THE INTERFERENCE OF A WIND TUNNEL ON A SYMMETRICAL BODY.

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Summary.—The effect of the boundaries of a wind tunnel on the flow in the neighbourhood of a symmetrical body (i.e. (*a*) in two dimensions, a cylinder having a plane of symmetry parallel to the axis of the tunnel : (*b*) in three dimensions a body of revolution coaxial with the tunnel), may be represented on the assumption of irrotational flow to a first approximation as an increase in magnitude of the velocity at any point near the body in a constant ratio $(u_1 + V)/V$. In two dimensions the value of u_1/V is proportional to the square of the ratio of a linear dimension of the body to the width of the tunnel, provided that this ratio is sufficiently small. In three dimensions u_1/V varies as $(s/S)^{3/2}$ where s is the maximum cross sectional area of the body and S that of the tunnel. It is therefore possible to write in two dimensions

$$u_1/V = \tau\, \lambda\, (s/S)^2,$$

and in three dimensions

$$u_1/V = \tau\, \lambda\, (s/S)^{3/2}\ ;$$

λ is a coefficient depending only on the shape of the body, being taken as unity for a circular cylinder or sphere ; τ has distinct values for an open or closed jet, and varies with the shape of cross section of the tunnel in three dimensions. In two dimensions theoretical values of λ have been worked out for a range of fineness ratios for the following four shapes of cross section :—Ellipse : Rankine Oval : Generalised symmetrical Joukowski : Simple symmetrical Joukowski ; in three dimensions for the two shapes :—Spheroid : "Rankine Ovoid." It may be worth noting that λ also determines the flow at a sufficiently great distance from the body in free air. Values of τ have been determined :—in two dimensions ; in three dimensions for square and circular cross section with open and closed jets, and for the closed "Duplex" tunnel. The limits of accuracy of the approximation were checked (*a*) in two dimensions by comparison with the exact calculations (for irrotational flow) by Fage in R. & M. 1223 of the flow past a Rankine Oval in a channel, (*b*) in three dimensions by the use of the solution given by Professor Lamb in R. & M. 1010 for the body generated by a source and equal sink ("Rankine Ovoid") in a closed tunnel of circular cross section. This analysis has been extended to the analogous case of an open circular jet. The results in two dimensions have also been compared with the observations of Fage in R. & M. 1223 of the drag of a two-dimensional body in a channel whose width could be varied. The comparison suggests that the "effective volume" of a body increases with its drag coefficient owing to the formation of a wake, the interference effect rising in the extreme case of a circular cylinder to rather over twice its theoretical value.

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1. *Introduction.*—The problem of determining the magnitude of the interference of wind tunnel walls on a symmetrical body has recently become more important owing to the increased size of models tested. Examples at the National Physical Laboratory are the two-dimensional Joukowski profiles tested by Fage and Falkner[1]* and the three-dimensional bodies of revolution tested by Ower, Townend and Hutton[2] for the Interference Sub-Committee. This increase of size has been made possible partly by the elimination of the pressure drop in the 7-ft. tunnel No. 3 ; previously the correction for pressure drop so greatly exceeded the direct effect of the tunnel walls on velocity that the latter could reasonably be ignored. It is the correction on velocity in the absence of pressure drop which forms the subject of the present report.

The two-dimensional case has been placed on a sound basis by Fage in R. & M. 1223.[3] This report contains the results of experiments in a model tunnel of varying width, which he analysed by means of the theory of a Rankine Oval (source and equal sink) in a channel, given by Sir Richard Glazebrook[4]. No experiments on the three-dimensional case are at present available, but the corresponding theory of a Rankine "Ovoid" (source and equal sink) in a closed circular tunnel was given by Professor Lamb[5]. This solution involves the use of an infinite series whose convergence becomes slow in certain cases. The three-dimensional case may be of some importance in the future owing to the increased desirability of using large size streamline models suggested by the researches of Professor Jones[6] and in connection with experiments on airscrew body interference.

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* A list of references is given at the end of the report.

In the present paper it is shown that a first approximation to the two-dimensional theoretical solution can be obtained by the method of images. This solution has the advantage that it can be applied to any symmetrical cylinder for which the solution in an infinite stream is known ; it also avoids the somewhat laborious calculations made by Fage in (3). These calculations have been used to check the accuracy of the approximate method and show that the approximation is adequate at least for the relative size of body and tunnel tested in (1). According to this approximation the interference for a given shape varies as the square of the ratio of a linear dimension of the body to the width of the tunnel.

A precisely analogous method can be applied to the three-dimensional case of a body of revolution in a tunnel of rectangular section (including square), subject to the labour of determining numerically the sum of a certain double series which is, however, a constant for a given shape of tunnel. As before, the method may be applied to any body of revolution for which the theoretical flow in an infinite stream is known. The accuracy has been checked by comparison with Lamb's solution for one shape only. According to the approximation, the interference now varies as the cube of the ratio of the linear dimensions of model and tunnel, or as the 3/2 power of the ratio of the cross-sectional areas.

Finally, a solution has been obtained analogous to Professor Lamb's for the case of a Rankine " Ovoid " in an open jet tunnel of circular cross-section.

2. *Method of images in two dimensions.*—Consider a symmetrical body placed in an infinite uniform stream of velocity V with its axis parallel to the direction of V. It may be shown that to a first approximation the flow at a large distance from the body is equivalent to that produced by a source and an equal sink on the axis of the body and depends only on the product of the strength of source or sink into the distance between them. In particular, at a large distance y at right angles to the stream, the velocity is to this approximation uniform and parallel to the stream of magnitude

$$V + u_0 = V + Q_0/y^2 \qquad \qquad (1)$$

where Q_0 is a constant which may be described as the strength of the doublet* (Fig. 1). The ratio Q_0/V which is of the dimensions of the square of a length is completely determined by the shape and size of the body.

Next consider the flow past an equal source and sink in a channel with straight parallel sides of breadth h. It may be shown that the flow is identical with that produced by the infinite series of images shown (Fig. 2). The images of the source (or sink) lie on a straight line normal to the channel, are all of equal strength and are

* The word " doublet " is used here for a combination of source and sink.

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at distances h apart. To the same order of approximation, the effect of the walls is equivalent to superposing an additional velocity u_1 in the neighbourhood of the original doublet, representing the effect of the sum of the images and given by the equations

$$u_1 = 2 \sum_{n=1}^{\infty} Q_0/n^2 h^2$$

$$= (2\, Q_0/h^2) \sum_{1}^{\infty} 1/n^2$$

$$= \frac{\pi^2}{3} \cdot \frac{Q_0}{h^2} \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

Under these conditions, however, the size and shape of the original boundary will be altered, and in order to return to the same boundary (to the first order) the strength Q_0 of the doublet must be replaced by Q such that

$$Q/Q_0 = (V + u_1)/V \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (3)$$

It follows that the magnitude of the velocity at any point in the neighbourhood of the body will be altered in this same ratio $(V+u_1)/V$. (In equation (2) it is immaterial to this order of approximation whether Q_0 is replaced by Q). Hence, it is to be expected that all differences of pressure as well as the drag of the whole body will be increased in the ratio $(u_1 + V)^2/V^2$. It may be noticed that the pressure at the stagnation point is equal to $p_0 + \frac{1}{2} \rho V^2$, where p_0 is the static pressure at a point at a distance upstream large compared with the dimensions of the body.

3. *Method of evaluating the interference in two dimensions.*—Writing $z = x + iy$, where x and y are the co-ordinates of a point relative to the axes shown in Fig. 3, the velocity field round any symmetrical cylindrical body is given by the formula

$$V + u - iv = V + f(z)$$

where $f(z)$ is an analytic function of z which for large values of z may be expanded in a series of inverse powers of z, of which the first term has the form

$$-Q/z^2$$

where Q is a real constant. The velocity at a large distance y normal to the stream is then given by

$$V + u = V + Q/y^2$$

in agreement with equation (1) of §2 above.

