R. & M. No. 3084
(15,977)
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

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LONDON: HER MAJESTY'S STATIONERY OFFICE
1958
PRICE 10*s.* 6*d.* NET

# The R.A.E. Hollerith Punched-Card Installation (1945–1952)

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Communicated by the Director-General of Scientific Research (Air), Ministry of Supply

*Reports and Memoranda, No.* 3084*

*December,* 1952

*Summary.*—This report sketches the scope and working of the punched-card installation in Mathematical Services Department, Royal Aircraft Establishment, during the period from 1945 to 1952. Types of work for which the equipment is best suited are indicated, and some representative computing problems that have been handled by the equipment discussed.

1. *Introduction.*—This report sketches the scope and working of the Hollerith punched-card installation in Mathematical Services Department, R.A.E., and indicates the kinds of work for which it is better suited.

Possible use of punched-card machines for matrix calculations in structural and aero-elastic work had been studied from 1938 onwards by Pugsley[1], Schmidt and Fairthorne[2] at the R.A.E., and by Frazer[3] and Bratt at the National Physical Laboratory. In 1940 the Oscillations Sub-Committee of the Aeronautical Research Council recommended establishment of a punched-card unit for this type of work. By 1944 there was prospect of apparatus becoming available for the purpose, and the Treasury (O. & M.), approved and aided its acquisition.

The Computing Laboratory was formed as a section in Structural and Mechanical Engineering Department in August 1945, and equipped, together with desk calculators and a National Accounting Machine, Class 3000, with a Hollerith installation made up of Sorter, Collator, Reproducer, Multiplying-Punch and an E6/6 Tabulator, the last named being a temporary substitute for a Senior Rolling-Total Tabulator with Summary Punch, which soon replaced it. The E6/6 had some sterling sections, so for a short time it was occasionally necessary to display results in pounds, shillings and pence.

In 1948 the Computing Laboratory joined forces with other units working in other fields of computation in other Departments to form Mathematical Services Department.

2. *General Principles.*—Information to be processed in punched-card machines is first converted into patterns of holes in standardized cards. The cards are fed into the machines, and electrical impulses ‘ read ’ the pattern of holes and convert them into a pattern of timed electric currents.

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* R.A.E. Report M.S. 52, received October, 1953.

The machine responds to these currents according to its specific design and to its wiring for the particular problem; it may punch new cards, print signals, transfer signals to new positions, etc. Packs of cards are moved from machine to machine by hand. One popular treatment of punched card and other automatic computing machines, together with a bibliography is Ref. 4 and some punched-card realizations of mathematical processes are illustrated in Ref. 5.

2.1. *The Punched Card.*—Punched cards are used to feed data into any Hollerith machine and for all permanent storage. A reproduction of a typical card is given in Fig. 1. They are made from special manilla paper, each card measuring $7\frac{3}{8}$ in. by $3\frac{1}{4}$ in. The length of a card is divided into eighty columns; the width into twelve levels referred to as the $X$, $Y$, 0, 1, 2, . . 9 positions (or levels) respectively. The $X$ and $Y$ are purely designatory positions for giving special instructions to the machines. For decimal arithmetics each column can store a signal corresponding to one decimal figure, so that if necessary one card could hold one 80-figure number or, for example, eight 10-figure numbers. The actual digit stored in any column is indicated by the position of the hole punched in that column. Thus the card illustrated in Fig. 1 holds the number 384 in columns 26 to 28 (a group of columns is known as a ' field ').

2.2. *Punching and Verification of Cards.*—The initial punching of cards is done on a hand punch, which has a separate key to correspond to each level on the card. The speed of punching by a skilled operator (if all 80 columns of each card are punched) is about 250 cards per hour; if only 30 columns in each card are punched, the speed would be about twice this.

The punching is checked by a second operator who may be regarded as punching the cards over again using another keyboard device known as a Verifier. This is really a hand punch, but with the sharp punching knives replaced by blunt plungers. The verifier is arranged to stop when there is no hole in the correct position to correspond with the key being pressed.

These punches are about the size of hand-cranked desk calculators and can be used, after a little training, by those outside Mathematical Services Department at the source of information. Thus experimental and similar data can be recorded directly on to cards more quickly and legibly than by handwriting without having to be copied from intermediate and possibly miscopied records. Such cards can be used not only for calculation, but also for automatic reproduction of their contents in typescript, with control over the lay-out and selection of data printed, for the Hollerith Tabulator is an automatic typist as well as an automatic computor.

Direct recording on to punched-cards is not always possible, but clearly is something to consider as a means to prevent waste of labour and time copying the same data into different forms.

Development of instruments directly recording on to punched cards and other interchangeable media is an important part of the joint work of Instrumentation and Mathematical Services Departments.

Recently a Programme-Board Punch and Programme-Board Verifier have been added to the installation. In these punching and verification of holes are electrically powered. The former device can be set before punching begins so that any common patterns are punched automatically into every card passing through the punch feed; such patterns being punched on a separate card held in a special pocket. The punch can also be set to skip automatically over any group of columns not to be punched by the operator. This facility also exists on the programme-board verifier.

3. *The R.A.E. Installation.*—The functions of the various machines making up the R.A.E. installation will now be described. Most of the machines, which are self-contained, cannot be purchased but are hired from the manufacturers, The British Tabulating Machine Co. of Letchworth, who have modified some machines to meet non-standard requirements.

3.1. *The Sorter*.—This selects cards, reading one column on each run, and places them in numbered pockets according to the values of the numerals punched in that column. Thus, to place in order a set of cards punched with three-figure numbers, three separate runs of all the cards through the machine would be required, one run for each figure, the cards being withdrawn from the pockets and the pack reassembled between each run. The speed of the machine is 400 cards per minute.

Fig. 2 is a diagrammatic sketch of the sorter. From the feed stacker, which can hold rather less than 1,000 cards, the cards are fed between a brass roller and a brush which can be set to read any one column of the cards. After leaving the reading station the cards pass through the selecting mechanism whence they go to the card pockets. There is one pocket corresponding to each of the levels 9, 8, . . ., 1, 0, $X$, $Y$ in a card, and a reject pocket corresponding to an unpunched column. The selecting mechanism consists of a set of spring-steel tongues, called chute blades, resting on a plate which is the armature of an electro-magnet, called the sorter magnet, underneath. The magnet is energised and pulls down its armature as soon as a circuit is completed through a hole in a card. The cards feed through the machine with their ' 9's '-edges leading. If a card at the reading station has a 9 punched in the column being read by the pre-set brush, the sorter magnet will be energised and the chute blades dropped before the card reaches them, and the card will slide over the top of all the shute blades and be conducted between rollers to the 9-pocket. If, however, the card be punched with say a 5, its leading edge will have passed under the chute blades corresponding to 9, 8, 7 and 6 before the armature drops. Subsequently as it is fed forwards between rollers the card will be conducted between the chute blades corresponding to 6 and 5 and dropped in the 5-pocket. If a card is completely blank in the column being read, it passes under all the chute blades into the reject pocket.

The sorter has a feature called a ' drum ', which rotates in synchronism with a card passing through the reading station. Round its periphery the drum has a set of twelve pre-set contacts labelled 9, 8, . . ., 1, 0, $X$, $Y$ corresponding to the levels on a card. If any of these contacts are set open, all cards punched with these numbers in the column being read will be treated as blank cards and sorted into the reject pocket. This feature is particularly useful when it is required to separate $X$-punched master cards from their accompanying detail cards (*see* section 3.3.2), the drum then being set with all the contacts open except the $X$ contact.

The sorter is also provided with a number of counters, which display the number of cards in each pocket, and the total number of cards sorted. Some sorters do not have this feature; those which do are called Counter-Sorters.

3.2. *The Reproducer*.—This consists of two parts, the reading and the punching units, each of which has one card feed and one output pocket. The reading unit has also three sets of brushes, and the punching unit two sets of brushes and one set of punching knives (*see* Fig. 3).

The reproducer has three distinct uses, which will now be described.

3.2.1. *Copying*.—This is the operation of copying the holes of one set of cards on to a second (blank) set. The cards are to be copied are fed to the reading unit and the blank cards to the punching unit. As a card feeds under the reading brushes (in the reading unit), a blank card is simultaneously fed under the punching knives (in the punching unit). All brushes, the punching mechanism, and both sides of a comparing mechanism are connected to the back of a plug board on which pluggable connections may be made between the various parts of the machine. Thus, for copying, the reading brushes and punching knives are connected, while the comparing brushes are brought to one side of the comparing mechanism and the punch brushes to the other. If the comparing mechanism detects a discrepancy the machine is stopped, a red light is illuminated, and an indicator shows in which column the discrepancy occurs.

This use of the reproducer is particularly valuable for replacing sets of worn cards by new ones.

3.2.2. *Gang punching*.—Gang Punching is the operation of transferring common patterns from master cards to accompanying groups of detail cards. The master cards are distinguished from the detail cards by an $X$-punching in one column. The Gang-Punching operation, apart from checking, involves the punching unit only. The punch brushes are connected through the plug board to the punching knives, so that any card under the brushes transfers its hole pattern to the following card, which is passing under the knives. This card, now punched, then passes under the punch brushes and transfers the newly punched pattern to the card following it. This process continues until the $X$-punching of the next master card is read by one of the $X$-brushes. This signal stops operation of the punching knives while the master card passes beneath them. On reaching the punch brushes the master card transfers its hole pattern to the following detail card and then this pattern is passed from card to card until the next $X$-punched master card is read by the $X$-brushes.

To check gang punching, the cards are passed through the reading unit, with the reading brushes plugged to one side of the comparing mechanism and the comparing brushes to the other side.

To check a block of cards gang punched from a single master, it is usually enough to check that the last card punched is the same as the master, because the hole pattern has passed down from each intermediate card to the next.

3.2.3. *Summary punching.*—Summary Punching is the operation of punching on to blank cards digital data stored on counters within the Tabulator (*see* Section 3.3). The output of the desired counter is connected through the plug board to the punching knives of the Reproducer. Here again only the punching unit is used.

Summary punching may be checked by listing (*i.e.*, printing direct from cards on to a roll of paper) the summary-punched cards during a separate run through the Tabulator (*see* Section 3.3), and comparing the printed results with those previously printed by the Tabulator during the summary punching. The two sets of numbers should be identical.

3.3. *The Tabulator.*—This is a combination listing, adding, subtracting and printing machine. The Senior Rolling Total Tabulator consists of six registers (' counters ' or ' accumulators '), each holding eleven figures, into which numbers can be added or subtracted; three plug boards (or panels) and a print unit. There are a single card feed with output pocket, and a separate paper feed in the print unit. Fig. 4 shows the connections associated with Plug Board No. 1. The connections shown from the brushes, counters and print unit, to the plug-board, are permanently soldered to the back of the board. Connections from the brushes to the counters or print unit or both may be made as desired by connecting together sockets on the front of the board with plugs and wires. Connections for addition and subtraction (by addition of complements) between counters, and from counters to the print unit are made on plug board No. 2. There are two sets of 80 reading brushes, the upper and lower, so placed that when a card is passing the lower brushes the following card is passing the upper brushes at the same relative position. The machine contains a comparing unit (called the control) by which the digits in not more than sixteen of any columns of the card at the lower brushes may be compared with those in the same number of columns of the card at the upper brushes. If the comparison shows a difference between the corresponding digits of the cards, card feed ceases and one of three types of break of control (minor, intermediate or major) takes place according to the arrangement of the plugging from the brushes to the control. During breaks of control the machine operates according to instructions set up on plug board No. 3. At any break of control, at most eight operative stages or ' cycles ', numbered from 8 down to 1, can be called into play one after the other, the actual number required being arranged by plugging. On any one cycle a counter may send out its data to other counters, to the print unit or, on the penultimate cycle only, to the summary punch. Alternatively a counter may receive signals, but it cannot both send out and receive signals in the same cycle unless it is adding its own numerical value into itself, thus doubling its contents.

