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Part I. —Calculations based on Lighthill's Method

Part II.—Calculations based on Stratford's Method

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Part I.—Calculations based on Lighthill's Method

Summary. Two methods are derived for calculating the heat transfer through a constant-property laminar boundary layer, that depend respectively on approximations valid at low and high values of σ, the Prandtl number. The high σ approach is a development of an earlier method, due to Lighthill; an additional term is now retained in the assumed form of the velocity profile near the wall. The low σ approach is effectively an extension to Prandtl numbers of order unity of the limiting solution of Morgan *et al*, valid as $\sigma \to 0$.

A certain amount of empirical fitting ensures good agreement with the 'similar' solutions. When the methods are used to calculate heat transfer in flow past a circular cylinder, a comparison with experiment suggests that the high σ approximation is at least as good as other methods of calculating heat transfer, and that the low σ approximation may be even better.

1. *Introduction and Outline of Method.* It is now some years since Lighthill[1] (1950) wrote his well-known paper on heat transfer through a laminar boundary layer, a paper which aimed mainly at two things: (i) the calculation of heat transfer in laminar flow at low Mach number for arbitrary main-stream velocity and wall temperature, (ii) the calculation of a distribution of wall temperature which, with constant supersonic main-stream speed, would yield a positive heat-transfer rate which balanced radiation of heat from the wall. This paper is concerned with the first of these two problems.

Physically Lighthill's work was based on the idea that when the Prandtl number, σ, is large the temperature boundary layer is much thinner than the velocity boundary layer, so that the velocity u is given approximately as

$$u = \tau_w(x)y/\mu \tag{1.1}$$

throughout the temperature boundary layer, an idea used earlier by Fage and Falkner[2] (1931). Here x and y are co-ordinates along and normal to the wall, μ is the viscosity and $\tau_w(x)$ is the skin friction at the wall. Naturally one would expect that at fixed Prandtl number of order unity the

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best results would be obtained for the case of zero pressure gradient, since the velocity is then closely linear for a considerable part of the (velocity) boundary layer. This, indeed, turned out to be the case, the predicted heat transfer being in error by only 3 per cent when $\sigma = 0{\cdot}7$. For main-stream velocities like $u_1 \propto x^c$, the error increases to $18\frac{1}{2}$ per cent when $c = 1$, that is at a stagnation point, and to 24 per cent for large values of c. On the other hand, the errors are even more severe for negative values of c, and become intolerable as $c \to -0{\cdot}0904$ (i.e., as a separation profile is approached), when the predicted heat transfer tends to zero, since the approximation (1.1) is then equivalent to the assumption that the velocity is everywhere zero.

Mathematically Lighthill's method works because from (1.1) we may express the velocity u in terms of ψ, the rate of mass flow between a given point and the wall, as

$$u = \left(\frac{2\tau_w}{\mu\rho}\psi\right)^{1/2}, \tag{1.2}$$

so that upon neglecting frictional heating, the von Mises form of the energy equation becomes

$$\frac{\partial T}{\partial x} = \frac{1}{\sigma}\{2\mu\rho\tau_w(x)\}^{1/2}\frac{\partial}{\partial\psi}\left(\psi^{1/2}\cdot\frac{\partial T}{\partial\psi}\right). \tag{1.3}$$

Lighthill integrated this equation, using the Heaviside operational method, and showed that the local heat-transfer rate to the wall is given by

$$Q_w(x) = k\left(\frac{\partial T}{\partial y}\right)_w = -k\left(\frac{\sigma\rho}{9\mu^2}\right)^{1/3}\frac{\{\tau_w(x)\}^{1/2}}{(1/3)\,!}\int_0^x\left(\int_{x1}^x\{\tau_w(z)\}^{1/2}dz\right)^{-1/3}dT_0(x_1), \tag{1.4}$$

where T_0 is defined as the difference between the wall temperature and the temperature T_1 of the main stream (constant at low Mach number),

$$T_0(x) = T_w(x) - T_1, \tag{1.5}$$

and the Stieltjès integral in (1.4) may be regarded as a shorthand notation for

$$\int_0^x\left(\int_{x1}^x\{\tau_w(z)\}^{1/2}dz\right)^{-1/3}dT_0(x_1) = \frac{T_0(+0)}{\left(\int_0^x\{\tau_w(z)\}^{1/2}dz\right)^{1/3}} + \int_0^x\frac{dT_0/dx_1}{\left(\int_{x1}^x\{\tau_w(z)\}^{1/2}dz\right)^{1/3}}\,dx_1\,. \tag{1.6}$$

It is fairly clear that a considerable all-round improvement could be effected by approximating the velocity as

$$\mu u = \tau_w(x)y + \frac{1}{2}\frac{dp}{dx}y^2, \tag{1.7}$$

but this cannot be expressed in the form

$$u(x, \psi) = A(x)B(\psi) \tag{1.8}$$

so that Lighthill's mathematical technique could not be used.

It is perhaps worth noting that Davies and Bourne[3] (1956) indicated a method yielding good heat-transfer predictions for the similarity solutions. This involves choosing u to be approximately given by

$$u(x, \psi) \sim A(x)\psi^n, \tag{1.9}$$

where $A(x)$ and n are chosen so that (1.9) gives a good representation of the true velocity over the major portion of the boundary layer. It is difficult to see how this approach can be precisely expressed for the case of arbitrary main-stream velocity, when the velocity profile is not known *a priori* in simple terms. An alternative (and better) method of improving the numerical accuracy of Lighthill's method was later given by Spalding[4] (1958), who attempted to account for the effects of the second term in equation (1.7) for the velocity. Spalding noted that a key parameter is the ratio of the two terms on the right-hand side of (1.7) as calculated at a representative position in the thermal boundary layer. By introducing such an additional parameter, good agreement is obtained with the similarity solutions over a much greater range of pressure gradients, with Prandtl numbers from 0·7 to infinity, although the method still breaks down very near to separation.

A further important development has recently appeared in the work of Liepmann[5] (1958), who derives Lighthill's formula (1.4) by an independent procedure which makes use of the thermal-energy integral equation. For low-speed flow, with frictional heating neglected, it is well known that this equation may be written as

$$\frac{d}{dx}\int_0^\infty u(T-T_1)dy = -\frac{Q_w(x)}{\rho c_p}, \tag{1.10}$$

$Q_w(x)$ being defined, as in (1.4), to be the heat-transfer rate to the wall. Liepmann uses an alternative approximation to the velocity, holding asymptotically as the wall is approached,

$$u = \frac{\tau_w(x)c_p(T-T_w)}{\sigma Q_w(x)}, \tag{1.11}$$

and by writing

$$dy = \frac{k}{Q}\,dT, \tag{1.12}$$

(1.10) becomes

$$Q_w(x) = -\frac{\sigma\rho}{\mu^2}k^3\frac{d}{dx}\left\{\frac{\tau_w}{Q_w^{\,2}}(T_w-T_1)^3\int_0^1\frac{\theta(1-\theta)}{Q/Q_w}\,d\theta\right\}, \tag{1.13}$$

