastic skin which will stabilise the boundary layer as far as possible. To eliminate the longitudinal compression wave and resonance modes we must at least make $u_\delta < c_4$, though this may not of itself be sufficient. Anticipating that values of θ will be so small that we are interested only in the part of Fig.6 where the variables are related by a cube law, we find that, to eliminate the Tollmien-Schlichting waves, we must make $\theta^3 c_4^2/\sigma u_\delta^2$ less than - in round numbers - say, 2,000. Then we would predict the minimum speed of propagation of the flexural mode to be c_f where

$$\left(\frac{c_f}{u_\delta}\right)^3 \simeq 0.42\left(\frac{\theta^3 c_4^2}{\sigma u_\delta^2}\right)^{1/3}\left(\frac{c_4}{u_\delta}\right)^{4/3}\frac{\sigma^{1/3}}{12+\sigma}$$

and the extent of the amplified region $c > c_f$ will evidently be less the higher the value of c_f. Of course, as pointed out before, we may erroneously predict that $c_f > u_\delta$, and this could not be taken to indicate the absence of this mode in more exact calculations; but as a simple criterion, this value of c_f (which is also inversely proportional to the

largest/

largest wavelength of amplified disturbances) may yet have some relevance to the extent of amplification.

Clearly, to make c_f large, we must take $(\theta^3 c_4^2/\sigma u_\delta^2)$ as close to its permissible upper bound (of 2,000) as possible, and make u_δ/c_4 as small as possible, and not merely just less than unity. The optimum value of σ appears to be 6. There is, no doubt, little significance in this precise figure; it can merely be inferred that σ must neither be too large, nor too small - the one alternative reducing the flexural wave speed, and the other reducing the effect of the modification to the Tollmien-Schlichting mode. In fact, the physical stiffness of the skin would be determined by a parameter proportional to the product of rigidity modulus and the square of the thickness, that is to

$$\lambda^2 = \frac{G\theta^2}{\rho u_\delta^2} = \left(\frac{\theta^3 c_4^2}{\sigma u_\delta^2}\right)^{2/3} \left(\frac{c_4}{u_\delta}\right)^{2/3} \Big/ 4\sigma^{1/3}\left(1 - \frac{c_2^2}{c_1^2}\right)$$

so that although the largest permissible $(\theta^3 c_4^2/\sigma u_\delta^2)$ and the smallest value of (u_δ/c_4) also help to stiffen the skin, so does choice of a small σ (i.e., high density skin).

The extent to which we could reduce u_δ/c_4 in practice would be governed by the limit to which we could reduce θ since, with fixed σ, the two are related by the criterion for the elimination of the Tollmien-Schlichting mode. If we suppose θ therefore fixed at some practical minimum, $\sigma u_\delta^2/c_4^2$ is automatically determined by that criterion, and therefore also the stiffness criterion λ^2 will have a definite value - being inversely proportional to the chosen θ, so that the thinnest skins which avoid the inflexible surface mode are the stiffest. The expression for c_f is then largest if we provide the required $\sigma u_\delta^2/c_4^2$ by choosing an elastic material with as large a value of c_4, and as small a density, as is compatible with this value.

9.671 Examples of thin stabilising surfaces

As an example, for movement through water with a value of $\frac{1}{2}\rho u_\delta^2$ of, say, 1000 lb/sq ft, the critical value of displacement thickness for the Tollmien-Schlichting mode of the flat plate boundary layer is only about 0.01 inches, and the least practical value of θ might be (say) 3. To eliminate this mode we must accordingly find a material with a modulus of rigidity of 40,000 lb/sq ft, which is in the range of the strongest vulcanized rubber. Assuming, for example, a specific gravity for the rubber of unity, the equivalent c_4 would be about 300 ft/sec, and c_f would apparently be $1.5\ u_\delta$.

In wind tunnel work, on the other, hand, dealing here with values of $\frac{1}{2}\rho u_\delta^2$ of (say) 30 lb/sq ft, the critical displacement thickness would be nearer 0.02 inches, and so taking now $\theta = 3/2$, we find the rigidity modulus required is about 10,000 lb/sq ft - in the range of the softer rubbers. Thus with c_4 about 300 ft/sec as before, but now a value of $\sigma = 0.005$, the value of c_f would be $0.35\ u_\delta$: using a lighter rubber which has the same modulus, so that c_4 is increased to 400 ft/sec (say) and $\sigma = 0.01$, would increase the value of c_f by 60%. Putting $\theta = 1/20$ takes us into the range of thin papers and metal foils, with rigidities of the order of 1000 tons/sq in.; and then c_f would apparently equal $2u_\delta$. However, although the stiffness of the skin is enhanced, compared with the rubber, its strength is certainly not, and panels of such a paper or foil skin could only carry a minute part of the breaking load the thicker rubber could withstand - though it is, of course, very doubtful whether full use could be made of the rubber's tenacity, due to the displacements produced in the surface.

9.672 Support spacing

Supposing that we call ℓ the distance between the supports of the skin, and imagining that the greatest applied pressures are of order $\frac{n}{2}\,\rho u_\delta^2$ (say), then to avoid fracture of the skin some such relation as

$$\ell/\delta^{\times} = \lambda(S/nG)^{1/2}$$

must be satisfied, where S is the elastic material's tenacity and $\delta^{\times}$ the displacement thickness of the boundary layer. However, the applicability of our theory depends, it will be recalled, on two assumptions about the magnitude of ℓ: first that the applied load is sufficiently small that the skin is not greatly tensioned or 'stretched' - i.e., that this load is small compared with the product of G and the square of skin thickness; and second, that the pitch of the supports is large compared with the wavelength of oscillations. In algebraic form

$$1/\delta << \ell/\delta^{\times} << \lambda/n^{1/2}\,.$$

Provided (S/G) is small, therefore, as it is for most materials except the rubbers, structural integrity will ensure that the skin is not unduly stretched. For skins just thin enough to avoid the Tollmien-Schlichting mode, λ is approximately $(45/\theta^{1/2})$ if θ is small (and reaches a minimum of 26 at $\theta = 6$). This is certainly greater than the values of $1/\delta$ for the mode of oscillation we are seeking to avoid, except at very high Reynolds numbers, and in the above examples it is possible that the metal foil or paper skin will behave essentially as if it were free of support, as far as the lower Reynolds number Tollmien-Schlichting waves are affected. For rubber skins which have tenacities of 10 or 100 times G, on the other hand, it would be possible only to use a fraction of the available strength if we are not to stretch the skin out of the proportions assumed in our analysis, but even so the permissible support distance can certainly be several times the wavelength without the skin tensions becoming prohibitively large. This is the essential virtue of rubber, or rubber-like substances, in the present context.

9.68 A thick stabilising skin

We have so far envisaged the 'thin skin' approach to the stabilisation of the boundary layer: the alternative thick skin approach is denied in aerodynamic applications because then the value of $\frac{1}{2}\rho u_\delta^2/G$ must not be small compared with unity, and since (to avoid the longitudinal wave mode) u_δ/c_4 must be less than unity, it demands the use of elastic materials having densities of the same size as, or smaller than, that of the air. For hydrodynamic applications where such relatively light materials are available, it would seem a better device; the largest values of c_f are obtained with $\sigma u_\delta^2/c_4^2 = 1/2$, (i.e., $\frac{1}{2}\rho u_\delta^2 \simeq G$), and with the skin density and θ as large as practical. Substances which have a rigidity modulus of the order of the 1000 lb/sq ft which we have quoted for $\frac{1}{2}\rho u_\delta^2$ are rubber foams: with their small densities we can calculate that c_f would be about $\frac{1}{2}\theta^{1/3}\,u_\delta$: it would thus need a skin of about 20 times the displacement thickness (i.e., 0.2 inches) to better the value of c_f for the thin skin quoted for the same set of conditions; furthermore, any value of $\theta > 10$ would produce a stiffer skin - i.e., a larger value of λ, - and permit a spacing of supports more likely to simulate a free surface in relation to the Tollmien-Schlichting waves for large R_δ.

Since, - as just suggested - many materials of interest may be porous, and it may be of interest to note that the method of approach[8] used in seismology to the study of the elastic properties of such a medium lies in the adjustment of the elastic constants to simulate an equivalent homogeneous and isotropic material. Thus there would appear to be a precedent for the application of our present results to such materials.

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9.7 The effect of fluid at the inner boundary

In the sense that the presence of a sufficient depth of a very dense and highly viscous fluid below the inner boundary of the elastic surface will make this boundary more or less rigid, and the presence of a very light inviscid fluid will leave it virtually free, our previous discussion will therefore be relevant to the effects of some kinds of fluid at the inner boundary: all that we have omitted is any account of wave propagation at the speed of sound within the fluid - in other words, we have assumed it to be incompressible. By the same token, we can consider the effects of a very heavy, inviscid (HI) fluid, or of a very viscous, light (LV) fluid - interpreting these broad descriptions to mean hypothetical substances, the HI fluid being heavy enough to inhibit normal displacements, and the LV fluid viscous enough to prevent tangential displacements of the inner boundary. Viewed as a limiting condition, the supposed presence of such fluids - assuming them to be incompressible - suggests the possible extent of the action of more realistic fluids at the inner boundary.

9.71 The effect of heavy inviscid fluid

The presence of an HI fluid will cause the Tollmien-Schlichting mode to be modified in the same way as if it were replaced by a fixed boundary, but enhances the effect particularly for elastic skins of small thickness: the effective density of the elastic surface is raised in the ratio $2c_1^2/c_4^2$ if θ is round about unity, but, of course, for a very thick elastic surface, the effect is negligible. The Rayleigh wave mode for thick surfaces is replaced on a thin surface exposed to HI fluid by the longitudinal wave mode discussed earlier in relation to thin free membranes. It now occurs on c-R_δ and δ-R_δ diagrams generally as two lobes (Fig.4(i)) extending from the extremes of large and small Reynolds numbers. If c_4/u_δ is sufficiently small, the higher Reynolds number lobe becomes 'absorbed' into the modified Tollmien-Schlichting mode (Fig.4(j)) and if $\rho c_2^2/c_1^2$ is large enough (certainly if it is greater than 1/2), the higher Reynolds number lobe becomes identified on the diagrams as a closed curve over a finite range of Reynolds number - ultimately vanishing altogether for large σ.

For very small values of c_4/u_δ another closed curve appears on the c-R_δ and δ-R_δ diagrams whose extent and position has not been found, although it is known that it involves speeds above c_4 (considerably above, if σ is small). In fact, there is a similar mode of oscillation for a free surface - though its presence was not noted in our discussion before; however, there are also, of course, resonance modes to be considered if c_4/u_δ is small enough.

