R. & M. No. 3068
~~(826)~~ 18 588
A.R.C. Technical Report

MINISTRY OF SUPPLY

AERONAUTICAL RESEARCH COUNCIL
REPORTS AND MEMORANDA

Calculated Derivatives for Rectangular Wings Oscillating in Compressible Subsonic Flow

By

DORIS E. LEHRIAN, B.Sc.,
of Aerodynamics Division, N.P.L.

LONDON : HER MAJESTY'S STATIONERY OFFICE
1958
PRICE 5*s*. 6*d*. NET

Calculating Derivatives for Rectangular Wings Oscillating in Compressible Subsonic Flow

By

Doris E. Lehrian, B.Sc.,
of the Aerodynamics Division, N.P.L.

*Reports and Memoranda No. 3068**
July, 1956

Summary.—Stability and flutter derivatives are obtained for rectangular wings describing plunging and pitching oscillations in subsonic flow. These are evaluated by applying the simple approximate 'equivalent' wing theory (R. & M. 2855) with the vortex-lattice method of downwash calculation. The derivatives for the wing of aspect ratio 4 at Mach number 0·866 are compared with values calculated by a method based on exact theory; at this high Mach number it is found that the present method is sufficiently accurate for only a very limited range of the frequency parameter. At very low values, the pitching-moment derivatives for this wing are in reasonable agreement with those calculated by the Multhopp-Garner method and with results from wind-tunnel tests at high subsonic Mach number.

1. *Introduction.*—The theory for a finite wing oscillating at any frequency in compressible subsonic flow is considered in Ref. 1 (W. P. Jones, 1951). It is shown that under certain conditions the problem is related to one of determining the lift distribution over a 'reduced wing' of lower aspect ratio, oscillating in incompressible flow. The problem is thus simplified by making an approximation to the integral equation which is theoretically only valid if the frequency-Mach-number parameter $M\nu/(1 - M^2)$ is small. In application, however[1], a similar approximation in the solution of the two-dimensional problem gives values for the plunging and pitching derivatives which are in good agreement with exact two-dimensional values, when the frequency parameter $\nu = fc/U \leqslant 0 \cdot 4$ and $M = 0 \cdot 7$. This suggests that the approximate form of the theory might give sufficiently accurate results in the three-dimensional case for a similar range of the frequency-Mach-number parameter.

In the present note this approximate method is applied to rectangular wings of aspect ratio 3, 4 and 6 describing simple harmonic plunging and pitching motion. Stability and flutter derivatives are evaluated by using, for the reduced-wing calculations, the vortex-lattice method for wings oscillating in incompressible flow[2,3] (1953, 1954). The computation is relatively simple and routine, and the method can be used for wings of general plan-form. The accuracy of derivatives obtained by the present scheme is therefore of considerable practical interest. At present, there is very limited information relating to the effects of frequency at subsonic Mach number. A general treatment of the problem is given by Acum[4] (1955), who has derived expressions for the downwash in terms of influence functions. This method involves very laborious computation which cannot readily be reduced to a routine, but it has been used to evaluate flutter derivatives for rectangular wings at high subsonic Mach number[4]. The effect of aspect ratio on the theoretical stability derivatives for rectangular wings in subsonic flow has been estimated by Multhopp's low-frequency method[5] (Garner, 1952).

* Published with permission of the Director, National Physical Laboratory.

Comparison of the results of the present method with those obtained in Refs. 4 and 5 indicates the accuracy within the limitations of linearised theory. The usefulness of results in which thickness and boundary-layer effects are neglected must be judged by comparison with experimental values. The only measured values available at present for rectangular wings at high Mach number are the pitching-moment derivatives for low frequencies which are reported in Ref. 7.