In all cases in which the form of the function $f(z)$ adapted to a particular shape of body is known, the value of Q and hence the above approximation to the tunnel interference can be determined. The following examples will be discussed :—(1) Ellipse ; (2) Rankine

Oval; (3) The series of generalised Joukowski forms employed in R. & M. 1241, Ref. (1); (4) Simple symmetrical Joukowski form with a cusp at the tail.

(1) *Elliptic Cylinder.*

The appropriate functions are given by

$$\phi + i\psi\, w = V\left\{z + c\sinh\xi_o e^{\xi_o - \zeta}\right\},$$
$$V + u - iv = dw/dz,$$
$$z = c\,.\cosh\zeta,$$

where the semi axes of the ellipse are $l = c\cosh\xi_o$, and $t = c\sinh\xi_o$.

For large values of z

$$(u - iv)/V \text{ approximates to } -\sinh\xi_o\, e^{\xi_o}\, c^2/2z^2$$

or

$$Q/V = \tfrac{1}{2}c^2\sinh\xi_o(\cosh\xi_o + \sinh\xi_o)$$
$$= \tfrac{1}{2}t\,(t+l)$$

and

$$Q/V\,t^2 = \tfrac{1}{2}(1 + l/t).$$

For the case when t is greater than l the formulæ become

$$w = V\left\{z - ic\cosh\xi_o\, e^{\xi_o - \zeta}\right\},$$
$$z = ic\cosh\zeta,$$
$$l = c\sinh\xi_o,\ t = c\cosh\xi_o,$$
$$Q/V = \tfrac{1}{2}c^2\cosh\xi_o(\cosh\xi_o + \sinh\xi_o)$$
$$= \tfrac{1}{2}t\,(t + l)$$

as before, so that this formula holds for all values of t/l.

(2) *Rankine Oval.*

$$\phi + i\psi = w = V\left\{z - A\log\frac{z - c}{z + c}\right\},$$
$$(V + u - iv)/V = 1 - 2cA/(z^2 - c^2),$$

where $2c$ is the distance between source and sink.

The thickness $2t$ and the length $2l$ are determined from the following considerations:—

(*a*) $\psi = o$ for $x = o$ and $y = t$,
(*b*) $V + u = o$ for $y = o$ and $x = l$.

(*a*) This leads to the condition

$$A = t/2\alpha,$$

where

$$\tan\alpha = c/t.$$

(*b*) This leads to the condition

$$l^2 - c^2 = ct/\alpha.$$

It is obvious that

$$Q/V = 2\,c\,A = ct/\alpha$$

and so

$$Q/Vt^2 = \tan\alpha/\alpha.$$

Also

$$l^2/t^2 = \tan^2\alpha + \tan\alpha/\alpha.$$

The shape is given explicitly by the equation

$$x^2 = c^2 - y^2 + 2\,cy\cot(2\alpha y/t).$$

In the limiting case

$$l/t = 1{\cdot}0$$

corresponding to $\alpha = 0$ we have

$$Q/V^2t^2 = 1{\cdot}0$$

agreeing with the corresponding result for the ellipse, the limiting shape being a circle in both cases.

(3) *General symmetrical Joukowski wing* (Ref. 1 and 8).

$$w = V\left\{t + \frac{a^2}{t}\right\},$$

where

$$t = \zeta - a + c,$$

$$\frac{z - nc}{z + nc} = \left(\frac{\zeta - c}{\zeta + c}\right)^n.$$

For large values of z

$$z \text{ approximates to } \zeta\left\{1 + (1/3)(n^2 - 1)\,c^2/\zeta^2\right\}.$$

$$\frac{dw}{dz} = (V + u - iv)/V \text{ approximates to } 1 - \left\{a^2 - (1/3)(n^2 - 1)\,c^2\right\}/\zeta^2,$$

$$Q/V = a^2 - (1/3)(n^2 - 1)c^2.$$

For the actual shapes used in R. & M.1241, which satisfy the relation $n = 2{\cdot}5 - 0{\cdot}5\,(a/c)$, the values of l and t are recorded and so Q/V^2 can be determined for given l/t.

(4) *Simple Joukowski form.* This corresponds to the simple case of the above for which $n=2$. For this case theoretical formulæ can be obtained for the maximum thickness as well as for the chord. The formulæ may be written in the following form. On the surface we may write $t = a\,e^{i\theta}$ from which the shape of the section is given by

$$x/c = (\alpha + \overline{1 + \alpha}\cos\theta)\,(1 + 1/F),$$

$$y/c = \overline{1 + \alpha}\,\sin\theta\,(1 - 1/F),$$

where α is a constant given by

$$\alpha = (a - c)/c,$$

$$F = 2\alpha(1 + \alpha)\,(1 + \cos\theta) + 1$$

and x and y are rectangular co-ordinates. Then the value θ_1 of θ corresponding to the maximum ordinate is given by the relation :—

$$F_1 = \sec \theta_1 - 1.$$

The equation for Q/V becomes

$$Q/V = c^2 \alpha(\alpha + 2).$$

The chord length is equal to the difference between the values of x which correspond to $\theta = 0$ and $\theta = \pi$ and reduces to

$$\text{Chord} = 4c\,(1 + \alpha)^2/(1 + 2\alpha).$$

4. *Results of calculations in two dimensions*—Values of $Q/V\,t^2 = \lambda$ have been calculated from the above formulæ for the four different shapes for various values of the fineness ratio l/t and are shown plotted against l/t in Fig. 4. The shapes 1, 2 and 4 all have the circular cylinder as a particular case and the formulæ are in agreement in making $\lambda = 1$ for $l/t = 1$. The first approximation to the tunnel interference is given in terms of λ by the formula

$$\begin{aligned} u_1/V &= (\pi^2/12)\,\lambda\,(2t/h)^2 \\ &= 0{\cdot}822\,\lambda\,(2t/h)^2 \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1) \end{aligned}$$

where $2t$ is the maximum thickness of the section and h the breadth of the tunnel (equation (2) of § 2).

To assist in the process of guessing a suitable value of λ for an arbitrary shape, the four profiles corresponding to the fineness ratio 3·67 have been calculated, and are shown in Fig. 5.* Comparison of Figs. 4 and 5 shows how the tunnel interference for a given aspect ratio increases with the space occupied by the section. From these curves it should be possible to interpolate with sufficient accuracy the tunnel interference for any intermediate shape.

It may be worth noting that the curves of Fig. 4 give a first approximation to the flow at a large distance from the body in free air where the effect of the body approximates to that of a circular cylinder of radius $t\lambda^{\frac{1}{2}}$.

5. *Theoretical accuracy of the approximation in two dimensions.*— The accuracy of the above first approximation to the theoretical tunnel interference may be checked in the case of the Rankine Ovals by comparison with the exact calculation given in R. & M. 1223 (3). (Table 1.) This table gives values of the velocity $V + u = m\,(V + u_0)$ just outside the surface for a Rankine Oval in a channel, where $V + u_0$ is the velocity at the corresponding point in free air. The value of m at the maximum ordinate $x = 0$, $y = \frac{1}{2}T\ (\equiv t)$ is denoted by m_1, and the values of $m_1 - 1\ (\equiv u_1/V$ to the approximation of the present report) are plotted in Fig. 6 against

*Shape 3 is taken from (Ref. 1) Table 6, and the value of 3·67 for the fineness ratio was chosen to correspond with this shape.

$(T/H)^2$ $\{\equiv(2t/h)^2\}$ for the various values of the fineness ratio C/T ($\equiv l/t$). On the same diagram are shown the values of u_1/V corresponding to the first approximation in the form of straight lines through the origin of slope $0\cdot822\ \lambda$ (see equation (1) of section 4). When l/t is not greater than $5\cdot4$ the discrepancy is less than $\frac{1}{2}$ per cent. on velocity within the limits of the calculations ($2t/h$ not greater than $0\cdot2$). For the two sections of fineness ratio $9\cdot8$ and $18\cdot3$ the discrepancy is less than $\frac{1}{2}$ per cent. so long as the chord length is less than the width of the channel ($2\,l/h$ less than $1\cdot0$).