The three types of break of control are named Minor, Intermediate, or Major because they correspond to changes in the minor, intermediate, or major groupings of the set of cards. As very simple examples: the printed copy may need a double space after every tenth line. The instruction for such spacing would be triggered by a change in tens digit of the serial number of incoming cards, so the minor control would be associated with a discrepancy of the digits in this position on consecutive cards. Also, after every hundredth card it may be necessary to print some sub-totals accumulated in the counters and to give a few more extra lines of spacing. This would be done by the appropriate instructions set up on the plug board by a discrepancy between the hundreds digits of consecutive cards. A major operation might be called for at every thousandth card, such as subtracting one sub-total from another and printing the balance in one place if positive and in another if negative, transmitting the accumulated values to and operating the summary punch to punch a new card, and so on.

There is a fourth level of control for carrying out special operations, needed only after the last card has been dealt with. This, the Final-Total level, comes into action only after being called for manually by a switch. A measure of manual control is possible on the other levels, but in general it is undesirable to plan jobs so that they demand ‘ machine-sitting ’.

The Tabulator contains five distributors (or selectors), available on panels 1 and 2. Signals sent to a distributor may be routed to one or other of two destinations plugged to its two alternative outputs. The use of a distributor is essential if, for example, the numbers read from the cards vary in sign. The sign designation is always punched in a distinct column, and the distribution is arranged to route the signals to the alternative output when a negative sign designation is read. The two outputs are then plugged to different counters, one accumulating all the positive values and the other all the negative. During a subsequent break of control the counter containing the sum of negative values is subtracted from that containing the positive sum.

The print unit, unlike a typewriter, is a parallel operating device. Up to 79 characters (usually either the decimal digits or the sign symbols, + and —) can be printed across the sheet in one stroke, at a maximum rate of about two lines a second. Spacing and lay-out of the digits is arranged by a combination of plugging and manual setting of hammer locks on the printer. Stored digits may be printed from any of the counters when required (this is called tabulating) or direct from each card (called listing) under the control of plugged connections. The speed of the Tabulator is about 150 cards/minute when simply accumulating the contents of the cards without printing, and about 120 cards/minute when listing. Arithmetical operations between counters take up more time according to their complexity.

The R.A.E. tabulator has recently been adapted to multiply and divide. For multiplication, the multiplier is entered direct from the card into one counter, and then its complement is entered into a second counter. The multiplicand is then repeatedly added from a third counter to a fourth, the products counter. For each such addition unity is added to the second counter until the negative multiplier counter has become zero in that column. The change to zero shifts the additions of multiplicand and unity one place to the left and the process continues till the negative multiplier has been reduced to zero. The example below shows the steps of multiplying 786 by 132.

| | *Negative multiplier counter* | *Multiplicand counter* | *Products counter* |
|---|---|---|---|
| 1. | 9 . . . 9868 | 0 . . . 0786 | 0 . . . 000000 |
| 2. | 9 . . . 9869 | | 0 . . . 000786 |
| 3. | 9 . . . 9870 | | 0 . . . 001572 |
| 4. | 9 . . . 9880 | | 0 . . . 009432 |
| 5. | 9 . . . 9890 | | 0 . . . 017292 |
| 6. | 9 . . . 9900 | | 0 . . . 025152 |
| 7. | 0 . . . 0000 | | 0 . . . 103752 |

To divide, the divisor is added to a negative dividend until the counter changes to positive, the number of additions of the divisor being recorded in another counter. From the simple multiplication example shown it will be appreciated that for larger factors more than eight cycles will be required, and the machine has been fitted with repetitive cycling up to the fourth cycle. The chosen cycle of operations is repeated by the machine until some condition has been fulfilled; in the example above, when the negative multiplier counter has become zero. With these new additions the Tabulator can obtain 20-digit products (using two counters as one for the product) and 10-digit quotients.

Multiplication and division are rather slower on the tabulator than are fully automatic multiplication and division on a modern desk calculator. On the other hand these operations are automatic and the recording and the transfer of these results are automatic and fast. No longer does the tabulator have to be stopped now and again in order to divide some intermediate results on a desk calculator, punch the quotient into a card, and inject this information into the tabulator. In desk work, actual computational time is often negligible compared with that needed for recording and communication.

Computational time is not negligible when multiplication or division are needed at each step or most steps of the procedure as, for instance, in the solution of a differential equation by recurrence formulae connecting values at intervals of the independent variable. Nevertheless, possibility of automatic internal multiplication, even if rather slow, widens the scope of the equipment so much as to make it worthwhile in many routines.

With standard equipment, products can be formed conveniently only by block multiplication of a pack of cards, each carrying the factors of the multiplication, and the products cannot interact until all multiplications are completed. As described in Section 3.4 below, the Multiplying Punch punches the product on the same card as the previously punched multiplicand, and the contents of one card cannot influence the contents of another except to provide a new multiplier, and this multiplier must be a number punched into the card before insertion into the Multiplying Punch.

Thus procedures in which multiples of numbers generated during the course of calculation influence their successors are not easy with standard equipment. In general it is better to change the method of solution altogether; *e.g.*, by use of orthogonal functions instead of step-by-step relations, but step-by-step procedures are occasionally necessary, sometimes expedient, and often unavoidable. In the past they have been achieved by exceedingly intricate set-ups of the tabulator so as to multiply by patterns of binary multiplications by repeated self-additions, decimal shifting transfers and cross additions and subtractions between registers. These left little of the machine available for anything else. The repeated cycle facility has greatly simplified this.

3.4. *The Multiplying Punch.*—The multiplying punch can form and punch the 16-digit product of two 8-digit numbers, or, if fewer figures are required, punch the product after rounding off. The two factors and the product are all punched on the same card. Furthermore, numbers punched on the same card, other than the factors, may be added to or subtracted from the product. Thus one can evaluate expressions such as $a \pm (bc + d + e)$ where the letters represent positive numbers punched on any one card.

Fig. 5 exhibits the chief parts of the machine, but omits the multiplication table which is connected internally with the multiplier and multiplicand counters. During multiplication each figure of the multiplier is read in turn and the product of the whole multiplicand by that figure is formed in two parts, the left hand component (L.H.C.) and right hand component (R.H.C.). The R.H.C. consists of the unit digit occurring in each product of a multiplicand figure by the multiplier figure, while the L.H.C. consists of the corresponding tens or 'carry' figures shifted one place to the left. Thus, if the multiplier digit were 7 and the multiplicand 1 2 3 4 5 6 7 8, the R.H.C. would be 7 4 1 8 5 2 9 6 and the L.H.C. 0 1 2 2 3 4 4 5. The R.H.C.'s and L.H.C.'s corresponding to each multiplier figure in turn are sent by the multiplication table to R.H.C. and L.H.C. counters respectively, and finally the R.H.C. is added to the L.H.C. to leave the product in the L.H.C. (product) counter.

The multiplier also has another counter, the summary counter. This has 10-place capacity and is normally used to sum (as a check) the individual products formed in the products counter. It may, however, be used to hold the number $a$ when such expressions as $a \pm bc$ or $a \pm (bc + d + e)$ are being evaluated, the quantities $bc$ or $(bc + d + e)$ being formed in the products counter and afterwards added to or subtracted from the summary counter.

A separate column of the card is generally used for holding the sign designation of each factor, any even number (usually zero) designating a positive, and any odd number (usually one) designating a negative number. The sign designation of the multiplier is fed to the L.H.C. counter and that of the multiplicand to the R.H.C. counter in such a way that these designations are added during the multiplication. The parity (*i.e.*, evenness or oddness) of the resulting number thus indicates the sign of the product. If the signs are dealt with in the same run through the machine as that on which the multiplications are performed, 7-figure factors are the largest that can be accommodated, but if a separate run is used the full capacity of the machine is preserved. The standard multiplying punch handles positive numbers only. Some extra relays added by Mathematical Services Department enable it to evaluate expressions such as $a \pm bc$ where $a$, $b$ and $c$ may be positive or negative. In this case the answer is given to a maximum of ten digits, as the summary counter has to be used to store intermediate results. Furthermore, the presence or absence of $X$-punchings in distinct columns is needed to indicate the signs of the numbers.

When several different multiplicands are associated with the same multiplier, ' group multiplication ' is used. In this process the leading card of each group contains the constant multiplier for that group and has an $X$-punching to distinguish it. The multiplier remains in the multiplier counter and multiplies each multiplicand as it appears, and is only cleared, to be replaced by the next multiplier, when the next $X$-punched card is read. Each product is punched on the card holding the multiplicand.

The time required for performing any multiplication is proportional to the number of non-zero figures in the multiplier factor. The overall rate of working of the machine depends on the complexity of the operations demanded of it, and varies between five hundred and fifteen hundred cards per hour.

3.5. *The Collator*.—This machine is essentially an electro-mechanical filing clerk, arranging or collating cards by actions controlled by comparisons between different cards. It has two separate card feeds called primary and secondary (*see* Fig. 6). There are four card-receiving pockets. The primary feed can eject into Nos. 1 or 2 of these and the secondary feed to Nos. 2, 3 or 4. The primary feed has two sets of eighty column-reading brushes, the sequence brushes and the primary brushes, so disposed that when a card is passing under the primary brushes the following card is passing under the sequence brushes in the same relative position as in the comparison unit of the tabulator. The secondary feed is equipped with one set of eighty column-reading brushes called the secondary brushes.

Also there are two units for comparing sixteen columns at a time. Each of these can be set by plugging on the plug board to make a comparison between the readings at any two of the reading stations.

The R.A.E. collator is equipped also with a pair of separate counters counting from 0 to 9. They can be coupled with carry-over so as to count from 00 to 99. The counting may be triggered by cards fed from either feed, or from the total number of cards from both. Thus one can achieve such operations as selecting every thirty-seventh card from a pack. If one or both of these counters are put in series with a control circuit, they can delay instructions conditionally: *e.g.*, ' If so and so, do so and so after $n$ cards have been fed, unless . . . .' With some ingenuity they can be used as a short range memory for the machine.

The speed of the collator is 240 cards through each feed a minute.

The collator is used chiefly for the following operations.

3.5.1. *Sequence checking.*—The machine checks whether or not a run of cards that should be in ascending (or descending) order has some cards out of place. Clearly if a card be displaced towards the beginning of the sequence, it will be followed by a card of lower rank than itself; if displaced towards the end, it will be preceded by a card of higher rank than itself. Thus displacement in an ascending sequence always causes a falling sequence at the displaced card, and similarly in a descending sequence. The collator can therefore detect a displacement and locate the errant card by comparing serial numbers of consecutive cards.