9.72 The effect of light, viscous fluid

The presence of the hypothetical LV fluid at the inner boundary will cause the Tollmien-Schlichting mode to be modified in the same way as for a free-boundary, with a disappearance of this mode altogether for a sufficiently thin elastic skin, thus showing that it is the freedom of the inner boundary to move up and down which is the essential property on which this complete stabilisation depends. The action of the LV fluid however inhibits the effect to some extent if θ is small, and the thickness of the skin has to be about 80% of that necessary on a completely free surface to produce stabilisation (Fig.6, lower curve). The flexural mode (noted already in connection with a free surface) makes its appearance for thin elastic surfaces in contact with such a fluid, in place of the Rayleigh wave mode, and in a slightly modified form. It appears on the c-R_δ and δ-R_δ diagrams in much the same form as in Fig.4(g) (where it appears as the lower speed mode): the speed of propagation and wavelength are both slightly increased above the values for a free-surface. This neutral oscillation may presumably be viewed again as a boundary of amplified waves of faster speed and smaller wavelength.

9.73 Solutions for a 'heavy' surface

At first sight the appearance of shear (flexural) waves and absence of compression waves for a thin surface exposed to a viscous - i.e., shear resistant - fluid, and the opposite combination for a thin surface exposed to heavy fluid, seems the wrong way round. However, the shearing oscillations are vertically polarised, and the compression waves are longitudinal, so that the result is plausible. It may be of interest to note that studies on the propagation of waves in a floating ice sheet[9], a problem analogous to ours if $\sigma = 0$ (so that the boundary-layer fluid is too tenuous to have any effect), show indeed that inviscid fluid in contact with a solid reduces the speed of propagation of long flexural waves, in proportion to $z/(1+z)$, where z is equal to $(\rho_s d/\rho_f)$, and ρ_f is the density of the internal fluid; consequently their speed would be zero in the limit $\rho_f \to \infty$ we here assume. (The fluid is also shown to add an attenuation to the compression waves, vanishing in the limit appropriate to an incompressible fluid such as we assume, but otherwise being proportional to c_4/a, where a is the speed of sound of the fluid.) Thus - supposing speeds greater than that of propagation of the flexural mode imply amplified disturbances - amplified flexural waves might occur at all finite speeds of propagation in the presence of HI fluid, neutral oscillations being standing waves: in other words, the absence of a neutral line for travelling waves clearly does not, in this instance, necessarily imply absence of the mode of oscillation. Unfortunately there appear to be no corresponding treatments allowing us to find the way in which a viscous fluid modifies longitudinal waves.

It will be also realised that, as we have here been dealing with internal fluids of indefinitely large, or indefinitely small, densities, we cannot expect to obtain any precise analogy with the Kelvin-Helmholtz mode of instability between layers of fluids if we allow θ to tend to zero, although for real fluids such effects would exist.

10. Conclusions

A laminar boundary layer in contact with a resilient surface may sustain various modes of neutral oscillation, quite apart from modified forms of the Tollmien-Schlichting waves (present on an inflexible surface). These modes can generally be identified as modified forms of the neutral oscillations which the surface could perform in the absence of the boundary layer, if it were composed of non-dissipative material (as we here assume). Thus there may exist waves propagated at speeds in excess of c_1 (the speed of compression waves within the material) representing resonances struck by reflection of compression waves from the inner boundary of the resilient skin; or at speeds in excess of the smaller value c_2 (the speed of shear waves) representing the reflection of shear waves from this boundary. A thick resilient surface may reveal many such modes, representing higher 'harmonics' of some 'fundamental' resonance. There may also exist oscillations, whose wavelength is small compared with the skin thickness, which are propagated at speeds close to c_3 - the speed of Rayleigh surface waves, which attenuate exponentially with distance into the solid material. For a thin skin which is free of stress at its inner boundary, or exposed to a fluid, Rayleigh waves degenerate into longitudinal waves travelling at speeds close to c_4 - the speed of waves on an extensive membrane - or into slowly propagated flexural waves.

In many conditions, modified forms of these neutral oscillations may exist over the complete range of Reynolds number, and this invariably happens if the fluid density is small compared with that of the resilient material - though rapidly amplified oscillations could not then exist. However, all but those of the flexural wave mode apparently disappear if the material is chosen so that the speed c_3 (which is less than c_1, c_2 and c_4) is larger than u_δ (the free-stream speed), merely because the

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conventional analysis is restricted to a consideration of oscillations propagated at speeds less than u_δ. It is not known whether this 'disappearance' can be taken to imply their absence in reality.

The flexural wave mode may be eliminated by restraining the normal displacement of the resilient skin at its inner boundary (e.g., by securing it to an inflexible structure, or exposing it to a heavy fluid). But the freedom of this boundary to move in this particular fashion has a special importance, as it eliminates Tollmien-Schlichting neutral oscillations at both high and low Reynolds number. (A resilient surface will otherwise eliminate this mode only at low Reynolds numbers.) Indeed, if such a 'free' surface were thin enough, or of sufficiently low rigidity, amplified oscillations of the Tollmien-Schlichting mode would be eliminated at all Reynolds numbers, - though only at the cost of the introduction of amplified flexural waves.

Assuming - tentatively - that the condition $c_s > u_\delta$ eliminates all other modes, and that amplification of the flexural wave mode may be reduced by increasing the speed of propagation of neutral flexural oscillations (thereby decreasing their wavelength), one is led to two alternative 'optimum' forms of surface which completely stabilise the Tollmien-Schlichting mode. If a material is available whose density is less than that of the fluid, but whose rigidity modulus is about equal to the free-stream dynamic pressure, then a thick resilient skin of such material virtually free at its interior surface (i.e., say, supported on a honeycomb, or on discrete studs) would be best for this purpose. No such material exists if air is the fluid in question, and the second alternative of a very thin skin of rubber-like material - again virtually free at its interior surface - is then the only practical possibility; here, the thinner the skin is, the faster the flexural waves propagate, provided the rigidity is increased in conformity with the relationship of Fig. 6.

List of Notation/

List of Notation

A_1, A_2, A_3, A_4	constants determining X and Y by equation (2.2)
C_1, C_2	constants determining ϕ by equation (4.8)
D_1, D_2, D_3, D_4	parameters of stability equation (4.27)
D_1', D_2', D_3', D_4'	parameters of stability equation (4.20)
D_2^*	parameter of stability equation, given by (4.28)
E	related to Φ by equation (4.19)
F	related to $f(\zeta_1)$ by equation (4.19)
$\mathcal{F}(\zeta_1)$	$= 1/(1+F)$
G	function of ζ_1 given by equation (4.14)
H	$= \mathcal{R}\{\mathcal{F}(\zeta_1)\}$
H*	$= H - [D_3/(\Delta_0+D_3)]$
I	$= \mathcal{I}\{\mathcal{F}(\zeta_1)\}$
I_0	$= I/7.14$
$\mathcal{I}\{\ \}$	imaginary part of $\{\ \}$
K	bulk modulus
R	Reynolds number, $\rho c/\alpha\mu$
R_δ	Reynolds number, $\rho u_\delta \delta/\mu$
$\mathcal{R}\{\ \}$	real part of $\{\ \}$
$U(\eta)$	steady x-component of velocity with boundary layer divided by c
U_1'	$= U'(\eta_1)$
X,Y	strain-displacements of particle of elastic surface at (x,y) due to applied stress
a_m, a_{mn}	coefficients of equation (4.7) listed in Table 1
b_{mn}	coefficients of equation (4.16)
c	complex velocity of wave propagation (real part, speed of propagation)
c_1	speed of compression waves, $= [K+(4G/3)]^{1/2}/\rho_s^{1/2}$
c_2	speed of shear waves, $= (G/\rho_s)^{1/2}$
c_3	Rayleigh surface wave speed, $= c_3(0)$
$c_3(\sigma)$	root of (Δ_0+D_3) given by equation (6.7)
c_4	speed of longitudinal waves on an extensive thin plate, $= 2[1-(c_2/c_1)^2]^{1/2}c_2$

$c_4(\sigma)$/

$c_4(\sigma)$	$= c_4/(1-\sigma^2)^{1/2}$ for $\sigma < 1$
c_5, c_6	given by equation (8.5)
d	ratio of skin thickness to wavelength of oscillation
$f(\zeta)$	function related to ϕ by equation (4.8)
p	fluid static pressure
r_n	$= [1 - (c/c_n)^2]^{1/2}$
s	$= \sigma u_\delta/c_1$
t	time
u,v	velocity components of fluid parallel to x- and y-axes (paragraphs 3 and 4 only)
u,v	real and imaginary parts of w (paragraph 5 et seq)
u_δ	free stream speed
w	$= 1/(1+E)$
x,y	cartesian axes parallel and perpendicular to surface (y measured into fluid, x downstream)
x,y	compendium symbols defined by (6.17) (paragraph 6 only)
Δ	the determinant $\lvert a_{mn} \rvert$
Δ_0	$= -\Delta_{56}$
Δ_{mn}	the first minor of Δ corresponding to a_{mn}
$\Phi(\eta)$	related to ϕ by equation (4.8)
α	wave-number of oscillation (reciprocal of wavelength)
β	$= \sigma u_\delta^2/c_1 c_2$
γ	$= c_2/c_1$
σ	boundary-layer displacement thickness divided by wavelength of oscillation
ζ	$= (\eta-\eta_1)(RU_1')^{1/3}$
ζ_1	$= -\eta_1(RU_1')^{1/3}$
η	$= \alpha y$
η_1	value of η such that x-component of fluid velocity equals c (i.e., $U(\eta_1) = 1$)
θ	thickness of elastic skin divided by (u_δ/k) where k is the surface velocity gradient of the boundary layer
λ	modulus of elongation, $= K - (2G/3)$ (paragraph 2 only)
δ	boundary-layer thickness divided by wavelength of oscillation

λ/

λ = $(\eta_1 U_1' - 1)$ (paragraph 4 only)

μ viscosity of fluid

ξ $\doteq$ $4c^2/d^2c_4^2$ (paragraph 7), = $c^2/d^2c_4^2$ (paragraph 8)

ϖ normal stress at surface exposed to fluid divided by $Ge^{i\alpha(x-\sigma t)}$

ρ fluid density

ρ_s elastic medium density

σ = ρ/ρ_s

τ shear stress at surface exposed to fluid divided by $Ge^{i\alpha(x-ct)}$

τ = $\theta u_\delta/c_2$ (paragraph 6 only)

$\phi(\eta)$ related to ψ by equation (3.2)

χ = $-(D_2^*/\Delta_0)(u_\delta/c)^2$

χ_1 given by equation (6.9b)

ψ stream function of fluid flow defined by (3.1) (paragraph 3 only)

ψ = cd/c_2 (paragraph 6 et seq)

Barred quantities refer to the Tollmien-Schlichting oscillation for an inflexible surface.