When the above first approximation begins to break down it is no longer exactly true that the velocity is altered by tunnel interference in the same ratio at all points of the surface. It appears from the remaining entries in R. & M. 1223 (Table 1) that the discrepancies between velocities at different parts of the surface are of the same order of magnitude as the error of the first approximation.

6. *Alternative approximation for a long body.*—When the fineness ratio is large and the chord length is large compared with the breadth of the tunnel, an alternative approximation is available (suggested by Professor Lamb in (Ref. 5)). This depends on the assumption that the velocity $V + u$ is constant across the space between the maximum ordinate of the body and the tunnel walls so that u_1 may be determined from the condition of continuity by the relations

$$hV = (h - 2t)\,(u + V)$$
$$u_1/V = u/(u_0 + V)$$

The value of u_1/V may then be determined by comparison with the values of $(V + u_0)$ V for infinite stream given in R. & M. 1223 (Table 1) (V/V_0). Values of u_1/V obtained in this way for the sections of fineness ratio $9\cdot8$, $18\cdot3$ (and ∞) are shown in Fig. 6 by dotted curves. It appears that the error of this approximation is not greater than 1 per cent. in either case provided that $2l$ is not less than h. Hence by combining the two approximations it should be possible to estimate u_1/V within $\frac{1}{2}$ per cent. for all cases in which $2t/h$ is not greater than $0\cdot2$.

7. *Comparison with experimental results.*—The results of experiments on various symmetrical shapes in a model channel of variable width are given in R. & M. 1223 (Table 3). The results are there analysed on the assumption that the drag of the body is increased by channel interference in the ratio $n : 1$ where

$$n - 1 = K_1\,(m_1^{\,2} - 1) \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (1)$$

so that $K_1 = 1$ gives an increase of drag corresponding to the theoretical increase of velocity. The same results have been reanalysed by the methods of the present report ; the values of n given in Table 3 of R. & M. 1223 were plotted against $(2t/h)^2$. With the exception of one or two points at large values of t/h they appear to lie on straight lines through the point ($n = 1$, $t/h = 0$) within the limits of accuracy of the observations, so that it is possible to write

$$n = 1 + q\,(2t/h)^2$$

If equation (1) is replaced by the approximation

$$n - 1 = 2\,K_1\,(m_1 - 1)$$

(since $(m_1 - 1)$ and $(n - 1)$ are small quantities), it follows from equation (1) of section 4, that

$$q = 0{\cdot}822\ \lambda \times 2\ K_1$$

The values of λ from Fig. 4 appropriate to the actual shapes tested :—Joukowski section and ellipse as well as Rankine oval and circle, have been used to determine the values of K_1 given in Table 1, and the differences between these values and those quoted from R. & M. 1223 (Table 4) are chiefly due to the theoretical difference between Joukowski section and Rankine oval. The effect on the curve of K_1 against k_D shown in Fig. 7, is mainly to raise the values for small k_D. This is due to the fact that e.g. the theoretical value of λ for the Joukowski section of largest fineness ratio is only $0{\cdot}71$ of the value for a Rankine oval of the same fineness ratio as used in R. & M. 1223. The modified points suggest that the value of K_1 may tend to unity as k_D tends to zero which would be expected on theoretical grounds since the excess of the value of K_1 above unity is due to the increase of the effective volume of the section by the presence of a turbulent wake whose volume should tend to zero with k_D. More probably the factor K_1 approximates to unity when the form drag vanishes as is suggested in drawing the curve.

8. *Three dimensions.*—The method of calculating the interference velocity for a body of revolution on the axis of a square or rectangular tunnel is very similar to the method already described for the case of two-dimensional flow. The image of a three-dimensional point source in an infinite rigid plane is an equal source symmetrically placed. It follows that a single source on the axis of a square or rectangular tunnel is equivalent to a doubly infinite system of images at the corners of rectangles equal to the cross-section of the tunnel.

Again, the velocity $V + u$ at a point at a large distance R (measured normal to the axis) from a body of revolution is given to a first approximation by the equation

$$V + u = V + Q/R^3 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where Q is a constant. Exactly the same form applies if the arbitrary body is replaced by a source and equal sink (doublet), the constant Q depending on the moment of the doublet.

It follows that the additional velocity at the origin due to the images in the absence of the original doublet is given by

$$u_1 = Q\,\Sigma 1/R^3$$

where the summation is taken for values of R equal to the distances of all the images from the origin.

The summation of this double series is discussed in the Appendix. For a square tunnel of unit side the numerical value of the sum is 9·04 ; for a Duplex tunnel of sides 1·0 and 2·0 the sum is 4·05. It follows that for a square tunnel of side h, the value of u_1 is given by

$$u_1 = 9{\cdot}04\,Q/h^3$$

and for a rectangular tunnel of any shape we shall write

$$u_1 = \sigma\,Q/h^3 \quad \ldots \quad (2)$$

By an argument precisely similar to that given for the case of two dimensions it follows that the drag and pressures on any body of revolution in a tunnel stream of velocity V are equal to this order of approximation to the corresponding values in an infinite stream of velocity $V + u_1$, where u_1 is given by the last equation.

9. *Determination of the tunnel interference for particular shapes.*—As in the two dimensional case the value of Q may be determined for any shape for which the flow in an infinite stream is known. The only two simple cases are (1) The Rankine Ovoid and (2) The Spheroid.

(1) *The Rankine Ovoid.*—This name may be given to the three-dimensional analogue of the Rankine Oval in two dimensions, being the surface of revolution which is equivalent in its external effect to a point source and equal sink of strength A at a distance $2c$ apart in a uniform stream of velocity V. The appropriate formulæ are given by Lamb in (Ref. 5).*

The axial component velocity at a point whose cylindrical co-ordinates relative to the body are x, r, is given by

$$V + u = V + \frac{A}{4\pi}\left\{\frac{x+c}{R_2^3} - \frac{x-c}{R_1^3}\right\} \quad \ldots \quad (1)$$

where

$$R_1^2 = (x-c)^2 + r^2, \quad R_2^2 = (x+c)^2 + r^2.$$

For $x = o$ we have

$$u = c\,A/2\pi R^3$$
$$\text{where } R^2 = c^2 + r^2 \quad \ldots \quad (2).$$

* The sign convention is opposite to that used in (Ref. 5) and in Lamb's Hydrodynamics.

The corresponding forms for the velocity potential ϕ*, and Stokes' stream function ψ are

$$\phi = Vx + (A/4\pi)(1/R_1 - 1/R_2) \quad \ldots \quad \ldots \quad \ldots \quad (3),$$

$$\psi = \tfrac{1}{2}Vr^2 + (A/4\pi)\left\{(x-c)/R_1 - (x+c)/R_2\right\} \quad \ldots \quad (4),$$

where

$$\left.\begin{aligned} V + u &= \frac{\partial\phi}{\partial x} = \frac{1}{r}\frac{\partial\psi}{\partial r}, \\ v &= \frac{\partial\phi}{\partial r} = -\frac{1}{r}\frac{\partial\psi}{\partial x}. \end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

If the maximum radius of the equivalent surface is t, and the maximum length $2l$, the value of t is determined by the condition that $\psi = 0$ for $x = o$, $r = t$, giving

$$0 = \tfrac{1}{2}Vt^2 - Ac/2\pi(c^2 + t^2)^{\frac{1}{2}}$$

or

$$A/\pi V c^2 = (t^2/c^2)(1 + t^2/c^2)^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6).$$

The value of l is determined by the condition that $u = -V$ for $r = o$, $x = l$ leading to

$$(l^2/c^2 - 1)^2 = (l\,t^2/c^3)(1 + t^2/c^2)^{\frac{1}{2}} \quad \ldots \quad \ldots \quad \ldots \quad (7).$$

This equation is a biquadratic for l/c which can be solved fairly rapidly by successive approximation, when c/t is not too small, so as to give l/t in terms of t/c. The corresponding values of the interference velocity u_1 can then be derived from equations (2) and (6) with equations (1) and (2) of §8 in the form

$$u_1/V = (\sigma t^3/h^3)\,\tfrac{1}{2}(1 + c^2/t^2)^{\frac{1}{2}}$$

It is convenient to write in general

$$u_1/V = \tfrac{1}{2}\sigma\lambda t^3/h^3$$

where λ is a function of shape which is unity for sphere.