Each feed can read up to sixteen digital positions. Usually only the primary feed is used for sequence checking, though this can be carried out in conjunction with other operations using the secondary feed. Appropriate brushes of the primary-feed-reading head and the sequence head, which reads the following card, are plugged into opposite sides of one of the comparison units. These are built up of simplified differentials. If the digital impulses fed into one side equal those fed into the other, there is no relative motion. If they differ, relative motion slides a contact from ' equal ' to ' high ' or ' low ' positions. Thus current passing from contact to contact, in the direction of most to least significant digit will be diverted into a ' high ' or ' low ' channel as soon as it reaches a digital position where compared digits differ. If not diverted it emerges from the ' equal ' channel. Relays can therefore be operated to initiate different actions according to the three possible results of the comparison.

In checking an ascending sequence, the instructions would be to carry on so long as the sequence reading was high or equal, but to take other action when the reading is low. When this checking is a guard on other collating operations, the action called for is stopping the machine and displaying a red light. When checking a pack as a separate operation, completely automatic action is needed. A pack of cards with tabs stands in the secondary feed and the collator is instructed to inject one of these into the primary-feed pocket whenever a low sequence is detected. After the pack has run through there will be a tabbed card in contact with every card out of place, and order can then be restored quickly by hand.

Sequence checking is a valuable check on marshalling.

When, as above, cards from packs in both feeds are collected, or ' merged ' into one pocket, the leading card of the primary pack is fed before that of the secondary pack. Hence the names. If the two packs were thus simply merged, the resulting single pack would consist of primary and secondary cards alternately with, of course, their relative order unaltered. Effectively the packs would have been given a perfect faro or riffle shuffle. The action of the collator can thus be regarded as a perfect riffle shuffle modified according to relations between numbers on the pair of cards being handled at the moment, and between one of these cards and that about to follow it. Detectable relations are ' greater than ', ' equal to ', or ' less than ', so nine combinations are possible. Modifying actions consist of interruptions to the shuffle on one or both sides, and of dealing off into separate pockets. A rather restricted memory of previous relations and operations is provided by the counters.

It must be emphasized that no operations of dissection or merging can alter the original relative order of cards.

3.5.2. *Reading from tables.*—If a computation uses tables punched on cards, these tables can be read on the collator.

The table cards, which are $X$-punched, are run through the primary feed while the cards containing the arguments for which function values are required are run through the secondary feed. The primary brushes are plugged to one side of a comparing unit and the secondary brushes to the other side. So long as a low primary condition lasts (*i.e.*, the primary reading being lower than the secondary one), cards are fed to pocket 1, but when equality occurs the primary card is fed to pocket 2, the corresponding secondary card following it into the same pocket. If this movement of cards still results in a low secondary condition because there are more than one secondary card having the same argument as the table card just fed, or in a condition of equality corresponding to there being just one such secondary card, the machine continues to feed secondary cards to pocket 2. It does this until a condition of low primary is again reached when the machine feeds the table cards to pocket 1 as before.

The merged cards from pocket 2 are fed through the reproducer which is set to gang-punch the function values from the table cards (which by virtue of their $X$-punching act as master cards) on to the detail cards. The table cards are then separated from the remainder by running all the cards through the sorter, the drum being set so that only the $X$-contact is closed, all the table cards collecting in the $X$-pocket and all the remainder in the reject pocket.

Return of the used table cards back to their original file uses the collator for merging in the same way as the table was read in the first place.

3.5.3. *Locating the intersections of functions.*—If the arguments and functional values of two functions or, what is the same, the abscissae and ordinates of two curves are available on punched cards, the intersections of these curves can be located automatically by the collator. For if one curve is lower than the other before intersection, it will be higher than the other after intersection. Thus the collator must be set up not only to compare numbers on two corresponding cards, but also to compare the results of two successive comparisons. If there is no change, the action continues, each pack ejecting into its own pocket. If there is a change, the pair of cards concerned are fed into a separate pocket or pockets. Cards that give an equality response are usually segregated into a pocket of their own for examination later, since equality here signifies equality of the first so many digits only of the abscissae.

A standard collator cannot do this, for it requires a certain amount of memory and also of functional reversibility or symmetry not normally available. This can be overcome by use of the built-in counters of the R.A.E. machine, and the operation has been carried out experimentally.

If a set of blank cards is filed ahead of one pack, the effect is to shift one curve relative to another and, by extracting cards according to some scheduled pattern, certain changes of the independent variable can be simulated. Use of the multiplying punch in conjunction with the collator makes possible, in theory, punched card equivalents to all Eucledian constructions involving unambiguous intersections. Projective and other geometries can be dealt with by similar methods, especially as the gang-punching unit of the Reproducer can be made to produce permutations automatically. In theory, when numbers are used to indicate order and identity only, the collator and gang punch and sorter combination can deal with the situation.

In practice, such techniques presuppose the existence of enough data in punched-card form, and enough intersections and the like to be located, to make the overhead time and labour of preparation worthwhile.

Nevertheless, it is certain that the combinatorial and ordinal prowess of the Collator have not been fully exploited in scientific computation. Traditional methods are based on serial computation, inductive processes in which each step influences the next, and they avoid those using overall parameters of sets of quantities taken in various combinations, because of the amount of transcription and storage they require unless material tables are used.

Problems that appear to involve the operations outlined above are welcomed. In general, however, their treatment must be experimental, and quick results cannot be guaranteed at the present stage.

4. *Work Suited to Punched-Card Machine.*—Solution of a problem with punched-card or any other automatic machines may be split into two parts; preparation, and the actual flow of work. The first is to decide upon the computational steps, the machines to be used at each stage, the design of the lay-out of information on the cards, and the plugging of the plug boards. For the second part one must consider the human labour demanded for collecting and transcribing data, transporting cards, replugging boards, intermediate or after-processing of results, and the machine time needed for the automatic parts of the computation, including checking. It is the total

labour and time accounted for by these two parts, not only that by the running times of the machines, that must be balanced against that required by alternative methods. The alternatives must themselves be analyzed with equal completeness, in particular as to the amount of human clerical labour they ignore or pass on to others.

4.1. *Unity of the Computing Chain.*—In all computation, all stages from origination of data to publication of results must be considered in unison, not only the stages that are interesting. The whole benefit of some high-speed device can be swallowed by the necessity of transcribing the output into some other form, and of checking this transcription, before the required information goes in the required form to those who require it. Still grosser loss can occur at the input stage. In general, if data have to be vetted and recorded manually, direct hand-punching into cards is quicker and less vulnerable to error than writing it down. Moreover, the punched cards can be used to produce automatically typewritten lists. Such direct recording cannot be done always, but careful consideration should be given to the possibility whenever original data are extensive and cumulative.

No more successfully than mass production can mass computation be substituted as an afterthought for some link in an established procedure. In experimental work particularly, what is to be measured, how it is to be recorded, how it is to be processed, and how the output should be presented, demand at least some matching before the experiment is made, not afterwards.

Information may have to be extracted by hook or by crook from past records without regard for extravagent use of people or machinery. Such action is sometimes necessary and is always a sign of past incompetence, but the alternative of ignoring the archives and making a fresh and properly balanced experiment should be considered first. Often it is cheaper, quicker, and more enlightening to extract information direct from nature than from unsuitable and heterogeneous records.

These considerations weigh more heavily with automatic computational processes than with those under direct human control. This is because, in computing sequences watched over or controlled by human beings, error or the possibility of error is often sensed by the operator through complex judgments based on experience, including experience outside the course of calculation. The automaton can make only such judgments as it has mechanism for and its user knows how to set on this mechanism. It is therefore extremely vulnerable to errors in the input, and usually these can be remedied only after the course of calculation. A punched-card installation is peculiar in using a large number of material tallies that carry the information, these tallies being recombined in various ways to generate new tallies. The input, being larger than that of sequence calculators, is more likely to include corrupt data, whilst the subsequent combinatorial operations allow more routes for the spread of infection within and between generations of tallies.

Corrupt initial data have stunted or stopped the development of certain important applications of punched cards to R.A.E. work, and have made others scarcely worthwhile. Corruption occurs not only through misreadings and transcription errors, but also, even less excusably, through erroneous and unchecked analytic derivation and desk calculation of initial parameters. This is not peculiar to the R.A.E., nor to this country alone, but unless remedied, it gravely hampers application of high-speed automata to experimental data. Higher powered methods demand higher standards of operation in all links of the process. Automatic processes of all kinds are at the mercy of the uniformity and purity of their raw materials.

It is essential also that the ends for which the computation is presumed necessary should be made explicit to those who will have to do it, and the means left for their devising. This is not requested so that computation may appear as great a mystery to the clients as it is to the computors, but to cut out the time and expense of a correct solution to the wrong problem. Mathematical stumbling blocks turn up even in the best approach, but more usually they trip up those who have strayed into the wrong path. The quickest way to dispose of a stumbling block may be to go round it, or to start again by another route. Either way one must know both starting point and destination, neither of which can be deduced by the computors from the stumbling block alone.

4.2. *Historical Development of Punched-Card Applications.*—These considerations apply to all manipulation of empirical data, but probably more to punched-card methods than to others currently available. For over sixty years punched card machines have been made for carrying out simple and rigid patterns of arithmetic on very numerous data capable of being grouped in many different ways. The first Hollerith installation was made for the U.S.A. Census of 1890, where its powers were confined to sorting and counting. Addition, subtraction, and printing of results followed later to meet the needs of accountancy. Machines for explicit multiplication did not become available for some time. This does not imply that multiplication could not be done, only that it was done differently. For instance, all cards to be multiplied by the same multiplier, or even having the same digit in the same position in their respective multipliers, could be segregated and treated the same, and then rearranged in an order appropriate to the next phase of the computation. This is typical of all tally systems, which marshal the tallies into groups requiring uniform treatment, put new labels on them, and distribute them again in new groupings. Formerly the system was used for pen and ink calculation when the data could be so set out so that by viewing them along rows, columns, or through gaps in stencils or masks, the required groupings could be picked out without rewriting. Schedules for harmonic analysis were a very high developed sample as, in another field, are the lay-outs of balance sheets and accounts.

There is a limit to the relations that can be exhibited topologically without chaos in such a static display. To exceed this, new arrangements can be made by recopying, or the existing lay-out can be made of movable components so that recopying is unnecessary. This is a reversion to the old system of calculating with pebbles (' calculi ') or counters, but with the profound difference that the counters carry inscriptions that may be altered with the course of calculation. The counters thus become tallies that can be of a very wide range of material, size, and use.

In the field of fare collection alone, application of tallies (tickets) seems capable of indefinite variation[6], as is their use in filing and indexing, particularly manually sorted edge-punched cards[7, 21]. In scientific computation their use is at the moment confined to the mechanically handled and sensed tallies of punched-card installations, with occasional use of hand-sorted and visually interpreted cards in conjunction with desk calculators and dissection of numerical tables into sets of mobile elements.

The original function of elaborate computational lay-outs was to substitute extensive transcription and complicated patterns of additions and subtractions for multiplication. Nowadays electrically driven desk calculators have reduced multiplication time to anything between one-fifth to one-twentieth of the time needed to get the factors on to and the product off the machine. The time consuming and error generating operations are now transfer and transcription.

