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Kinematic Rectification in Damped Single-Axis Gyros

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Summary.

In a single-axis gyro a rotation about the input axis forces the spin axis to rotate about the output axis. The true input axis is oscillating about its nominal position and the gyro sees a varying component of the angular velocity about the nominal spin axis. This may be rectified, thereby causing a drift, if the components of angular velocity along the input and spin axes are correlated. The drift rate is evaluated in terms of the components of angular velocity suffered by the gyro.

LIST OF CONTENTS

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1. *Introduction.*

This report deals with the drift caused in damped single-axis gyros by rectification of angular vibrations.

The function of a gyro is to preserve a reference frame fixed in space. With a two-axis (or 'free') gyro, the spin axis can, in principle, remain fixed in space, any motion of the platform being taken up by the gimbal system. On the other hand with a single-axis gyro a rotation about the input axis, which is the direction orthogonal to the output and spin axes, forces the spin axis to rotate about the output axis (*see* Fig. 1). In response to angular vibrations, the spin and input axes are wobbling about their nominal positions. Thus the gyro 'sees' a varying component of the angular velocity about its nominal spin axis, and under certain conditions this varying component may be rectified.

Arutyunov[1] has given a similar calculation of the drift rate in a damped single-axis gyro for sinusoidal inputs. Cannon[2] has dealt with kinematic rectification in undamped single-axis gyros, and has given quantitative experimental verification of the effect for this case using abnormally high angular vibrations. A somewhat similar phenomenon in two-axis gyros, due to inertia effects when the gimbals are not orthogonal, has been discussed by Plymale and Goodstein[3].

2. *Equation of Motion for a Damped Single-Axis Gyro.*

2.1. *The Fluid-Floated Gyro.*

The treatment to be given applies particularly to the 'fluid-floated' type of gyro (*see* Fig. 1). Basically this consists of a rotor mounted with its spin axis across a cylindrical float chamber, which in turn is enclosed in a cylindrical outer case. The float is pivoted to the case about their common axis, and the small radial gap between them, as well as the larger spaces at the ends, is filled with a fluid of fairly high density and viscosity. The viscous drag of the fluid in the gap supplies the torque resisting motion of the float relative to the case. Also the float chamber is designed to have the same mean density as the fluid, so that the gravitation and acceleration forces on the float are small. This allows a fine suspension to be used which locates the axis of the float very precisely and gives only very small frictional torques.

2.2. *Choice of Axes, etc.*

In operation the gyro outer case is clamped to the platform to be stabilised. The true output axis and also the *nominal* positions of the spin and input axes, which are determined by the null position of the output pick-off, are fixed in the gyro outer case. These give a convenient basic frame of reference fixed relative to the platform, which we will take as the axes 0123 (Fig. 1) coinciding with the nominal input, output and spin axes respectively. They are moving axes to the extent that the platform moves.

We will suppose the platform rotates through the small angles θ_1 about 01, θ_2 about 02, and θ_3 about 03. This representation is permissible provided the angles are small, which is true here, since only then do rotations about different axes commute. We will also suppose the float rotates relative to the case through the angle ϕ about the output axis, and define a further set of moving axes 01′2′3′ which are fixed in the float and coincide with the *true* input, output and spin axes respectively (02′ coincides with 02).

The angular velocity of the frame 0123 has components $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ along 0123. Thus the angular velocity of the frame 01′2′3′ (fixed to the float) is $\boldsymbol{\omega}$ with components ω_1, ω_2, ω_3 along 01′2′3′, where

$$\left.\begin{aligned} \omega_1 &= \dot{\theta}_1 \cos\phi - \dot{\theta}_3 \sin\phi \\ \omega_2 &= \dot{\theta}_2 + \dot{\phi} \\ \omega_3 &= \dot{\theta}_1 \sin\phi + \dot{\theta}_3 \cos\phi . \end{aligned}\right\} \qquad (1)$$

Also the gyro rotor is kept spinning at constant angular velocity Ω with respect to the frame 01′2′3′ by a synchronous motor attached to the float. The axes 01′2′3′ are principal axes of inertia for the rotor and the float, and we will write the moments of inertia about 01′, 02′, 03′ as I_{01}, I_{02}, I_{03} for the rotor and I_{11}, I_{12}, I_{13} for the float. Then the angular momenta $\mathbf{H}_0$ and $\mathbf{H}_1$ of the rotor and float respectively are

$$\mathbf{H}_0 = \{I_{01}\omega_1,\ I_{02}\omega_2,\ I_{03}(\Omega+\omega_3)\} \qquad (2a)$$

$$\mathbf{H}_1 = \{I_{11}\omega_1,\ I_{12}\omega_2,\ I_{13}\omega_3\} \qquad (2b)$$

where the components are given in the moving axes 01′2′3′.

2.3. *Dynamical Equations.*

We will write the reaction torque on the rotor from the float as $\mathbf{R}_0$ with components R_{01}, R_{02}, R_{03} along 01′2′3′. The component R_{03} is such as to keep the rotor running at constant speed and is the resultant of the torque supplied by the synchronous motor, and the torques due to bearing friction, drag due to residual gas in the float chamber, etc. The other components R_{01}, R_{02} are due to lateral reactions at the pivots. In addition there is an external torque $\mathbf{T}_0$ on the rotor due to its mass unbalance (and possibly other effects).