(2) *Spheroid.*—The expressions for the velocity potential and stream function for a prolate spheroid are given in Lamb's Hydrodynamics, section 105. The corresponding formula for the axial component velocity is given in R. & M. 1239 (Ref. 7) in the form

$$u/V = C\left\{\log\coth\eta/2 - \cosh\eta/(\sinh^2\eta + \sin^2\theta)\right\}$$

where

$$C = 1/\left\{\cosh\eta_0/\sinh^2\eta_0 - \log\coth\eta_0/2\right\}$$

and $x = k\cos\theta\cosh\eta$, $r = k\sin\theta\sinh\eta$ are the axial and radial co-ordinates of any point; $l = k\cosh\eta_0$, $t = k\sinh\eta_0$, being the semi-axes of the spheroid. For a point at a large distance normal to the

* The sign convention is opposite to that used in Lamb's Hydrodynamics.

axis, $\theta = \pi/2$, and u/V may be expanded in powers of $e^{-\eta}$* the first term of the expansion being :—

$$u/V = \frac{8}{3}\, Ce^{-3\eta}$$

To the same approximation, $r = \frac{1}{2}\, k\, e^{\eta}$ and so

$$u/V = \frac{1}{3}\, C\, k^3/r^3$$

$$= \frac{1}{3}\, C \operatorname{cosech}^3 \eta_0\, t^3/r^3$$

and in the notation of the last section

$$\lambda = \frac{2}{3}\, C \operatorname{cosech}^3 \eta_0$$

η_0 being given by

$$\tanh \eta_0 = t/l.$$

Finally, it may be noted that (as in §4) the three dimensional curves of Fig. 4 give a first approximation to the flow at a large distance from the body in free air, where the effect of the body approximates to that of a sphere of radius $t\lambda^{\frac{1}{3}}$.

10. *Closed tunnel of circular cross-section.*—An exact solution for the flow in a closed tunnel of circular cross-section past a particular body of revolution was given by Lamb in R. & M. 1010 (Ref. 5) in the form of an infinite series of Bessel functions. The body is that generated by a point source and equal sink which has been described as a Rankine Ovoid ; as in the two-dimensional case the shape of the body is slightly modified by the presence of the tunnel walls. To conform to the sign conventions of the present paper, the sign of ϕ and ψ must be changed and V, $V + u$, written in place of $-U$, u, respectively.

The formulæ determine the axial component velocity just outside the largest cross-section of the body, and this may be compared with the corresponding value $V + u_0$ in the absence of the tunnel walls. From analogy with the solution for a square tunnel, it is to be expected that for sufficiently small values of the ratio t/a (a is the radius of the tunnel) the value of $(u - u_0)/(u_0 + V)$ can be expressed as an expansion in ascending powers of t/a of which the first term is of the form Bt^3/a^3, where B is a constant. To a first approximation the flow round the body would be identical with that in a free stream of velocity $V + u_1$ where

$$u_1/V = (u - u_0)/(u_0 + V).$$

* It is convenient to expand in powers of $\operatorname{cosech}^2 \eta/2$.

Again the value of B would vary with the fineness ratio in the same proportion as that already calculated for a Rankine Ovoid in a square tunnel, so that it should be possible to write as in Section 12 below

$$u_1/V = \tau \lambda (s/S)^{3/2}$$

where s/S is the ratio of maximum cross-sectional area of body to that of tunnel, λ is the same function of shape of body as in §9, and τ is an absolute constant for a circular tunnel which probably differs from the value for a square tunnel.

It has not been found possible to establish these results analytically or to obtain an analytical formula for the constant τ. The following method has therefore been adopted. Taking the particular fineness ratio corresponding to $t/c = 1/5$, the calculations given by Lamb in R. & M. 1010 (Ref. 5) were extended to additional values of t/a. The results are given in Table 2 and plotted in Fig. 8 in the form $(a^3/t^3) \{(u - u_0)/(u_0+V)\}$ against t/a. The chief arithmetical and analytical difficulty arises from the fact that the series for u/V converges more and more slowly the smaller the value of t/a. This limits the smallness of the value of t/a for which calculations can usefully be made, but the points plotted in Fig. 8 are sufficient to verify that the ordinate in Fig. 8 approaches a constant value as t/a tends to zero and to establish the magnitude of the limit as 4·03 with an error not greater than 1 per cent. The value of λ for this fineness ratio is given by Fig. 4 as 5·01, and since s/S the ratio of areas is equal to t^2/a^2, the value of τ may be calculated as 0·804, from the equation $\lambda \tau (s/S)^{3/2} = 4{\cdot}03\ (t/a)^3$.

It appears from Fig. 8 that the error of the first approximation increases with increase of t/a rather more rapidly than in the corresponding two-dimensional case, but the error when the length becomes equal to the diameter of the tunnel is still slightly less than 1 per cent. on the body drag (i.e. on V^2). When the length of the body is appreciably greater than the tunnel diameter the approximation described in R. & M. 1010 (Ref. 5) and in §6 above is available, and is shown by the dotted curve. It appears that an estimate of the tunnel correction could be made with an error less than 1 per cent. on the drag, from a knowledge of the dotted curve and of the limit as t/a tends to zero.

11. *Open Jet Tunnel.*—A solution for the case of a free jet of circular cross-section can be obtained by a method analogous to that given by Lamb for a closed jet. The boundary condition in this case is that the pressure, and so the resultant velocity, are constant over the surface of the jet. As explained in R. & M. 723,* this condition is equivalent, to a first approximation, to the condition that the axial component velocity is constant and equal to V over

* R. & M. 723. Aerofoil Theory by H. Glauert.

the whole of a cylindrical surface of radius a coinciding with the undisturbed surface of the jet. It is therefore necessary to find the solution for a source (and equal sink) such that the axial component velocity is zero on this surface. Trying as before a solution as a series of terms of the form

$$\phi = \sum_{1}^{\infty} a_n J_o(k_n r) e^{-k_n x} \quad .. \quad .. \quad .. \quad .. \quad (1)$$

(for $x > o$) the required boundary condition is now

$$u \equiv \frac{\partial \phi}{\partial x} = o$$

for $r = a$, and therefore the values of k_n must now be taken as roots of the equation

$$J_o(k_n a) = o \quad .. \quad .. \quad .. \quad (2).$$

The remaining conditions for a single source at the origin are the same as before, viz. :—(for $x = o$)

$$u \equiv \frac{\partial \phi}{\partial x} = o$$

for all but infinitesimal values of r, and that the total flow outwards from the origin is equal to the strength $\underset{\sim}{Q}$ of the source, so that the limit as $x \to o$ of

$$\int_o^a u \,.\, 2\pi r dr = \pm \tfrac{1}{2} \underset{\sim}{Q} \quad .. \quad .. \quad .. \quad (3)$$

according as $x \gtrless o$.

