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The 'Newtonian' Theory of Hypersonic Flow for any Three-Dimensional Body

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SUMMARY

The 'Newtonian' theory of hypersonic flow ($M = \infty$, $\gamma \to 1$) originated by Busemann (1933) and developed by the author is extended to flow past any three-dimensional body shape. Mathematical complexity however limits the results obtained to those for slightly yawed axially symmetric bodies and in particular the cone is considered in some detail. Pressure distribution and shock shape are obtained to a first approximation in $(\gamma - 1)/(\gamma + 1)$, and to a second approximation in $\sin \delta$ where δ is the angle of attack.

1. Introduction

In a recent paper (1956) the author discusses the 'Newtonian' theory of hypersonic flow for two-dimensional and axially symmetric bluff bodies. The theory enables the pressure distribution and the shock wave shape to be obtained for these bluff bodies under inviscid flow conditions at infinite free stream Mach number when the ratio of the specific heats is near unity. Although the theory was developed for a perfect gas with constant specific heats, it was shown that it is possible to develop the theory for a gas of arbitrary thermodynamic properties. The author has also used the theory (1957) in the case of thermodynamic non-equilibrium to investigate the effects of dissociation rates on the flow pattern.

In the present paper* the author wishes to point out that this theory (and the above developments of it) can be generalised to give the pressure distribution and shock shape for any three-dimensional body shape in an explicit, though complicated, form. These results are of course only valid under the conditions stated above, viz., that the free stream Mach number is infinite and the ratio of specific heats near one. The theory will be set out in the present paper only for the case of a perfect gas, although as in the first paper (1956) the extensions to arbitrary thermodynamics are immediate. In principle, the extension of the theory is obvious, although to set it down in mathematical terms is a little more difficult. In fact, the procedure is exactly the same as for the two-dimensional and axially symmetric cases and is the approach originally used by Busemann (1933).

First/

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*The author would refer the reader to a recent publication by Hayes and Probstein (1959) for a more detailed study of this problem.

First we proceed to build up an expression for the pressure on the body surface. This is composed of two parts. We have the initial rise of pressure at the shock wave which is reduced by the fall in pressure across the layer between shock and body. The only contribution to this fall which we consider to this approximation is that due to the curvature of the body surface. As a first approximation the shock wave is assumed to have the shape of the body itself. Physically, therefore, we consider that the flow impinges upon the body surface and there loses its normal component of momentum and then the particles skid along the curved surface causing a lower pressure than would be given were the surface plane.

Secondly, we calculate the distance of the shock wave from the body by integrating across the stream tubes from the body to the shock wave at each point on the surface.

2. The Shape of the Streamlines

If we consider an arbitrary body shape, the streamlines no longer remain in a single plane. The first requirement therefore is to decide the form of the curves traced out by the streamlines themselves on the body surface. We assume that the normal component of momentum of the fluid is destroyed at the body surface and then that the particle of fluid moves along the smooth surface under the influence of the reaction normal to the surface only. Its path is then a geodesic of the surface (Grimminger et al. 1950, Whittaker 1927). Let the outward normal to the surface be in a direction z and choose curvilinear co-ordinates (x, y) on the body surface (Fig. 1). The elements of length in the x, y and z directions are denoted by $h_1 dx$, $h_2 dy$ and dz. The curvatures of the co-ordinate axes are denoted by κ_1, κ_2 and 0.

Now the particle will proceed at constant velocity along the surface since there are no forces acting along the surface. Thus the time taken to pass along a curve C is

$$t = \frac{1}{q_0}\int_C ds = \frac{1}{q_0}\int_{x_0}^{x}\left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) h_1\,dx \qquad \ldots(2.1)$$

and this will be a minimum along the particle path (Whittaker 1927). The point (x_0, y_0) is where the particle strikes the surface, and $q_0 = (u_0, v_0)$ is its velocity there. By the calculus of variations, we require for a minimum

$$\frac{d}{dx}\left\{\frac{h_1\left(\frac{h_2}{h_1}\right)^2\frac{dy}{dx}}{\sqrt{1+\left(\frac{h_2}{h_1}\frac{dy}{dx}\right)^2}}\right\} - \frac{\partial}{\partial y}\left[h_1\sqrt{1+\frac{h_2}{h_1}\left(\frac{dy}{dx}\right)^2}\right] = 0 \qquad \ldots(2.2)$$

with the boundary conditions

$$\frac{h_2}{h_1}\frac{dy}{dx} = \frac{v_0}{u_0}$$

at $x = x_0$, $y = y_0$.

This/

This will give us an equation for the curve C of the form

$$G(x, y, x_0, y_0) = 0. \qquad \ldots(2.3)$$

Also, the velocities in the directions x and y can be obtained by resolving the velocity q_0 along the curve in the directions x and y and hence we obtain $u = u(x, y, x_0, y_0)$ and $v = v(x, y, x_0, y_0)$ where

$$u = q_0(x_0, y_0) \cos\left\{\tan^{-1} \frac{h_2\, dy}{h_1\, dx}\right\}$$
$$v = q_0(x_0, y_0) \sin\left\{\tan^{-1} \frac{h_2\, dy}{h_1\, dx}\right\} \qquad \ldots(2.4)$$

Alternatively, we can look at equation (2.3) in a slightly different way and say that it represents the locus C of all points (x_0, y_0) on the shock wave (or, to this approximation, the body) from which a streamline passes through the point (x, y). These streamlines are arranged in some order above the point (x, y) which is, however, not immediately obvious. The contribution of the streamline to the pressure depends on the curvature of the body in the direction of that streamline, and thus it is necessary to integrate the contributions of these streamlines to obtain the surface pressure. It would seem reasonable therefore to try to convert the integral across the layer between shock and body at the point (x, y) to an integral along curve C of all the points where the particles following the streamlines through (x, y) strike the body. This corresponds to the introduction of a stream function and Stokes' stream function for the two-dimensional and axially symmetric cases respectively. It can also be shown that, approximating to the full inviscid Navier-Stokes equations in three dimensions in a similar way to that of the previous paper (1956) (i.e., by assuming a thin "shock-layer" near the body), the corresponding streamline curves are the geodesics. This derivation is left to Appendix A.

3. The Pressure Distribution

The equation of momentum in the direction normal to the body surface (Appendix A) becomes to this approximation

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = u^2 \kappa_x(x, y) + v^2 \kappa_y(x, y) \qquad \ldots(3.1)$$

or

$$p = p_S(x, y) + \kappa_x(x, y) \int \rho u^2 \, dz + \kappa_y \int \rho v^2 \, dz \qquad \ldots(3.2)$$

where $p_S(x, y)$ is the pressure on the shock wave at the point (x, y) assuming that shock and body coincide. p and ρ are the density and pressure at the point (x, y, z) between the shock and body. κ_x and κ_y are the curvatures of the body in x and y directions. We can further, following our suggestion in § 2, write (3.2) in the form

p/

$$p = p_S + \kappa_x \int_C \rho u^2 \left(\frac{\partial z}{\partial s_0} \right) ds_0 + \kappa_y \int_C \rho v^2 \left(\frac{\partial z}{\partial s_0} \right) ds_0 \qquad \ldots(3.3)$$

where ds_0 is the element of length along the curve C defined by equation (2.3) assuming (x, y) are constant.