where

$$\theta = \frac{T-T_w}{T_1-T_w}. \tag{1.14}$$

Liepmann then considers the special case in which the wall temperature is equal to its zero heat-transfer value T_1 when $x \leqslant x_0$, say, and increases step-wise to a new constant value downstream of this position. In such a special case Q/Q_w is approximately a universal function of θ alone, so that (1.13) becomes

$$Q_w(x) = -a\frac{\sigma\rho k^3}{\mu^2}(T_w-T_1)^3\frac{d}{dx}\left(\frac{\tau_w}{Q_w^{\,2}}\right), \tag{1.15}$$

where

$$a = \int_0^1\frac{\theta(1-\theta)}{Q/Q_w}\,d\theta, \tag{1.16}$$

is a constant. When τ_w is known as a function of x, equation (1.15) is easily integrated to yield $Q_w(x)$. After doing this, Liepmann derives the value of $Q_w(x)$ for an arbitrary wall temperature by integrating the contributions from a distribution of elementary charges similar to the above, and shows that his result is identical to (1.4) with the constant $\{9^{1/3}(\frac{1}{3})\,!\}^{-1} = 0{\cdot}538$ replaced by $(\frac{2}{3}a)^{1/3}$. Upon assuming the value $a = 0{\cdot}215$, obtained by setting $Q/Q_w = (1-\theta^2)^{1/2}$ in (1.16), Liepmann's formula agrees with Lighthill's to within 3 per cent. Liepmann also derives a formula, similar to (1.4), which is valid in the vicinity of separation.

The present paper shows how the improved approximation (1.7) to u may be incorporated into a method similar to Liepmann's. The result of doing this is an equation

$$-Q_w(x) = \frac{\sigma\rho}{\mu^2} k^3 \frac{d}{dx}\left\{\frac{a\tau_w(T_w - T_1)^3}{Q_w^2} - \frac{b\frac{dp}{dx}(T_w - T_1)^4 k}{2Q_w^3}\right\}, \tag{1.17}$$

which replaces (1.15). Here a and b are constants which can be determined either by making a similarity approximation (following Liepmann) or empirically so that (1.17) agrees well with exact solutions. This latter was done, using known results for the exact similarity solutions, and it was found that with

$$a = 0{\cdot}2226, \quad b = 0{\cdot}1046 \tag{1.18}$$

the predicted heat-transfer values are correct to within about 1 per cent for values of σ of order unity and ten. This accuracy is attained over the whole range from stagnation point to separation, and indicates the considerably greater accuracy than that obtained from Lighthill's formula.

By integrating (1.17) we find

$$\frac{\sigma\rho}{\mu^2}\frac{k^3(T_w - T_1)^3}{Q_w^3}\left\{a\tau_w Q_w - \tfrac{1}{2}b\frac{dp}{dx}k(T_w - T_1)\right\} = -\int_0^x Q_w(\xi)d\xi. \tag{1.19}$$

The presence of two terms on the left-hand side of this integral equation precludes its solution in terms of a simple quadrature, but the integral equation is not in practice excessively more difficult to deal with, as will be seen.

The possibility of calculating heat-transfer rates for Prandtl numbers of order unity by proceeding from a low Prandtl number solution has also been investigated in this paper. This corresponds, physically, to a temperature boundary layer which is much thicker than the velocity layer, so that most of the temperature boundary layer is outside of the velocity layer. Thus, corresponding to the use of (1.1) as an approximation to the velocity when σ is large, there is the approximation

$$u(x, y) \doteq u_1(x), \tag{1.20}$$

valid as $\sigma \to 0$. By substituting this approximation in (1.10) and proceeding exactly in the manner indicated by Liepmann for the high σ approach, the equation

$$\frac{d}{dx}\left(\frac{u_1}{Q_w}\right) = \frac{\nu Q_w}{a_0 k^2 \sigma (T_w - T_1)^2} \tag{1.21}$$

is obtained, corresponding to (1.15), for the special case when the wall temperature distribution is given by a step-function. The constant a_0 is determined empirically, as before. This equation may be integrated exactly, and the solution further generalised to the case of arbitrary wall temperature, yielding the result

$$Q_w = -k\left(\frac{a_0\sigma}{2\nu}\right)^{1/2} u_1(x)\int_0^x\left\{\int_\xi^x u_1 dx\right\}^{-1/2} dT_0(\xi). \tag{1.22}$$

An equation of this form, with the constant $(\frac{1}{2}a_0)^{1/2} = \pi^{-1/2}$, was given by Morgan *et al*[6] (1958), who substituted the approximation (1.20) into the von Mises form of the temperature equation, so finding that

$$\frac{\partial T}{\partial x} = \frac{\mu\rho}{\sigma} u_1 \frac{\partial^2 T}{\partial \psi^2}, \tag{1.23}$$

an equation which they solved by operational techniques.

If, alternatively, the value of $(\frac{1}{2}a_0)^{1/2}$ is obtained empirically by reference to exact solutions with values of σ close to unity, an accuracy of $\pm$ 25 per cent is obtained, but this may be considerably improved if a_0 is chosen to depend upon a pressure-gradient parameter, and a method of doing this will be indicated.

The methods are applied (Section 4) to the case of sub-critical flow past a circular cylinder for which extensive experimental work has been done. Comparisons between the results of the two methods of the present paper and the theoretical results of Squire[7] (1942) and Eckert and Drewitz[8] (1942) are made. These indicate that the low Prandtl number approximation shows very good agreement with experiment, and that the high Prandtl number solution, though less accurate, is still to be preferred to the other methods with which comparison is made. It is suggested that, although the high Prandtl number approximation is extremely accurate ($\pm$1 per cent) for the case of the similar solutions, it is likely to be less accurate in more realistic cases, and that in general the low Prandtl number approximation is likely to prove most reliable, as well as being fairly simple to apply.

2. *High Prandtl Number Approximation.* When speeds are sufficiently low that terms of order the square of the Mach number can be neglected, so that frictional heating may be ignored, the temperature T in a laminar boundary layer is given by solution of the equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{\nu}{\sigma}\frac{\partial^2 T}{\partial y^2}, \tag{2.1}$$

provided that temperature differences are not too great. Upon integration from $y = 0$ to $y = \infty$ the thermal-energy integral equation is obtained

$$\frac{d}{dx}\int_0^\infty u(T-T_1)dy = -\frac{\nu Q_w(x)}{k\sigma}. \tag{2.2}$$

We now seek a solution of (2.1) and (2.2) subject to the approximation that the velocity u is taken as

$$\mu u = \tau_w y + \frac{1}{2}\frac{dp}{dx}y^2, \tag{2.3}$$

which approximation is relevant to flows where the Prandtl number is large. Upon substitution into (2.2) we find

$$\frac{d}{dx}\left\{\tau_w\int_0^\infty y(T-T_1)dy+\frac{1}{2}\frac{dp}{dx}\int_0^\infty y^2(T-T_1)dy\right\} = -\frac{\mu^2}{\rho k\sigma}Q_w(x). \tag{2.4}$$