For $x > o$, the value of u given by (1) tends to the limiting form

$$u = -\sum_{1}^{\infty} k_n a_n J_o(k_n r).$$

Multiplying both sides by $J_o(k_m r) \,.\, r\, dr$ and integrating from o to a we have

$$-k_m a_m \int_o^a J_o^2(k_m r) \,.\, r\, dr = Lt_{x \to o} \int_o^a J_o(k_m r)\, u \,.\, r\, dr$$

or since u is zero except for infinitesimal values of r,

$$-\tfrac{1}{2} a^2 k_m a_m J_1^2(k_m a) = J_o(o) \,.\, Lt_{x \to o} \int_o^a u \,.\, r\, dr = \underset{\sim}{Q}/4\pi$$

Hence (1) becomes (for $x > o$)

$$\phi = -\frac{\underset{\sim}{Q}}{2\pi a^2} \sum_{1}^{\infty} \frac{J_o(k_n r) e^{-k_n x}}{k_n J_1^2(k_n a)} \quad .. \quad .. \quad .. \quad (4)$$

The corresponding value of the stream function ψ satisfying the relations (5) of §9 is

$$\left.\begin{aligned}(x>o),\ \psi &= \frac{Q}{2\pi a^2}\Sigma\frac{r\,J_1(k_n r)\,e^{-k_n x}}{k_n J_1^2(k_n a)}\\ (x<o),\ \Psi &= -\frac{Q}{2\pi a^2}\left\{\Sigma\frac{r\,J_1(k_n r)\,e^{k_n x}}{k_n J_1^2(k_n a)} - a^2\right\}\end{aligned}\right\} \quad \ldots \quad (5).$$

The term a^2 is inserted so as to make $\psi \to o$ as $x \to +\infty$, $\psi \to Q/2\pi$ as $x \to -\infty$.

For a uniform stream V with a source Q at $(-c, o)$ and a sink $-Q$ at (c, o) and for $-c<x<+c$,

$$\psi = \tfrac{1}{2}\,V\,r^2 + \frac{Q}{2\pi a^2}\left\{2\Sigma\frac{rJ_1(k_n r)\,e^{-k_n c}\cosh k_n x}{k_n J_1^2(k_n a)} - a^2\right\} \quad (6).$$

If t is the radius of the streamline surface corresponding to $\psi = o$ at $x = o$ we have putting $r = t$, $x = o$ in (6)

$$\frac{\pi a^2 V}{Q} = \frac{a^2}{t^2} - 2\sum_1^{\infty}\frac{J_1(k_n t)\,e^{-k_n c}}{k_n t\,J_1^2(k_n a)} \quad \ldots \quad \ldots \quad \ldots \quad (7),$$

and the axial component velocity at the corresponding point is given by

$$\frac{V+u}{V} = 1 + \frac{Q}{\pi a^2 V}\sum_1^{\infty}\frac{J_0(k_n t)\,e^{-k_n c}}{J_1^2(k_n a)} \quad \ldots \quad \ldots \quad (8).$$

Values of the velocity u_0 at the corresponding point when the body is in a free stream are given by R. & M. 1010 (Ref. 5) equation (4),

$$\frac{V+u_o}{V} = 1 + \frac{t^2}{2(t^2+c^2)}.$$

It appears from the formulæ that the equivalent free air speed in an open jet tunnel is lower than the tunnel speed, so that the correction is of the opposite sign to that in a closed jet. Values of $(a^3/t^3)\ \{(u_0-u)/(u_0+V)\}$ were calculated as before for the case of a body of fineness ratio 5·50 for which $t/c = 1/5$ and are given in Table 2 and plotted in Fig. 8 against a/t. The limiting value of the ordinate as determined by extrapolation has the value 1·03 and so the interference is almost exactly 1/4 of the corresponding interference in a closed jet, but of opposite sign. As the size of the body is increased the ratio becomes smaller still. The error of the first approximation when the length of the body is equal to the tunnel diameter though forming a greater proportion of the total correction than in the case of the closed jet is, as before, less than 1 per cent. on the body drag.

An approximation similar to that obtained for the closed jet for a body whose length is greater than the tunnel diameter is given by assuming that the velocity outside the maximum section of the body is equal to the tunnel velocity. The approximation is shown by the dotted curve.

Open Jet Square Tunnel.—As a check on the result for an open jet circular tunnel, the case of an open jet square tunnel has been worked out by the method of images, although this case is of no practical importance. The only difference as compared with the closed tunnel is that the sign of every alternate image is changed. The resulting calculated value of σ is $-2 \cdot 65$ as compared with $9 \cdot 04$ for a closed tunnel. The ratio $-0 \cdot 29$ (open jet to closed jet) is not very different from the value $-0 \cdot 25$ for the case of a circular section.

12. *Summary of Results.*—It is convenient to summarise the results obtained for tunnel interference.

The first approximation for all cases may be expressed in the following forms. If $u_1 + V$ is the free air speed corresponding to a tunnel speed V, then :—

In two dimensions

$$u_1/V = \tau\lambda\,(s/S)^2.$$

In three dimensions

$$u_1/V = \tau\lambda\,(s/S)^{3/2}.$$

Here s/S is the ratio of maximum cross-sectional area of body to area of tunnel. λ is a coefficient of shape independent of the shape of cross section of tunnel. All calculated values of λ for both two and three dimensions are collected in Fig. 4. τ is a constant depending only on the shape of cross-section of the tunnel. Values of τ are collected in the following table :—

Table of values of τ.

	Closed Jet.	*Open Jet.*
Two dimensions	0·82	—0·62
Three dimensions, Circular	0·80	—0·20
,, ,, Square	0·81	—0·24
,, ,, Duplex	1·03	

To illustrate the method the following two examples have been worked out :—

Examples.

(1) Body of revolution : diameter 23 inches in a 7 ft. square closed tunnel : fineness ratio $l/t = 5 \cdot 45$. (Shape given in R. & M. 1030.)

The shape approximates more closely to a Rankine Ovoid than to a spheroid, hence from Fig. 4, $\lambda = 5 \cdot 0$. Also from the above data, $\tau = 0 \cdot 812$, $(s/S)^{3/2} = 0 \cdot 0146$. Again the fineness ratio is approximately that of Fig. 8, and so the first approximation to (u_1/V) corresponding to $t/a = o$ in Fig. 8 may be corrected by multiplying by the ratio $(3 \cdot 2/4 \cdot 0)$ of the ordinates for $t/a = 0 \cdot 24$ and $t/a = o$. Hence, finally

$$u_1/V = \frac{3 \cdot 2}{4 \cdot 0} \cdot \tau\lambda \, (s/S)^{3/2}$$

$$= 0 \cdot 0473$$

and

$$\left(\frac{u_1 + V}{V}\right)^2 = 1 \cdot 098$$

giving a correction of 10 per cent. on the drag coefficient.

(2) Body of same diameter in same tunnel but of fineness ratio $3 \cdot 0$ (shape given in R. & M. 1271 (Ref. 2)).

The shape approximates to a spheroid and so

$$\lambda = 2 \cdot 25, \; \tau = 0 \cdot 812, \; (s/S)^{3/2} = 0 \cdot 0146$$

and the first approximation is adequate. Hence

$$u_1/V = 0 \cdot 0267$$

or

$$(u_1 + V)^2/V^2 = 1 \cdot 055.$$

13. *Conclusions*.-- Theoretical formulæ have been obtained which give a first approximation to the equivalent free air speed for a symmetrical body in a wind tunnel. In two dimensions results are given for cylinders whose cross-sections are the ellipse, Rankine Oval and symmetrical Joukowski wing. From these results it should be possible to guess a figure with sufficient accuracy for all ordinary shapes. The results in two dimensions should also assist in the process of guessing a figure in three dimensions where theoretical figures are available for the spheroid and Rankine Ovoid only.

In three dimensions the effect of change of shape is independent of the shape of cross-section of the tunnel. Values of the constant determining the effect of shape of tunnel section are given to a sufficient approximation for practical purposes for open jet circular section, closed jet circular section, square and " Duplex " ; it could be evaluated for any rectangular section.

The theoretical accuracy of the approximation in two dimensions is checked by comparison with the exact calculation of Fage for the case of an equal source and sink (Rankine Oval). The comparisons suggest that the accuracy is sufficient provided that the thickness is less than one-fifth the breadth of the tunnel and the chord length less than the breadth of the tunnel. In three dimensions the case of a source and sink in a circular closed jet has been worked out by Professor Lamb; the calculated results at present available are much less complete, but the limits of application of the approximation appear to be about the same as in two dimensions. It may be remarked that when the approximation breaks down it is no longer necessarily true that the same correction will apply even approximately to all parts of the body; but in the case of an elongated body appreciably longer than the breadth of the tunnel a different approximation is available which may have some practical application.