Let us now consider the continuity of flow along a stream tube which starts at (x_0, y_0) and passes over the point (x, y). The amount of fluid flowing across the shock wave is approximately

$$\rho_\infty(\underline{U}_\infty, \underline{n}_0)\, dS_0 \qquad \ldots(3.4)$$

where ρ_∞ is the free stream density, $\underline{U}_\infty$ is the vector of the free stream velocity, $\underline{n}_0$ is the normal and dS_0 the element of area at the point (x_0, y_0). At the point (x, y), the amount of fluid is

$$\rho q\, dS$$

$$= \rho(u\, dS_{yz} + v\, dS_{zx} + w\, dS_{xy}) \qquad \ldots(3.5)$$

where q is the speed, dS the element of area of the stream tube; dS_{yz}, dS_{zx} and dS_{xy} the elements of dS in the various co-ordinate planes, and (u, v, w) are the velocities in the (x, y, z) directions. To this approximation $w\, dS_{xy}$ is of smaller order than the other terms and may be neglected. Thus, by continuity,

$$\rho_\infty(\underline{U}_\infty \cdot \underline{n}_0)\, dS_0 = \rho(u\, dS_{yz} + v\, dS_{zx}). \qquad \ldots(3.6)$$

Since however the normal to the stream tube cross-section is in the direction of $\underline{q}$, we also have

$$\frac{dS_{yz}}{u} = \frac{dS_{zx}}{v}, \qquad \ldots(3.7)$$

from which (3.6) may be written as either

$$dS_{yz} = \frac{u\rho_\infty(\underline{U}_\infty . \underline{n}_0)\, dS_0}{\rho q^2} \qquad \ldots(3.8)$$

or

$$dS_{zx} = \frac{v\rho_\infty(\underline{U}_\infty . \underline{n}_0)\, dS_0}{\rho q^2}. \qquad \ldots(3.9)$$

Alternatively, this is

$$\frac{\partial(y, z, x)}{\partial(x_0, y_0, x)} = \frac{u\rho_\infty(\underline{U}_\infty . \underline{n}_0)}{\rho q^2} \qquad \ldots(3.10)$$

and

$$\frac{\partial(z, x, y)}{\partial(x_0, y_0, y)} = \frac{v\rho_\infty(\underline{U}_\infty . \underline{n}_0)}{\rho q^2} \qquad \ldots(3.11)$$

where $\frac{\partial(\ \ )}{\partial(\ \ )}$ denotes a Jacobian.

Now/

Now
$$\frac{\partial z}{\partial s_0} = \frac{\partial(z, x, y)}{\partial(s_0, x, y)} = \frac{\partial(z, x, y)}{\partial(x_0, y_0, y)} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)}$$

$$= \frac{u\rho_\infty(\underline{U}_\infty . \underline{n}_0)}{\rho q^2} \frac{\partial(x_0, y_0, x)}{\partial(s_0, x, y)} \qquad ...(3.12)$$

or
$$= \frac{v\rho_\infty(\underline{U}_\infty . \underline{n}_0)}{\rho q^2} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)}. \qquad ...(3.13)$$

And hence

$$p = p_S + \kappa_1 \rho_\infty \int_0^{s_0} \frac{u^3(\underline{U}_\infty . \underline{n}_0)}{q^2} \frac{\partial(x_0, y_0, x)}{\partial(s_0, x, y)} ds_0 + \kappa_2 \rho_\infty \int_0^{s_0} \frac{v^3(\underline{U}_\infty . \underline{n}_0)}{q^2} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)} ds_0 \qquad ...(3.14)$$

where the integrands are taken as functions of x, y and s_0 and the integration is at constant x and y. From equation (2.3) we know the curve C. in terms of (x_0, y_0) for any point (x, y). It is therefore possible by the methods of differential geometry to obtain s_0 in terms of x_0 or y_0 at constant x and y. Inverting these relations we have x_0 and y_0 as functions of s_0, x and y. The integrands in (3.14) can thus be expressed as functions of s_0, x and y, and the integrations made at constant x and y. The pressure on the body is then obtained by putting $s_0 = s$, where s is the total length of the curve C, as the streamline closest to the surface is the one originating at the stagnation point. The length s is that of the curve C from the point (x, y) to the stagnation point which is given by

$$\underline{n} = -\left(\frac{\underline{U}_\infty}{U_\infty}\right)$$

where $\underline{n}$ is the outward normal to the surface and $\underline{U}_\infty$ is the free stream velocity vector. This is only the case for a bluff body in which the shock is detached. For a body with an attached shock wave where there is, strictly speaking, no stagnation point, the corresponding point will be the point of attachment of the shock wave.

It will be noted that in the previous theory the quantity s_0 plays the part of the stream function (or Stokes' stream function) in the two-dimensional (or axially symmetric) flow. In place of the usual von Mises' variables x and ψ used in the two-dimensional and axially symmetric theories we have the variables x, y and s_0 in the three-dimensional theory.

4. The Shock Shape

When the pressure distribution has been determined it is then relatively straightforward to obtain the shock shape. Bernoulli's equation may be written to this approximation

$$\frac{\gamma}{\gamma - 1} \frac{p}{\rho} + \tfrac{1}{2}(q^2) = \frac{\gamma}{\gamma - 1} \frac{p_0}{\rho_0} + \tfrac{1}{2}(q_0^2) \qquad ...(4.1)$$

and/

and thus since $q = q_0$, we have

$$\frac{p}{\rho} = \frac{p_0(x_0, y_0)}{\rho_0(x_0, y_0)}. \qquad \ldots(4.2)$$

Using (3.13) we have

$$z = \int_s^{s_0} \frac{v\rho_\infty(\underline{U}_\infty . \underline{n}_0)}{\rho q^2} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)} ds_0$$

$$= \epsilon \int_s^{s_0} \frac{p_0 v(\underline{U}_\infty . \underline{n}_0)}{p . q^2} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)} ds_0 \qquad \ldots(4.3)$$

where $\epsilon = (\gamma - 1)/(\gamma + 1)$. The integral (4.3) can be evaluated by expressing the integrand in terms of x, y and s_0 and integrating with respect to s_0 at constant x and y. The shock shape is then obtained by putting $s_0 = 0$.