Thus the ability of desk calculators to display and store their results must be utilized if the machines are to be operated efficiently; that is, so that further operations can go on without transcription or resetting.

4.3. *Comparison of Sequence Calculators and Punched-Card Machines.*—Once a computing schema has been devised that does not call for transcription from the machine, the next step is to make the controls of the machine automatically follow a sequence of pre-assigned operations that modify its contents step-by-step to arrive at the result. This is the sequence computer. Because it modifies signals more than it shuffles material objects it can operate far more quickly than can a punched-card machine. Because of its ephemeral signals and its relative lack of storage, it has need to.

In their extreme forms, sequence and punched-card machines are reciprocal. The problem of the sequence computer is to get the right signals in the right places at the right times so as to perform a sequence of different operations on them. This is the central problem of desk computation also. The problem of the punched card installation is to find uniformities of procedure so as to perform them on a sequence of different signals (sign events), here materialized as a set of mobile and permanent inscriptions. Broadly speaking, sequence computers materialize their operational patterns (as tapes, etc.) but tally systems materialize the patterns to be operated on (as counters, tickets, punched cards, etc.), in as much as these types of pattern are distinct.

Extreme forms of sequence and tally computers are rare; punched-card installations have sequential components, and sequence computers have components for storing, shunting, and marshalling. Balanced use of the two will come in time but, because punched-card machines were relatively unknown to scientific workers whilst desk calculators were not, for the last decade the bulk of effort and prestige has gone to automatic-sequence computers and their use.

4.4. *Combinatorial Possibilities.—' Monte-Carlo ' Methods.*—Thus till now for scientific work, other than statistics, punched-card machines have been used mainly as rather inadequate sequence computers, and the combinatorial possibilities of mobile inscriptions have been little explored. This criticism is as valid for the R.A.E. installation as for others, as will be shown by perusal of the descriptions of jobs in the following sections.

The main reasons for this are that the mathematical basis of sequence computation is far more developed, and is more direct, than that using overall parameters (' functionals '), and that the equipment, such as punched-card tables, essential to the latter, is also sparse. Thus methods using functionals (moments, orthogonal functions, spectra, etc.) have till now entailed both fundamental theoretical investigation and the calculation and manufacture of special punched-card tables except in fields, such as statistics, where the techniques are developed already. Of recent years, however, more attention has been given to functional, as well as step-by-step approaches.

Such, for instance, are the techniques sometimes named ' Monte Carlo ' methods[8]. These were foreshadowed by Arbuthnot and by the Count de Buffon in the eighteenth century. The latter derived an approximate value for Pi, not by numerical evaluation of an infinite series, but from observation of the intersections of a randomly thrown needle with a grid of parallel lines. In the 1930's punched-card machines were used similarly to determine statistical parameters awkward to get analytically. This they did by generating samples, and then calculating the parameters of these samples. The main novelty of later developments lies mainly in reversion to Buffon's use of this method for solution of problems in non-statistical contexts, such as the numerical solution of integral and partial-differential equations.

That it can be so applied is plausible enough; the laws governing the statistical behaviour of populations can be stated as integral or differential equations, so the process may be inverted by inducing statistical behaviour of the type required. A game of chance is played according to rules contrived to make the mathematical expectation of the outcome equal to the solution of the equation. If the game is played often enough, the average score will converge to the required value, to any required degree of precision and confidence.

It is essential that the probability distribution of the error of the average of a finite number of plays* should be estimated, so that no less and no more than the number necessary for given precision should be played. Broadly speaking, the precision increases with the square root of the number of plays, so that to increase the precision by one decimal place, the number of plays must be increased one hundredfold. Against this the ' moves ' of the game are usually more simple than those of conventional calculation, mainly composed of generation of random numbers (' dice throwing '), observation of coincidence or non-coincidence, followed by elimination, substitution, or rearrangement.

Thus the method is in general well adapted to punched-card machines, which can generate many plays simultaneously, each component of each move being completed for each card before going on to the next. Also the past history of operations on a card is usually carried on it, so that time and ' ensemble ' averages are readily available. Nevertheless, much research is needed

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* In an accepted convention, ' plays ' are individual instances of the abstract ' game ', corresponding to the French use of the words ' parties ' and ' jeu '.

to make good use of ' Monte Carlo ' methods. Very strongly convergent ' games ' must be devised if the volume of moves is not to overwhelm the advantage of their simplicity. Straightforward models of the physical situation giving rise to the problem are usually too weakly convergent.

Fundamentally, the principle of the stochastic, ' Monte Carlo ', techniques is a simple one. It gives an alternative method of multiplication, in which accuracy is increased by repetition, based on the fact that the frequency of joint occurrence of independent events approaches asymptotically the product of their probabilities. In ordinary multiplication, information is lost unless all digits in the product are retained. The number of digits in a product is the sum of the number of digits in its factors so, sooner or later, successive multiplications will fill the machine. Thereafter curtailed, ' rounded-off ', products must be used with consequent loss of information. The effects of such digital attrition, particularly in matrix calculations, are notorious[9]. In one computation (not described below) in Mathematical Services Department, attempts at numerical application of a lengthy but analytically correct procedure resulted in an overall accuracy of minus four significant figures. That is, the cumulative effects of digital attrition were ten thousand times greater than the quantity sought.

An immediate, but expensive, method of combating this is to retain large numbers of digits throughout all stages of a protracted calculation. Even this demands estimation, often very difficult, of how many digits to retain. Also it makes clear the unfortunate fact that in protracted calculations rough calculations do not lead to rough results, but to completely wrong ones. ' Monte Carlo ' methods give a hope of compensating for limited digital capacity in space by increasing the capacity in time; that is, by repetition.

4.5. *Classification of Types of Work.*—Little in the way of novel mathematical techniques will appear in the representative jobs described below. If repeated now they would be done differently in the light of experience and newer knowledge. They were done by such methods as seemed at the time to offer reasonable chance of achieving the correct solution quickly. For this no apology is needed.

In subsequent sections some typical jobs will be described. They are classified here from the point of view of the originator of the problem. Mathematical and operational points of view would each demand a different dissection. Fundamentally all punched-card operations are those of combinatorial and matrix algebras, but this outlook does not help anyone seeking the solution to a particular problem arising from specific happenings in the outside world. Our grouping is as follows:

(i) Problems already in matrix form (Section 5 below).

(ii) Computation of numerical values of functions at equal intervals of the argument (Section 6 below).

(iii) Calculations involving linear expressions with assigned coefficients (Section 7 below).

(iv) Combinatorial problems (Section 8 below).

(v) Statistical problems (Section 9 below).

The jobs described have been chosen to display clearly to prospective clients the kinds of work that can be done. If they were chosen to display ingenuity, or its lack, of computational method, or to illustrate historically, mathematically, or technically important work, the selections would differ.

5. *Matrix Multiplication Problems* (*Class* i).—Before describing the solution of a particular problem of this kind, we will detail the process of matrix multiplication on punched-card equipment.

5.1. *Matrix Multiplication with Punched Cards.*—Consider premultiplication of a matrix $A$, with elements $a_{ij}$ and $m$ rows and $n$ columns, by a matrix $B$, with elements $b_{ij}$ and $p$ rows and $m$ columns, to give the product matrix $BA$, equal to $C$, say, with elements $c_{ij}$ and $p$ rows and $n$ columns. Then

$$c_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj} \qquad (i = 1, 2, \ldots p. \quad j = 1, 2, \ldots n) .$$

A separate card is punched for each element $a_{ij}$, $b_{ij}$, of each matrix to contain the value of the element, with sign designation if necessary, and its co-ordinates, $i\,j$. The values of the $a$'s are punched in a field distinct from and not overlapping that of the $b$'s while the co-ordinates are punched in the same fields for the $a$'s as for the $b$'s, but in the reverse order, so that the row numbers $i$ of the $a$'s are in the same field as the column numbers of the $b$'s. The cards may be punched in either row or column order. The $b$ cards are labelled with an $X$-punching and their row numbers $i$, are punched again in a separate field where they are called 'set' numbers, each row being thus associated with a distinct set number. $p$ complete sets of $a$ cards are then reproduced, the number of the set being punched at the same time by the reproducer emitting in the same field as that used for the set numbers of the $b$ cards.

All the $b$ cards are sorted ahead of all the sets of $a$ cards, first in order of the row number of the $a$'s (and therefore in column order for the $b$'s), and then in order of set number. The products $b_{ik}a_{kj}$ may then be obtained immediately by group multiplication, because each $b_{ik}$ will be followed by the $a_{kj}$'s which it has to multiply. Alternatively the $b_{ik}$'s can be gang-punched into the $a_{kj}$'s they lead, the $b_{ik}$ cards removed, and the factors now on each card can be multiplied in the usual way without having to preserve the order of the pack. With current apparatus the latter method is necessary if the $b$ elements are of variable sign, for then segregation of signs, with consequent re-ordering, are necessary.

After the cards leave the multiplier the products they contain are checked. This is usually done by summing the multiplicands $a_{kj}$ and the products $b_{ik}a_{kj}$ in groups in the tabulator, the groups corresponding to constant values of the multiplying factors $b_{ik}$ so that each group consists of $n$ cards. The check is that for each group the sum of the products must equal the product of the sum of the multiplicands with the constant multiplier for that group.

When the checking is completed the cards are sorted according to column number (during which operation the $X$-punched $b$ cards are removed if group multiplication were used). They are then summed in the tabulator in groups of constant set number, the minor control being called on change of set number. The elements $c_{ij}$ of the product matrix are produced in column order, and can be printed and summary punched in this order.

5.2. *An Air-Traffic-Control Problem.*—An air traffic problem which was solved for the Operational Research Division of the Ministry of Civil Aviation provides an example of the use of the above technique, though in a simpler form since the matrices involved were square. The problem was to determine the probabilities of different numbers from zero to nine of aircraft landing or waiting to land during succeeding intervals of time as a consequence of given numbers from zero to nine of aircraft having been landing or waiting to land during the initial interval.

Thus, let $P(i,m)$ be the probability of $i$ aircraft landing or waiting to land during the $m$th interval, and let $a_i$ be the probability of $i$ aircraft arriving during a unit interval. If the mean arrival rate is $\varepsilon$, and the $a_i$ are assumed to follow a Poisson distribution, their values are $(\varepsilon^i e^{-\varepsilon})/i\,!$. Because any aircraft during landing is assumed to block the use of the runway to other aircraft, we have the series of relations:

$$P(0, m + 1) = a_0P(0, m) + a_0P(1, m)$$

$$P(1, m + 1) = a_1P(0, m) + a_1P(1, m) + a_0P(2, m)$$

$$P(2, m + 1) = a_2P(0, m) + a_2P(1, m) + a_1P(2, m) + a_0P(3, m)$$

$$P(i, m + 1) = a_iP(0, m) + a_iP(1, m) + a_{(i-1)}\,P(2, m) + \ldots + a_1P(i, m) + a_0P(i + 1, m).$$

This series of equations may be written

$$A.P(m) = P(m+1)\,, \qquad \text{(1)}$$

where

$$A \equiv \begin{bmatrix} a_0 & a_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1 & a_1 & a_0 & \cdot & \cdot & \cdot & \cdot & \cdot & \\ a_2 & a_2 & a_1 & a_0 & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & \\ \cdot & \cdot & \cdot & \cdot & & & & & \\ a_i & a_i & a_{i-1} & a_{i-2} & & a_0 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \end{bmatrix}$$

is an infinite matrix, and

$$P(m) \equiv \{P(0, m), P(1, m), \ldots, P(i, m), \ldots\}$$

is an infinite column vector.