Turning to the torques on the float, the reaction from the rotor is $-\mathbf{R}_0$. Similarly the reaction torque on the float from the case is $\mathbf{R}_1$ with components R_{11}, R_{12}, R_{13} in 01′2′3′. Here R_{11} and R_{13} are only lateral reactions at the pivots and the important component is R_{12}. The main part of R_{12} is due to fluid drag and we will assume this obeys a linear law so that we may write

$$R_{12} = -L\dot{\phi} + \text{unwanted parts} \qquad (3)$$

where L is a (positive) constant. The unwanted parts conceal a multitude of sins including torque due to the motor leads, friction at the pivots and various other possible imperfections, e.g. bubbles in the fluid, bending of the float and inertia effects in the fluid, etc. In addition there is the external torque $\mathbf{T}_1$ due to mass unbalance of the float.

Using the theorem of moving axes, the equations of motion of the rotor and the float are

$$\mathbf{R}_0 + \mathbf{T}_0 = \frac{d}{dt}\mathbf{H}_0 = \frac{\partial}{\partial t}\mathbf{H}_0 + \boldsymbol{\omega} \wedge \mathbf{H}_0 \qquad (4a)$$

$$\mathbf{R}_1 - \mathbf{R}_0 + \mathbf{T}_1 = \frac{d}{dt}\mathbf{H}_1 = \frac{\partial}{\partial t}\mathbf{H}_1 + \boldsymbol{\omega} \wedge \mathbf{H}_1 . \qquad (4b)$$

When resolved into components along 01′2′3′, these equations give

$$\left.\begin{aligned} R_{01} + T_{01} &= I_{01}\dot{\omega}_1 - (I_{02} - I_{03})\omega_2\omega_3 + I_{03}\Omega\omega_2 \\ R_{02} + T_{02} &= I_{02}\dot{\omega}_2 - (I_{03} - I_{01})\omega_3\omega_1 - I_{03}\Omega\omega_1 \\ R_{03} + T_{03} &= I_{03}\dot{\omega}_3 - (I_{01} - I_{02})\omega_1\omega_2 \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} (R_{11} - R_{01}) + T_{11} &= I_{11}\dot{\omega}_1 - (I_{12} - I_{13})\omega_2\omega_3 \\ (R_{12} - R_{02}) + T_{12} &= I_{12}\dot{\omega}_2 - (I_{13} - I_{11})\omega_3\omega_1 \\ (R_{13} - R_{03}) + T_{13} &= I_{13}\dot{\omega}_3 - (I_{11} - I_{12})\omega_1\omega_2 . \end{aligned}\right\} \tag{6}$$

The equations (6) are the usual Euler equations for a rotating rigid body with respect to axes fixed in the body, and the right-hand sides consist of the angular acceleration terms $I_{1i}\dot{\omega}_i$ and the terms $-(I_{1j} - I_{1k})\omega_j\omega_k$ which couple angular velocities about different axes. The first set of equations (5) contains all these terms and in addition a gyroscopic term in the first and second equations. It will turn out that the kinematic rectification term, which is a part of the gyroscopic term, is of the same form as the cross-coupling terms. Moreover the kinematic rectification term is larger, simply because Ω is larger than $|\boldsymbol{\omega}|$, and we will discard the cross-coupling terms without further ado.

Since $R_{12}(\simeq -L\dot{\phi})$ is the only known internal reaction the only equation of interest is got by adding the second equation of set (5) and the second equation of set (6), viz:

$$R_{12} + (T_{02} + T_{12}) = (I_{02} + I_{12})\dot{\omega}_2 - I_{03}\Omega\omega_1 . \tag{7}$$

The remaining five independent equations only determine the five unknown internal reactions. We can put equation (7) in a neater form by writing:

$I = I_{02} + I_{12}$ = total moment of inertia of the rotor and the float about the output axis.

$H = I_{03}\Omega$ = spin angular momentum of the rotor.

$T = T_{01} + T_{02} + (R_{12} - L\dot{\phi})$ = total unwanted torque on rotor and float about the output axis (due to mass unbalance, pivot friction, ligament torques, etc.).

After substituting for ω_1 and ω_2 from (1), we have the equation

$$I\ddot{\phi} + L\dot{\phi} = N(t) \tag{8}$$

where the forcing function is

$$N(t) = H(\dot{\theta}_1 \cos\phi - \dot{\theta}_3 \sin\phi) + T - I\ddot{\theta}_2 . \tag{9}$$

2.4. *Physical Significance of the Input.*

The whole of the first term of $N(t)$ in (9) describes the purely gyroscopic effects. At any instant the gyro 'sees' the component of angular velocity along the true instantaneous input axis 01′ which is at an angle ϕ to the nominal input axis 01. As far as the rotor is concerned the nominal input axis, which is determined by the pick-off null, is a fiction.

The gyroscopic term splits naturally into two parts, and these and the remaining two terms of $N(t)$, may be classified as follows:

(i) The principal term*

$$H\dot{\theta}_1 \cos \phi \simeq H\dot{\theta}_1$$

which is of first order in small angles and gives the effect of rotation about the nominal input axis. This term by itself would be the ideal input to a single-axis gyro. The remaining terms all represent false inputs and give rise to drifts.