5. Particular Cases

In the previous three sections, the procedure to be followed in working out the pressure and shock shape for any bluff body has been outlined. In theory, therefore, we should be able to do this. In practice however the task is rather formidable. The first difficulty arises from the fact that only in a few particular cases can the equations of the geodesics be obtained explicitly. In view of the necessity to work out the quantity s_0 as a function of (x, y) and (x_0, y_0) this is virtually a necessity. The function s_0, however, is still difficult to obtain explicitly. In view of these difficulties with the differential geometry of the theory, the only practical approach seems to be to consider the solution for a slightly yawed bluff body and, in particular, a slightly yawed body of revolution for which the geodesics can always be found. Equation (2.2) becomes, for a body of revolution, with $h_2 = h_2(x)$ and $h_1 = h_1(x)$

$$\frac{h_1 \left(\frac{h_2}{h_1}\right)^2 \frac{dy}{dx}}{\sqrt{1 + \left(\frac{h_2 \, dy}{h_1 \, dx}\right)^2}} = \text{constant} = a \text{ (say)}. \qquad \ldots(5.1)$$

Thus

$$\frac{dy}{dx} = \sqrt{\frac{a^2 h_1^2}{h_2^2 (h_2^2 - a^2)}} \qquad \ldots(5.2)$$

where

$$a = \frac{h_{2o}^2}{h_{1o}} \frac{\left(\frac{dy}{dx}\right)_o}{1 + \left[\frac{h_{2o}}{h_{1o}}\left(\frac{dy}{dx}\right)_o\right]^2}.$$

Let/

Let us replace (x, y) by a new system of curvilinear co-ordinates (x, ϕ) on the body. ϕ measures the angle of a plane of symmetry to a fixed plane and x is the co-ordinate in the plane of symmetry (Fig. 2). The angle of the tangent to the body in a plane of symmetry to the axis of the body is $\Phi(x)$, and the distance of the point from the axis $\eta(x)$. Let $h_1(x) = h(x)$ and $h_2 dy = \eta(x)d\phi$.

We will now treat the pressure distribution as in three parts - the contribution due to the pressure at the shock wave (p_S), that due to centrifugal effects in the x direction (p_x), and that due to centrifugal effects in the ϕ direction (p_ϕ), then

$$p = p_S + p_x + p_\phi. \qquad \ldots(5.3)$$

Written in terms of the new co-ordinates the equation of the geodesics becomes

$$\frac{d\phi}{dx} = \sqrt{\frac{a^2 h^2(x)}{\eta^2(\eta^2 - a^2)}} \qquad \ldots(5.4)$$

or

$$\phi - \phi_0 = \int_{x_0}^{x} \frac{h(x)dx}{\eta\sqrt{\frac{\eta^2}{a^2} - 1}} \qquad \ldots(5.5)$$

and

$$a = \frac{\eta_0^2 (d\phi/dx)_0}{h(x_0)\sqrt{1 + \frac{\eta_0^2}{h_0^2}\left(\frac{d\phi}{dx}\right)_0^2}}$$

where the suffix o denotes that the quantity is evaluated at the point $x = x_0$. If the body is placed symmetrically to the oncoming stream, the geodesics required are then the curves made by the intersection of the plane through the axis with the body surface or $\phi = \phi_0$. If the body is slightly yawed, however, they will deviate slightly from these curves. Let us suppose that the body is yawed at a small angle δ to the oncoming stream in the plane $\phi = 0$. Then $\left(\frac{\eta d\phi}{h dx}\right)_0 = 0\ (\sin\delta)$ since this is the tangent of the angle on the surface which the oncoming stream makes with the x direction. Thus $a = 0\ (\sin\delta)$ and we have

$$\phi - \phi_0 = \frac{\eta^2(x_0)}{h(x_0)}\left(\frac{d\phi}{dx}\right)_0 \int_{x_0}^{x} \frac{h(x)}{\eta^2(x)}\,dx + 0\ (\sin^3\delta)$$

$$= -\left\{\frac{\eta(x_0)\sin\phi_0 \sin\delta}{\cos\phi_0 \sin\delta \sin\Phi_0 + \cos\Phi(x_0)}\right\}\int_{x_0}^{x} \frac{h}{\eta^2}\,dx + 0\ (\sin^3\delta). \qquad \ldots(5.6)$$

The latter step follows by simple geometry as described in Appendix B.

The/

The part of the pressure p_S is deduced directly from the shock conditions which to this approximation give

$$p_S = \rho_\infty U_n^2$$

where U_n is the component of the free-stream velocity normal to the body.

Thus
$$\frac{p_S}{\rho_\infty U^2} = (\sin \Phi \cos \delta - \cos \Phi \cos \phi \sin \delta)^2 \qquad \ldots(5.7)$$

(see Appendix B).

The pressure contribution from the cross-flow in the ϕ direction is again simple to evaluate. We have $\kappa_\phi = (\eta(x))^{-1}$ and hence

$$p_\phi = (\eta(x))^{-1} \rho_\infty \int_0^{s_0} \frac{v^3 (\underline{U}_\infty . \underline{n}_0)}{q^2} \frac{\partial(x_0, y_0, y)}{\partial(s_0, x, y)} ds_0. \qquad \ldots(5.8)$$

Now
$$v \approx q_0 \frac{h_2}{h_1} \frac{dy}{dx} \approx q_0 \frac{\eta d\phi}{h dx}$$

where q_0 is the velocity at (x_0, y_0).

$$\approx v_0 \left(\frac{\eta}{h} \frac{d\phi}{dx}\right) \Big/ \left(\frac{\eta}{h} \frac{d\phi}{dx}\right)_0$$

$$\approx v_0 \frac{\eta(x_0)}{\eta(x)} \qquad \ldots(5.9)$$

by equation (5.6) for the geodesics. Thus the first term in equation (5.8) will be of order $\sin^2 \delta$, since v_0 is $O(\sin \delta)$ and the Jacobian will be $O((\sin \delta)^{-1})$. It will be found more convenient, however, to introduce the identity obtained by equating equations (3.12) and (3.13) to obtain

$$p_\phi = (\eta(x))^{-1} \rho_\infty \int_0^{s_0} \left(v_0 \frac{\eta_0}{\eta}\right)^2 \frac{(\underline{U}.\underline{n}_0)u}{q^2} \frac{\partial(x_0, \phi_0, x)}{\partial(s_0, x, \phi)} ds_0 \qquad \ldots(5.10)$$

where (5.9) has also been introduced. To order $\sin^2 \delta$ it is then sufficient to substitute the values of the factors for $\delta = 0$ after substituting for v_0. The equation (5.10) then becomes, since $s_0 = x - x_0$

$\dfrac{p_\phi}{\rho_\infty U^2} /$

$$\frac{p_\phi}{\rho_\infty U^2} = -[\eta(x)]^{-1}\int_{x_0}^{x}\frac{\sin^2\phi\sin^2\delta}{\cos^2\Phi_0}\left(\frac{\eta_0}{\eta}\right)^3\cos\Phi_0\sin\Phi_0\,dx_0 + O(\sin^3\gamma)$$

$$= \frac{\sin^2\delta\sin^2\phi}{\eta^4}\int_{x_0}^{x}\eta_0^3\tan\Phi_0\,dx_0. \qquad \ldots(5.11)$$

This equation gives the contribution to the pressure from the cross-flow component of the stream. It now remains to evaluate the contribution to the pressure from the component in the x-direction which is given by equation (3.14) as

$$p_x = \kappa_x\rho_\infty\int_0^{S_0}\frac{u^3(U_\infty.n_0)}{q^2}\frac{\partial(x_0,\phi_0,x)}{\partial(S_0,x,\phi)}dS_0. \qquad \ldots(5.12)$$