The only satisfactory experimental data on tunnel interference at present available are those of Mr. Fage in (3) and apply to the two-dimensional case. It appears that the theoretical correction is to be multiplied by an empirical factor which increases from unity for a low drag form to 2·0 or 3·0 for a high drag form. Evidently this represents the ratio of the actual disturbance of the fluid to the theoretical disturbance and may therefore be assumed, as by Mr. Fage, to be a function of the drag coefficient and as such applied to the three-dimensional case in default of further evidence.

The author wishes to record his appreciation of the assistance of Miss D. Yeatman in the calculations.

APPENDIX.

Sum of the double series

$$4 \times \left\{ \frac{1}{(0^2+1^2)^{3/2}} + \frac{1}{(1^2+1^2)^{3/2}} + \frac{1}{(0^2+2^2)^{3/2}} + \frac{2}{(1^2+2^2)^{3/2}} + \ldots \right\}$$

No formula for the sum of this series has been discovered and the series has, therefore, been summed numerically. A practical difficulty arises from the extremely slow convergence of the series, e.g., the error of the sum to 47 terms is $3\frac{1}{2}$ per cent. To get over this difficulty use was made of the following artifice. Take the typical term as $1/R_{mn}^3$ where

$$R_{mn}^2 = m^2 + n^2$$

so that R_{mn} is the radius vector of a point whose Cartesian co ordinates are m and n. Each term of the series may then be considered as occupying a unit square with its centre at the point (m, n). Hence, for sufficiently large values of R_{mn} the sum of all terms of the series for which the points (m, n) lie outside a certain curve approximates to the surface integral of $1/R^3$ taken over the whole area outside this curve. In particular, taking the circle radius R_0, the sum of all terms for which

$$m^2 + n^2 \geq R_0^2$$

approximates to

$$\int_{R_0}^{\infty} \frac{2\pi r \, dr}{r^3} = \frac{2\pi}{R_0}$$

The sum of the series was calculated by adding this value of the remainder to the sum of all terms for which $m^2 + n^2 \leq R_0^2$. Taking in succession values $R_0 = 10$ and $R_0 = 20$, the difference between the results in the two cases was only 1 in 2,000, although the two remainders were $3\frac{1}{2}$ per cent. and 1·75 per cent. respectively.

LIST OF REFERENCES.

1. R. & M. 1241.—Experiments on a series of symmetrical Joukowski sections. Fage and Falkner.
2. R. & M. 1271.—Investigation of the boundary layers and the drags of two streamline bodies. Ower and Hutton.
3. R. & M. 1223.—On the two-dimensional flow past a body of symmetrical cross-section mounted in a channel of finite breadth. Fage.
4. Sir Richard T. Glazebrook. *Trans. Inst. Nav. Arch,* 1909 (p. 155).
5. R. & M. 1010.—On the effect of the walls of an experimental tank on the resistance of a model. H. Lamb.
6. R. & M. 1199.—Skin Friction and the drag of streamline bodies. B. M. Jones.
7. R. & M. 1239.—The application of the theoretical velocity field round a spheroid to calculate the performance of an airscrew near the nose of a streamline body. Lock.
8. R. & M. 911.—A generalised type of Joukowski aerofoil. Glauert.

TABLE 1.

Analysis of observations recorded in Table 3 of R. & M. 1223.

Section.	l/t	λ	K_1	k_D	K_1 R. & M. 1223.
1. Joukowski ..	9·6	4·53	1·15	0·00444	0·87
2. Joukowski ..	4·8	2·50	1·82	0·00750	1·29
3. Joukowski ..	3·05	1·77	1·83	0·0106	1·81
4. Rankine Oval ..	10·3	6·76	1·67	0·0108	1·78
5. Ellipse	5·4	3·20	1·58	0·0184	1·55
6. Circle	1·0	1·00	3·13	0·620	2·33

TABLE 2.

Rankine Ovoid in a tunnel of circular cross-section. $c/t = 5·0$, $l/t = 5·503$, $V/(u_0 + V) = 0·98113$.

t/a	Closed Jet. $(u-u_0)/(V+u_0)$	Open Jet. $(u-u_0)/(V+u_0)$
0·06	0·00086	—0·00021
0·08	0·00196	—0·00053
0·10	0·00383	—0·00094
0·12	0·00651	—0·00153
0·20	0·0271*	—0·00442
0·30		—0·01069
0·40	0·168*	—0·01511

* Taken from R. & M. 1010.

R. & M. 1275.

Figs. 1–3.

TUNNEL INTERFERENCE.

Fig. 1.

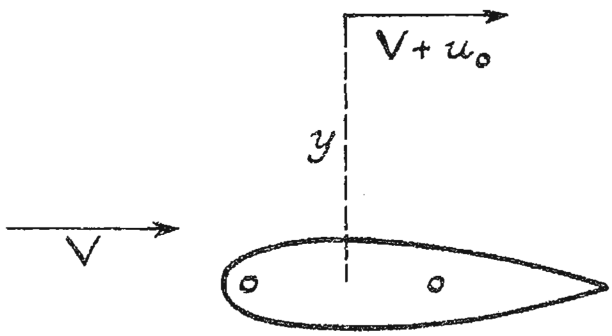


Fig. 2.

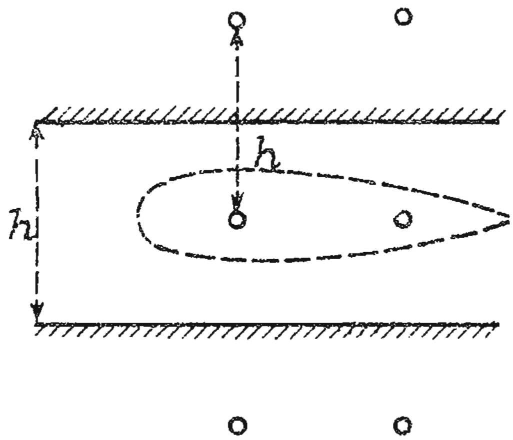
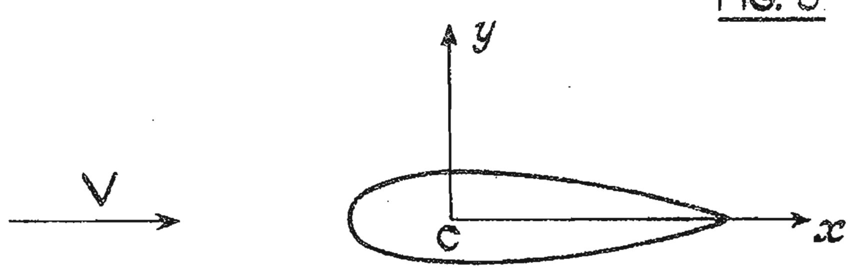


Fig. 3.