Primes denote differentiations.

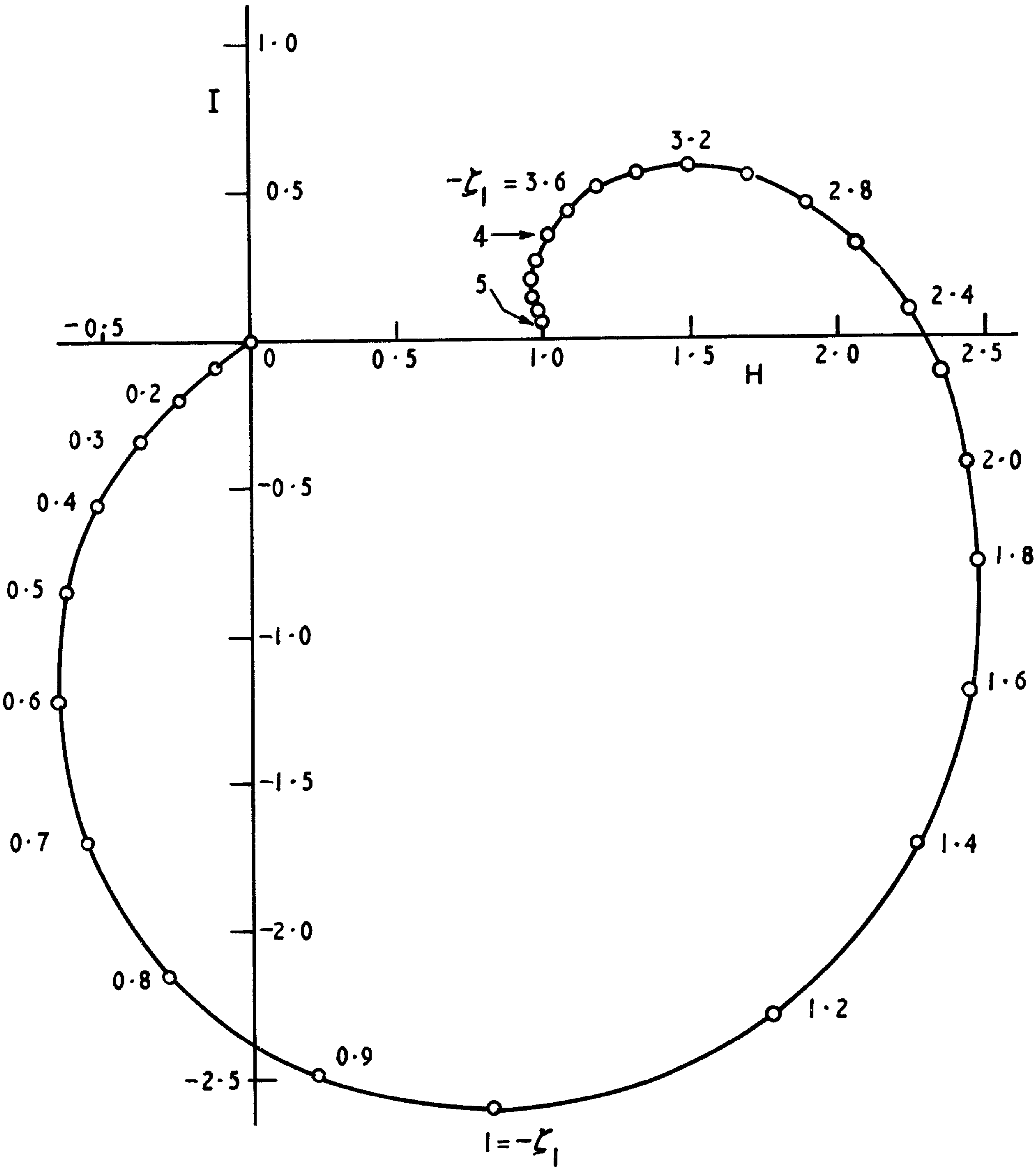
Subscript ϕ denotes oscillatory components.

References/

References

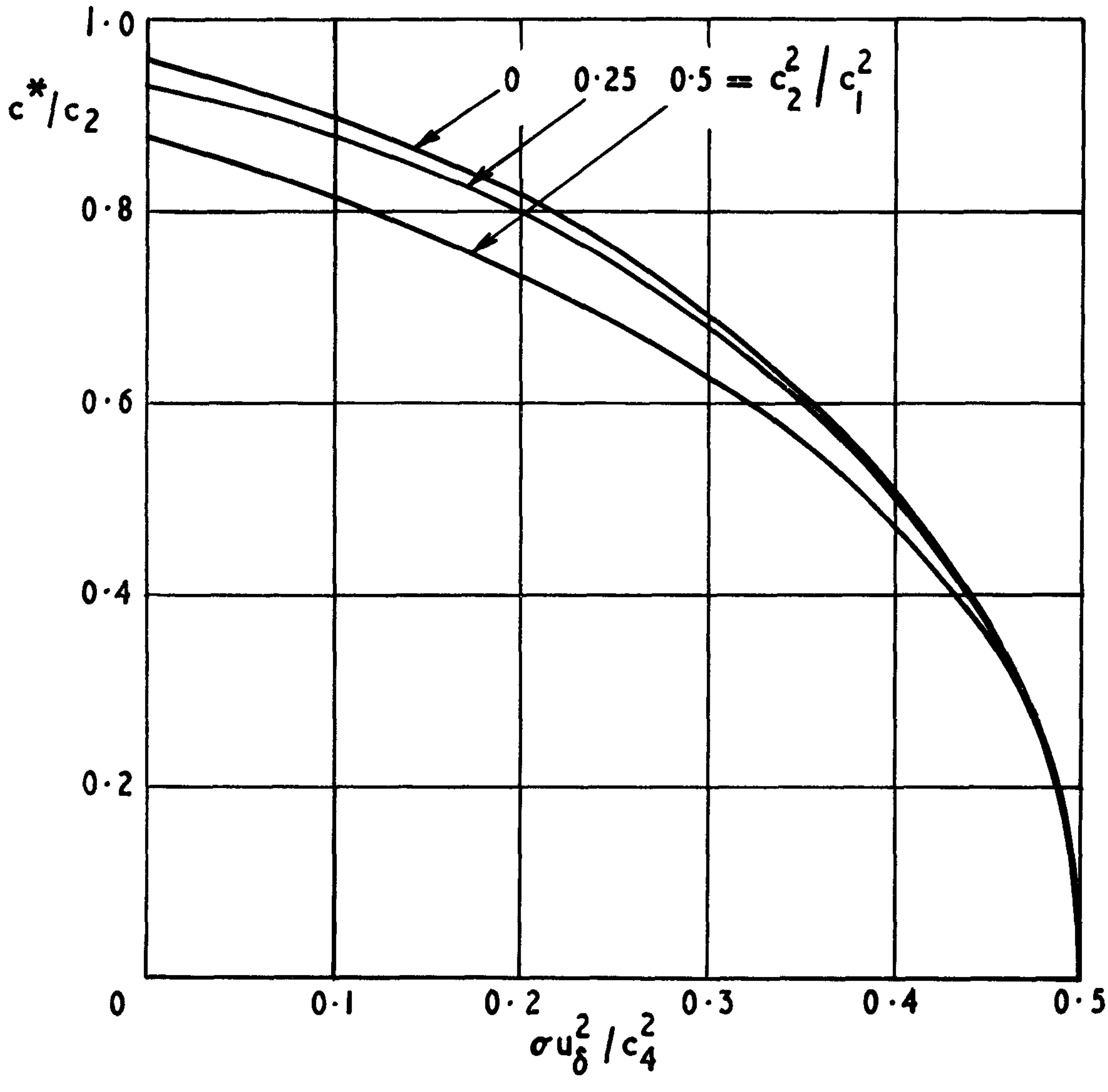
No.	Author(s)	Title, etc.
1	C. C. Lin	The theory of hydrodynamic stability. Cambridge University Press, 1955.
2	T. Brooke Benjamin	Effects of a flexible boundary on hydrodynamic stability. J. Fluid Mech., Vol.9, Part 4, p.513. December, 1960.
3	A. E. H. Love	A treatise on the mathematical theory of elasticity. Dover, New York, 1944.
4	W. M. Ewing and W. S. Jardetzky	Elastic waves in layered media. McGraw Hill, 1957.
5	W. Voigt	Über innere Reibung fester Körper. Ann. Physik, 47, 671, 1892. (See also H. Kolsky: Stress waves in solids, Clarendon Press, Oxford, 1953).
6	M. O. Kramer	Boundary layer stabilisation by distributed damping. Reprinted from "Journal of the American Society of Naval Engineers Inc., February, 1960, Vol.72, No.1, pp.25-33". A.R.C.21,965. 30th May, 1960.
7	I. Tolstoy and E. Usdin	Dispersive properties of stratified elastic and liquid media: A ray theory. Geophysics, 18, 844, 1953.
8	J. K. Mackenzie	The elastic constants of a solid containing spherical holes. Proc. Phys. Soc. B., Vol.63, p.2. 1950.
9	F. Press and W. M. Ewing	Propagation of elastic waves in a floating ice sheet. Trans. Am. Geophys. Union, 32, 673. 1951.