Now, from Appendix B,

$$q_0 = U\{(\cos\delta\cos\Phi_0 + \sin\delta\sin\Phi_0\cos\phi_0)^2 + \sin^2\delta\cos^2\phi_0\}^{\frac{1}{2}} \ldots(5.13)$$

and

$$u = \frac{q_0}{\sqrt{1+\left(\frac{\eta d\phi}{h dx}\right)^2}}.$$

Hence (5.12) may be written

$$p_x = \kappa_x\rho_\infty\int_0^{S_0}\frac{q_0(U_\infty.\underline{n_0})}{\left(\sqrt{1+\left(\frac{\eta}{h}\frac{d\phi}{dx}\right)^2}\right)^3}\frac{\partial(x_0,\phi_0,x)}{\partial(s_0,x,\phi)}. \qquad \ldots(5.14)$$

Now $\underline{U}_\infty.\underline{n}_0$ is the component of velocity normal to the surface outside the shock wave, and is from Appendix B (equation (4)),

$$U_\infty(\sin\delta\cos\Phi_0\cos\phi_0 - \cos\delta\sin\Phi_0). \qquad \ldots(5.15)$$

Also

$$\frac{\eta}{h}\frac{d\phi}{dx} = -\frac{1}{\eta}\left\{\frac{\eta_0\sin\phi_0\sin\delta}{\cos\Phi_0}\right\} + O(\sin^2\delta) \qquad \ldots(5.16)$$

from equation (5.6).

It now remains to evaluate the Jacobian in (5.12) to the required approximation. Now,

$$\frac{\partial(x_0, \phi_0, x)}{\partial(s_0, x, \phi)} = \frac{1}{\dfrac{\partial(s_0, x, \phi)}{\partial(x_0, \phi_0, x)}}$$

$$= \frac{\partial(x_0, x, \phi)}{\partial(s_0, x, \phi)} \cdot \frac{\partial(x_0, \phi_0, x)}{\partial(x_0, x, \phi)}$$

$$= \frac{\partial(x_0, x, \phi)}{\partial(s_0, x, \phi)} \cdot \frac{\partial\phi_0}{\partial\phi} \cdot \frac{\eta_0}{\eta} \qquad \ldots(5.17)$$

and

$$\frac{\partial\phi_0}{\partial\phi} = 1 + \frac{\partial}{\partial\phi}\left\{\frac{\sin\delta \sin\phi\,(\cos\Phi_0 + \cos\phi \sin\delta . g\,(x, x_0))}{\cos\Phi_0\,(\cos\phi \sin\Phi_0 \sin\delta + \cos\Phi_0)}\right\} . g\,(x, x_0) \qquad \ldots(5.18)$$

from equation (5.6). Here we have written

$$g\,(x, x_0) = \eta \int_{x_0}^{x} \frac{h}{\eta^2}\, dx.$$

Thus, we may write (5.14) in the form

$$\frac{p_x}{\rho_\infty U_\infty^2} = \kappa_x \int_{x_0}^{x} [(\cos\delta \cos\Phi_0 + \sin\delta \sin\Phi_0 \cos\phi_0)^2 + \sin^2\delta \cos^2\phi_0]^{\frac{1}{2}}$$

$$.(\sin\delta \cos\Phi_0 \cos\phi_0 - \cos\delta \sin\Phi_0).\left(1 - \frac{3}{2}\frac{1}{\eta^2}\left(\frac{\eta_0 \sin\phi_0 \sin\delta}{\cos\Phi_0}\right)^2\right)$$

$$.\frac{\eta_0}{\eta}\left\{1 + \sin\delta . g(x, x_0)\left(\frac{\cos\phi_0}{\cos\Phi_0} - \frac{\sin\delta \cos 2\phi_0 \tan\Phi_0}{\cos\Phi_0}\right)\right\} dx_0$$

$$+ 0\,(\sin^3\delta) \qquad \ldots(5.19)$$

where

$$\phi_0 = \phi + \frac{\sin\delta\,(\cos\Phi_0 + \sin\delta \cos\phi . g\,(x, x_0)) \sin\phi}{\cos\Phi_0\,(\cos\phi \sin\Phi_0 \sin\delta + \cos\Phi_0)} + 0\,(\sin^3\delta)$$

obtained by inverting (5.6). This becomes, after some tedious but straightforward algebra,

$$\left.\frac{p_x}{\rho_\infty U_\infty^2}\right|$$

$$\frac{p_x}{\rho_\infty U_\infty^2} = \frac{\kappa_x}{\eta}\int_{x_0}^{x} \eta_0 \cos\Phi_0 \sin\Phi_0 \left\{ 1 + \sin\delta\cos\phi\left[\frac{\tan\Phi_0}{\cos\Phi_0} - \cot\Phi_0 + \frac{g(x, x_0)}{\cos\Phi_0}\right] + \sin^2\delta\left[\frac{1}{2}\frac{\cos^2\phi}{\cos^2\Phi_0}(2 - 2\cos\Phi_0 - \cos^2\Phi_0) - \frac{1}{2} - \frac{3}{2}\frac{\eta^2}{\eta_0^2}\frac{\sin^2\phi}{\cos^2\Phi_0} + g(x, x_0)\left(\frac{\tan\Phi_0}{\cos^2\Phi_0} - \frac{\tan\Phi_0}{\cos\Phi_0} - \frac{1}{\sin\Phi_0}\right)\cos 2\phi - [g(x, x_0)]^2\frac{\sin^2\phi}{\cos^2\Phi_0}\right]\right\} dx_0 + 0(\sin^3\delta)$$

where
$$g(x, x_0) = \eta_0\int_{x_0}^{x}\frac{h}{\eta^2}\,dx. \qquad \ldots(5.20)$$

Hence we have the pressure to order $\sin^3\delta$ as a function of x, ϕ and x_0. The pressure on the body surface is obtained by putting x_0 equal to its value on the body. This will be the value of x_0 at the stagnation point which is the point where the inward normal to the body coincides with the free stream direction. This normal is given by

$$-(-\sin\Phi, \cos\Phi\cos\phi, \cos\Phi\sin\phi)$$

and this is in the direction of the free stream, i.e., $(\cos\delta, \sin\delta, 0)$ when

$$\sin\Phi = \cos\delta, \quad \cos\Phi\cos\phi = -\sin\delta$$

and
$$\cos\Phi\sin\phi = 0. \qquad \ldots(5.21)$$

Thus $\phi = \pi$ and

$$\Phi = \frac{\pi}{2} - \delta. \qquad \ldots(5.22)$$

Hence if $x = X(\cos\Phi)$ we have

$$x = X(0) + X'(0)\sin\delta + \tfrac{1}{2}X''(0)\sin^2\delta + 0(\sin^3\delta) \qquad \ldots(5.23)$$

as the co-ordinate of the stagnation point to this approximation.