(ii) The kinematic rectification term*

$$-H\dot{\theta}_3 \sin \phi \simeq -H\dot{\theta}_3\phi$$

which is of second order in small angles. Since ϕ is mainly determined by the input $H\dot{\theta}_1$, the two factors in the kinematic rectification term will be correlated if $\dot{\theta}_1$ and $\dot{\theta}_3$ are correlated.

(iii) The torque term T which is the component about the output axis of the total extraneous torque on the rotor and float. The gyro must be designed so that this is small.

(iv) The output-axis term $-I\ddot{\theta}_2$ which gives the effect of rotations about the output axis. When the outer case rotates, the float must be partly carried with it. The float cannot remain at rest since the fluid-drag torque would be unbalanced, nor can it rotate with the case since there would be no torque to start the rotation.

Here we will be concerned only with kinematic rectification, and the torque and output-axis terms will be ignored.

Although the angular velocities $\dot{\theta}_1$, $\dot{\theta}_2$, $\dot{\theta}_3$ suffered by the platform and gyro give a convenient stage at which to isolate the problem, the situation is rather more complex. A complete stabilised system would consist of a platform with three single-axis gyros mounted with their input axes orthogonal, the whole platform being mounted in a gimbal system. Any rotations suffered by the platform give rise to outputs from the appropriate gyros, which are sensed and restoring torques applied to the platform. The motion of the outer gimbal pivots and the gimbal frictions, and also inertia effects when the gimbals are not quite orthogonal, determine the external torques applied to the platform. The restoring torques depend on the input rotations and the response of the gyros and of the pick-off/torque motor servo systems. Finally the actual rotations are given by the resultant torques applied, knowing the moments of inertia of the platform, and taking into account the gyroscopic reactions of the gyros.

3. *Drift Rate for Sinusoidal Inputs.*

3.1. *Perturbation Solution.*

Fortunately we may solve equation (8) perturbation-wise since the ideal input $H\dot{\theta}_1$ is of first order in small angles, and is very much larger than the kinematic rectification term $-H\dot{\theta}_3\phi$ which is of second order. Setting

$$\phi = \phi_1 + \phi_2 + \text{terms of third order and higher,}$$

where ϕ_1 and ϕ_2 are of first and second order in small angles, we have

$$I\ddot{\phi}_1 + L\dot{\phi}_1 = H\dot{\theta}_1 \qquad (10)$$

$$I\ddot{\phi}_2 + L\dot{\phi}_2 = -H\dot{\theta}_3\phi_1. \qquad (11)$$

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* The gyro is operated so that ϕ always remains small, and we will freely replace cos ϕ by 1, and sin ϕ by ϕ.

The sinusoidal inputs to the gyro will be taken as

$$\theta_1 = a_1 \cos(\omega t + \eta_1), \quad \theta_3 = a_3 \cos(\omega t + \eta_3) \tag{12}$$

where the difference in phase is arbitrary, but constant. We have used angles rather than angular rates in the belief that amplitudes will be more useful parameters. For this input the steady-state solution of (10) can be written as

$$\phi_1 = a_1 \omega H |K(\omega)|^{-1} \sin(\omega t + \eta_1 + \delta) \tag{13}$$

where

$$K(\omega) = -\frac{1}{I\omega(\omega - i\omega_0)} \tag{14}$$

is the transfer function of the gyro, and $\omega_0 = L/I$ is the characteristic frequency. Also the phase shift δ is given by $\tan\delta = \omega_0/\omega$.

For the kinematic rectification term we have from (11), (12) and (13)

$$\begin{aligned} I\ddot{\phi}_2 + L\dot{\phi}_2 &= a_1 a_3 \omega^2 H^2 |K(\omega)|^{-1} \sin(\omega t + \eta_3) \sin(\omega t + \eta_1 + \delta) \\ &= \tfrac{1}{2} a_1 a_3 \omega^2 H^2 |K(\omega)|^{-1} \{\cos(\eta_3 - \eta_1 - \delta) - \cos(2\omega t + \eta_1 + \eta_3 + \delta)\}. \end{aligned}$$

The steady-state solution of this equation for the *rate* $\dot{\phi}_2$ is

$$\dot{\phi}_2 = \tfrac{1}{2} a_1 a_3 \omega^2 H^2 L^{-1} |K(\omega)|^{-1} \cos(\zeta - \delta) + \text{a term of frequency } 2\omega.$$

Here ζ is the phase difference $(\eta_3 - \eta_1)$. The gyro interprets this output as due to a rotation about the input axis and the opposite rotation is applied so as to null the output. Dividing by the gain factor $G(= H/L)$ at zero frequency, the mean drift rate $\dot{\chi}$ is

$$\dot{\chi} = -\tfrac{1}{2} a_1 a_3 \omega^2 H |K(\omega)|^{-1} \cos(\zeta + \delta). \tag{15a}$$

Remembering that $\tan\delta = \omega_0/\omega$, this result can also be written as

$$\dot{\chi} = -\tfrac{1}{2} G a_1 a_3 \omega \omega_0 (\omega^2 + \omega_0^2)^{-1} (\omega \cos\zeta + \omega_0 \sin\zeta). \tag{15b}$$

3.2. *Discussion.*

When the phase difference ζ is zero the angular motion is an oscillation about a fixed axis, but when the components are not in phase the instantaneous axis of rotation is itself performing a coning motion at the same frequency ω. The spiralling motion of a rocket might give rise to just such an oscillation of the stable platform.