Iteration of (1) gives

$$A^k P(0) = P(k)\,.$$

The condition that there are actually $i$ aircraft waiting to land (or $i - 1$ waiting and one landing) during the initial interval is expressed by

$$P(i, 0) = 1\,; \quad P(j, 0) = 0, \quad j \neq i\,.$$

Then the consequent probabilities $P(j, k)$ of $j$ aircraft waiting to land during the $k$th interval are given by $A^k Q(i)$, where $Q(i)$ is an infinite column vector with all elements zero, except the $(i + 1)$th, which is equal to unity. These probabilities are then given by the elements of the $(i + 1)$th column of $A^k$. We thus arrive at the conclusion that the successive elements in the $(r + 1)$th column of the $k$th power of the matrix $A$ give in order the probabilities of 0, 1, 2, . . . aircraft waiting to land during the $k$th interval as a consequence of there having been $r$ aircraft waiting to land during the initial interval.

Since all aircraft are assumed to be diverted if a concentration of ten or more waiting aircraft occurs, $P(i, m)$ is zero when $i$ is not less than ten for all values of $m$. Hence only the square matrix formed by the first ten rows and columns of $A$ need be considered.

To make the work with the punched-card machines as straightforward as possible the values of $k$ were first taken as 2, 4, 8, 16, 32, *i.e.*, as powers of two, so that the operation at each stage was that of squaring the matrix obtained at the preceding stage. The values 6, 12, 24 of $k$ were then found, these powers of the matrix $A$ being obtained from the first set by multiplication.

5.3. *Latent Roots and Latent Vectors.*—A fundamental matrix multiplication problem arising, for instance, in structural and aero-elastic computations, is to determine the latent roots and latent vectors of a given matrix. A method well suited to existing punched-card equipment was the well-known iterative method based on the Liouville-Neuman expansion, with various accelerating techniques due to Aitken[10]. In outline, an arbitrary column vector is chosen and is repeatedly pre-multiplied by the given matrix until the corresponding elements in the successive vectors obtained are in geometrical progression. Then the common ratio of the progression is the largest latent root, and any one of the vectors in the progression is the corresponding latent vector. In this simple form the method is unduly extravagant, but Aitken's devices for accelerating the convergence are readily applied. In the years since the problem was first tackled here much has been published on matrix computation and many powerful methods developed. Nevertheless, the punched-card equivalent of Aitken's procedures remains well suited to machines of conventional type, being simply organized and needing little machine minding.

5.4. *The Hilbert Matrix Problem.*—One example of mathematical interest was the calculation of the 'best possible' constant $k_N$ occurring in the inequality

$$\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n-1} \leqslant k_N \sum_{r=1}^{N} a_r^2$$

for the set of values $N = 1(1)6(2)10, 20$. The value for infinite $N$ is known to be $\pi$ (Hilbert's Double Series Theorem)[11] but this is not the best value for finite $N$.

The required value is the maximum, for arbitrary $a$'s, of the function

$$f \equiv \sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n-1} \Big/ \sum_{r=1}^{N} a_r^2 .$$

Now

$$\frac{\partial}{\partial a_p}\sum_{m=1}^{N}\sum_{n=1}^{N}\frac{a_m a_n}{m+n-1} = 2\sum_{n=1}^{N}\frac{a_n}{p+n-1}$$

and

$$\frac{\partial}{\partial a_p}\sum_{n=1}^{N} a_n^2 = 2a_p$$

so that

$$\tfrac{1}{2}\frac{\partial f}{\partial a_p} = \left\{\sum_{n=1}^{N}\frac{a_n}{p+n-1} - a_p f\right\} \Big/ \sum_{n=1}^{N} a_n^2$$

Thus the set of $a$'s that maximize $f$ is the solution of the set of equations

$$\sum_{n=1}^{N}\frac{a_n}{p+n-1} - a_p f = 0 \qquad (p = 1, 2, \ldots, N) .$$

These equations show that the maximized $f$ is the largest latent root, and the maximizing $a$'s the corresponding vector, of the $N$ by $N$ matrix whose element in the $m$th row and $n$th column is $1/(m+n-1)$.

Details of this computation have been published[12] and so will not be repeated here.

5.5. *Vibration of a Swept-Back Wing.*—Another matrix problem investigated was the estimation of frequencies and modes of vibration of a swept-back wing with six degrees of freedom, the equivalent matrix thus being of order $6 \times 6$. Here the higher frequencies and modes were required as well as the fundamental, which corresponds to the dominant latent root.

It was found, as would be expected, that with such small matrices the work could be handled more expeditiously by human computers using desk machines. Only if there had been many of these small matrices to be analysed in a batch would punched-card treatment have been worthwhile.

6. *Evaluation of Functions by the Method of Differences* (*Class* ii).—If a function is such that its finite differences of some order may be regarded as known, then that function may be evaluated for evenly spaced values of its argument by a pattern of cumulative additions.

To appreciate this method, consider the difference table of a function, $f(x)$, in central difference notation.

| x | f(x) | Δ' | Δ'' | Δ''' | Δ^iv | Δ^v | Δ^vi |
|---|---|---|---|---|---|---|---|
| $x_{-3}$ | $f_{-3}$ | | | | | | |
| | | $\Delta'_{-2\frac{1}{2}}$ | | | | | |
| $x_{-2}$ | $f_{-2}$ | | $\Delta''_{-2}$ | | | | |
| | | $\Delta'_{-1\frac{1}{2}}$ | | $\Delta'''_{-1\frac{1}{2}}$ | | | |
| $x_{-1}$ | $f_{-1}$ | | $\Delta''_{-1}$ | | $\Delta^{iv}_{-1}$ | | |
| | | $\Delta'_{-\frac{1}{2}}$ | | $\Delta'''_{-\frac{1}{2}}$ | | $\Delta^{v}_{-\frac{1}{2}}$ | |
| $x_0$ | $f_0$ | | $\Delta''_0$ | | $\Delta^{iv}_0$ | | $\Delta^{vi}_0$ |
| | | $\Delta'_{\frac{1}{2}}$ | | $\Delta'''_{\frac{1}{2}}$ | | $\Delta^{v}_{\frac{1}{2}}$ | |
| $x_1$ | $f_1$ | | $\Delta''_1$ | | $\Delta^{iv}_1$ | | |
| | | $\Delta'_{1\frac{1}{2}}$ | | $\Delta'''_{1\frac{1}{2}}$ | | | |
| $x_2$ | $f_2$ | | $\Delta''_2$ | | | | |
| | | $\Delta'_{2\frac{1}{2}}$ | | | | | |
| $x_3$ | $f_3$ | | | | | | |

This table exhibits the general relation

$$\Delta^r_{n+1} = \Delta^r_n + \Delta^{r+1}_{n+1/2}$$

That is, any element of the table is the sum of the element above it and the element to the right. Thus, if we know the elements of one diagonal running upwards from left to right (*e.g.*, from $f_2$, to $\Delta_{-1/2}^{v}$) and the right-hand element of the diagonal below it (here $\Delta_0^{vi}$), we can find all the elements of the latter diagonal in turn. For $\Delta_0^{vi}$ is added to $\Delta_{-1/2}^{v}$ to give $\Delta_{1/2}^{v}$, this in turn is added to $\Delta_0^{iv}$ to give $\Delta_1^{iv}$, and so on, the ' snowball ' being rolled until $\Delta_{5/2}'$ is added to $f_2$ to give $f_3$.

The tabulator has six registers in which cumulative totals may be stored and rolled from one to another. Therefore it can evaluate any function for evenly spaced arguments if it is fed with the corresponding sixth differences and initial values. One complete cycle of additions is needed for each value tabulated.

If the function is a polynomial of exact degree $n$, the $n$th differences are constant and proportional to $n!$. Thus any polynomial of sixth or less degree can be evaluated in one run using the tabulator only.

When differences are not constant, a set of cards carrying the sixth differences are run through the tabulator, a break of control being arranged to occur at each card passage. During a break of control, the cumulative totals are snowballed from one counter to the next, as described above, leaving them with the next diagonal of differences and the next value of the function. Any one or all of these values can be punched into new cards in the reproducer, as well as being printed. When the function is a polynomial of sixth or less degree the constant difference is stored in a counter, and blank cards are fed through the machine to produce the breaks of control.

The operations described are those carried out by Babbage's ' Difference Engine ' of 1833[13], still extant. This, however, although designed for twenty digit differences up to the sixth, was abandoned after one digital place of the third difference had been completed, and can compute directly only functions whose third differences do not exceed nine units.

In theory, tables can be extended indefinitely from given initial differences if some order of differences is constant. In practice, this can be done only if the counters are large enough to contain the differences exactly. Otherwise the cumulative additions will magnify the round-off errors factorially; unit error in the leading $r$th difference, $\Delta_{r/2}$, causes an error of $\binom{n}{r-1}$ units in the $n$th value of the function.

For relatively short tables to few significant figures this difficulty is removed by the crude but readily applied method of clearing the counters before infection has spread to significant digits of tabulated values. Fresh, independently calculated leading differences are then injected and the tabulation continued. To deal with possible extensive tabulations in the future, a method has been devised whereby the 'constant' difference is varied periodically so that its average effect approximates more closely to the true value, at the expense of a periodic error in the output. The periodic pattern can be chosen to minimize the amplitude of the periodic error and, at the same time, to give the equivalent of a greater number of significant digits in the differences, thus lengthening considerably the intervals between digital disinfection.

Of the two examples of table making quoted below, in the first separate tables were small enough to deal with round-off by the direct method; the second had differences that could be represented exactly.

6.1. *Normalised Orthogonal Deflexion Functions for Beams.*—A set of normalised orthogonal polynomials connected with the deflexion of doubly built-in beams was tabulated for Duncan[14].

The work was in two parts. The first was to evaluate to six significant digits the five polynomials

$$S_n(x) = \frac{\sqrt{(4n+1)}}{(2n)!}\frac{d^{2n-2}}{dx^{2n-2}}\left\{x^{2n}(1-x)^{2n}\right\} \qquad n = 1, 2, 3$$

$$A_n(x) = -\frac{\sqrt{(4n+3)}}{(2n+1)!}\frac{d^{2n-1}}{dx^{2n-1}}\left\{x^{2n+1}(1-x)^{2n+1}\right\} \qquad n = 1, 2,$$

and their first and second derivatives for argument ranging from zero to unity by increments of one hundredth. (These polynomials can in fact be expressed in terms of Legendre Polynomials.)

Three of these functions, $A_2(x)$, $S_3(x)$ and its derivative, were of more than sixth degree, so their evaluation could not be completed in one run. During the first run values of some convenient difference of order not more than the sixth were formed from the known constant difference, and summary punched into cards through the reproducer. These cards were then fed through the tabulator in the second run to provide the differences from which the tabulation could be completed.