6. The Case of a Slightly Yawed Cone

In the case of a cone, the results of the previous section are simplified considerably since $\Phi = \beta$, where β is the half-angle of the cone. Also $\kappa_x \equiv 0$, and we have therefore $p_x \equiv 0$. Thus

$\frac{p}{\rho_\infty U^2}$/

$$\frac{p}{\rho_\infty U^2} = (\sin\beta\cos\delta - \cos\beta\cos\phi\sin\delta)^2$$

$$- (x\sin\beta)^{-1}\int_{x_0}^{x} \frac{\sin^2\phi\sin^2\delta}{\cos^2\beta}\left(\frac{x_0}{x}\right)^3 \cos\beta\sin\beta\, dx_0 + O(\sin^3\delta)$$

$$= (\sin\beta\cos\delta - \cos\beta\cos\phi\sin\delta)^2$$

$$- \frac{1}{x^4}\frac{\sin^2\phi\sin^2\delta}{\cos\beta}\left(\frac{x^4 - x_0^4}{4}\right) + O(\sin^3\delta)$$

$$= (\sin\beta\cos\delta - \cos\beta\cos\phi\sin\delta)^2$$

$$- \frac{\sin^2\phi\sin^2\delta}{4\cos\beta}\left(1 - \left(\frac{x_0}{x}\right)^4\right) + O(\sin^3\delta). \qquad \ldots(6.1)$$

Hence the pressure on the body is given by

$$\frac{p_B}{\rho_\infty U^2} = (\sin\beta\cos\delta - \cos\beta\cos\phi\sin\delta)^2 - \frac{\sin^2\phi\sin^2\delta}{4\cos\beta} + O(\sin^3\delta). \qquad \ldots.(6.2)$$

This result is shown in Fig. 3 for the particular case $\beta = 60°$, $\delta = -5°$. The first term is denoted by $p_S/\rho_\infty U_\infty^2$, and the second by $p_C/\rho_\infty U_\infty^2$. The variation for $\phi = 0$ to $90°$ only is shown as the portion of the body $90° < \phi < 270°$ is in the 'shadow' where the theory is no longer applicable. The shock shape is (from (4.3))

$$z = \epsilon\int_S^0 \frac{p_0 u(U_\infty . n_0)}{pq^2}\frac{\partial(x_0, \phi_0, x)}{\partial(s_0, x, \phi)} ds_0 = \epsilon\int_{x_S}^{x} \frac{p_0 u(U_\infty . n_0)}{pq^2}\frac{\partial(x_0, \phi_0, x)}{\partial(x_0, x, \phi)} dx_0$$

$$= -\epsilon\int_{x_S}^{x} \frac{p_0 u(U_\infty . n_0)}{pq^2}\left(\frac{\eta\partial\phi}{\eta_0\partial\phi_0}\right)^{-1} dx_0, \qquad \ldots(6.3)$$

and here

$\dfrac{p}{\rho_\infty U_\infty^2}$/

$$\frac{p}{\rho_\infty U_\infty^2} = (\sin\beta\cos\delta - \cos\beta\cos\phi\sin\delta)^2 - \frac{\sin^2\phi\sin^2\delta}{4\cos\beta}\left(1-\left(\frac{x_0}{x}\right)^4\right) + 0\,(\sin^3\delta)$$

$$\frac{p_0}{\rho_\infty U_\infty^2} = (\sin\beta\cos\delta - \cos\beta\cos\phi_0\sin\delta)^2$$

$$(\underline{U}_\infty . \underline{n}_0) = U_\infty(\sin\delta\cos\beta\cos\phi_0 - \cos\delta\sin\beta), \qquad \ldots(6.4)$$

$$\frac{q_0^2}{U_\infty^2} = (\cos\delta\cos\beta + \sin\delta\sin\beta\cos\phi_0)^2 + \sin^2\delta\cos^2\phi_0$$

and $$u = \frac{q_0}{\sqrt{1+\left(\dfrac{\eta d\phi}{h dx}\right)^2}} = q_0\left(1 - \frac{1}{2}\left[x\sin\beta\frac{d\phi}{dx}\right]^2\right) + 0\,(\sin^3\delta).$$

The value of ϕ_0 as a function of ϕ, x and x_0 may be deduced from equation (5.6) as

$$\phi_0 = \phi + \frac{\sin\delta\sin\phi\left(\cos\beta\sin\beta + \cos\phi\sin\delta\left[1-\dfrac{x_0}{x}\right]\right)}{\sin^2\beta\cos\beta\,(\cos\phi\sin\beta\sin\delta + \cos\beta)}\left(1-\frac{x_0}{x}\right).$$

$$\ldots(6.5)$$

Thus

$$\frac{\partial\phi_0}{\partial\phi} = 1 + \left[\frac{\cos\phi\sin\delta}{\sin\beta\cos\beta} + \frac{\sin^2\delta\cos 2\phi}{\sin^2\beta\cos^2\beta}\left(\left[1-\frac{x_0}{x}\right] - \sin^2\beta\right)\right]\left(1-\frac{x_0}{x}\right) + 0\,(\sin^3\delta)\ldots(6.6)$$

after a little manipulation. In a similar manner, the following expressions may be obtained

$$\frac{p_0}{p} = 1 + \frac{2\sin^2\delta\sin^2\phi}{\sin^2\beta}\left(1-\frac{x_0}{x}\right) + \frac{\sin^2\phi\sin^2\delta}{4\cos\beta\sin^2\beta}\left(1-\left[\frac{x_0}{x}\right]^4\right),$$

$$\frac{\underline{U}_\infty\,\underline{n}_0}{U_\infty} = -\sin\beta\left(1 - \sin\delta\cot\beta\cos\phi + \frac{\sin^2\phi\sin^2\delta}{\sin^2\beta}\left(1-\frac{x_0}{x}\right) - \tfrac{1}{2}\sin^2\delta\right),$$

$$(q_0)^{-1} = (\cos\beta)^{-1}\left[1 - \sin\delta\tan\beta\cos\phi + \sin^2\delta\left\{\frac{\sin^2\phi}{\cos^2\beta}\left(1-\frac{x_0}{x}\right) + \frac{1}{2} - \frac{1}{2}\frac{\cos^2\phi}{\cos^2\beta} + \tan^2\beta\cos^2\phi\right\}\right]$$

and

$$\frac{u}{q_0} = 1 - \frac{x_0^2}{2x^2}\,\frac{\sin^2\phi\sin^2\delta}{\cos^2\beta\sin^2\beta} \qquad \ldots(6.7)$$

where/

where we have neglected terms of order $\sin^3\delta$. Substituting these expressions in equation (6.3) and noting that $x_S = 0$ for a cone, we obtain

$$z = \epsilon\int_0^x \left\{1 - \frac{x_0}{x}\frac{\sin\delta\cos\phi}{\cos\beta\sin\beta} + \frac{\sin^2\delta}{\sin^2\beta\cos^2\beta}\left[\left\{1 + \frac{x_0}{x} - \frac{5}{2}\left(\frac{x_0}{x}\right)^2 + \frac{1}{4}\cos\beta\left(1 - \left(\frac{x_0}{x}\right)^4\right)\right\}\sin^2\phi + \cos^2\phi\left\{\frac{1}{2}\sin^2\beta - \left(1 - \frac{x_0}{x}\right)\right\} + \left(1 - \frac{x}{x_0}\right)^2 + \left(1 - \frac{x_0}{x}\right)\sin\beta\right]\right\}\frac{x_0}{x}\,dx_0 + O(\sin^3\delta). \quad ...(6.8)$$