Following Liepmann, we consider the special case in which the temperature is uniform upstream of $x = x_0$ (which could in practice be the leading edge), and a true temperature boundary exists downstream of it, with the wall at a constant temperature $T_w + T_1$. The temperature profile will not vary much over a reasonable range of pressure gradients, and it is likely that it can be represented fairly accurately by a universal shape dependent upon a boundary-layer thickness δ which varies with position. This assumption was made by both Squire and Liepmann, and is a direct consequence of any attempt to specify the temperature profile by means of a Pohlhausen approach using a few boundary conditions. It follows that

$$\left.\begin{aligned} \int_0^\infty y(T-T_1)dy &= 0\{(T_w-T_1)\delta^2\}, \\ \int_0^\infty y^2(T-T_1)dy &= 0\{(T_w-T_1)\delta^3\}, \\ \text{and} \quad \left(\frac{\partial T}{\partial y}\right)_w &= 0\left(\frac{T_w-T_1}{\delta}\right), \end{aligned}\right\} \tag{2.5}$$

where δ is a measure of the thermal boundary-layer thickness. Thus

$$\begin{aligned} \int_0^\infty y(T-T_1)dy &= \frac{a(T_w-T_1)^3}{(\partial T/\partial y)_w^2} \\ &= \frac{ak^2(T_w-T_1)^3}{Q_w^2}, \end{aligned} \tag{2.6}$$

and

$$\int_0^\infty y^2(T-T_1)dy = -\frac{bk^3(T_w-T_1)^4}{Q_w^3}, \tag{2.7}$$

where a and b are positive constants, to be determined later. Substitution from (2.6) and (2.7) into (2.4) yields

$$\frac{d}{dx}\left\{\frac{ak^2\tau_w(T_w-T_1)^3}{Q_w^2} - \frac{b\dfrac{dp}{dx}k^3(T_w-T_1)^4}{2Q_w^3}\right\} = -\frac{\mu^2}{\rho k\sigma}Q_w, \tag{2.8}$$

which may be compared with Liepmann's equation (1.15). The presence of two terms on the left-hand side makes it impossible to solve (2.8) in closed form, but it may be integrated once to yield an associated integral equation

$$-\int_0^x Q_w dx = \frac{\rho\sigma}{\mu^2}\left\{\frac{a\tau_w k^3(T_w-T_1)^3}{Q_w^2} - \frac{b\dfrac{dp}{dx}k^4(T_w-T_1)^4}{2Q_w^3}\right\}. \tag{2.9}$$

An alternative form of (2.9), more suitable for many purposes, is obtained by introducing a representative velocity U_0 and a representative length l, and defining a local Nusselt number

$$Nu = -\frac{l}{T_w-T_1}\left(\frac{\partial T}{\partial y}\right)_w = -\frac{lQ_w}{k(T_w-T_1)}, \tag{2.10}$$

and a representative Reynolds number

$$R_0 = \frac{U_0 l}{\nu}. \tag{2.11}$$

Then (2.9) can be reduced to

$$\int_0^x Kd\left(\frac{x}{l}\right) = \left\{\frac{A}{K^2} - \frac{B}{K^3}\right\}, \tag{2.12}$$

where

$$K = NuR_0^{-1/2}, \tag{2.13}$$

$$A = a\sigma\frac{\tau_w}{\mu}\left(\frac{\nu l}{U_0^3}\right)^{1/2}, \tag{2.14}$$

and

$$B = -\tfrac{1}{2}b\sigma\frac{l}{\rho U_0^2}\frac{dp}{dx}. \tag{2.15}$$

It now remains to estimate the constants a and b. There are many ways of doing this, but perhaps the most useful one from a practical point of view is to choose a and b so that (2.9) or (2.12) yield good predictions of heat transfer for accurately known cases. Following Lighthill we determine a and b to give good agreement with the similarity solutions.

If we consider a flow in which the main-stream velocity is

$$u_1 = U_c x^c, \tag{2.16}$$

for some value of c, then the velocity distribution is (Falkner and Skan[9], 1930)

$$\left.\begin{aligned} u &= u_1 f'(\eta), \qquad \eta = \left(\frac{u_1}{\nu x}\right)^{1/2} y, \\ v &= -\frac{1}{2}\left(\frac{u_1\nu}{x}\right)^{1/2}\{(c+1)f(\eta) + (c-1)\eta f'(\eta)\}. \end{aligned}\right\} \tag{2.17}$$

With this velocity distribution, a solution of (2.1), with *constant* wall temperature T_w and main-stream temperature T_1, is

$$\frac{T_w - T}{T_w - T_1} = \alpha(c, \sigma)\int_0^\eta \left[\exp\left(-\frac{c+1}{2}\sigma\int_0^\eta f d\eta\right)\right] d\eta, \tag{2.18}$$

where

$$\alpha(c, \sigma) = \left\{\int_0^\infty \left[\exp\left(-\frac{c+1}{2}\sigma\int_0^\eta f d\eta\right)\right] d\eta\right\}^{-1}. \tag{2.19}$$

It is seen from (2.17) and (2.18) that

$$-Q_w(x) = -k\left(\frac{\partial T}{\partial y}\right)_w = \alpha(c, \sigma)k(T_w - T_1)\left(\frac{u_1}{\nu x}\right)^{1/2}, \tag{2.20}$$

or

$$\alpha(c, \sigma) = -\frac{Q_w(x)}{k(T_w - T_1)}\left(\frac{\nu x}{u_1}\right)^{1/2}. \tag{2.21}$$

The exact values of $\alpha(c, \sigma)$ can be readily computed from (2.19), and the results are already known for a wide range of values of c and σ. By substituting for Q_w from (2.20) into (2.8) or (2.9) the following equation for $\alpha(c, \sigma)$ is obtained after some algebra.

$$\alpha^4 = \tfrac{1}{2}\sigma(c+1)\{af''(0)\alpha - \tfrac{1}{2}bc\}, \tag{2.22}$$

where

$$f''(0) = \frac{\tau}{\mu u_1}\left(\frac{\nu x}{u_1}\right)^{1/2} \tag{2.23}$$

is a function of m which was tabulated by Hartree[10] (1937). The values of a and b are chosen so that solutions of (2.22) agree as nearly as possible with the exact values given by (2.19) for the physically important range of values of c and σ.

A number of representative cases were chosen, namely, $c = 1$ (stagnation point), $c = 0$ (Blasius layer) and $c = -0{\cdot}0904$ (separation), and values of the Prandtl number $\sigma = 0{\cdot}7$ (typical of gases) and $\sigma = 10$ (typical of liquids). It was found possible to choose a and b so that the heat-transfer rate was predicted by the present method to within about ± 1 per cent for all these cases, the values chosen being

$$a = 0{\cdot}2226, \qquad b = 0{\cdot}1046. \tag{2.24}$$

The considerable accuracy of the method is apparent.

In general, the presence of two terms on the right-hand side of (2.12) makes it impossible to integrate the equation analytically, even for the above simple case of a region of zero heat transfer followed by a step in wall temperature. Accordingly, numerical methods must be used. A suggested method is to replace the integral in (2.12) by its Simpson's rule value, an idea used earlier by Thwaites[11] (1949), so that when $K(x)$ and $K(x+h)$ are known, $K(x+2h)$ can be determined by solution of a quartic algebraic equation. The solution for arbitrary wall temperature is then obtained by adding the contributions (to the heat transfer) from a distribution of elementary steps in wall temperature.