The second part of the work was to evaluate, and print as a double-entry table, the deflexion functions $G_1(x, t)$, $G_2(x, t)$, for an isolated load and an isolated bending couple respectively. These give the deflexion at the point $x$ of a unit load or couple applied at point $t$.

Explicitly the functions are

$$G_1(x, t) = \tfrac{1}{6}x^2(1-t)^2[3t - x(1+2t)] \qquad \text{when } 0 \leqslant x \leqslant t \leqslant 1$$
$$= \tfrac{1}{6}t^2(1-x)^2[3x - t(1+2x)] \qquad \text{when } 0 \leqslant t \leqslant x \leqslant 1$$

$$G_2(x, t) = \tfrac{1}{2}x^2(1-t)[1 - t(3-2x)] \qquad \text{when } 0 \leqslant x \leqslant t \leqslant 1$$
$$= \tfrac{1}{2}t(1-x)^2[2x - t(1-2x)] \qquad \text{when } 0 \leqslant t \leqslant x \leqslant 1\,.$$

Both these functions are of degree less than six, and could be tabulated in one run. The main difficulty was to print the computed values in the form of a double-entry table. For this the combinatorial prowess of punched-card equipment in re-marshalling mobile tallies proved its worth. Other evaluations of these polynomials were carried out by Kirkby[14] at the College of Aeronautics, Cranfield.

6.2. *Computation of Binomial Coefficients.*—At the request of the Royal Society Mathematical Tables Committee exact values of some Binomial Coefficients of high order were computed on punched-card machines and printed in a form directly acceptable as printers copy.

The coefficients were those designated by ${}_nC_r$, $C_r^n$ , ${}^nC_r$ or $\binom{n}{r}$, and have the value $n!/(n-r)!r!$. These were computed for all values of $n$ for 201 to 1000, and of $r$ from 2 to 12, eight thousand and eight hundred coefficients in all.

The method used was based on the relation

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$$

which is displayed by the well-known construction of Pascal's Triangle[15], and is a special case of the relation between differences given in the introduction to this section, the downward and upward diagonals being renumbered as $n$ and $r$ respectively.

Details of this computation are given elsewhere[16]. It suffices to point out the interesting occurrence of integers with up to twenty-five digits, making necessary linking of numbers contained in two or three separate tabulator counters.

Values of the coefficients for all eleven values of $r$ and every hundredth value of $n$ were calculated individually on a desk calculator. Concordant results on the tabulator after a hundred cumulative additions was taken as proof of the correctness of intermediate values.

6.3. *Checking by Differencing*.—The preceding examples have illustrated how numerical sequences can be built by cumulative addition of differences. The inverse procedure, finding the differences of a given sequence, is a well-known method for finding errors, and is a routine operation for tables computed or used in punched-card operations in Mathematical Services Department.

Consideration of the difference schema at the beginning of this section will show that a unit positive error in any item will produce unit positive error in the next higher order of difference to its upper right, and unit negative error in the one to its lower right. These errors in their turn induce others fanning outwards to the right. The errors resulting in the $r$th column to the right will equal, in magnitude and sign, the binomial coefficients in the expansion of $(1-1)^r$, and the largest error or errors will be on or will straddle the horizontal line containing the original error. Thus differencing magnifies errors so that they become visible by inspection and, at the same time, locates them and indicates their magnitude.

Only exceptionally will correctly computed values in a sequence be exact. Usually they are subject to round-off errors, ranging at random (or so one hopes) half a unit above or below the exact values. These produce random variations of known average amplitude in the successive orders of differences. Sequences are therefore checked by differencing to an order at which the amplitudes due to unit error will swamp those due to round-off together with the systematic differences to be expected from the true sequence.

The method is powerful, but virtually impossible to use with ordinary desk calculators. For, to derive the $r$th differences of $n$ values, $\frac{1}{2}r(2n-r-1)$ results have to be recorded and $\frac{1}{2}(r-1)(2n-r)$ of these have to be fed back into the machine. Almost certainly there will be transcription error.

L. J. Comrie recognized multiple-register cross-footing accounting machines to be disguised Babbage 'Difference Engines', and brought to a fine art their application to finite-difference procedures[17]. With such machines only the values of the sequence need to be set up manually and, with punched cards, this transcription is done by the machine itself, which produces a printed list of values and differences. Thus, to check a set of values generated on a pack of cards, the pack is just run through the tabulator. A set of control boards is kept permanently wired for this routine differencing operation.

The property used to obtain differences can be followed from the difference schema at the beginning of this section. The value of a difference is the difference between the two entries vertically above one another to its left. The value of the lower of these two entries is again the difference between the two values immediately to its left; and so on. Finally we have the required value as the difference between the sum of all entries along the downward diagonal starting above and to left and the next tabular value. For instance,

$$\begin{aligned} \Delta_{1/2}{}^{v} &= \Delta_1^{iv} - \Delta_0^{iv} \\ &= \Delta_{3/2}{}''' - \Delta_{1/2}{}''' - \Delta_0^{iv} \\ &= f_3 - \{f_2 + \Delta_{3/2}{}' + \Delta_1'' + \Delta_{1/2}{}''' + \Delta_0^{iv}\}\,. \end{aligned}$$

The machine prints one line of differences at each card passage. Here, $f_2$, $\Delta_{3/2}{}'$, . . . $\Delta_0^{iv}$ would be printed, and would be stored in separate registers. Another register would hold their negative sum; the bracketed expression above. The new tabular value, $f_3$, would be read off the card into the last register, forming $\Delta_{1/2}{}^{v}$ there. This is added on to the stored $\Delta_0^{iv}$ to form $\Delta_1^{iv}$, and so on. At the same time these values are accumulated negatively as formed to provide the starting value when the next tabular value is read in.

Since the tabulator has six registers, up to the fifth order of differences can be evaluated in one run. For higher orders, the fifth differences would be summary punched into new cards on the first run, and these would be differenced on a second run.

7. *Evaluation of Linear Expressions with Assigned Coefficients* (*Class* iii).—7.1. *Computation of True Deflexions of Wings in Stiffness Tests.*—Between January, 1945, and January, 1946, there was a series of routine calculations to correct the crude readings from wing-stiffness tests. The wings had been loaded as cantilevers and the deflexions at various stations were needed relative to the wing roots. The deflexions had been measured relative to the floor and had to be corrected for the ' leverage ' displacement caused by shifting of the centre section.

The required results were of the form $r + ax + by + c$ where $r$, the reading, varied with station and load, as did $a$, $b$ and $c$, which were functions of wing-root deflexions. $x$ and $y$ depended on wing geometry, and varied with station only. The results for each test had to be presented as a typescript tabulation of readings, corrections, and net displacements grouped within stations in order of ascending load.

The leverages $x$, $y$ were known before the tests; the other quantities had to be punched or derived from the test readings. Multiplication was done on the multiplying punch but, although $a$, $b$, $c$ could have been evaluated on the tabulator, it was more convenient to use a National Accounting Machine. This was efficient because there were relatively few leverages to calculate, and these had to be transcribed on to a keyboard in any case. The National Accounting Machine carried out the necessary patterns of addition and subtraction, with a currently typed record of the operations as a check. This last is essential when evaluating basic parameters upon which large computing programmes will hinge.

In outline, the machine part of the punched-card work was to sort cards into groups with common multipliers, multiply, and then reassemble by test, station and increasing load, and combine and print the result on the tabulator. During tabulation various check sums were produced also, and desk-calculator operations on them served to check the output. Normal checks on the punched operations had been applied, of course, during the machine processes.

Though the procedure was technically satisfactory, it fell far short of its potential value. This was because of various defects in recording and collection of the original data. Attempts to rectify them absorbed more time and labour than did the computation. In particular, some original data was recorded on backs of envelopes and similar stationery. Decipherment and identification of these inscriptions required much joint consultation, and even then the cost of information lost was between £300 and £700 in each test.

Nevertheless, the fact that not only the computations, but also their printing in acceptable form were carried out automatically saved much time and staff. With current procedures for collection and recording of data and co-ordination of experiment, computation, and presentation, the savings could have been much greater. Any job involving humble but over-numerous arithmetical operations with much rearrangement will benefit from punched-card treatment if it is large enough and planned early enough.

7.2. *Iterative Solution of a Partial Differential Equation in the Theory of Heat Exchangers.*— The basic numerical operation for solving partial differential equations is to estimate the value at a point as the weighted mean of the values at other points. In general this is well suited to punched cards because the weights are predetermined and constant for points at assigned positions relative to the point investigated.

Before such a sequence of operations is performed the assignment of points within the domain of integration, the way in which boundary values and derivatives are represented, the order in which new values are to be assimilated, methods for accelerating the convergence, and methods for extrapolating to infinitesimal mesh, all need careful study. At all stages it is necessary also to confirm analytically that the quantized procedures will converge and, if so, that they will converge on the solution of the original equation. A correct solution to the wrong equation is as undesirable as a wrong solution to the correct equation. For instance, if a circle be represented by a hexagon, and calculations were made with absolute accuracy, the resulting value of $\pi$ would be exactly 3. Beyond this no advance could be made without further assumptions about relations between the original and quantized problems.

Elliptical equations such as those of Laplace and Poisson are numerically stable enough for their solutions to emerge from even incompetent formulation and procedures, and for this reason should not be chosen or accepted as test equations for numerical procedures. Hyperbolic and parabolic equations are not so obliging and much research into their numerical treatment has yet to be done. A particular idiosyncrasy of such equations is that the numerical stability or otherwise of their quantized counterparts is sensitive to mesh shape, as well as to mesh size. This is easy to see in the hyperbolic wave equation, in particular, because there the duration between consecutive epoch points must not exceed the time required for a disturbance to travel from one space point to the next.

The numerical stability of such quantized equations is also very sensitive to initial conditions. These are not usually known, because the initial data involves derivatives, which cannot be represented on a finite mesh. The initial differences that are required in place of the derivatives must therefore be approximated to with some care.

With these theoretical requirements are numerous empirical but equally imperative constraints, such as limited serial arithmetical facilities and the need to conform, to some extent, with the client's ideas about how computation should be done. Even limited success in this field owes something to good fortune. This was almost certainly so for the job to be described, for very little theoretical knowledge about numerical stability was available when it was done.

7.2.1. The problem was that of a heat exchanger through which cool air is blown for a given period. Hot air is then blown through from the other end for another given period. This cycle is repeated until a steady state is reached; that is, the temperature distribution is the same before and after a cycle of cooling and heating.

In non-dimensional variables the basic equation involved is

$$\frac{\partial^2 T}{\partial x\, \partial y} + \frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} = 0\,,$$

with suitable change of signs when the flow is reversed.

The clients required that this problem be solved by a punched-card model of the physical course of events. That is, an arbitrary temperature distribution would be given, and computational processes supposedly equivalent to hot and cold blows carried out until the initial and final temperature distributions were the same within the degree of accuracy called for. Blindly iterative modelling of physical phenomena are usually extravagent of computational labour and time. Here solutions were originally required for sixteen values of the parameter corresponding to length of exchanger, and nine values of the ratio of durations of hot to cold blow, a gross of iterations in all. This programme would have demanded the consumption of some millions of cards. The requirements were often changed during the progress of the work, which strained the organization, but they were eventually reduced. By using accelerating techniques to hasten convergence of the iterations from knowledge of previous results, consumption of time and cards was much reduced. Even then over half a million cards were used.