2. *Theory of the Reduced Wing.*—The linearised theory for a thin wing of finite span which oscillates with small amplitude in inviscid subsonic flow is considered by W. P. Jones in Ref. 1*. In that paper, the problem of a wing of aspect ratio A, describing simple harmonic oscillations of frequency $f/2\pi$ in a stream of Mach number M, is transformed to that of a reduced wing of aspect ratio $A_r = \beta A$ which oscillates in a different mode at a frequency $p/2\pi = f/2\pi\beta^2$, where $\beta = \sqrt{(1 - M^2)}$. The lift distribution $l\, e^{ift}$ over the wing A is related to the lift distribution $\rho U \Gamma\, e^{ipt}$ over the reduced wing A_r such that

$$l = \rho U \Gamma \exp\left(i \frac{fxM^2}{U\beta^2}\right) \quad \ldots \quad (1)$$

Here the point $(x', y', z' = 0)$ on the wing A is transformed to a point $(x, y, z = 0)$ on the reduced wing A_r by

$$x' = x, \qquad \beta y' = y \,. \quad \ldots \quad (2)$$

For plunging and pitching oscillations of respective amplitudes $c\bar{z}$ and α, the motion of a rectangular wing defined in terms of the normal downward displacement z', is

$$z' = (c\bar{z} + x'\alpha)\, e^{ift}, \quad \ldots \quad (3)$$

where c is the chord of the wing and x' is measured downstream from the leading edge. Then if $w\, e^{ift}$ is the downwash at the point $(x', y', 0)$, the tangential flow condition

$$w\, e^{ift} = \frac{\partial z'}{\partial t} + U \frac{\partial z'}{\partial x'}$$

becomes

$$w = U\left[\alpha + i\nu\left(\bar{z} + \frac{x'}{c}\alpha\right)\right], \quad \ldots \quad (4)$$

where the frequency parameter $\nu = fc/U$. Equation (4) is related to the downwash $W\, e^{ipt}$ at the point $(x, y, 0)$ on the reduced wing A_r by the equation

$$w = \beta W \exp\left(i \frac{fxM^2}{U\beta^2}\right) . \quad \ldots \quad (5)$$

Therefore W has to satisfy the boundary condition defined by (2), (4) and (5), that is

$$\frac{W}{U} = \frac{1}{\beta}\left[\alpha + i\nu\left(\bar{z} + \frac{x}{c}\alpha\right)\right] \exp\left(-\, i \frac{\nu M^2}{\beta^2}\frac{x}{c}\right) . \quad \ldots \quad (6)$$

The downwash W is given by the complicated integral of Ref. 1, equation (16)*. In the present note, this integral is taken to first-order accuracy in the parameter $M\nu/\beta^2$, by assuming that the integral I_0 of Ref. 1, equation (18), is zero. This simplifying approximation is made for all frequencies and Mach numbers so that in the reduced-wing problem W becomes W_s. As shown in Ref. 1, W_s is readily identified with the downwash on a wing oscillating in incompressible flow. Thus the lift distribution $\rho U \Gamma\, e^{ipt}$ over the reduced wing A_r in incompressible flow is determined so that W_s satisfies equation (6). Then by (1) the actual lift distribution on the wing A is

$$l\, e^{ift} = \rho U \Gamma\, e^{ipt} \exp\left[i \frac{\nu M^2}{1 - M^2}\left(\frac{x - Ut}{c}\right)\right] . \quad \ldots \quad (7)$$

---

* The basic equations of Ref. 1 are given in the Appendix.

3. *Method of Calculation*.—The results given in this note are evaluated by using the vortex-lattice method of calculating the downwash on the reduced wing oscillating in incompressible flow[2,3]. The lift distribution $\rho U\Gamma\, e^{ipt}$ over the reduced wing A_r is assumed to be represented by the finite series

$$\Gamma = U \sum_m (\Gamma_0 C_{0m} + \Gamma_1 C_{1m}) A_m \,, \qquad (8)$$

where

$$\left.\begin{aligned} \Gamma_0 &= 2 \cot \tfrac{1}{2}\theta \\ \Gamma_1 &= -2 \sin\theta + \cot \tfrac{1}{2}\theta + \frac{i\omega}{2}\left(\sin\theta + \frac{\sin 2\theta}{2}\right) \\ cA_m &= s\eta^{m-1}\sqrt{(1-\eta^2)} \\ m &= 1, 3, 5 \text{ for symmetrical motion} \end{aligned}\right\} , \qquad (9)$$

and

$$\left.\begin{aligned} x &= \tfrac{1}{2}c(1-\cos\theta) & 0 \leqslant \theta \leqslant \pi \\ y &= s\eta & -1 \leqslant \eta \leqslant 1 \\ \omega &= pc/U & \end{aligned}\right\} . \qquad (10)$$