3. *Low Prandtl Number Approximation.* When the Prandtl number is extremely small, the thermal boundary layer is much thicker than the velocity boundary layer, so that the velocity is equal to its free-stream value throughout most of the thermal boundary layer. Thus, upon neglecting frictional heating, the von Mises form of the energy equation becomes

$$\frac{\partial T}{\partial x} = \frac{\mu\rho}{\sigma} u_1 \frac{\partial^2 T}{\partial \psi^2}, \tag{3.1}$$

which is immediately soluble. This equation has been derived by Morgan *et al*[6] (1958), and integrated by operational techniques. An alternative method of deriving their result for the heat transfer is as follows. We begin with the thermal-energy integral equation (1.10), which approximates to

$$\frac{d}{dx}\left\{u_1 \int_0^\infty (T - T_1)dy\right\} = -\frac{\nu Q_w(x)}{k\sigma}. \tag{3.2}$$

Considering, as a starting point, the special case of a wall temperature distribution with a single step, we may assume that

$$\int_0^\infty (T - T_1)dy \doteqdot -\frac{a_0 k (T_w - T_1)^2}{Q_w}, \tag{3.3}$$

where a_0 is positive and is almost constant. Thus (3.2) becomes

$$\frac{d}{dx}\left(\frac{u_1}{Q_w}\right) = \frac{\nu Q_w}{a_0 k^2 \sigma (T_w - T_1)^2}, \tag{3.4}$$

and upon multiplying by u_1/Q_w, this equation integrates to yield

$$\frac{u_1^2}{Q_w^2} = \frac{2\nu}{a_0 k^2 \sigma (T_w - T_1)^2} \int_0^x u_1 dx,$$

or

$$Q_w(x) = -k\left(\frac{a_0\sigma}{2\nu}\right)^{1/2} u_1(x) \left\{\int_0^x u_1 dx\right\}^{-1/2} (T_w - T_1). \tag{3.5}$$

We remember that this represents the heat transfer to the wall at a station x, due to a step in wall temperature of magnitude $T_w - T_1$ at position $x = 0$. More generally, with $T_w - T_1$ written as T_0 {as in (1.5)}, the heat transfer when there is a step $\Delta T_0(\xi)$ at $x = \xi$ is

$$\Delta Q_w(x) = -k\left(\frac{a_0\sigma}{2\nu}\right)^{1/2} u_1(x)\left\{\int_\xi^x u_1 dx\right\}^{-1/2} \Delta T_0(\xi), \tag{3.6}$$

and for an arbitrary distribution of wall temperature

$$Q_w(x) = -k\left(\frac{a_0\sigma}{2\nu}\right)^{1/2} u_1(x)\int_0^x\left\{\int_\xi^x u_1 dx\right\}^{-1/2} dT_0(\xi), \tag{3.7}$$

where the integral is to be interpreted in the Stieltjès sense (1.6). The form of this result agrees with that obtained by Morgan *et al*, who show that the asymptotically exact value of $(\frac{1}{2}a_0)^{1/2}$ as $\sigma \to 0$ is

$$(\tfrac{1}{2}a_0)^{1/2} = \pi^{-1/2} = 0\cdot5642. \tag{3.8}$$

We now examine the possibility of using an equation such as (3.7) when the Prandtl number is of order unity. It is clear that the approximation to the velocity in (3.2) should be amended when the velocity and thermal boundary layers have the same order of thickness, and it would seem appropriate to set the velocity equal to say βu_1, where β will be less than unity, tending to unity as σ tends to zero. Accordingly a_0 must be replaced by βa_0 in equations (3.4) to (3.7). The problem is to choose the value of βa_0 in the revised form of (3.7)

$$Q_w(x) = -k\left(\frac{\beta a_0\sigma}{2\nu}\right)^{1/2} u_1(x)\int_0^x\left\{\int_\xi^x u_1 dx\right\}^{-1/2} dT_0(\xi). \tag{3.9}$$

When $\sigma = 0\cdot7$, the values of $(\frac{1}{2}\beta a_0)^{1/2}$ required to obtain exact agreement with the similarity solutions are (i) $0\cdot418$ for a stagnation point, (ii) $0\cdot350$ for zero pressure gradient, and (iii) $0\cdot250$ at separation. The significance of these differences is fairly easy to see. The asymptotic value (3.8) is obtained by assuming zero viscous boundary-layer thickness. At finite σ the viscous boundary layer is of order $\sigma^{1/2}$ times the thermal boundary layer in thickness, the precise ratio being further affected by the influence of pressure gradient upon the viscous boundary-layer thickness. Accordingly we expect the required correction to the asymptotic value to be greater near separation than when there is zero pressure gradient, and to be smaller again near to a stagnation point, in view of the relative boundary-layer thickness at these positions. This is in fact what is found.

As a first attempt to account for these effects empirically when σ is of order unity, it would seem reasonable to take

$$(\tfrac{1}{2}\beta a_0)^{1/2} = 0\cdot5642 - \sigma^{1/2} g(m), \tag{3.10}$$

where m is the pressure-gradient parameter $u_1'\delta_2^2/\nu$. Assuming that m is calculated by the method of Thwaites[12] (1949), so that

$$m = -0\cdot45\, u_1' u_1^{-6}\int_0^x u_1^5 dx, \tag{3.11}$$

it is found that (3.9) agrees with the three values cited above provided

$$g(-0\cdot075) = 0\cdot174, \quad g(0) = 0\cdot257, \quad g(0\cdot074) = 0\cdot376. \tag{3.12}$$

A particularly simple expression, agreeing with these three values to $\pm$ 4 per cent, is

$$g(m) = 0\cdot265\,(1+5m), \tag{3.13}$$

and this will be accepted in all that follows.

It is rather difficult to estimate precisely what accuracy would be expected from the empirically corrected formula. It may be remarked, however, that the choice of a constant value, $(\frac{1}{2}\beta a_0)^{1/2} = 0\cdot 313$, yields values of the heat transfer agreeing with the similar solutions to $\pm$ 25 per cent. Presumably the rough correction suggested above should account for most of this difference. It will be noted that numerical calculations require nothing more difficult than the quadratures in (3.9) and (3.11).

4. *Some Calculated Results for Flow Past a Circular Cylinder.* The methods developed in the preceding sections have been applied to the calculation of the heat transfer in sub-critical flow past a circular cylinder which is heated to a constant temperature. This case is particularly appropriate, since it has been extensively studied both experimentally and theoretically. It was found by Heimenz[13] (1911) that for a cylinder of diameter d and an oncoming stream of velocity U_0, the local velocity u_1 at the edge of the boundary layer could be expressed as

$$\frac{u_1}{U_0} = 3\cdot 628\,\frac{x}{d} - 2\cdot 164\,\frac{x^3}{d^3} - 1\cdot 507\,\frac{x^5}{d^5} \tag{4.1}$$

throughout the region from the forward stagnation point to separation.

Since the excess of wall temperature T_w over that of the main stream T_1 was constant, equations (2.12) and (3.5) were relevant to calculations of heat transfer by the two approximations. For the high Prandtl number approximation (2.12), the function $A(x)$ was estimated by the method of Thwaites[12] (1949), and $B(x)$ by direct substitution from (4.1). For the low Prandtl number approximation the relevant value of the parameter $(\frac{1}{2}\beta a_0)^{1/2}$ was given by (3.10) and (3.13), the values of $m(x)$ having been already determined by Thwaites's method when calculating $A(x)$.