Results could be got to agree in the first three figures but not, by any amount of iteration, in the fourth. This was in part due to the rather crude method, which was not employed entirely voluntarily, and in part to the need for ' weights ' that could be applied in the tabulator without resource to the relatively slow multiplying punch. This necessity would not press so heavily nowadays, when direct multiplication is available in the tabulator.

Thus, instead of finding the weights to be applied to points of the mesh to give a value of specified accuracy, it was necessary to find a mesh whose points would require given weights to produce values of that accuracy. Moreover, the spacings of these points must be reasonable aliquot parts of unity, since the values were to be tabulated.

Values were calculated successively along one row at a time, the value required, $T_4$, being calculated as a linear function of the value last calculated $T_3$, and the values $T_2$, $T_1$ in the positions corresponding to $T_4$, $T_3$ of the row last calculated. In other words, the value at $T_4$, the upper right-hand corner of a rectangle, was calculated as a linear function of the value $T_3$ at the upper left-hand corner, which had just been calculated and of the values $T_2$, $T_1$ at the bottom left and right corners. These last had been calculated and punched into cards when their row was being calculated, and these cards fed into the machine to give information for this new row. The starting values for the first row were either the final values of the preceding hot or cold blow, or some distribution derived from preceding blows and punched into cards. The values at the intake end of the row at future times were calculated from the exponential low of cooling. Thus the values along a row at given time and at one end for different times were available, and from these all other values could be calculated successively from a relation of the form

$$T_4 = aT_1 + bT_2 + cT_3 .$$

The first relation tried was

$$T_4 = 0{\cdot}9(T_2 + T_3) - 0{\cdot}8T_1$$

applied to a rectangular grid of time interval 0·05, and space interval 0·5. These could be calculated entirely on the tabulator, the first term being formed by adding $T_2$ to $T_3$, ' staggered ' rolling of this total into another register to produce its tenth, and subtraction of the latter from the original total. The second term was formed by staggered rolling, doubling by self-addition, and subtraction from the original.

Even after the ninth iteration and after various accelerating procedures the successive results would not agree beyond three figures, and fourth figures oscillated. The mesh chosen was therefore tending to produce instability and the relation was changed to

$$T_4 = 0{\cdot}8(T_3 + T_2) - 0{\cdot}6T_1$$

applied to a square grid of interval 0·25. Coarsening of the mesh was also forced by the size of the proposed programme, and by the danger of accumulated round-off errors.

Once again states ' steady ' only to three figures were obtained. Relations using coefficients demanding the multiplying punch were debarred, because it would both slow down the arithmetic and also the convergence. For information just obtained could not be put into use immediately as it could when values were being generated serially in the tabulator.

The work was therefore stopped at this stage. Although the target was not reached, the results that were obtained were of considerable value to the clients. It is clear that with close analytical as well as computational study of the problem, together with the fuller knowledge and better arithmetical facilities now available, punched-card methods can attack successfully many parabolic as well as elliptic partial differential equations.

8. *Combinatorial Problems* (*Class* iv).—Here the emphasis is on sorting, selecting and re-assembling decks of cards according to prescribed numerical criteria. All statistical analysis could be classed here and the job for which punched-card machines were first built, census analysis, is an example of pure selection and enumeration. Much conventional statistical work has been done by the section, not only for the R.A.E., but for other Ministries and institutions, but the field is well documented, so the examples have been chosen to illustrate either less obvious applications or applications of particular value within this Establishment.

As has been mentioned earlier, the combinatorial prowess of punched-card systems is their outstanding feature, and still awaits full analysis and application.

8.1. *Gear-Ratio Tables for the Differential Analyzer.*—In Differential Analyzers (multiple integraphs) the scale factors needed to keep magnitudes within the scope of the machine are realized by gear trains. Nowadays the gear ratios, apart from the obviously necessary powers of two and five, are chosen on an Equal Temperament system. That is, the ratios available cover an octave up or down in a given number of logarithmic steps, equal within a given error. Thus the relative error can be made uniform, and the sets of wheels are minimal and systematic, being assigned in much the same way as the weights in Bachet's weighing problem. The wheels needed to approximate a given ratio most nearly can be found systematically from what is virtually a table of logarithms to an appropriate base.

In earlier days a rather Pythagorian outlook was usual, inspired by musical associations and the mistaken belief that logarithmic tables were generated from computed values of the logarithms of prime numbers.

A stock of gear pairs gave ratios involving small primes, and from those ratios of larger denominator could be produced. These ratios were of course exact, but were scattered very unevenly over the range. Moreover, fairly elaborate calculations were needed to find what particular train would give the ratio closest to the required one, if that were not realizable exactly. Thus some sort of dictionary became essential in order to look up the gear combination to give the nearest available ratio.

Compilation of this dictionary by the Hollerith section inspired the latter to suggest the Equal Temperament system mentioned above. For, together with the objections to the Pythagorian system already given there was the additional one of the size of dictionary needed. Even with a stock in trade of seven basic ratios only, other than unity, five hundred and eighty-three distinct compound ratios could be formed, using three or less gear pairs in series.

The punched-card procedure was first to punch sets of cards with the basic ratios, 1/1, 4/11, 5/13, 3/7, 1/2, 2/3, 4/5, 7/8 and the reciprocals of all but the first; fifteen in all. From these the multiplier generated two-hundred and twenty-five cards carrying all possible compound ratios of two basic ratios in series.

The compound ratios, marshalled in order of ascending magnitude on the sorter, were winnowed on the collator to remove mere changes in order of assembly and basic ratios in series with their reciprocals. The first was done by rejecting all but the first of cards bearing the same compound ratio, the second by rejecting all cards whose compound ratio lay between 0·9999999 and 1·0000001 (because decimal equivalents to the basic ratios had been punched to seven decimal places). Exempted from these rejections was the unit ratio produced by the product of two unit ratios.

The distinct compound ratios remaining were numbered serially from 1 to 114, in ascending order of magnitude, and from them were generated and selected in the same way all distinct compound ratios made from three basic ratios in series. Because compounds containing one and two unit ratios were retained, this final list included ratios compounded from one or two basic ratios also. It therefore contained all possible compound ratios once and once only.

The final list contained the compound ratios in ascending order from 0·0480841, the cube of 4/11ths, to 20·796875, the cube of 11/4ths. Against each compound ratio were quoted the components of the first two ratios, the serial number of their product, and the third component ratio. Because of certain listing difficulties the first two ratios were quoted as fractions; *e.g.*, 0703 1305 represented 7/3rds and 13/5ths. The third ratio was quoted to seven decimal places.

The greater part of the work, and the most difficult, was the essential task of checking multiplications, and guarding against loss of cards or of order. Both were achieved mainly by forming check sums of multipliers, multiplicands, products, and serial numbers, during all segments of the work. The basic principles are that in linear processes grand totals are invariant under change of order, but sub-totals are not, and that with common multipliers the sum of the products equals the product of the sum. As a check on desertion or gate-crashing of cards, additional to the standard counting devices on the sorter, G.P.O. counters had been fitted to the feeds of the reproducer and multiplier. Visual indication of card totals are very valuable, giving immediate warning of grosser mishaps to card packs during complicated handling.

8.2. *Analysis of Records of Aircraft Accelerations from Counting Accelerometers*.—The work was in two stages:

(*a*) Analysis of vertical accelerations recorded on a Hughes recorder.

(*b*) Listing and sorting of the accelerations as recorded on the counting accelerometer.

The object of the first stage was to check the operation of the counting accelerometers[18]. The acceleration records were examined and consecutive maxima and minima of accelerations noted. These were punched on Hollerith cards together with identification data (the date of flight, aircraft speed and height, time) and the number of maxima and minima occurring in any one minute. One card was prepared for each minute of flight, but since the total number of accelerations that could conveniently be punched on a card was 11, more than one card was sometimes needed for a single minute of flight. The last acceleration recorded was the first value on the following card, thus giving an uninterrupted record. The cards were then listed by the tabulator, the presentation being arranged, and the printing being made on translucent paper, to permit direct reproduction of the results for distribution in reports. After printing, the cards were sorted and counted by the sorter on all the maxima and then on all the minima, so that tables could be constructed of the number of occurrences of associated maximum and minimum accelerations. The corresponding number of thresholds of acceleration crossed was then computed by a desk machine, and checked against the counting accelerometer readings.

In the second stage, the accelerations as recorded on the counting accelerometer at ten minute intervals, together with the aircraft height, speed and weight are punched on the cards. The cards are listed and summed in groups, printing again being arranged for direct reproduction. They are then sorted on speed, weight and height, and summed in the tabulator, every change in group height or weight giving a control change. As a total is printed it is also punched into a new card by the summary punch and these new cards are printed afterwards, again for direct reproduction, to give a table of total counts at different heights and weights.

For this job all people concerned co-operated before actual experiments began. Thus the recording instruments themselves, the records and their monitoring and measurement, the analysis and its final presentation were all designed to match. The total time from trial to publication was reduced to less than one-thousandth of that previously needed.

Such a reduction can be achieved very rarely indeed. Here not only was the nature of the work peculiarly suitable, but also previous methods of tackling it must have fallen something short of perfection. Nevertheless, if matching the various stages of a routine operation only halves the overall time, it is worthwhile.

8.3. *Analysis of Aircraft Speeds and Accelerations from V-g Recorders.*—V-g records are produced as markings on small glass slides. These are examined and measured by trained staff of Structures Department, who punch the measurements direct on to cards by means of hand punches. The cards are then checked by repetition of the same procedure by another observer using a verifier instead of a punch. Thus unnecessary transcription is cut out, with its risk of error. Also, because punching is quicker and more legible than writing, and because information common to a set of cards can be punched in automatically, the remaining unavoidable transcription is speeded up.

The cards are punched with speed, divided into speed bands each of which covers a range of 25 m.p.h., and the extreme accelerations in each band, aircraft identification, slide number, date of flight, weather, etc. The aircraft identification and weather require alphabetical as well as numerical printing so a special tabulator (the Pierce machine) at the National Physical Laboratory is used for listing the results. The alphabetical printing is obtained by double punching the particular column, *e.g.*, for *A* punch *Y* and 1 in the same column, or for *J* punch *X* and 1 in the same column.

8.4. *Absentee Records for R.A.E.*—Cards are punched monthly, one for each person who has had leave during the month, to contain the different types of leave, *i.e.*, sick, uncertified, unpaid, etc. These cards are then sorted into Departments, and tabulated to give Departmental totals. Each group of cards is then sorted into four groups: industrial and staff grades, male and female in each case. The amount of leave in each sub-group is tabulated and totalled, thus giving a comprehensive absentee record for each Department.