FIG. I



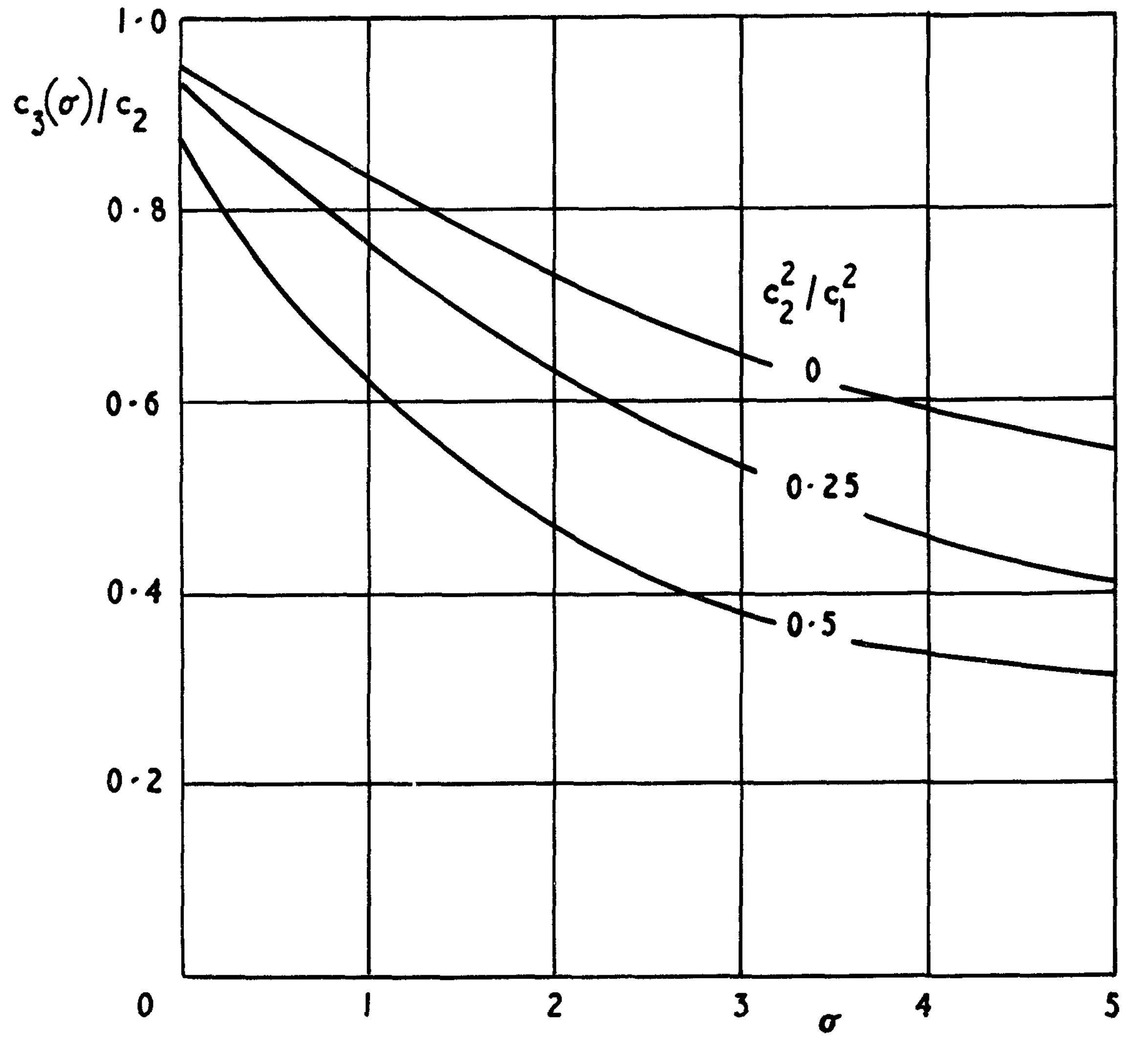
Variation of the real and imaginary parts of the complex function $F(\zeta_o)$

FIG. 2



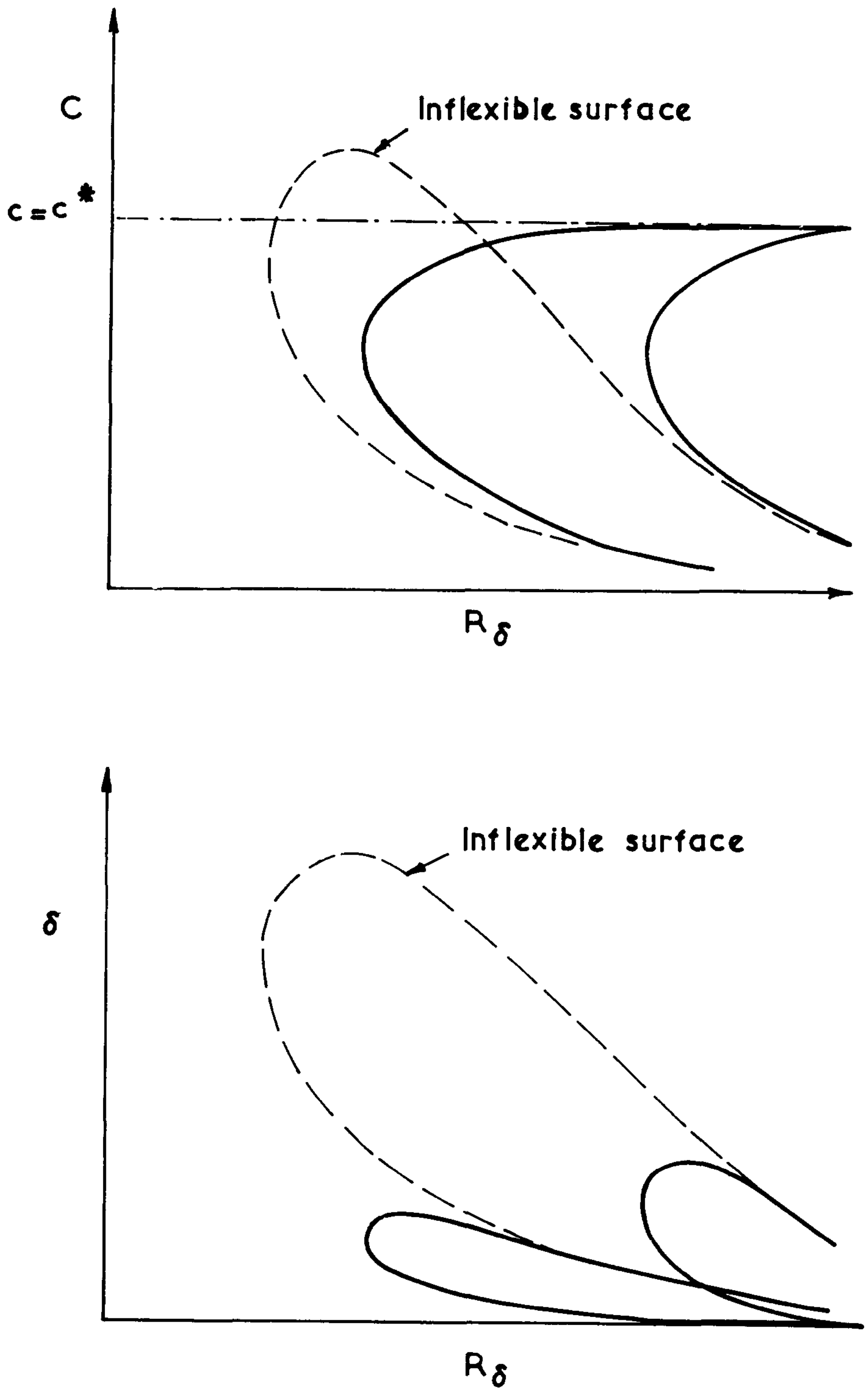
Variation of speed c^*

FIG. 3



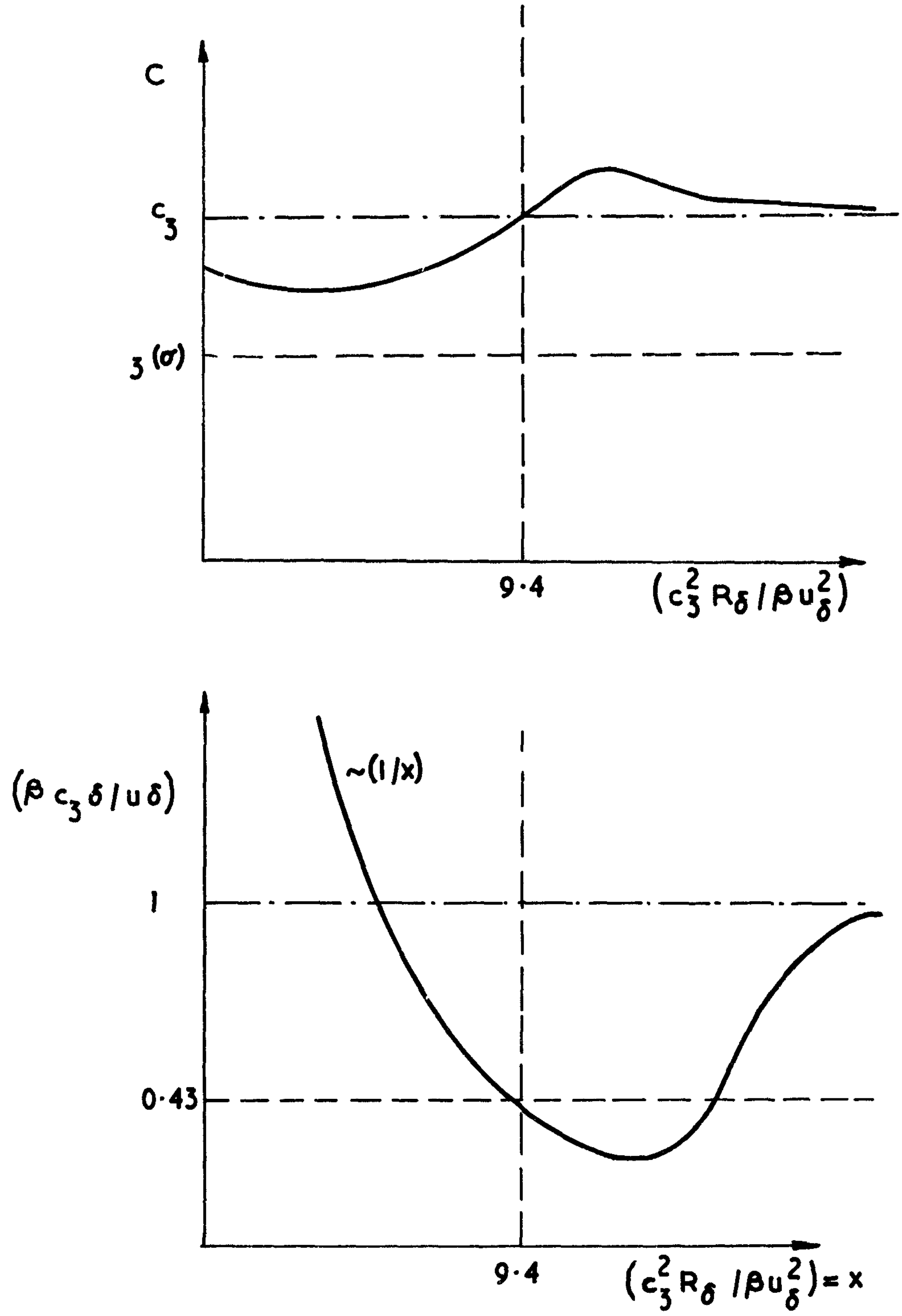
Variation of speed $c_3(\sigma)$

FIG. 4(a).



Modification of Tollmein Schlichting mode if $\theta\delta >> 1$, $c^* << u_\delta$.