The ratio of the drift rate when $\zeta = 90°$ to that when $\zeta = 0°$ is ω_0/ω, so that the out-of-phase condition will be more damaging than the in-phase at low frequencies $(\omega < \omega_0)$. This is obvious from (13) since at low frequencies $\delta \simeq 90°$, so that ϕ is out of phase with θ_1 and in phase with $\dot{\theta}_1$. Conversely the in-phase condition is more damaging at high frequencies $(\omega > \omega_0)$. At a given frequency the maximum drift rate occurs when

$$\zeta = \delta = \tan^{-1}(\omega_0/\omega).$$

To put this in perspective consider numerical values for a typical fluid-floated gyro. Typical values of the constants (actually for Kearfott types T 2500 and T 2502—2C/3C) are

$$I = 5{\cdot}628 \times 10^3 \text{ gm.cm}^2$$

$$H = 6{\cdot}05 \times 10^6 \text{ gm.cm}^2.\text{sec}^{-1}$$

$$L = 2{\cdot}039 \times 10^6 \text{ gm.cm.}^2 \text{ sec}^{-1}.$$

Also $\omega_0(= L/I)$ is $362 \cdot 3$ sec^{-1}, giving a characteristic frequency $\omega_0/2\pi = 57 \cdot 7$ c/s and a characteristic time lag $1/\omega_0 = 2 \cdot 76$ millisec. The gain factor $G(= H/L)$ is $2 \cdot 97$.

For simplicity a mean amplitude 'a' is defined as $(a_1 a_3)^{1/2}$. The values of a for sinusoidal platform oscillation required to produce a standard drift rate of $0 \cdot 1°$/hr for the two conditions $\zeta = 0°$ and $\zeta = 90°$ and at various frequencies are shown in the following table:

Frequency $\omega/2\pi$ (c/s)	$a(\zeta=0°)$ minutes of arc	$a(\zeta=90°)$ minutes of arc
0·1	59·5	2·47
0·2	29·8	1·75
0·5	11·9	1·11
1	5·95	0·785
2	2·98	0·555
5	1·20	0·352
10	0·604	0·252
20	0·315	0·186
50	0·158	0·147
100	0·119	0·157

The assumption of sinusoidal oscillations is possibly unrealistic at very low frequencies (< 2 c/s). In practice such an oscillation would probably appear at the platform as a series of low-frequency kicks in alternate directions.

It must be emphasised that these results apply to an oscillation of constant amplitude and phase difference. In practice one mode of oscillation dies away to be replaced by another mode. If the phase difference changes by 180° the gyro will drift in the opposite sense.

To return to vibration about a fixed axis, when the components along different axes are in phase, the kinematic drift rate vanishes if the component of angular velocity along either 01 or 03 vanishes. It has one sign if the axis of vibration (strictly its projection on the plane 013) lies in the first or third quadrant, and the other sign if in the second or fourth quadrant. To some extent this is reminiscent of anisoelastic drift[4]. If the vibration environment of the platform is symmetric the axis of vibration is equally likely to lie in any direction, and the gyro is equally likely to drift in either sense giving zero mean drift.

4. *Statistical Treatment for Random Inputs.*

A more general treatment can be given which is possibly of more academic than practical interest. Here the angular vibrations suffered by the gyro, i.e. the angular velocities $\dot{\theta}_i(t)$ about the axes $0i$, $i = 1, 2, 3$, are treated as stationary random noise with auto-correlation and cross-correlation functions* given by

$$\langle \dot{\theta}_i(t)\dot{\theta}_i(t') \rangle = \sigma_i^2 \psi_i(t-t') \tag{16a}$$

$$\langle \dot{\theta}_i(t)\dot{\theta}_{i'}(t') \rangle = \rho_{ii'}\sigma_i\sigma_{i'}\psi_{ii'}(t-t') . \tag{16b}$$

where $i, i' = 1, 2, 3$. Here σ_i is the r.m.s. value of the angular rate, and the correlation functions are normalised to unity, i.e.

$$\psi_i(0) = \psi_{ii'}(0) = 1 .$$

* All means are thought of as ensemble averages.

Also $\rho_{ii'}$ ($= \rho_{i'i}$) is the correlation coefficient between the angular vibrations about the axes $0i$ and $0i'$. We also have the symmetry properties

$$\psi_i(-t) = \psi_i(t); \quad \psi_{ii'}(-t) = \psi_{i'i}(t)$$

as a direct consequence of the definitions.

First of all we must develop a slightly more general form for the solution of equations (10) and (11). It is readily shown by various methods that the general solution of equation (8) can be written in the form

$$\phi(t) = \int_{-\infty}^{\infty} k(t-t')N(t')dt' \tag{17}$$

where the weighting function $k(t)$ is

$$k(t) = \begin{cases} \frac{1}{L}(1-e^{-\omega_0 t}), & t > 0 \\ 0, & t < 0 \end{cases} \tag{18}$$

and $\omega_0 = L/I$ is the characteristic frequency of the gyro. With the weighting function defined as in (18) to be zero for $t < 0$, the upper limit of the integral in (17) has been taken as $+\infty$ rather than $t' = t$. The weighting function builds up exponentially from zero at $t = 0$, approaching the value $1/L$ for t large with a characteristic time lag of $1/\omega_0$.