8.5. *Location of Bibliographical References.*—There is a very old and widespread principle for locating small changes in complex environments by contriving that items unchanged or unwanted cancel themselves out. Such is the practice of printing a white silhouette of a small tool behind its proper hook, so that the absence of tools may be positively indicated. Similarly the number of rivets inserted during a shift has been determined by photographing the structure from the same point before and after the shift, and superimposing the negative of one photograph on a positive, of equal contrast, from the other. If there had been no change a uniform grey area would result, but parts that have changed upset this balance, and are immediately distinguishable. A more sophisticated application is in subtractive colour printing, where colours are formed as the logical products (' overlaps ') of complementary primaries, and still more so to the procedures of ' masking ' used to obtain separation negatives. On the other hand we have numerous ' stencil ' techniques, where information is isolated by superposition of suitably pierced opaque cards.

For this last, Hollerith punched cards are a most convenient and compact medium, and many applications have been made. In sorting, for instance, the operator always checks that all cards in a pocket are perforated in the appropriate place by holding them up to the light in register, or by probing them with a rectangularly sectioned ' needle '.

Operations with cards as stencils must eventually be formally equivalent to Boolean algebra. For from two cards two others can be produced readily; one, the logical sum, punched in every position punched on either card; the other, the logical product, punched in all those and only those positions punched in both cards. Because the last, the logical product or ' highest common factor ', of any number of cards can be found by simply superposing the cards in register, it is the operation most usually exploited. The other the logical sum, is more often exploited in edge-punched cards[19,7] where the code patterns for various entities are punched in superposition on the edge of a single card.

Punched-card stencils for solving, without any machinery, number-theoretic problems have been used for some time. The ' sieve ' method of Eratosthenes (275–194 BC) made explicit the stencil method for finding prime numbers, and many factor tables have used material ' sieves ' for this, including rotating perforated discs photo-electrically scanned. All number-theoretic applications pivot on the fact that the common parts of systematic patterns of holes can be interpreted as the solutions of various congruences. A fairly recent pack of punched-card stencils for solving congruences by superposing selected cards is that of R. M. Robinson[20].

The overlap of superposed cards has been exploited efficiently by W. E. Batten[7] to find such patent specifications as might claim to satisfy pre-assigned complicated specifications. The field was that of plastic technology in which at least four broad attributes would have to be considered simultaneously; chemical nature, process or treatment, application, and source (inventor or manufacturer). Each of these aspects needed extensive subdivision. One card was used for each subdivision of aspect, and carried numbered sites which could be perforated. Patent specifications were assigned a serial number when received and, after study, marked with the aspects appropriate to its content. The cards corresponding to these aspects were punched in the site corresponding to the serial number of the document. Thus each card was a list of all patents dealing with a particular aspect. To find what patents, if any satisfied an arbitrary specification, the appropriate aspect cards would be superimposed and held to the light. The common holes would indicate the serial numbers of the required patents. If there were no holes in common, the least important aspect card would be removed, and so on.

Originally the cards were about ten by eight inches, with four hundred sites. This was unwieldy and needed a special punching jig. On the advice of one of the writers of this report, Hollerith cards and hand-punch were used, the firm concerned already having these available in its accountancy section. Each card then had eight hundred punching sites, and was very much more easily handled. This last is important, because as sites are exhausted, new cards must be used to accommodate new serial numbers. Thus several sets of cards may need superimposition, though only cards bearing similar ranges of serial numbers need be used together.

The system was used in the R.A.E. to deal with documents on fuel technology. The Hollerith section, after explaining the scheme, merely punched the cards once a week in the places indicated by the user, a matter of a few minutes. Not all the card surface was used, so there was room for ordinary inscriptions. A fully perforated template was used to identify the common serial numbers without difficulty.

Semantic mapping of the topic was, of course, left to the client, who was trained in such matters. Clearly no syntactical device such as this can be useful unless the semantic and pragmatic aspects as well are competently handled.

The method is meant to deal with relatively few items that may have very many simultaneously relevant properties. If used for very numerous items with few properties it becomes slow and laborious. Correctly applied, as it was here, it can be very useful.

Much has to be done yet in exploiting these and similar methods, where the Boolean and lattice properties of punched cards are used. Clearly there is much to be done also in adapting such properties, essentially binary, to existing combinatorial and arithmetic equipment. The Department will always help in testing or developing ideas for making use of these or any other currently unconventional techniques.

9. *Statistical Problems* (*Class* v).—Hollerith equipment is well suited to handle statistical computations: both of the commercial type involving sorting, listing, subtabulation, averaging, etc.; also problems of the more scientific type involving computation of variances, correlation coefficients and the other well-known statistical parameters. This class of work is of such wide occurrence and is so well documented that it need not be discussed here. It is in fact the work for which punched-card machines were originally devised. Baehne's collective work[21] is still a good survey of punched-card applications over this wide field, and Hartley[22] outlines punched-card treatment of conventional statistical computations.

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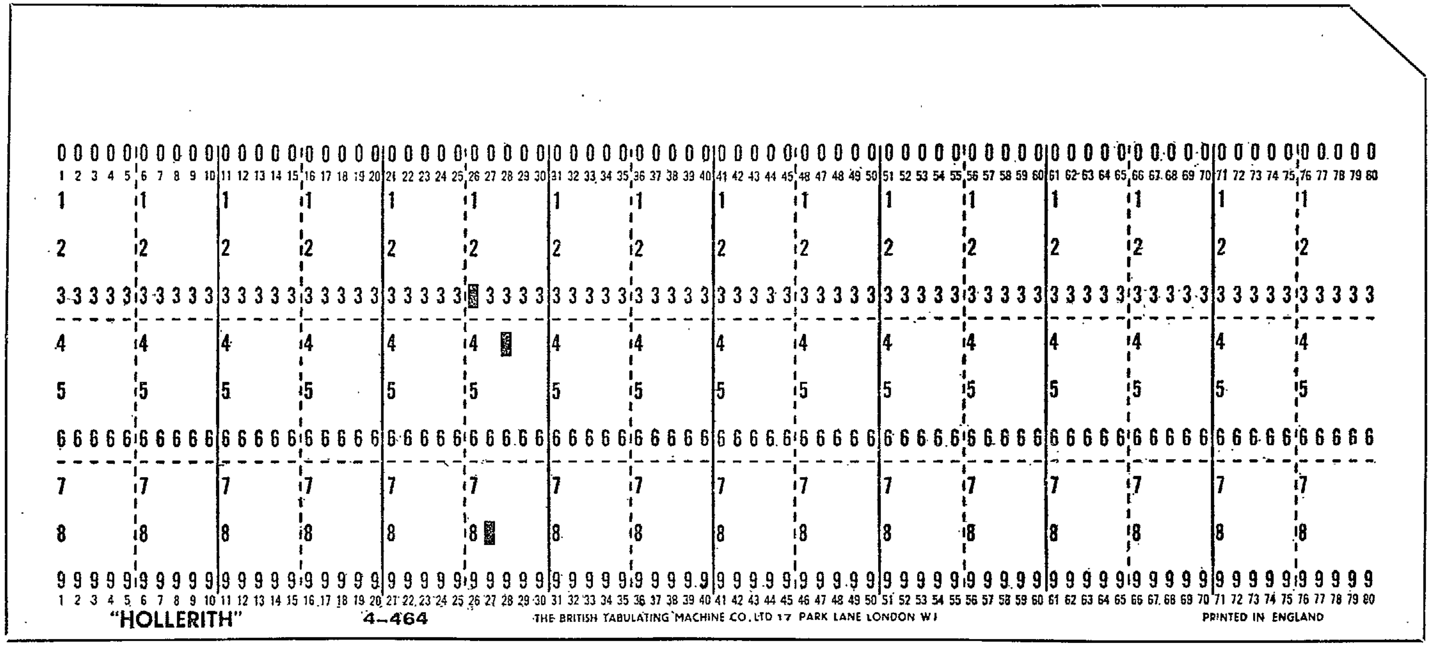


FIG. 1. A Hollerith punched card.

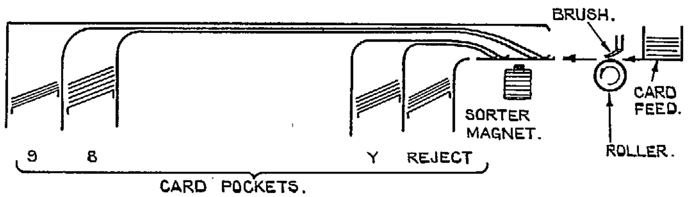


FIG. 2. Sorter.

(72956) Wt. 52/8943 K.7 8/58 Hw.

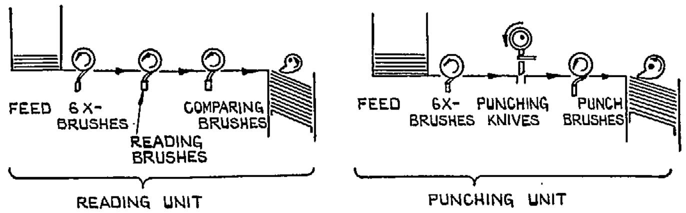


Fig. 3. Reproducer.

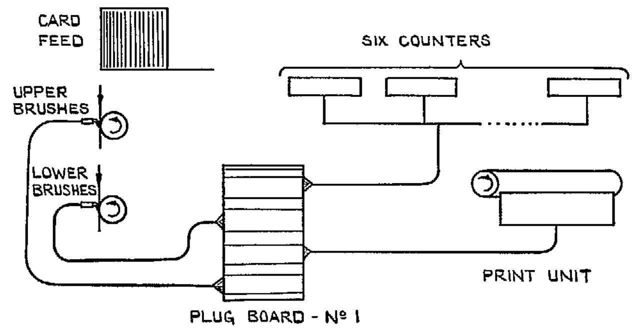


Fig. 4. Tabulator.

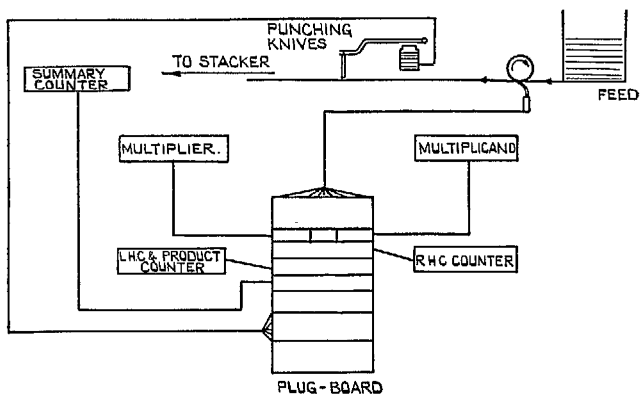


Fig. 5. Multiplying punch.

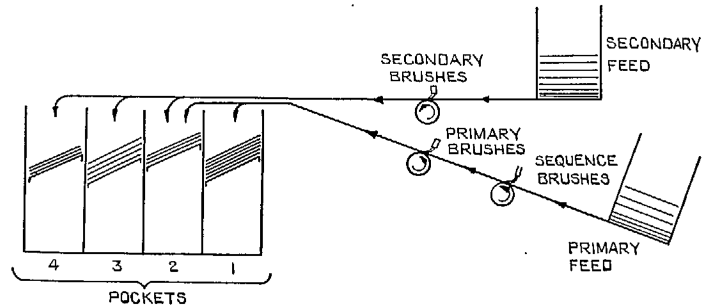


Fig. 6. Collator.



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