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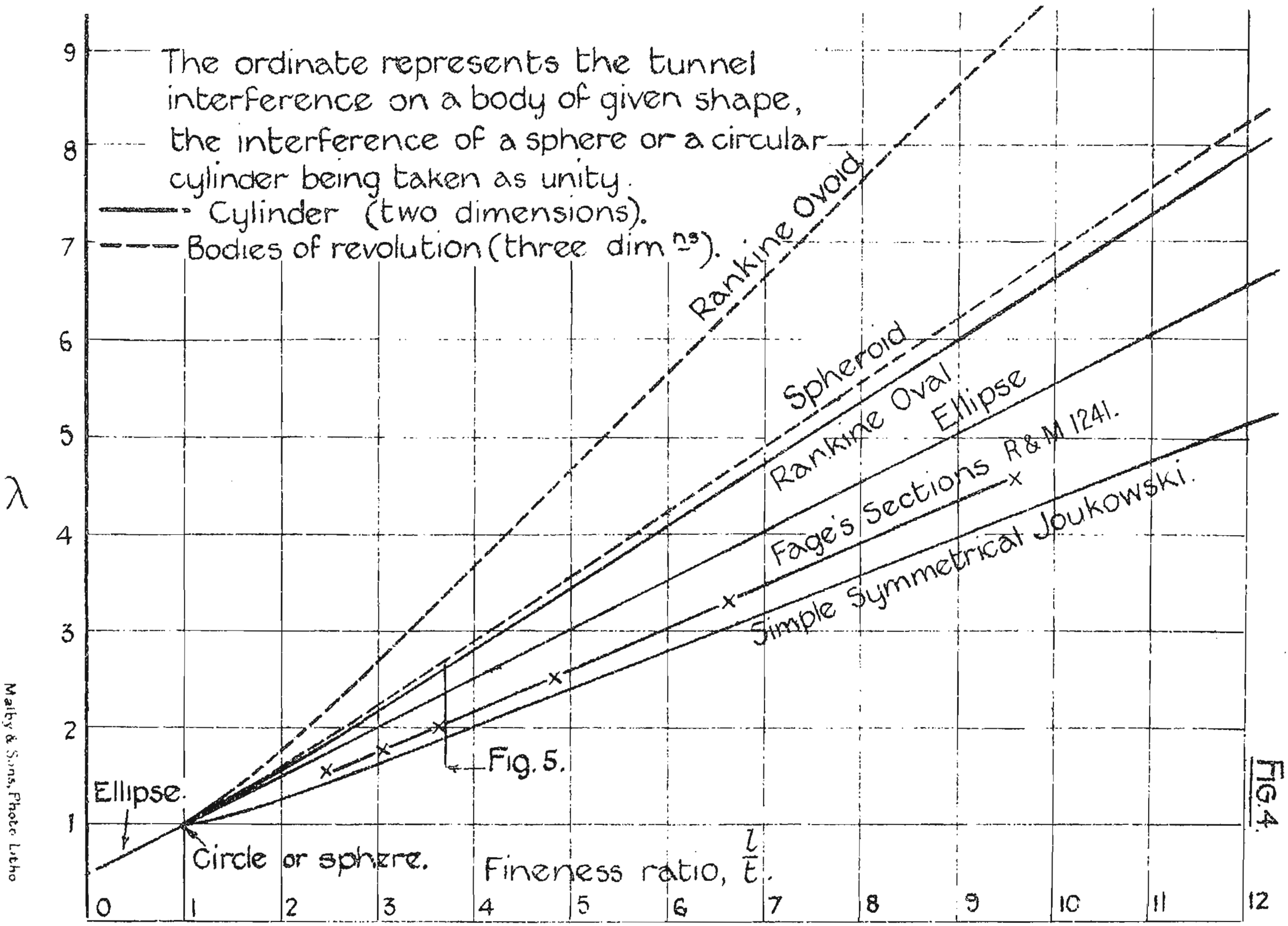
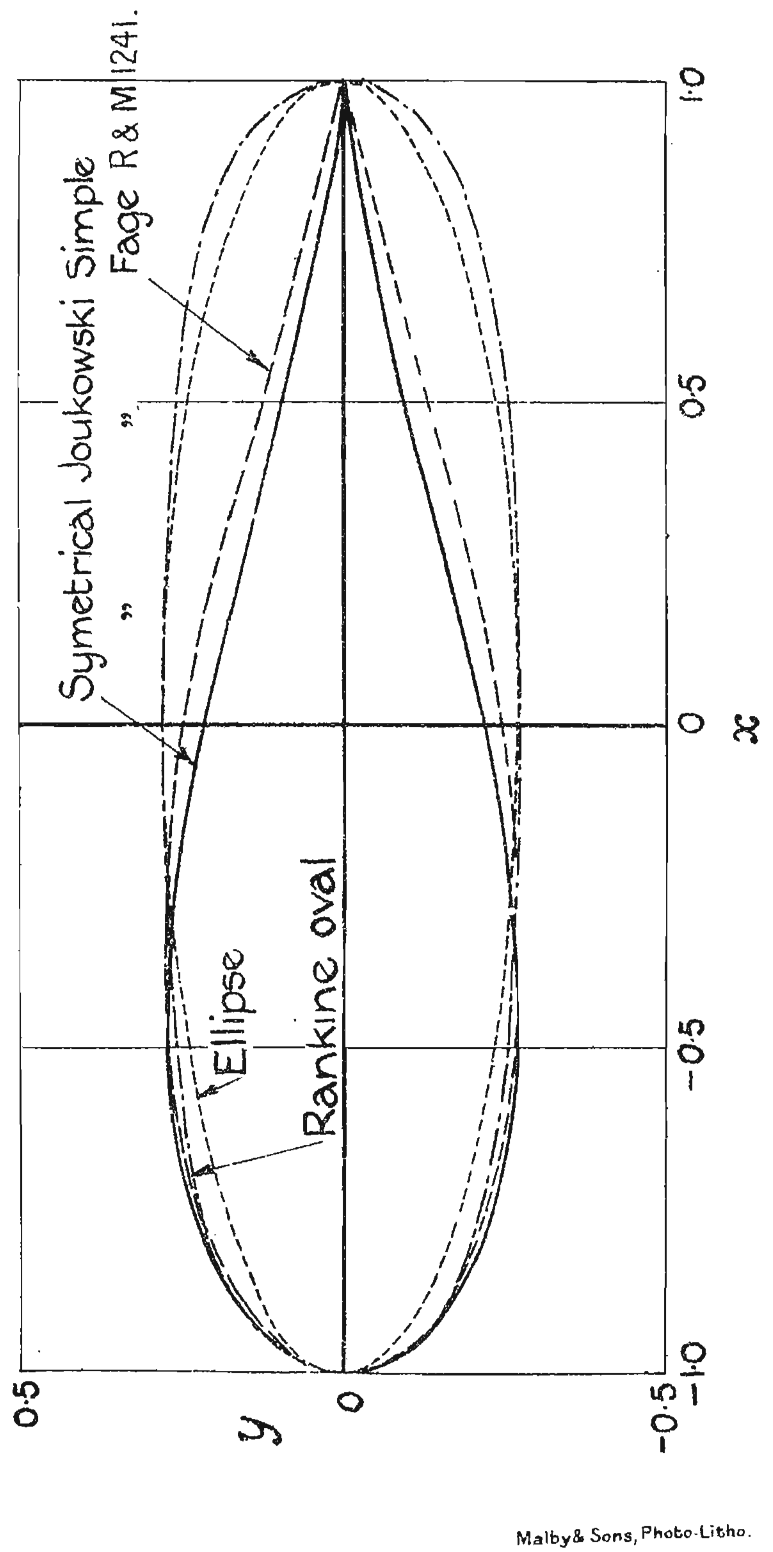


The ordinate represents the tunnel interference on a body of given shape, the interference of a sphere or a circular cylinder being taken as unity.
Cylinder (two dimensions).
Bodies of revolution (three dimns).
Rankine Ovoid
Spheroid
Rankine Oval
Ellipse
R&M 1241.
Fage's Sections
Simple Symmetrical Joukowski.
Fig. 5.
Ellipse.
Circle or sphere.
Fineness ratio, $\frac{l}{t}$.
λ
0
1
2
3
4
5
6
7
8
9
10
11
12

FIG. 4.

FIG. 5.

Four theoretical shapes all of fineness ratio 3·67 to 1.



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FIGS. 6 & 7.

FIG. 6.