The frequency response $K(\omega)$ as previously defined in (14) is the Fourier transform of $k(t)$ defined in the usual way, *viz.*

$$k(t) = \frac{1}{2\pi}\int K(\omega)e^{i\omega t}d\omega \tag{19a}$$

with inverse

$$K(\omega) = \int k(t)e^{-i\omega t}dt. \tag{19b}$$

A slight difficulty is that $K(\omega)$ has a pole at $\omega = 0$, and the integral in (19b) is not convergent. This is easily overcome either by replacing the definition of $k(t)$ by

$$k_\epsilon(t) = \frac{1}{L}(e^{-\epsilon t} - e^{-\omega_0 t}), \quad t > 0 \tag{18'}$$

or by evaluating $K(\omega - i\epsilon)$. In either case ϵ is a small positive quantity that is eventually made to approach zero.

Using the weighting function $k(t)$, the solution of (10) may be written as

$$\phi_1(t) = H\int_{-\infty}^{\infty} k(t-t')\dot{\theta}_1(t')dt'$$

and the solution of (11) as

$$\phi_2(t) = -H^2\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\xi' k(t-t')k(t'-\xi')\dot{\theta}_3(t')\dot{\theta}_1(\xi').$$

We will take the corresponding drift angle and drift rate as $-\phi_2/G$ and $-\dot{\phi}_2/G$ respectively, where $G = H/L$ is the gain factor at zero frequency. This ignores the delay introduced by the pick-off/torque motor loop. Thus the drift rate $\dot{\chi}$ can be written as

$$\dot{\chi} = H^2/G\int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\xi' j(t-t')k(t'-\xi')\dot{\theta}_3(t')\dot{\theta}_1(\xi'). \tag{20}$$

While the drift angle $\chi(t)$ at time t after the gyro is released at $t = 0$ is

$$\chi(t) = H^2/G \int_0^t dt' \int_{-\infty}^{\infty} d\xi' k(t-t')k(t'-\xi')\dot{\theta}_3(t')\dot{\theta}_1(\xi'). \tag{21}$$

In (20) the weighting function $j(t)$ is the derivative of $k(t)$, i.e.

$$j(t) = k'(t) = \frac{\omega_0}{L} e^{-\omega_0 t} = \frac{1}{I} e^{-\omega_0 t} \tag{22a}$$

while the corresponding transfer function is

$$J(\omega) = \frac{1}{I(\omega_0 + i\omega)}. \tag{22b}$$

Now consider the mean value of the drift rate. Taking the ensemble average throughout (20) using (16b),

$$\langle \dot{\chi} \rangle = (H^2/G)\rho_{13}\sigma_1\sigma_3 \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} d\xi' j(t-t')k(t'-\xi')\psi_{13}(\xi'-t'). \tag{23a}$$

This result, as well as others to be found later, can be transformed to an integral over frequency involving the frequency-response function of the gyro and the power spectra of the noise. The noise power spectra $\Psi_i(\omega)$ and the mixed power spectra $\Psi_{ii'}(\omega)$ are the Fourier transforms, defined exactly as for $K(\omega)$ in (19), of the correlation functions $\psi_i(t)$ and $\psi_{ii'}(t)$. Expressing k and ψ_{13} in terms of their Fourier transforms and performing the integration over ξ', (23a) takes the form

$$\langle \dot{\chi} \rangle = (H^2/G)\rho_{13}\sigma_1\sigma_3 \int_{-\infty}^{\infty} j(t-t')dt'(2\pi)^{-1} \int_{-\infty}^{\infty} K(\omega)\Psi_{13}(\omega)d\omega.$$

The integral over t' now falls out. Remembering that $J(0) = 1/L$ and $G = H/L$ the final result is

$$\langle \dot{\chi} \rangle = H\rho_{13}\sigma_1\sigma_3(2\pi)^{-1} \int_{-\infty}^{\infty} K(\omega)\Psi_{13}(\omega)d\omega. \tag{23b}$$

The drift rate is constant, and will be zero unless ρ_{13} is non-zero. Also the mean value of the drift angle at time t after release at $t = 0$ is simply $\langle \dot{\chi}(t) \rangle \simeq t\langle \dot{\chi} \rangle$. If this is not obvious it may be calculated formally by taking the ensemble average throughout (21).

Even if the mean value of the drift rate and angle are zero, the r.m.s. value of the drift angle after time t will not be zero. We will show this by calculating var $\chi(t)$ for a general value of ρ_{13}. Thus square (21) and take the ensemble average throughout:

$$\langle \chi^2(t) \rangle = (H^2/G)^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \int_{(2)} d\xi' d\xi'' k(t'-\xi')k(t''-\xi'') \times$$

$$\times \langle \dot{\theta}_3(t')\dot{\theta}_1(\xi')\dot{\theta}_3(t'')\dot{\theta}_1(\xi'') \rangle. \tag{24}$$

Here the subscript 2 in brackets signifies double integration, and the limits are $-\infty$ to $+\infty$ for both variables.