The results are shown in Fig. 1, together with the theoretical results of Squire[7] (1942) and Eckert and Drewitz[8] (1942). In calculating these results the Prandtl number σ was taken to be $0\cdot 715$, the representative length l and velocity U_0 being the diameter d of the cylinder and the velocity U_0 of the oncoming stream, respectively. Also shown are the limits between which the relevant experimental results of Schmidt and Wenner[14] (1941) lie.

It will be noted that of the four sets of theoretical results the low Prandtl number approximation of the present paper is clearly superior to the others; it is also, of course, comparatively simple to apply. The high Prandtl number approximation is also seen to give better results than either of the other methods.

A feature of these comparisons which is at first sight surprising is that the low Prandtl number approximation gives better results than the high Prandtl number approximation. On the basis of the expected accuracies, as discussed in Sections 2 and 3, we would have expected these to have been reversed. Possible reasons for this are as follows.

(i) The high σ approximation was shown above to give agreement to within $\pm$ 1 per cent with the 'similar' solutions. It is not clear, however, that such good agreement will be given for more realistic cases. For example, calculations of the development of the velocity boundary layer by methods based upon the similar solution (Walz[15], 1941), with separation when $m \simeq 0\cdot 07$ are well known to be less accurate in general than methods with separation occurring when $m \simeq 0\cdot 09$ (Thwaites[12], 1949, Curle and Skan[16], 1957). It would not be surprising, therefore, if related discrepancies arise in calculating thermal boundary layers by methods based on the similar solutions. This point is illustrated by the results obtained by the method of Eckert and Drewitz[8] (1942), which is based wholly on the similar solutions, since Fig. 1 indicates that it is less accurate than the other methods.

(ii) The exceedingly high accuracy of the high σ approximation was achieved in cases for which the skin friction was precisely known. In general this is only known approximately, and the accuracy of the predicted heat transfer is reduced accordingly. To illustrate this point we remark that when the function $A(x)$ is calculated by Thwaites's method, with separation on the circular cylinder at an angle of 78·6 deg from the stagnation point, the predicted value of K at separation is about 0·48. When, however, $A(x)$ is slightly altered, so as to give separation at an angle of 81 deg (the experimentally obtained value) the value of K at this position is 0·52.

It will be seen, then, that though we have every reason to expect the predicted accuracy of say $\pm$ 5 per cent to be achieved in general by the low Prandtl number approximation, a more realistic estimate of the accuracy of the high Prandtl number approximation awaits the calculation of exact solutions for physically more acceptable cases. The results of the comparison with the experimental results in the case of the circular cylinder suggest that the low Prandtl number approximation may well be the more accurate.

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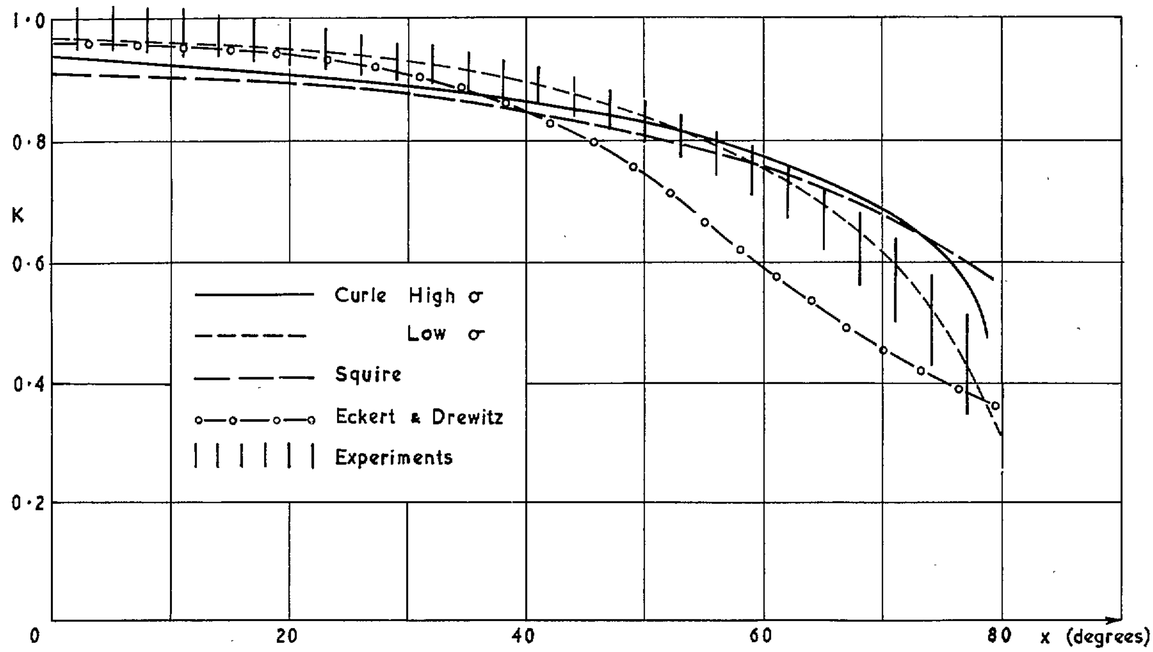

FIG. 1. Comparisons between theories and experiments for circular cylinder.

Part II.—Calculations based on Stratford's Method

Summary. The now well-known technique of Stratford is used to calculate heat-transfer rates through a constant-property laminar boundary layer with an arbitrary adverse pressure gradient. The velocity profiles are assumed to have been calculated by Stratford's method, linking solutions valid respectively in the inner and outer parts of the boundary layer, and a similar division is made in analysing the temperature profiles. A particularly simple formula is obtained for the heat-transfer rate at the wall.

The accuracy of the method, estimated by comparison with such exact solutions as are available, is found to be reasonable.

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1. *Introduction and Outline of Method.* In two-dimensional low-speed flow a very accurate (and physically enlightening) method of calculating laminar boundary-layer development in an adverse pressure gradient is that due to Stratford[1] (1954). Stratford developed the idea, originally due to Kármán and Millikan[2] (1935), of dividing the boundary layer into two regions. In the inner region near the wall, where inertia forces are small, the velocity profile is determined principally by the balance between viscous and pressure forces. In the outer part of the boundary layer, where viscous effects are small, changes in velocity profile are determined mainly by the balance between pressure and inertia forces, as expressed in Bernoulli's equation that the total head does not vary along a streamline. By expressing these conditions appropriately, allowing approximately (in effect) for second-order effects, and linking together the solutions for the outer and inner regions, Stratford derives a criterion for boundary-layer separation. This criterion was later extended by the author (Curle[3], 1960) and a method derived for calculating the distribution of skin friction in a boundary layer with an adverse pressure gradient, including the effects of distributed suction.