Completing the integration, the shock shape can be written in the form

$$z = \epsilon x F(\phi) \quad ...(6.9)$$

where

$$F(\phi) = \tan\beta\left\{\frac{1}{2} - \frac{1}{3}\frac{\sin\delta\cos\phi}{\sin\beta\cos\beta} + \frac{\sin^2\delta}{\sin^2\beta\cos^2\beta}\left[\frac{1}{8}\sin^2\phi\left(1 + \frac{2}{3}\cos\beta + 2\cos^2\beta\right) - \frac{1}{12}(7 + 5\cos^2\beta)\right]\right\}. \quad ...(6.10)$$

The first two terms correspond to a conical shock wave with axis inclined slightly to the cone at an angle $\frac{1}{3}\frac{\epsilon\sin\delta}{\cos^2\beta}$ in the plane of the stream and cone axis but in the opposite direction to the stream. The half-angle of this cone is greater than that of the body cone by an amount $\frac{1}{2}\epsilon x\tan\beta\sec\beta$. This is the approximation used by Ferri (1951) in a more general treatment of flow past slightly yawed cones where the complete equations are used.

The function $F(\phi)$ is plotted in Fig. 4 for the case $\beta = 60°$, $\delta = -5°$. The portion of the body $-90° < \phi < 90°$ is then presented to the flow.

Conclusion

To the approximation of this paper, the surface pressure distribution and shock shape for any three-dimensional body can be derived, although mathematical complexity of the problem in the general case does not allow explicit expressions to be given. In the case of slightly yawed bodies, however, a series solution in powers of the angle of attack is feasible. The simplest case is that of a slightly yawed cone where complete solutions of the problem to the first order in angle of attack have been derived by Ferri (1951). The approximate solution of this paper is given to second order in angle of attack.

The limitations on the problem of assuming γ near one and Mach number infinite remain a cause of concern about the applicability of the results to the practical case. The author believes, however, that the understanding of the general subsonic-supersonic flow field can best be achieved by consideration of these essentially simple first approximations.

APPENDIX A

The equations of motion of a fluid in the orthogonal curvilinear three-dimensional co-ordinates (x, y, z) may be written in the form

$$\frac{u}{h_1}\frac{\partial u}{\partial x} + \frac{v}{h_2}\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - v^2\left[\frac{1}{h_1 h_2}\frac{\partial h_2}{\partial x}\right] + uv\left[\frac{1}{h_1 h_2}\frac{\partial h_1}{\partial z}\right] + \frac{uw}{h_1}\frac{\partial h_1}{\partial z} + \frac{1}{h_1 \rho}\frac{\partial p}{\partial x} = 0. \qquad \ldots(A.1)$$

$$\frac{u}{h_1}\frac{\partial v}{\partial x} + \frac{v}{h_2}\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + vw\left[\frac{1}{h_1}\frac{\partial h_2}{\partial z}\right] - \frac{u^2}{h_2 h_1}\frac{\partial h_1}{\partial y} + \frac{vu}{h_2 h_1}\frac{\partial h_2}{\partial x} + \frac{1}{h_2 \rho}\frac{\partial p}{\partial y} = 0. \qquad \ldots(A.2)$$

$$\frac{u}{h_1}\frac{\partial w}{\partial x} + \frac{v}{h_2}\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - u^2\left[\frac{1}{h_1}\frac{\partial h_1}{\partial z}\right] - \frac{v^2}{h_2}\frac{\partial h_2}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0. \qquad \ldots(A.3)$$

Assuming that the shock wave is close to the body, and that the velocity component w normal to the body is small we have $\frac{\partial}{\partial z} \gg \frac{\partial}{\partial x}, \frac{\partial}{\partial y}$. However $u\frac{\partial}{\partial x}$, $v\frac{\partial}{\partial y}$, $w\frac{\partial}{\partial z}$ are of the same order. The elements of length h_1 and h_2 may be written $h_1'(x, y) + H_1(x, y, z)$ and $h_2'(x, y) + H_2(x, y, z)$ where H_1 and H_2 are small. Small in this context means that the functions are of order $\left(\frac{\rho_0}{\rho}\right)$. $\frac{\partial}{\partial x}$, and $\frac{\partial}{\partial y}$ are then $O(1)$, $\frac{\partial}{\partial z} = O\left(\frac{\rho_0}{\rho}\right)$ and $p = O(1)$. Substitution in (A.1), (2) and (3) then gives

$$\frac{u}{h_1'}\frac{\partial u}{\partial x} + \frac{v}{h_2'}\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - v^2\left(\frac{1}{h_1'h_2'}\frac{\partial h_2'}{\partial x}\right) + uv\left(\frac{1}{h_1'h_2'}\frac{\partial h_1'}{\partial y}\right) = 0. \qquad ..(A.4)$$

$$\frac{u}{h_1'}\frac{\partial v}{\partial x} + \frac{v}{h_2'}\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + uv\left(\frac{1}{h_1'h_2'}\frac{\partial h_2'}{\partial x}\right) - u^2\left[\frac{1}{h_1'h_2'}\frac{\partial h_1'}{\partial y}\right] = 0. \qquad ..(A.5)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = u^2\left[\frac{1}{h_1}\frac{\partial h_1}{\partial z}\right] + v^2\left[\frac{1}{h_2}\frac{\partial h_2}{\partial z}\right] = 0 \qquad \ldots(A.6)$$

where we have neglected terms of order (ρ_0/ρ). That the first two equations ((A.4) and (A.5)) do in fact give the equations of the geodesics can be easily shown by considering the motion of a particle on the given surface. Such a particle will have kinetic energy

$$T = \frac{m}{2}\left(h_1'^2\,\dot{x}^2 + h_2'^2\,\dot{y}^2\right) \qquad \ldots(A.7)$$

where/

where $\dot{x} = \frac{dx}{dt}$, $\dot{y} = \frac{dy}{dt}$ and m is its mass. For the integral of this to be a minimum along a curve C, which is an alternative definition of a geodesic, we require by the calculus of variations

$$\frac{d}{dt}(h_1^2 \dot{x}) - \frac{1}{2}\left(\frac{\partial}{\partial x}\right)_{\dot{x},\dot{y}} (h_1^2 \dot{x}^2 + h_2^2 \dot{y}^2) = 0 \qquad \ldots(A.8)$$

plus a similar equation in the y-direction. This may be written

$$\left(\frac{u}{h_1}\frac{\partial}{\partial x} + \frac{v}{h_2}\frac{\partial}{\partial y}\right)u + \frac{uv}{h_1h_2}\frac{\partial h_1}{\partial y} - \frac{v^2}{h_1h_2}\frac{\partial h_2}{\partial x} \qquad \ldots(A.9)$$

by noting that $\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$ and $h_1\dot{x} = u$, $h_2\dot{y} = v$.