Provided we further assume that all the variables are Gaussianly distributed, the mean of the product of four variables in (24) can be expanded in terms of the correlation functions of pairs of the variables. This is carried through in the Appendix where it is found that

$$\operatorname{var} \chi(t) = \langle \chi^2(t) \rangle - \langle \chi(t) \rangle^2 = V_2 + V_3 \tag{25}$$

where V_2 and V_3 are the integrals

$$V_2 = (H^2/G)^2 \sigma_1{}^2 \sigma_3{}^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \times$$
$$\times \int_{(2)} d\xi' d\xi'' k(t'-\xi') k(t''-\xi'') \psi_1(\xi'-\xi'') \psi_3(t'-t'')$$
$$V_3 = (H^2/G)^2 \rho_{13}{}^2 \sigma_1{}^2 \sigma_3{}^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \times$$
$$\times \int_{(2)} d\xi' d\xi'' k(t'-\xi') k(t''-\xi'') \psi_{13}(\xi'-t'') \psi_{13}(\xi''-t').$$

The further evaluation of these integrals is also performed in the Appendix, where it is shown that

$$V_2 \simeq t H^2 \sigma_1{}^2 \sigma_3{}^2 (2\pi)^{-1} \int_{-\infty}^{\infty} |K(\omega)|^2 \Psi_1(\omega) \Psi_3(\omega) d\omega \tag{26a}$$

$$V_3 \simeq t H^2 \rho_{13}{}^2 \sigma_1{}^2 \sigma_3{}^2 (2\pi)^{-1} \int_{-\infty}^{\infty} K^2(\omega) \Psi_{13}{}^2(\omega) d\omega . \tag{26b}$$

If the angular vibrations about the input and spin axes are uncorrelated ($\rho_{13} = 0$), the mean drift rate and angle are zero. However the variance, which is then also the mean-square drift angle, is given by V_2 alone, and increases linearly with time. This is the continuous analogue of 'random walk'; the gyro drifts sometimes in one sense, sometimes in the other, in such a way that the mean drift is zero. The other integral V_3 is a further term of the same form which must be added when $\rho_{13} \neq 0$, in which case however the mean drift angle itself increases linearly with time.

5. *Conclusions.*

In a single-axis gyro a rotation about the input axis forces the rotor to rotate about the output axis, so that the true input axis is oscillating about its nominal position and the gyro sees a varying component of the angular velocity about the nominal spin axis. This may be rectified, thereby causing a drift, if the components of angular velocity along the input and spin axes are correlated. The drift rate has been evaluated for sinusoidal angular vibrations of the platform.

At low frequencies (less than the characteristic frequency of the gyro, which is say 50 c/s for a typical gyro), a 'coning' type of motion of the platform, in which the components of angular velocity along the input and spin axes are out of phase, is the most damaging motion. Such a vibration could arise from a spiralling motion of a rocket. The sense in which the gyro drifts would depend on whether the coning was clockwise or counterclockwise. However the effect would only be embarrassing if it built up, i.e. if the platform had a tendency to vibrate in one mode rather than another. Conversely for frequencies above the characteristic frequency, the in-phase type of vibration, which corresponds to an oscillation about a fixed axis, is more damaging. The drift is in one sense if the axis (strictly its projection on the plane of the input and spin axes) lies in the first or third quadrant, and in the other sense if in the second or fourth quadrant. Again the effect would be embarrassing only if the platform tended to vibrate in one type of mode rather than another. The most reasonable approach would seem to be to find experimentally the vibration characteristics of the platform, and to change the platform mounting or servo control if there was any tendency for one type of vibration to be favoured.

It might be desirable to investigate the effect experimentally, e.g. by mounting the gyro directly on an oscillation table or by clamping the platform to the table. The only difficulty might be in having sufficiently small amplitude (say $< 1°$) that the gyro float did not hit its end-stops. Other possible methods would be to degrade the performance of the platform servos or to introduce extra friction at the platform mounting gimbals.

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LIST OF SYMBOLS

Only the principal symbols are included. The dot notation is used for time derivatives. The axes 0123 are fixed in the gyro outer case, and coincide with the nominal input, output and spin axes, while the axes 01′2′3′ are fixed in the gyro float and coincide with the true input, output and spin axes.