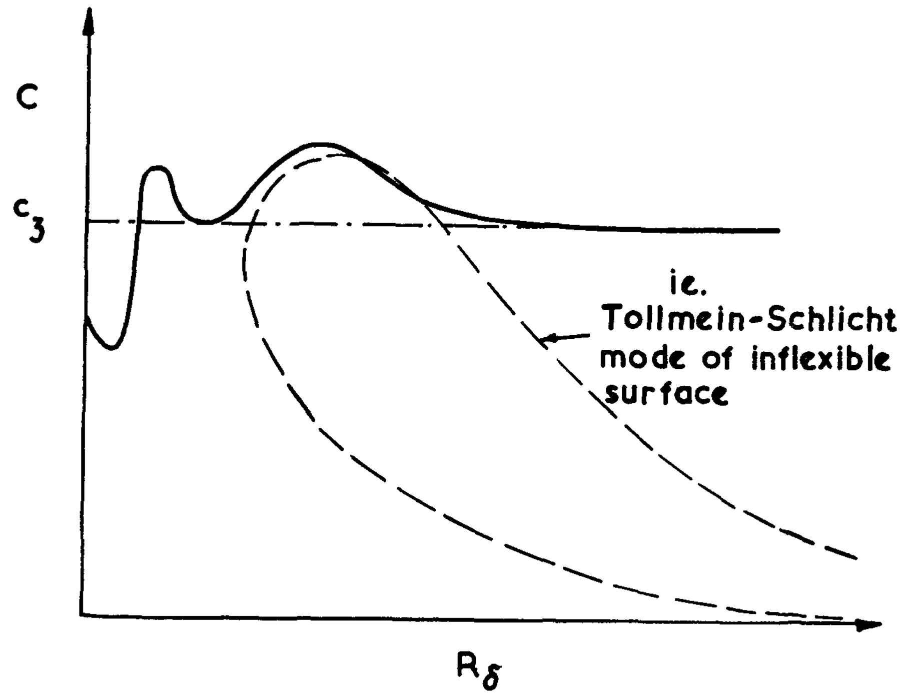
FIG. 4(b).

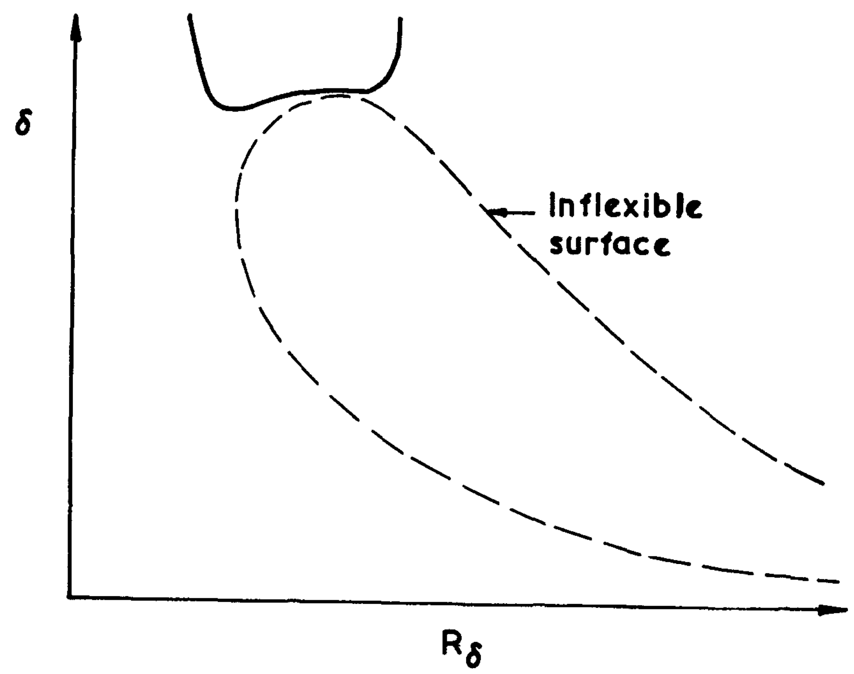


Rayleigh wave mode $\theta\delta >> 1$

$$\left[\beta = \beta_1 (c_2/c_1) + \sigma \beta_2 (c_2/c_1) \quad \beta_1(0) = 1{\cdot}88, \ \beta_2(0) = 1{\cdot}32\right]$$

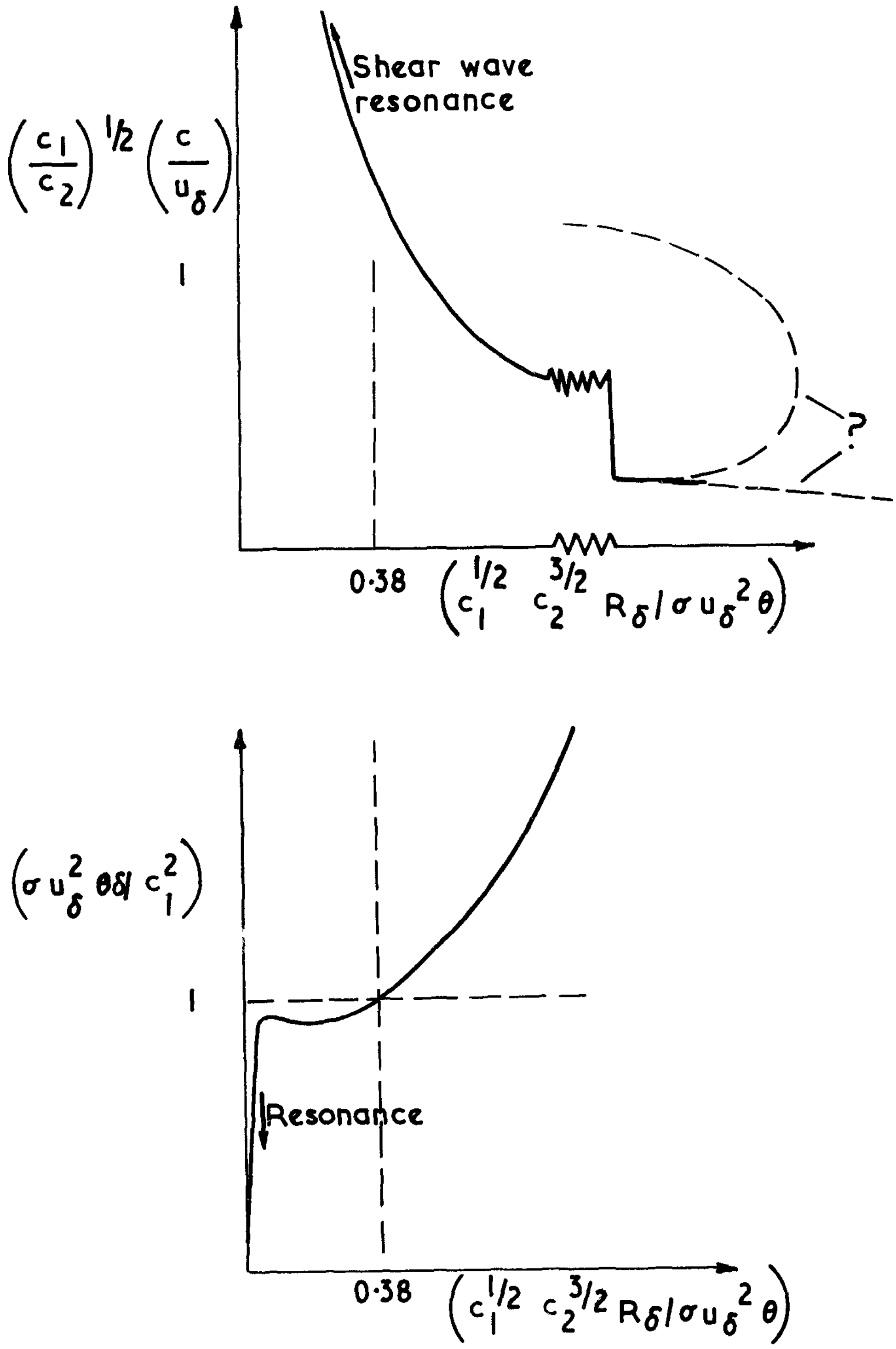
FIG. 4.(c).





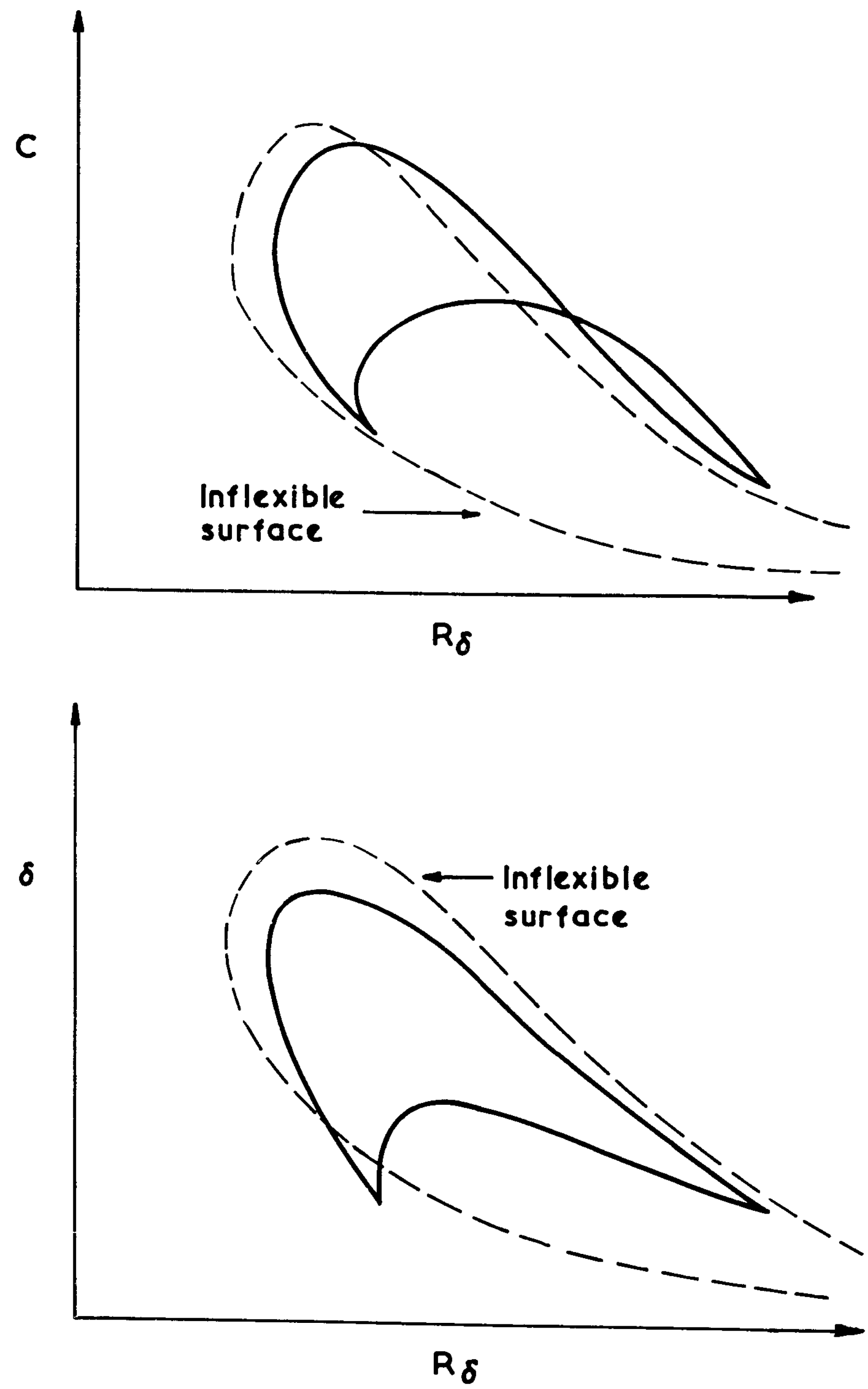
Rayleigh wave mode modification in region of Tollmien-Schlichting mode.