This is exactly equation (A.4). In a similar manner, the other equation reduces to (A.5). The equation (A.6) can be written to this approximation

$$\frac{1}{\rho}\frac{\partial p}{\partial z} = u^2 \kappa_1(x, y) + v^2 \kappa_2(x, y) \qquad \ldots(A.10)$$

where κ_1 and κ_2 are the curvatures of the surface in the x and y directions respectively.

The energy equation may be written

$$\left(\frac{u}{h_1}\frac{\partial}{\partial x} + \frac{v}{h_2}\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)S = 0 \qquad \ldots(A.11)$$

as the constancy of entropy S along streamlines.

APPENDIX B/

APPENDIX B

Geometry of the Axially Symmetric Case

Let the stream be inclined at an angle δ to axis of symmetry in the plane $\phi = 0$ (Fig. 2). Choose axes (X, Y, Z) such that X lies along the axis of symmetry and Y is in the plane $\phi = 0$. If the speed of the stream is U_∞ then its velocity in this system of co-ordinates is

$$(U_\infty \cos\delta, U_\infty \sin\delta, 0). \qquad \ldots(B.1)$$

The x direction at a particular point on the surface is given by the unit vector,

$$(\cos\Phi, \sin\Phi\cos\phi, \cos\Phi\sin\phi); \qquad \ldots(B.2)$$

the ϕ direction by

$$(0, -\sin\phi, +\cos\phi); \qquad \ldots(B.3)$$

and the z direction by

$$(-\sin\Phi, \cos\Phi\cos\phi, \cos\Phi\sin\phi) \qquad \ldots(B.4)$$

and hence at impact

$$\frac{u}{U_\infty} = (\cos\delta\cos\Phi + \sin\delta\sin\Phi\cos\phi) \qquad \ldots(B.5)$$

$$\frac{v}{U_\infty} = \sin\delta\cos\phi \qquad \ldots(B.6)$$

$$\frac{w}{U_\infty} = \sin\delta\cos\Phi\cos\phi - \cos\delta\sin\Phi. \qquad \ldots(B.7)$$

Note also that

$$\underline{U} \cdot \underline{n_0} = w_0 .$$

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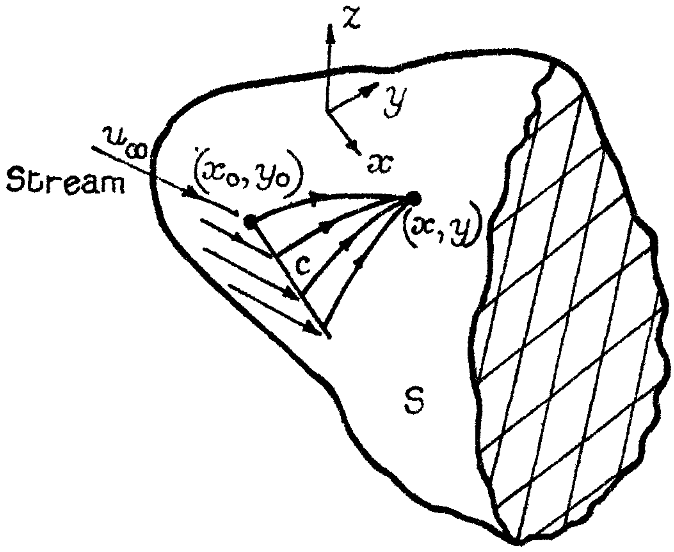
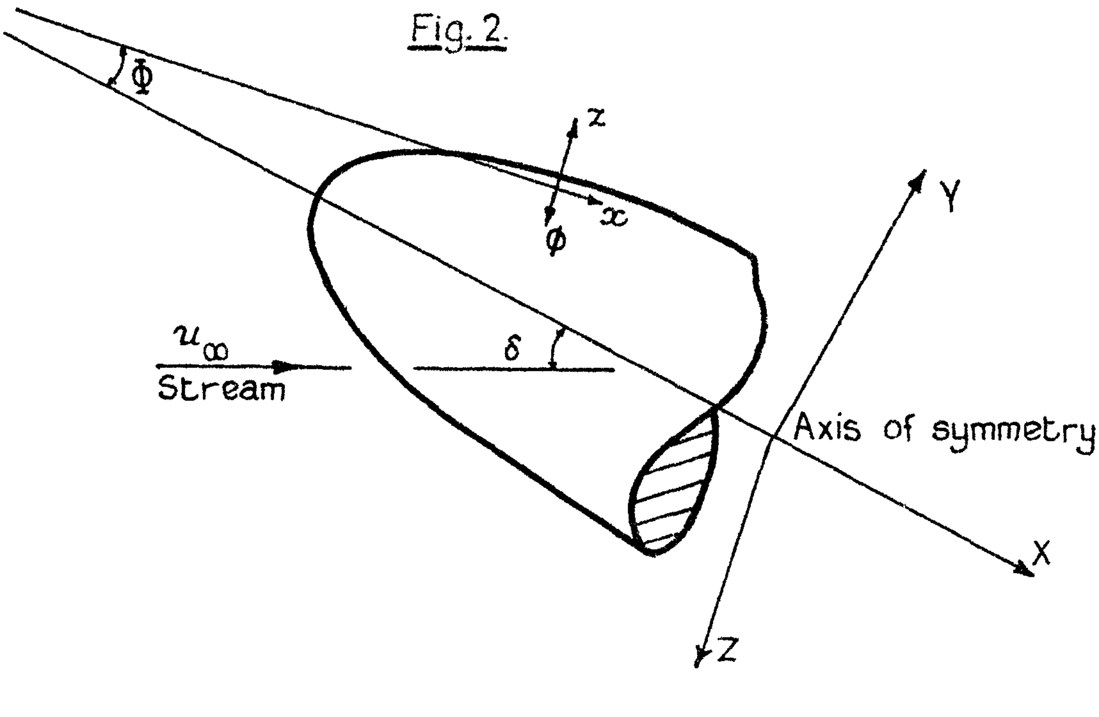
FIGS. 1 & 2

Fig. 1.