The purpose of this paper is to indicate how similar considerations may be used to provide a rapid, convenient and reasonably accurate solution for the thermal boundary layer. It is assumed that the viscous boundary-layer problem has been solved previously by Stratford's method. For constant-property flows the equation for the temperature profile takes the form

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{1.1}$$

where x, y, are measured parallel and normal to the wall, the associated velocity components are u and v, and κ is the thermometric conductivity. The method of solution (as for the viscous layer) is to link two solutions, valid in the inner and outer parts of the thermal boundary layer respectively. By this procedure an explicit expression is obtained for the heat-transfer rate at the wall,

$$Q_W(x) = \sigma^{1/3}\frac{k(T_1 - T_W)\tau_B}{\mu u_0}(0\cdot 6642 + 0\cdot 3358\, T_x)^{1/2}, \tag{1.2}$$

where k is the thermal conductivity of the fluid, σ its Prandtl number, μ its viscosity, T_W and T_1 are respectively the absolute temperatures at the wall and in the main stream, u_0 is the velocity at the leading edge,* $\tau_B(x)$ is the skin friction in a Blasius boundary layer with external velocity u_0, and T_x is the ratio τ/τ_B of local skin friction to the local zero pressure-gradient value.

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* Note that when there is an initial region of favourable pressure gradient, u_0 is equal to the maximum external velocity, and x is measured from a fictitious origin.

The predictions of (1.2) are compared with accurate values in two particular cases. Firstly, at the position of separation, where $T_x = 0$, (1.2) yields

$$Q_W(x) = 0 \cdot 72 \frac{k(T_1 - T_W)\tau_B}{\mu u_0} \tag{1.3}$$

when $\sigma = 0 \cdot 7$. Now in an exact solution of the boundary-layer equations the value of this numerical coefficient would presumably depend upon the pressure gradient. By an argument based on the 'similar' profiles a value of about $0 \cdot 57$ is obtained at a Prandtl number of $0 \cdot 7$. Secondly, for the case

$$u_1 = u_0 \left(1 - \frac{x}{c}\right), \tag{1.4}$$

it is shown that the exact solution for small values of x/c is

$$Q_W = \frac{k(T_1 - T_W)}{\mu u_0} \tau_B \left(1 - 1 \cdot 86 \frac{x}{c} \ldots\right), \tag{1.5}$$

when $\sigma = 1$, whereas the value of the coefficient of x/c is given as $2 \cdot 13$ by the present method.

It is deduced that in the presence of an adverse pressure gradient the predicted value of Q_W will decrease initially with distance slightly more rapidly than the accurate value. On the other hand, by the time separation has been reached, the predicted Q_W is likely to be too high, but possibly not by as much as the 20 per cent found for the similar profiles.

Finally the various limitations of the method are discussed. It is suggested that these are not unduly serious, and that the accuracy of the method bears favourable relationship to the small amount of work involved.

2. *General Theory.* We take co-ordinates x, y, measured along and normal to the wall, with associated velocity components u and v, and absolute temperature T. Then for constant-property flows, the continuity, momentum and energy equations of the laminar boundary layer become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.1}$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = u_1 \frac{du_1}{dx} + \nu \frac{\partial^2 u}{\partial y^2}, \tag{2.2}$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \kappa \frac{\partial^2 T}{\partial y^2}. \tag{2.3}$$

The boundary conditions at the wall, $y = 0$, are that u and v are zero, with T taking the value T_W, usually known. Far enough away from the wall u tends to the known main-stream value $u_1(x)$ and T tends to the (constant) value T_1.

We assume that equations (2.1) and (2.2) have already been solved by the author's generalization (Curle[3], 1960) of the method of Stratford[1] (1954). It must be remembered that this method is not applicable in the vicinity of a stagnation point; for $x \leqslant x_0$ we assume that the external velocity $u_1(x)$

takes the constant value u_0, and the velocity profile is of the Blasius type. Then when $x \geqslant x_0$ the solution of (2.1) and (2.2) in the region $y \leqslant y_j$, near the wall, may be written (Curle[3], 1960)

$$\mu u = \tau y + \tfrac{1}{2}\frac{dp}{dx}y^2 + a(x)y^n, \tag{2.4}$$

where the coefficients are

$$\tau = \mu\left(\frac{\partial u}{\partial y}\right)_W, \tag{2.5}$$

$$\tfrac{1}{2}\frac{dp}{dx} = \tfrac{1}{2}\rho u_1\frac{du_1}{dx}, \tag{2.6}$$

satisfying exactly a boundary condition at the wall, and $a(x)$, determined by the relationships at the join between (2.4) and an outer solution; the constant n is taken to equal $3\cdot043$. The value of τ is given by the equation

$$x^2C_p\left(\frac{dC_p}{dx}\right)^2 = 0\cdot0104(1-T_x)^3(1+2\cdot02T_x) \tag{2.7}$$

where

$$T_x = \frac{\tau}{\tau_B}, \tag{2.8}$$

τ_B being the skin friction for the two-dimensional Blasius boundary layer at station x, and C_p is the pressure coefficient, $1 - u_1^2/u_0^2$.

Having obtained this solution, we now turn to the solution of equation (2.3). In the outer part of the boundary layer we write this equation in the alternative form

$$u\frac{\partial T}{\partial s} = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{2.9}$$

where $\partial/\partial s$ denotes differentiation at constant ψ, where

$$\psi = \int_0^y u\,dy. \tag{2.10}$$

Following Stratford[1] (1954) we now expand $T(x, \psi)$ as

$$T(x, \psi) = T(x_0, \psi) + (x-x_0)\left(\frac{\partial T}{\partial s}\right)_{x_0, \psi} + 0(x-x_0)^2. \tag{2.11}$$

Now the pressure gradient, which is present only downstream of $x = x_0$, will affect the shape of the outer part of the boundary layer only some way downstream of $x = x_0$ since the effects will diffuse outwards from the wall. Accordingly we deduce from (2.9) that $\partial T/\partial s$ is unaffected in the outer part of the layer, so it follows from (2.11) that when $\psi > \psi_j'$, say, $T(x, \psi)$ is exactly as it would have been in the absence of the pressure gradient, provided terms of order $(x-x_0)^2$ are neglected. The solution $T_B(x, \psi)$ of equation (2.1) in the absence of the pressure-gradient term was given by Pohlhausen[4] (1921). It may easily be shown from his analysis (anticipating the approximation below) that when ψ is sufficiently small

$$T_B(x, \psi) \simeq T_W + \sigma^{1/3}(T_1 - T_W)\left(\frac{2\tau_B\psi}{\mu u_0^2}\right)^{1/2}\ldots \tag{2.12}$$

where σ is the Prandtl number ν/κ, and the factor $\sigma^{1/3}$ is a good approximation to a numerically-defined function. It must be stressed that though the value of ψ_j' will be of the same order of magnitude as ψ_j (which determined the join in the u-profile), there is no *a priori* reason why they should be equal. We shall assume, however, that ψ_j' is sufficiently small for $T_B(x, \psi)$ to be approximated by equation (2.12) whenever $\psi \leqslant \psi_j'$.