22,670.
FIG. 4 (d).



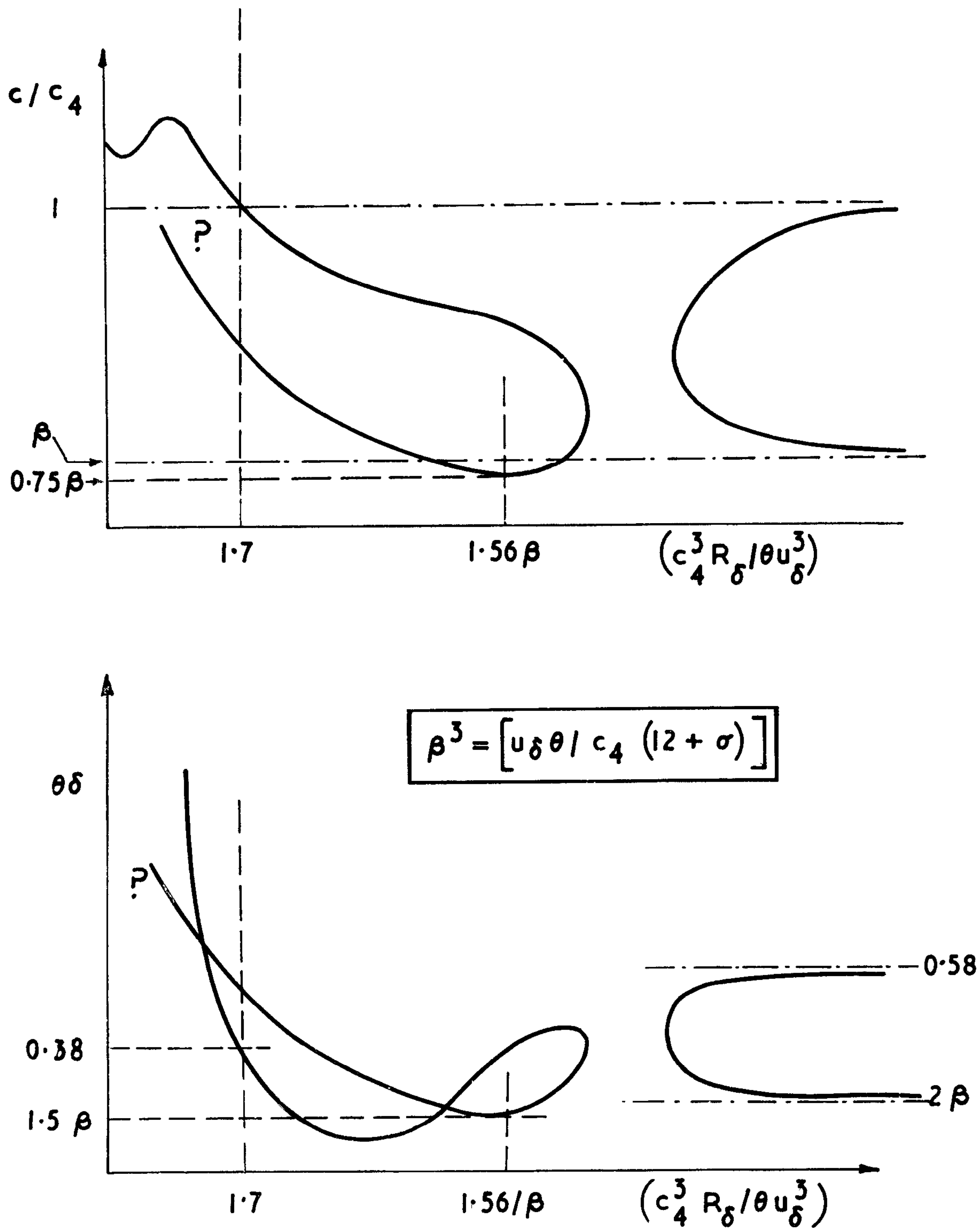
Rayleigh wave mode (?) for thin fixed surface.
[$\theta = O(1)$]

FIG. 4(e).



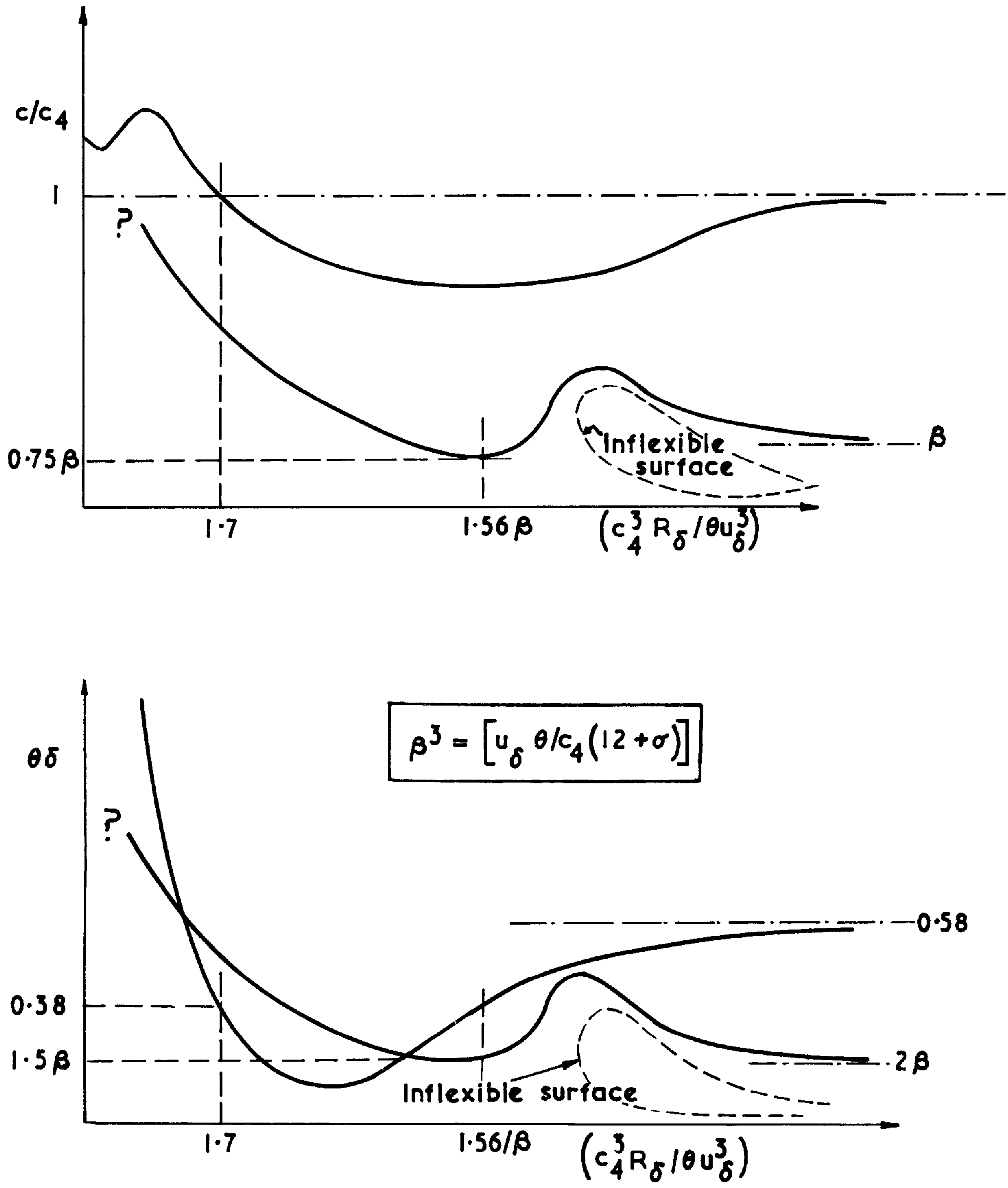
Modification to Tollmien-Schlichting mode with moderately thick free surface.

FIG. 4(f).