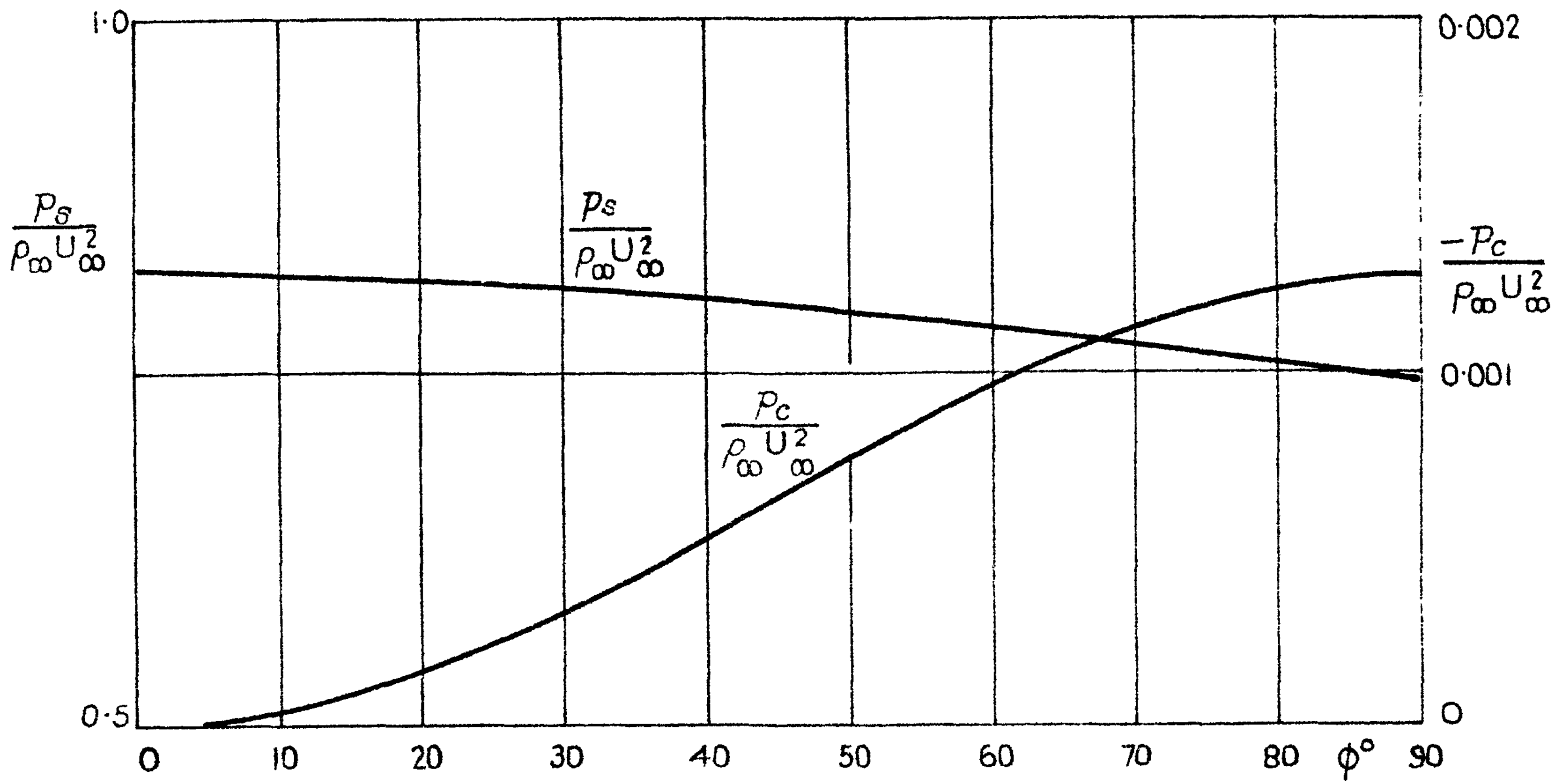
Coordinate system

Fig. 2.



Coordinate system for axially symmetric body.

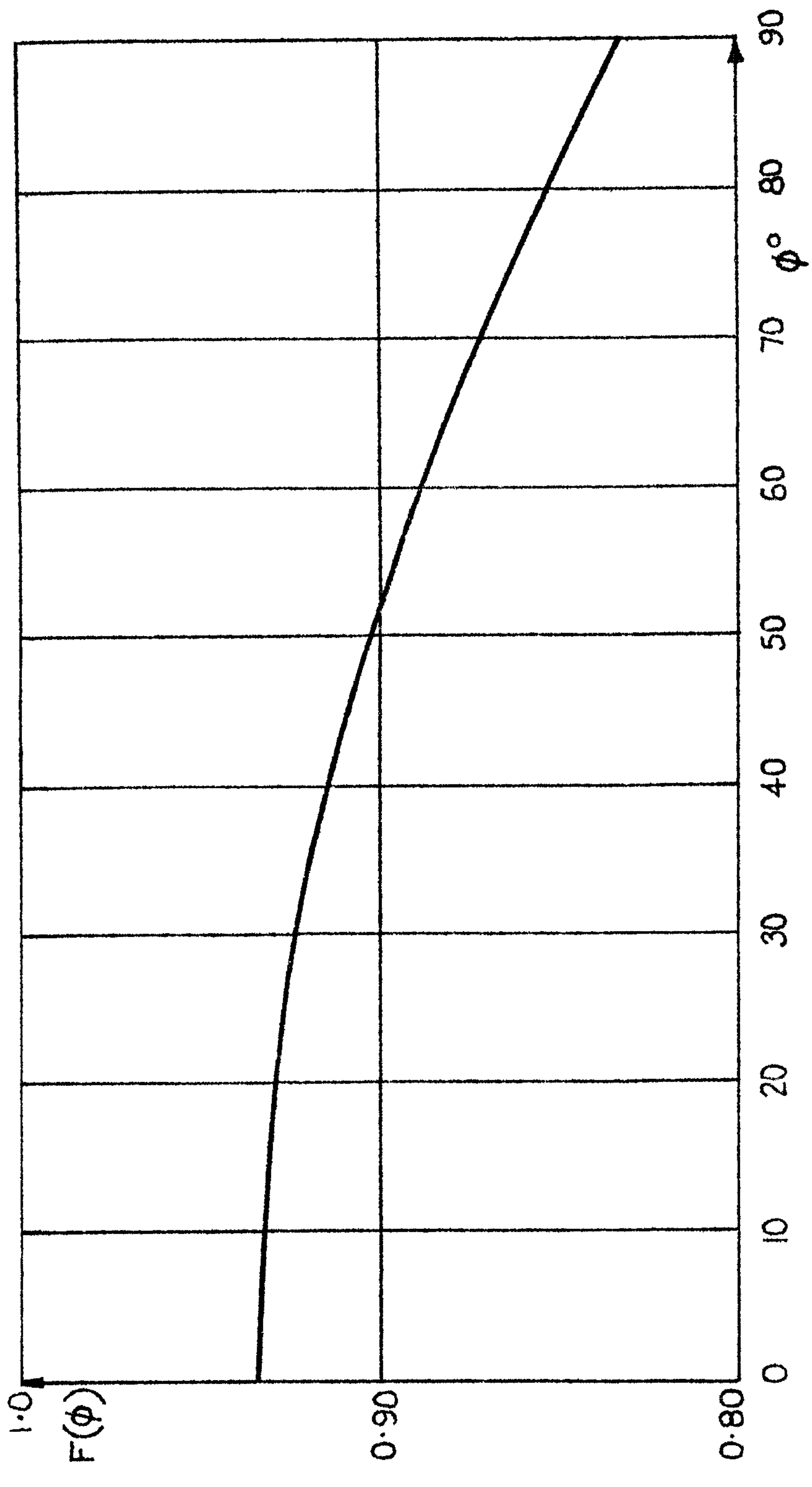
FIG. 3.



Pressure distribution on body surface; p_s = shock pressure and p_c = centrifugal pressure.

$p = p_s \pm p_c$

FIG. 4.



The shock shape. $z = \epsilon x F(\phi)$

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