Turning now to the inner region, $\psi \leqslant \psi_j'$, we assume a form

$$T = T_W + \frac{Q_W}{k} y, \tag{2.13}$$

where Q_W is the heat-transfer rate to the wall per unit area. This linear form should be a good approximation to T throughout a considerable region near the wall, since $T(0) = T_W$, and it further follows from (2.3) that both $\partial^2 T/\partial y^2$ and $\partial^3 T/\partial y^3$ are zero at the wall itself. We now link together the outer and inner solutions, (2.12) and (2.13), making T and $\partial T/\partial y$ continuous at the join $y = y_j'$, $\psi = \psi_j'$. Thus we have

$$\frac{Q_W}{k} y_j' = \sigma^{1/3}(T_1 - T_W)\left(\frac{2\tau_B}{\mu u_0^2}\psi_j'\right)^{1/2} \tag{2.14}$$

by continuity of T. Further, by differentiating (2.14) with respect to ψ, it follows from (2.13) that

$$\frac{Q_W}{k} = \sigma^{1/3}(T_1 - T_W)u_j'\left(\frac{\tau_B}{2\mu u_0^2 \psi_j'}\right)^{1/2}, \tag{2.15}$$

by continuity of $\partial T/\partial y$. These last two equations yield Q_W and any one of ψ_j', y_j', $u_j' = u(x, \psi_j')$, these three being related through the velocity profile $u(x, \psi)$. It is assumed, for simplicity, that the formula (2.4) may be used when $\psi \leqslant \psi_j'$. This means, strictly speaking, that ψ_j' must be less than ψ_j, but since the inner and outer solutions for u have continuous values of u, $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ at $\psi = \psi_j$, (2.4) may in practice be adequate for values of ψ somewhat greater than ψ_j. Assuming this holds, then

$$\mu u_j' = \tau y_j' + \tfrac{1}{2}\frac{dp}{dx} y_j'^2 + a(x) y_j'^n, \tag{2.16}$$

and

$$\mu \psi_j' = \tfrac{1}{2}\tau y_j'^2 + \tfrac{1}{6}\frac{dp}{dx} y_j'^3 + \frac{a(x)}{n+1} y_j'^{n+1}. \tag{2.17}$$

We now eliminate u_j', ψ_j', y_j', between (2.14) to (2.17). By squaring (2.14) and substituting for $\mu\psi_j'$ from (2.17), we find that

$$\frac{Q_W^2}{k^2} = \sigma^{2/3}\frac{(T_1 - T_W)^2}{\mu^2 u_0^2}\tau_B\left\{\tau + \tfrac{1}{3}\frac{dp}{dx} y_j' + \frac{2a(x)}{n+1} y_j'^{n-1}\right\}. \tag{2.18}$$

Also, by multiplying (2.14) and (2.15), we find that

$$\frac{Q_W^2}{k^2} = \sigma^{2/3}\frac{(T_1 - T_W)^2}{\mu^2 u_0^2}\tau_B\left\{\tau + \tfrac{1}{2}\frac{dp}{dx} y_j' + a(x) y_j'^{n-1}\right\}. \tag{2.19}$$

These equations are compatible provided

$$\tfrac{1}{6}\frac{dp}{dx} + \frac{n-1}{n+1} a(x) y_j'^{n-2} = 0, \tag{2.20}$$

whence

$$\frac{Q_W^2}{k^2} = \sigma^{2/3}\frac{(T_1 - T_W)^2}{\mu^2 u_0^2}\tau_B\left\{\tau + \frac{n-2}{3(n-1)}\frac{dp}{dx} y_j'\right\}. \tag{2.21}$$

Thus, defining $a(x)$ from Ref. 3 as

$$\left.\begin{aligned} a(x)y_j^{n-1} &= -\frac{\tau_B - \tau}{n(n-2)} \\ \frac{dp}{dx}y_j &= \frac{(n-1)(\tau_B-\tau)}{n-2} \end{aligned}\right\}, \tag{2.22}$$

we can calculate y_j' from (2.20) and then Q_W from (2.21).

Now we may deduce from (2.20) and (2.22) that

$$(y_j'/y_j)^{n-2} = \tfrac{1}{6}n(n+1), \tag{2.23}$$

and, from (2.21) and (2.22), that

$$\frac{Q_W^2}{k^2} = \sigma^{2/3}\frac{(T_1-T_W)^2}{\mu^2 u_0^2}\tau_B\left\{\tau + \tfrac{1}{3}(\tau_B-\tau_x)\frac{y_j'}{y_j}\right\}, \tag{2.24}$$

or

$$Q_W = \sigma^{1/3}\frac{k(T_1-T_W)}{\mu u_0}\tau_B\left\{T + \tfrac{1}{3}(1-T_x)\frac{y_j'}{y_j}\right\}^{1/2}. \tag{2.25}$$

Upon setting $n = 3\cdot043$ in (2.23) we deduce that

$$y_j'/y_j = 1\cdot9926, \tag{2.26}$$

so that (2.25) becomes

$$Q_W = \sigma^{1/3}\frac{k(T_1-T_W)}{\mu u_0}\tau_B\{0\cdot6642 + 0\cdot3358\,T_x\}^{1/2}. \tag{2.27}$$

This result is, of course, exact when there is no pressure gradient, and is expected to become less accurate as $T_x \to 0$, at the separation position, where it predicts that

$$Q_W = 0\cdot815\sigma^{1/3}\frac{k(T_1-T_W)}{\mu u_0}\tau_B, \tag{2.28}$$

the numerical coefficient $0\cdot815\sigma^{1/3}$ being equal to $0\cdot72$ when $\sigma = 0\cdot7$. Now the exact value of Q_W will presumably depend upon the particular pressure gradient, and it is not known just how widely it varies in typical cases. It can be said here, however, that for the case of the similarity solutions (possibly a rather extreme case) the accurate value of the numerical coefficient is $0\cdot600\,(u_1/u_0)^{1/2}$, where u_1/u_0 is the ratio of the value of the external velocity at separation to its maximum value. For a fair comparison we should perhaps set $(u_1/u_0)^{1/2} = 0\cdot95$ or thereabouts, so that a value of $0\cdot57$ results, about 20 per cent less than the predicted value.

We turn now to an example in which the external velocity $u_1(x)$ is a power series in x/c, so that (Tifford[5], 1954) the solutions for both u and T may be expanded in a power series in x/c, whose coefficients are functions of the distance y normal to the wall. In particular, for the case

$$u_1 = u_0\left(1-\frac{x}{c}\right), \tag{2.29}$$

it may be shown that

$$\begin{aligned} T &= T_1 + \frac{(T_1-T_W)u_0}{u_1}\left\{\phi_0 - \frac{x}{c}\phi_1\ldots\right\} \\ &= T_1 + (T_1-T_W)\left\{\phi_0 - \frac{x}{c}(\phi_1-\phi_0)\ldots\right\}, \end{aligned} \tag{2.30}$$

where ϕ_0, ϕ_1, satisfy the equations

$$\frac{1}{\sigma}\phi_0'' + f_0\phi_0' = 0, \tag{2.31}$$

$$\frac{1}{\sigma}\phi_1'' + f_0\phi_1' - 2f_0'\phi_1 = -2f_0'\phi_0 - 24f_1\phi_0', \tag{2.32}$$

with boundary conditions

$$\phi_0(0) = -1, \quad \phi_0(\infty) = 0, \quad \phi_1(0) = -1, \quad \phi_1(\infty) = 0. \tag{2.33}$$

Similarly the functions f_0, f_1, satisfy

$$f_0''' + f_0f_0'' = 0, \tag{2.34}$$

$$f_1''' + f_0f_1'' - 2f_0'f_1' + 3f_0''f_1 = -1, \tag{2.35}$$

subject to the boundary conditions

$$f_0(0) = f_0'(0) = 0, \quad f_0'(\infty) = 2, \quad f_1(0) = f_1'(0) = 0, \quad f_1'(\infty) = \tfrac{1}{4}. \tag{2.36}$$

In all the above, ϕ_0, ϕ_1, f_0 and f_1 are all functions of

$$\eta = \tfrac{1}{2}y\left(\frac{u_0}{\nu x}\right)^{1/2}, \tag{2.37}$$

and primes denote differentiation with respect to η.