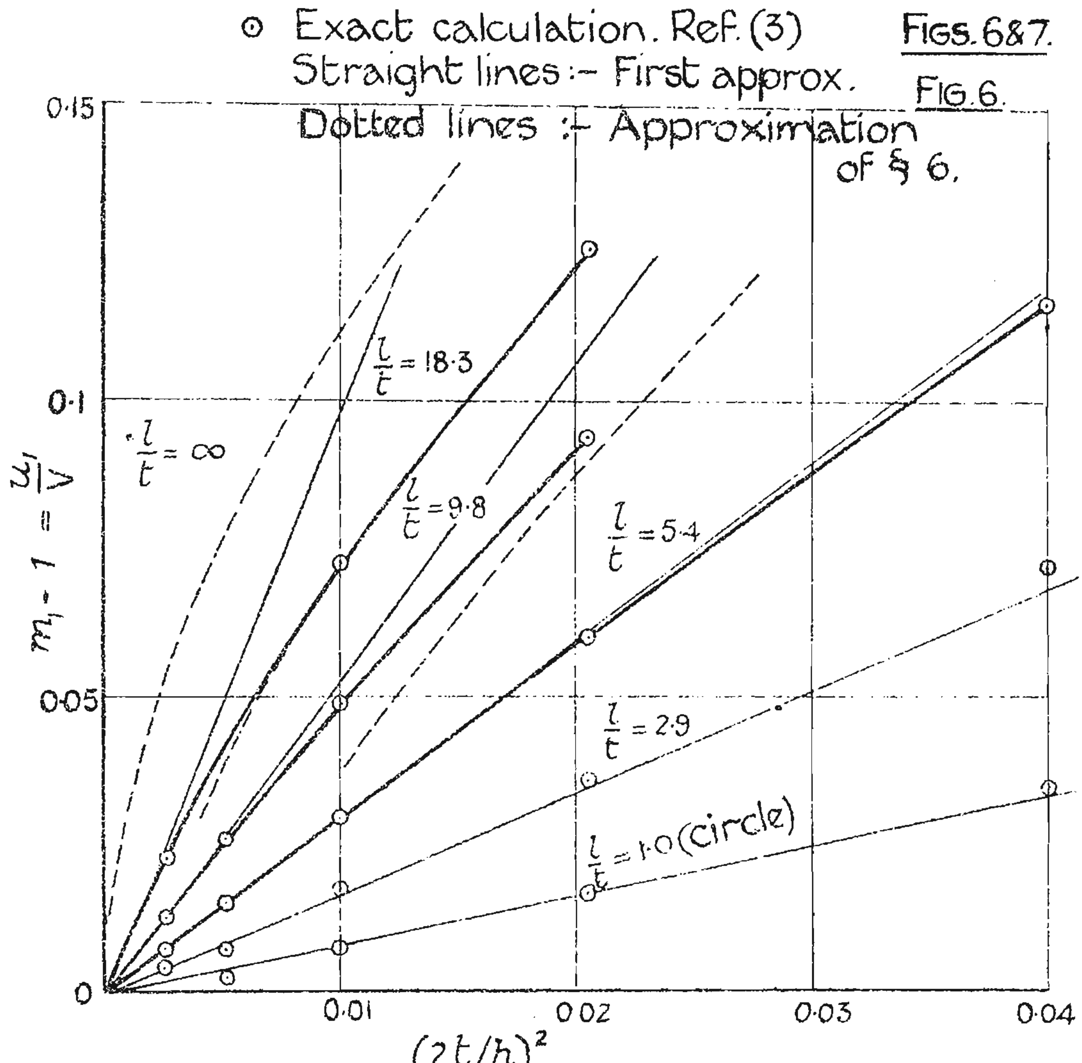
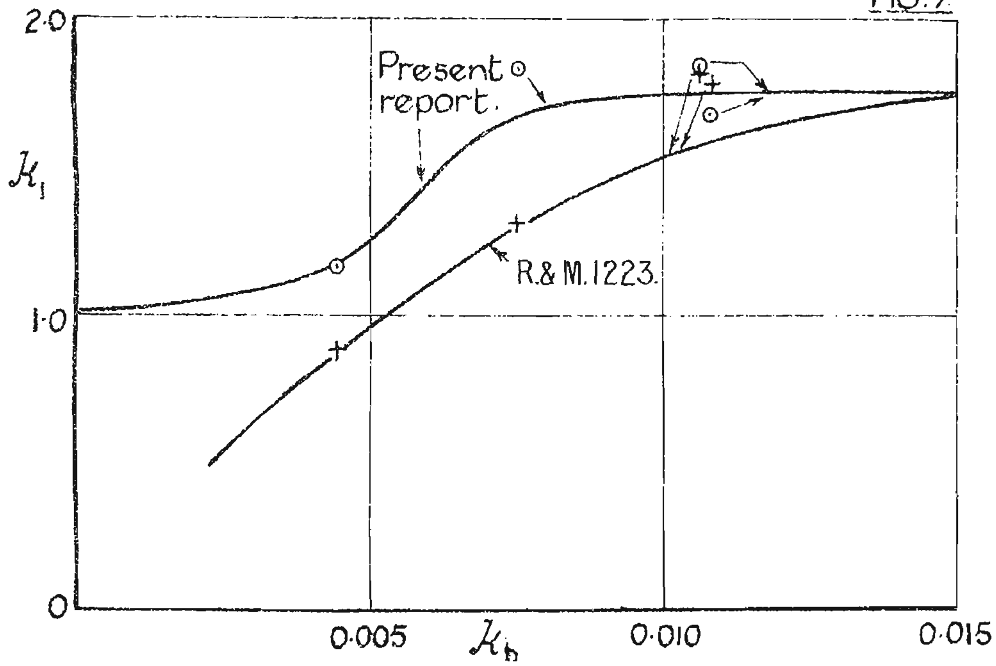


FIG. 7



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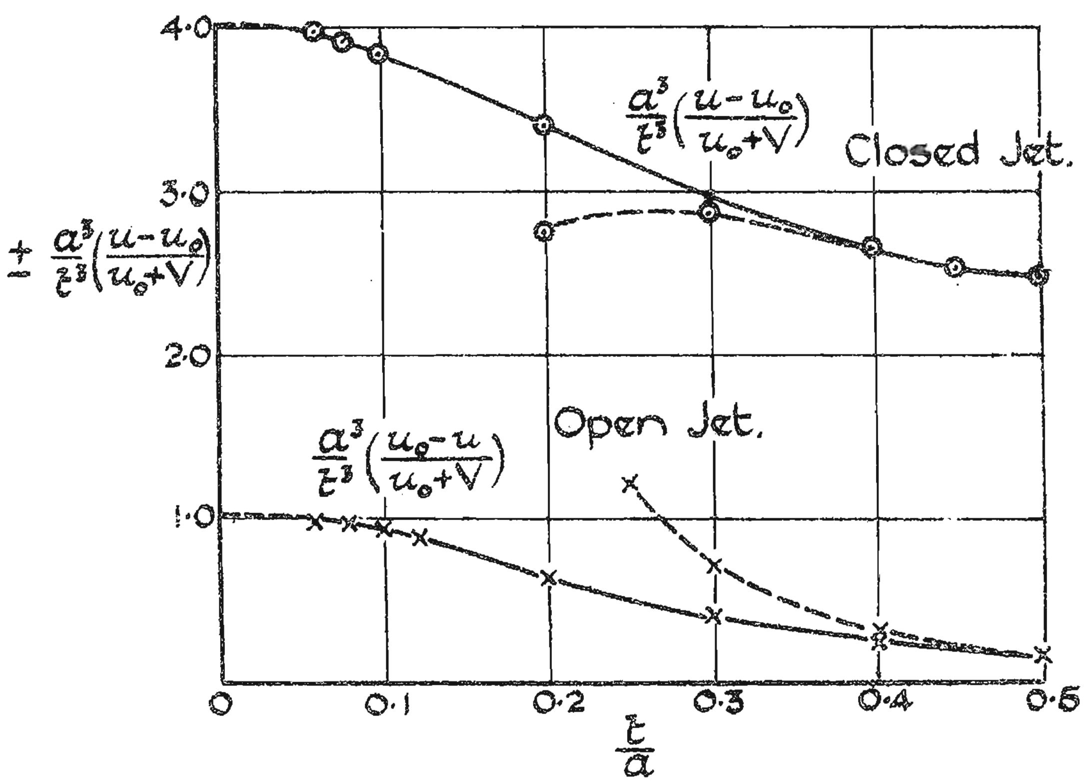
R. & M. 1275.

Fig. 8.

Circular Jet Tunnel.

Body of Fineness Ratio 5·503



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