a, a_1, a_3	Amplitudes of platform angular motion
G	Gain factor of gyro at zero frequency, $G = H/L$
H	Spin angular momentum of the rotor
I	Total moment of inertia of the rotor and the float about the output axis
$j(t)$	Subsidiary weighting function; the derivative of $k(t)$
$J(\omega)$	Frequency response corresponding to $j(t)$, $J(\omega) = i\omega K(\omega)$
$k(t)$	Weighting function of the gyro, defined by (18)
$K(\omega)$	Frequency response of the gyro, i.e. the Fourier transform of $k(t)$
L	Constant defined by: fluid-drag torque $= -L\dot{\phi}$
$N(t)$	Input function, defined by (8) and (9)
T	Total unwanted torque on rotor and float about output axis, due to mass unbalance, bearing friction, etc.
$t, t', t'', \xi' \xi''$	Time variables
δ	Phase shift given by $\tan \delta = \omega_0/\omega$
η_1, η_3	Phase angles defined in (12)
ζ	Phase difference $(\eta_3 - \eta_1)$ between two axes
$\psi_i(t), \psi_{ii'}(t)$	Correlation functions of the $\dot{\theta}_i(t)$
$\Psi_i(\omega), \Psi_{ii'}(\omega)$	Power spectra and mixed power spectra of the noise, the Fourier transforms of the ψ's
$\rho_{ii'}$	Correlation coefficient between angular vibrations about different axes
σ_i	r.m.s. value of the angular rate $\dot{\theta}_i$
$\dot{\theta}_1, \dot{\theta}_2, \dot{\theta}_3$	Components of angular velocity of gyro outer case along 01, 02, 03
ϕ	Output angle, the angle through which float has rotated relative to the case
ϕ_1, ϕ_2	First- and second-order parts of ϕ
$\chi, \dot{\chi}$	Drift angle and drift rate
$\omega_1, \omega_2, \omega_3$	Components of angular velocity of the float along 01′, 02′, 03′
$\omega, \omega', \omega''$	Frequency variables
ω_0	Characteristic frequency of the gyro, $\omega_0 = L/I$

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APPENDIX

Evaluation of the Variance

Provided the variables are Gaussianly distributed, the mean of the product of four variables in (24) can be expanded in terms of the correlation functions of pairs of the variables by the following theorem given for instance by Middleton[5]:

Suppose $x_1, \ldots x_{2n}$ is a set of Gaussian variables each with zero mean. The $2n$ variables may be grouped in pairs in $(2n)!/2^n n!$ ways, e.g. four variables may be grouped as (x_1, x_2), (x_3, x_4) or as (x_1, x_3), (x_2, x_4), or as (x_1, x_4), (x_2, x_3). Then the mean of the product of all $2n$ variables is the sum, over the $(2n)!/2^n n!$ pairings, of the products of the means of the n pairs for each particular pairing, e.g. for $n = 2$:

$$\langle x_1 x_2 x_3 x_4 \rangle = \langle x_1 x_2 \rangle \langle x_3 x_4 \rangle + \langle x_1 x_3 \rangle \langle x_2 x_4 \rangle + \langle x_1 x_4 \rangle \langle x_2 x_3 \rangle .$$

Applying this to the mean in (24), we find that

$$\begin{aligned}\langle \dot\theta_1(\xi')\dot\theta_3(t')\dot\theta_1(\xi'')\dot\theta_3(t'')\rangle &= \langle \dot\theta_1(\xi')\dot\theta_3(t')\rangle \langle \dot\theta_1(\xi'')\dot\theta_3(t'')\rangle + \\ &\quad + \langle \dot\theta_1(\xi')\dot\theta_1(\xi'')\rangle \langle \dot\theta_3(t')\dot\theta_3(t'')\rangle + \langle \dot\theta_1(\xi')\dot\theta_3(t'')\rangle \langle \dot\theta_1(\xi'')\dot\theta_3(t')\rangle \\ &= \rho_{13}{}^2\sigma_1{}^2\sigma_3{}^2\psi_{13}(\xi'-t')\psi_{13}(\xi''-t'') + \sigma_1{}^2\sigma_3{}^2\psi_1(\xi'-\xi'')\psi_3(t'-t'') + \\ &\quad + \rho_{13}{}^2\sigma_1{}^2\sigma_3{}^2\psi_{13}(\xi'-t'')\psi_{13}(\xi''-t')\end{aligned}$$

from (16a) and (16b). Using this in (24) we find that

$$\langle \chi^2(t)\rangle = V_1 + V_2 + V_3 \tag{27}$$

where

$$\begin{aligned}V_1 &= (H^2/G)^2\rho_{13}{}^2\sigma_1{}^2\sigma_3{}^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \times \\ &\quad \times \int_{(2)} d\xi' d\xi'' k(t'-\xi')k(t''-\xi'')\psi_{13}(\xi'-t')\psi_{13}(\xi''-t'') \\ V_2 &= (H^2/G)^2\sigma_1{}^2\sigma_3{}^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \times \\ &\quad \times \int_{(2)} d\xi' d\xi'' k(t'-\xi')k(t''-\xi'')\psi_1(\xi'-\xi'')\psi_3(t'-t'') \\ V_3 &= (H^2/G)^2\rho_{13}{}^2\sigma_1{}^2\sigma_3{}^2 \int_0^t dt' k(t-t') \int_0^t dt'' k(t-t'') \times \\ &\quad \times \int_{(2)} d\xi' d\xi'' k(t'-\xi')k(t''-\xi'')\psi_{13}(\xi'-t'')\psi_{13}(\xi''-t')\end{aligned} \tag{28}$$

The integrations over ξ' and ξ'' in V_1 can be performed independently so that V_1 splits into two parts each of which is $\langle\chi(t)\rangle$ as in (23a). Thus

$$\operatorname{var} \chi(t) = \langle \chi^2(t)\rangle - \langle \chi(t)\rangle^2 = V_2 + V_3 . \tag{29}$$