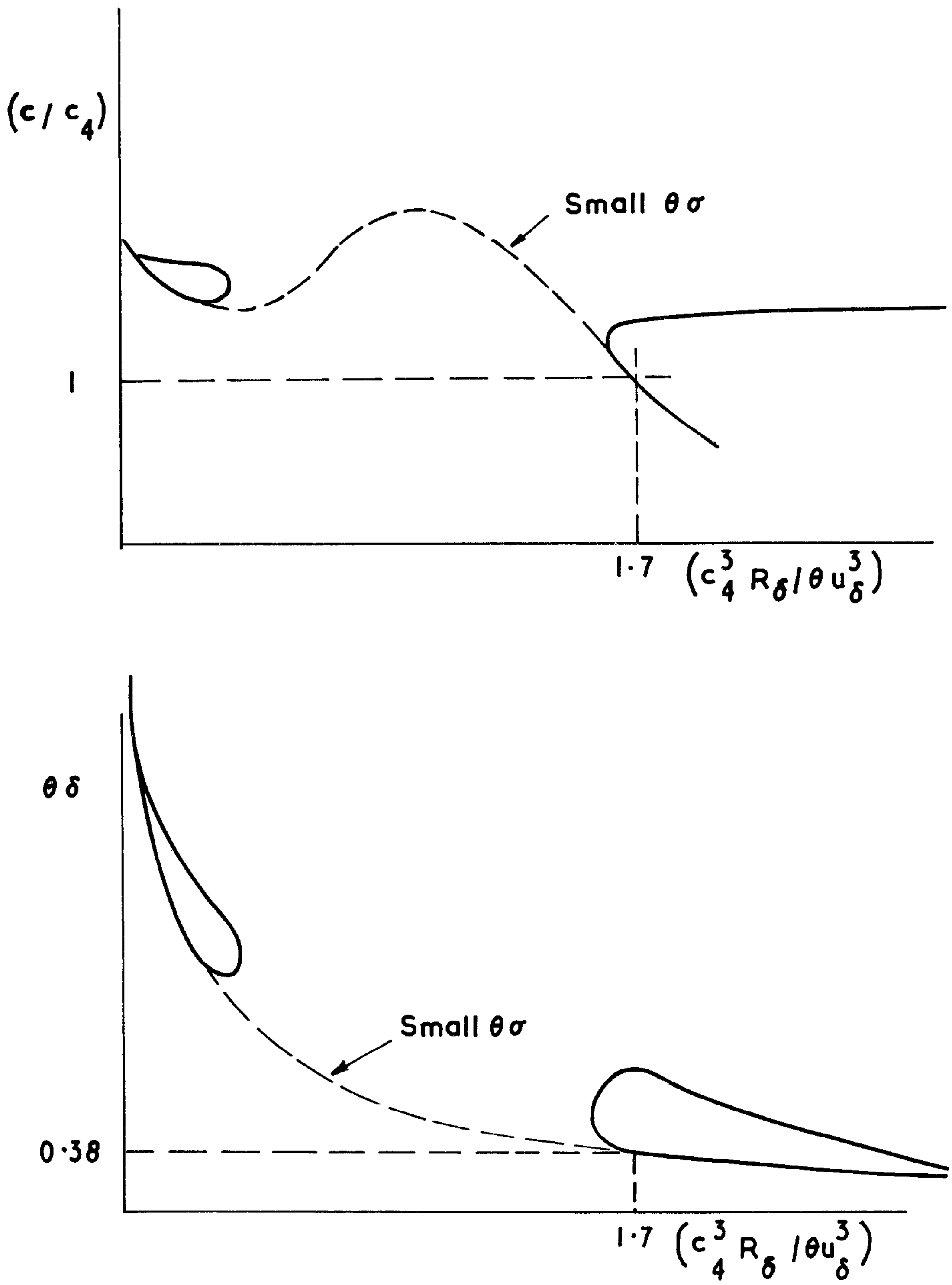
Longitudinal and flexural modes for thin free membrane — small (c_4/u_δ).

FIG. 4(g.)



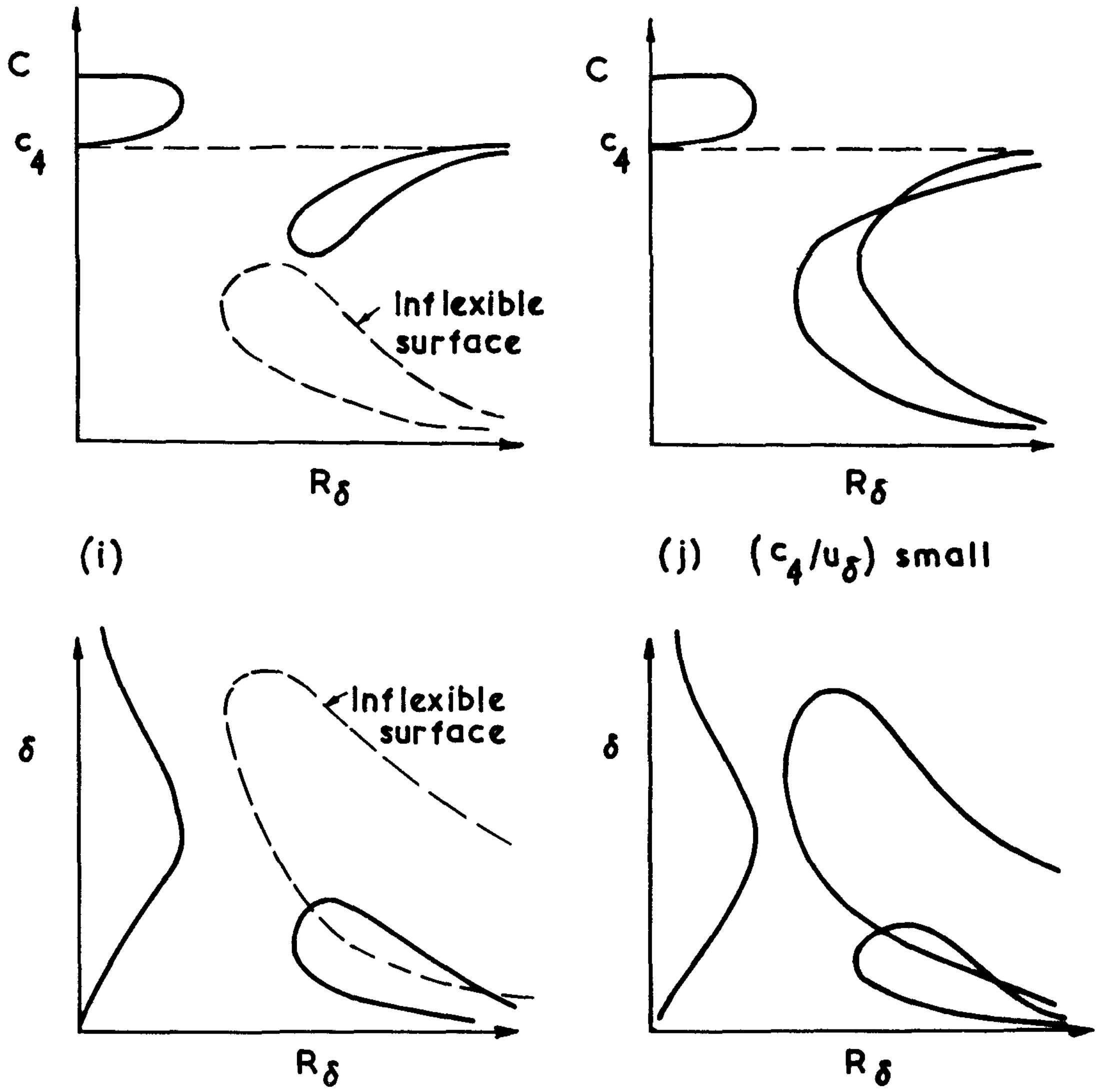
Longitudinal and flexural modes for thin free membrane — large σ

FIG. 4.(h).



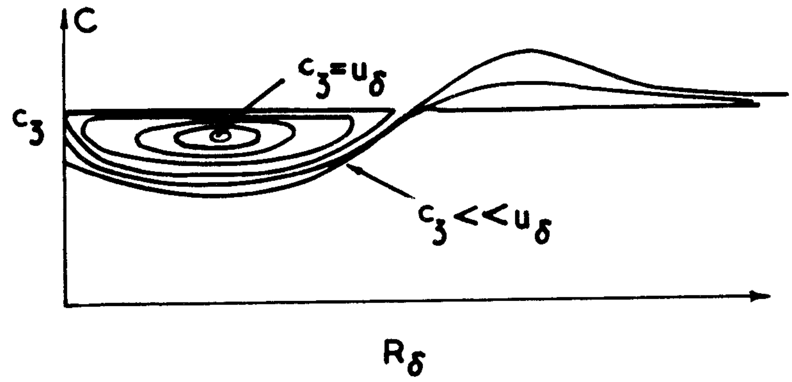
Anomalous behaviour of flexural wave mode for larger values of $\theta\sigma$

FIG. 4(conc).

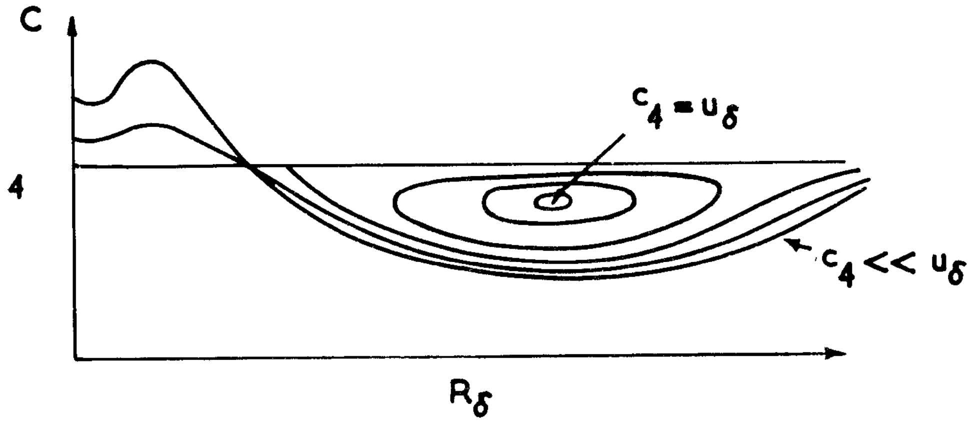


Modes of oscillation for a thin surface exposed to a heavy inviscid fluid at inner boundary.

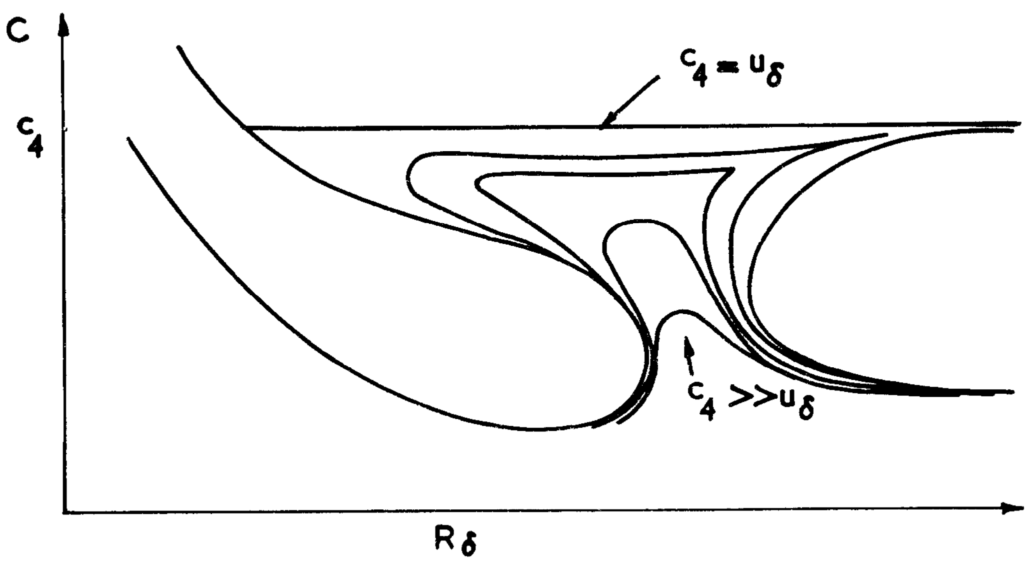
FIG 5(a-c).