Now the functions f_0 and f_1 have been calculated by Howarth[6] (1938), and it may be seen by inspection that when $\sigma = 1$ the solution of (2.31) is

$$\phi_0 = \tfrac{1}{2}f_0' - 1. \tag{2.38}$$

It remains to calculate ϕ_1, which requires the solution of (2.32). By writing

$$\phi_1 = \phi_0 + 4f_1' - G, \tag{2.39}$$

it is found that G must satisfy the equation

$$G'' + f_0G' - 2f_0'G = -4, \tag{2.40}$$

with boundary conditions

$$G(0) = 0, \quad G(\infty) = 1. \tag{2.41}$$

This equation has been integrated on the DEUCE by Mathematics Division, N.P.L., and the solution is shown in Table 1. The method of solution was to replace (2.40) by its finite-difference equivalent, which was solved by a standard linear-equations programme.

Substituting now from (2.38) and (2.39) into (2.30), we find that

$$T = T_W + (T_1 - T_W)\left\{\tfrac{1}{2}f_0' - \frac{x}{c}(4f_1' - G)\ldots\right\}, \tag{2.42}$$

so that

$$Q_W = k\left(\frac{\partial T}{\partial y}\right)_W = \tfrac{1}{2}k\left(\frac{u_0}{\nu x}\right)^{1/2}\left(\frac{\partial T}{\partial \eta}\right)_W$$

$$= \tfrac{1}{2}k\left(\frac{u_0}{\nu x}\right)^{1/2}(T_1 - T_W)\left\{\tfrac{1}{2}f_0'' - \frac{x}{c}(4f_1'' - G')\ldots\right\}_W. \tag{2.43}$$

Upon dividing by

$$\tau_B = \tfrac{1}{4}\mu u_0 \left(\frac{u_0}{\nu x}\right)^{1/2} f_0''(0), \tag{2.44}$$

this yields

$$Q_W = \frac{k(T_1 - T_W)}{\mu u_0}\,\tau_B \left\{1 - \frac{2x}{c}\,\frac{4f_1''(0) - G'(0)}{f_0''(0)} \cdots\right\}, \tag{2.45}$$

or, upon substituting for $f_0''(0)$, $f_1''(0)$, $G'(0)$,

$$Q_W = \frac{k(T_1 - T_W)}{\mu u_0}\,\tau_B \left(1 - 1\cdot 86\,\frac{x}{c} \ldots\right). \tag{2.46}$$

This, then, is the accurate solution immediately downstream of the leading edge. We compare with the solution given by the approximate method of this paper. In (2.7) we set

$$C_p = 1 - \frac{u_1^2}{u_0^2} \sim 2\,\frac{x}{c} + \ldots, \tag{2.47}$$

so that it yields

$$T = 1 - 6\cdot 338\,\frac{x}{c} \ldots . \tag{2.48}$$

Then upon substituting into (2.27) we find that

$$Q_W = \frac{k(T_1 - T_W)}{\mu u_0}\,\tau_B \left(1 - 2\cdot 13\,\frac{x}{c} \ldots\right). \tag{2.49}$$

We may regard the term in x/c as representing the effects of transverse pressure gradient upon the heat transfer. By comparison of (2.46) and (2.49) it is clear that the present method overestimates these effects a little, so that the predicted heat-transfer rate Q_W is slightly low. We note that the error is of opposite sign to that found earlier for the separating similarity solution, suggesting again that the predicted heat-transfer rate at separation may not be greater than the exact value by as much as the 20 per cent found for the similarity solutions.

3. *Limitations of the Method.* We examine first the theoretical limitations of the method, that is those which arise because of the various approximations made in the theory.

It is clear that since $u(x, \psi)$, which was assumed known in this paper, was itself calculated by an approximate method, the results for $T(x, \psi)$ will presumably be less accurate in general. This, however, is a limitation which applies to any approximate method of dealing with this type of problem, and one would imagine that less accurate methods than Stratford's of calculating $u(x, \psi)$ would suffer even graver deficiencies when applied to the calculation of $T(x, \psi)$.

It follows from the above that some consideration must be given to the theoretical limitations of Stratford's method. When applied in the form used in the present paper, to calculations of two-dimensional boundary layers, it is necessary to assume that the pressure coefficient is less than 0·11 at most, and it should be considerably smaller, strictly speaking. Nevertheless, the method has been found to give accurate predictions of the skin friction when the formula (2.7) has been applied beyond this range, even to values of C_p as great as 0·25. This can only be due to a convenient cancelling of errors. Now it is by no means clear that a similar cancellation will occur when calculating $T(z, \psi)$, so it may be necessary to limit the range of pressures to which the method is applied.

A further assumption, made in the analysis of this paper, was that the join of the outer and inner T-profiles occurred within the inner u-profile, so that $y_j' \leqslant y_j$. In fact the calculations indicate that this does not hold, for we have shown in equation (2.26) that

$$y_j' \simeq 2y_j. \tag{3.1}$$

It is probable, however, that this is not a serious source of error. In joining the inner and outer u-profiles it was specified that ψ, u, $\partial u/\partial y$ and $\partial^2 u/\partial y^2$ should be continuous, so the inner solutions for ψ and u may be fairly accurate for a sufficient distance beyond the join.

Turning now to the practical limitations of the method, the obvious one is that the method predicts only the heat-transfer rate at the wall, and not the details of the temperature profiles. Again this does not seem to be too great a drawback, since it is usually the heat-transfer rate which is of interest, rather than the various integral measures of boundary-layer thickness, which may be estimated fairly well by methods of the Pohlhausen type.

In conclusion it should be stressed that the amount of work involved in applying this method to a particular case is very small, and that the accuracy obtained is a good return for one's effort.

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TABLE 1

The Function $G(\eta)$

η	$G(\eta)$	η	$G(\eta)$
0·0	0·0000	2·0	1·0369
0·2	0·4894	2·2	1·0204
0·4	0·8271	2·4	1·0104
0·6	1·0339	2·6	1·0049
0·8	1·1385	2·8	1·0021
1·0	1·1715	3·0	1·0008
1·2	1·1610	3·2	1·0003
1·4	1·1298	3·4	1·0001
1·6	1·0936	3·6	1·0000
1·8	1·0614	3·8	1·0000

$$G'(0) = 2{\cdot}847$$

(85653) Wt. 64/1857 K5 12/62 Hw.

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