The remaining integrals V_2 and V_3 are rather more complicated, and the first step in their evaluation is to perform the integrations over ξ' and ξ'' by introducing the appropriate Fourier transforms. This procedure gives

$$V_2 = (H^2/G)^2\sigma_1^2\sigma_3^2 \int_0^t dt'k(t-t') \int_0^t dt''k(t-t'') \times$$

$$\times (2\pi)^{-2} \int_{(2)} d\omega' d\omega'' K(\omega')K(-\omega')\Psi_1(\omega')\Psi_3(\omega'') \exp\{i(\omega'+\omega'')(t'-t'')\}$$

$$V_3 = (H^2/G)^2\rho_{13}^2\sigma_1^2\sigma_3^2 \int_0^t dt'k(t-t') \int_0^t dt''k(t-t'') \times$$

$$\times (2\pi)^{-2} \int_{(2)} d\omega' d\omega'' K(\omega')K(\omega'')\Psi_{13}(\omega')\Psi_{13}(\omega'') \exp\{i(\omega'-\omega'')(t-t'')\}. \tag{30}$$

The next step is to perform the integrations over t' and t'', and to this end we define the function

$$K(\omega, t) = \int_0^t k(t')e^{-i\omega t'}dt'. \tag{31}$$

As the upper limit of integration approaches infinity, the function $K(\omega, t)$ tends to $K(\omega)$ as defined by (19b), and given explicitly in (14). A similar explicit form is easily found for $K(\omega, t)$ from (31), preferably using the modified $k_\epsilon(t)$ of (18′) to ensure convergence. It is found that $K(\omega, t)$ is a narrow function of ω concentrated around the value $\omega = 0$ {compare $K(\omega)$ of (14)}.

In terms of $K(\omega, t)$ the integrals V_2 and V_3 become

$$V_2 = (H^2/G)^2\sigma_1^2\sigma_3^2(2\pi)^{-2} \int_{(2)} d\omega' d\omega'' K(\omega')K(-\omega')\Psi_1(\omega')\Psi_3(\omega'')K(\omega'+\omega'', t)K(-\omega'-\omega'', t)$$

$$V_3 = (H^2/G)^2\rho_{13}^2\sigma_1^2\sigma_3^2(2\pi)^{-2} \int_{(2)} d\omega' d\omega'' K(\omega')K(\omega'')\Psi_{13}(\omega')\Psi_{13}(\omega'')K(\omega'-\omega'', t)K(-\omega'+\omega'', t). \tag{32}$$

Both integrals contain a product of the form

$$K(z, t)K(-z, t) = |K(z, t)|^2$$

which is negligible outside a narrow band centred on $z = 0$. Suppose we first integrate over ω'': the function $|K(z, t)|^2$ is so narrow that the remainder of the integrand may be considered constant, ω'' being replaced by $-\omega'$ in V_2 and $+\omega'$ in V_3. Thus we find that

$$V_2 \simeq (H^2/G)^2\sigma_1^2\sigma_3^2(2\pi)^{-2} \int d\omega' K(\omega')K(-\omega')\Psi_1(\omega')\Psi_3(-\omega') \int_{-\infty}^{\infty} |K(\omega, t)|^2 d\omega \tag{33}$$

with a similar result for V_3. Using the inverse of (31),

$$\int_{-\infty}^{\infty} |K(\omega, t)|^2 d\omega = \int_0^t k^2(t')dt' \simeq 2\pi L^{-2}t \tag{34}$$

for $t \gg 1/\omega_0$, since for t large, $k(t)$ approaches the constant value $1/L$ (sensibly attained for $t > 1/\omega_0$). This now leads to the results of (26a) and (26b) on using the appropriate symmetry properties.

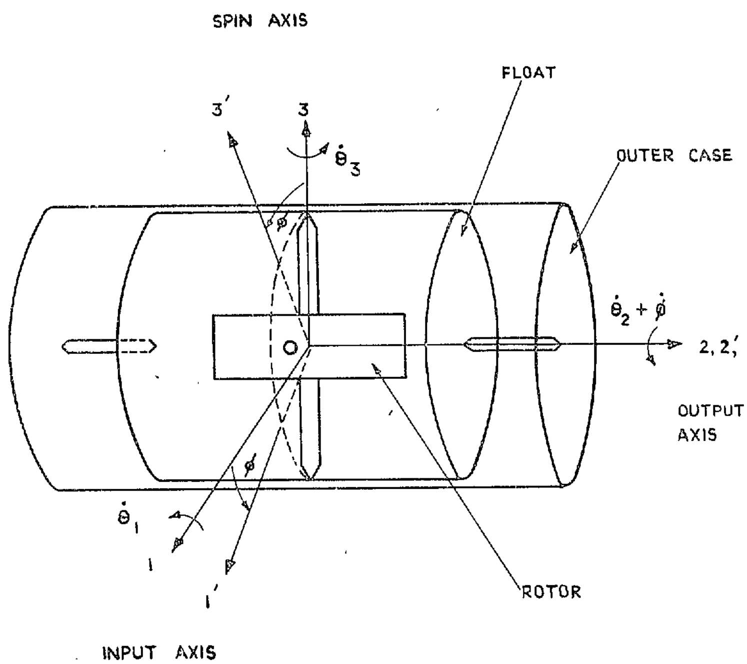

FIG. 1. Fluid-floated gyro: notation and choice of axes.

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