(a) Development of Rayleigh wave mode - Fig.4.(b).



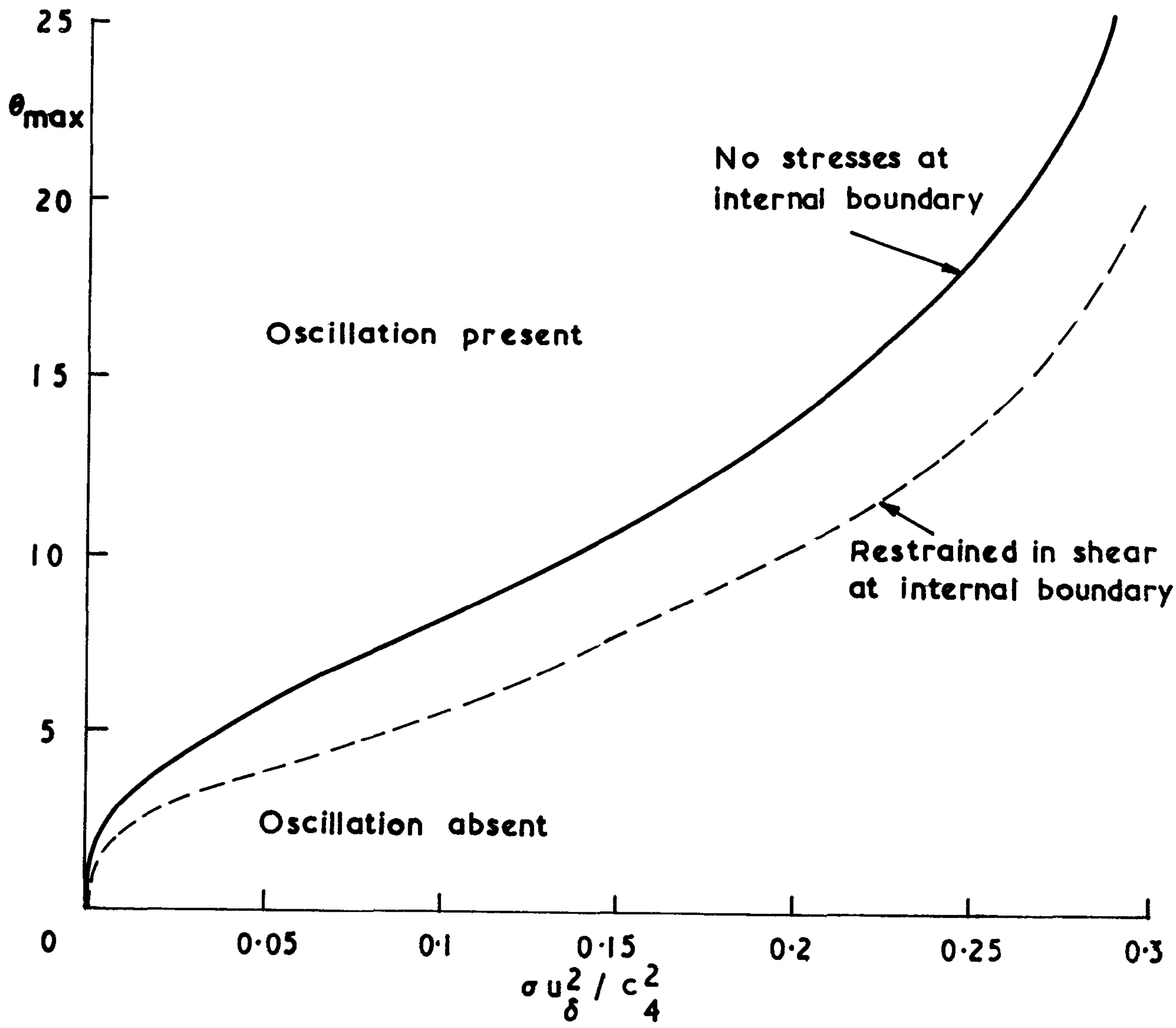
(b) Devolopment of flexural wave mode of Fig.4(g),($\sigma < 2$)



(c) Development of thin membrane modes of Fig.4.(f),($\sigma > 2$)

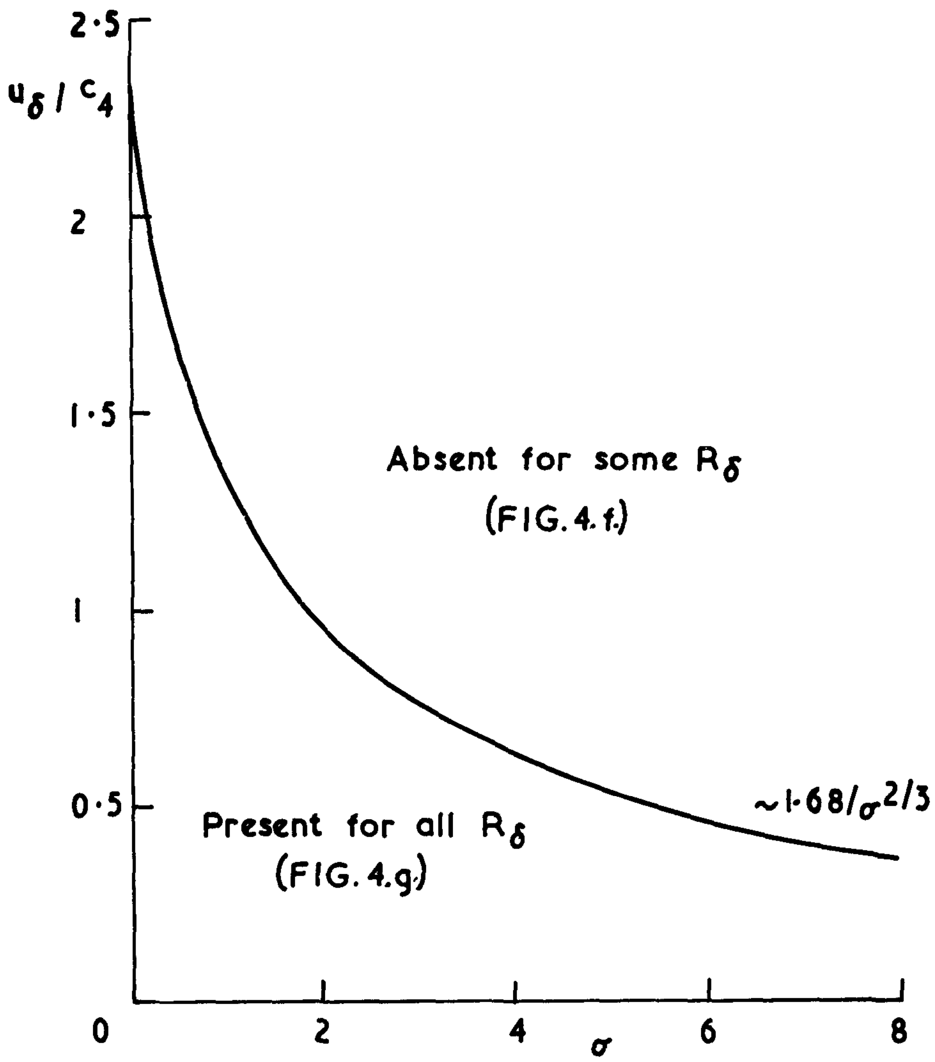
Conjectured formation of modes as natural wave speeds are varied relative to u_δ.

FIG. 6.



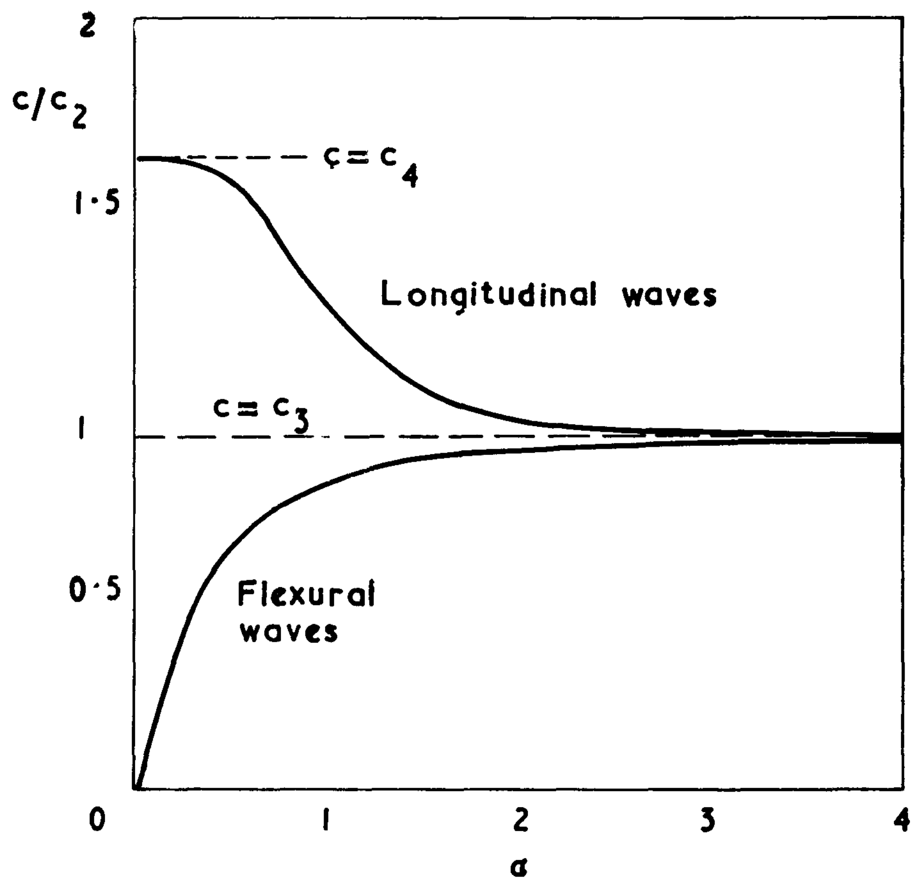
The maximum thickness of a freely mounted skin for which the Tollmein-Schlichting mode is absent.

FIG 7.



Condition governing form of modes of oscillation for a free membrane.

FIG. 8.



Wave propagation over a free surface in vacuo
(After Tolstoy and Usdin, for $c_2/c_1 = 1/\sqrt{3}$)

DS 89798/1/Wt.58 K4 11/62 XL/CL.

A.R.C. C.P. No.622. March, 1961
Nonweiler, T. - Univ. of Belfast

QUALITATIVE SOLUTIONS OF THE STABILITY EQUATION FOR A BOUNDARY LAYER IN CONTACT WITH VARIOUS FORMS OF FLEXIBLE SURFACE

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