The downwash $W_s\, e^{ipt}$ corresponding to (8) is given by

$$W_s = U \sum_m (W_{0m} C_{0m} + W_{1m} C_{1m}) \qquad m = 1, 3, 5, \qquad (11)$$

where W_{0m} is a complex quantity dependent on the frequency parameter ω and W_{1m} is a real quantity independent of ω. By using the 21×6 lattice as outlined in Refs. 2 and 3, the values of W_{0m} and W_{1m} are calculated for the six collocation points positioned at $\frac{1}{2}$ and $\frac{5}{6}$ chord of the spanwise sections $\eta = 0{\cdot}2$, $0{\cdot}6$ and $0{\cdot}8$. The six arbitrary coefficients C_{nm} are determined in terms of $\bar{z}$ and α so that W_s satisfies the boundary condition (6) at each collocation point. The lift distribution $\rho U\Gamma$ is then given by (8), so that the actual lift distribution $l\, e^{ift}$ over the wing A can be determined from (7) for the particular values of the frequency parameter ν and Mach number M.

If the leading edge of the rectangular wing A is taken as pitching axis, the lift $L\, e^{ift}$ and moment $\mathscr{M}\, e^{ift}$ are given by

$$\left.\begin{aligned} L &= \iint_S l\, dx'\, dy' \\ \mathscr{M} &= -\iint_S l\, x'\, dx'\, dy' \end{aligned}\right\} , \qquad (12)$$

where

$$S = \text{wing area.}$$

By (2) and (10), these are transformed into integrals with respect to θ and η. The integrations in η are simple but those in θ lead to the following relations

$$\left.\begin{aligned} \int_0^\pi \Gamma_0 \exp(-i\lambda'\cos\theta)\sin\theta\, d\theta &= 2\pi[J_0 - iJ_1] \\ \int_0^\pi \Gamma_1 \exp(-i\lambda'\cos\theta)\sin\theta\, d\theta &= \pi\left(\frac{1-M^2}{M^2}\right)[J_2 + iJ_1] \\ \int_0^\pi \Gamma_0 \exp(-i\lambda'\cos\theta)(1-\cos\theta)\sin\theta\, d\theta &= \frac{2\pi J_1}{\lambda'} \\ \int_0^\pi \Gamma_1 \exp(-i\lambda'\cos\theta)(1-\cos\theta)\sin\theta\, d\theta &= -\frac{\pi}{\lambda'}\left[J_1 - i\left(\frac{3-2M^2}{M^2}\right)J_2\right] \end{aligned}\right\} , \qquad (13)$$

where $J_n = J_n(\lambda')$ are Bessel functions of the first kind and $\lambda' = (\nu M^2/2\beta^2)$. Equations (1), (8), (9), (12) and (13) lead to final expressions for the lift and moment for the rectangular wing A,

$$\left.\begin{aligned} \frac{L}{\rho U^2 S} &= \frac{\pi^2 A \beta\, e^{i\lambda'}}{8}\left[(J_0 - iJ_1)D_0 + \left(\frac{1-M^2}{2M^2}\right)(J_2 + iJ_1)D_1\right] \\ \frac{\mathcal{M}}{\rho U^2 Sc} &= \frac{\pi^2 A \beta\, e^{i\lambda'}}{16\lambda'}\left[-J_1 D_0 + \left\{\frac{J_1}{2} - i\left(\frac{3-2M^2}{2M^2}\right)J_2\right\}D_1\right] \end{aligned}\right\}, \quad \ldots \quad \ldots \quad (14)$$

where

$$D_n = C_{n1} + \tfrac{1}{4}C_{n3} + \tfrac{1}{8}C_{n5}\,.$$

In the case of the solution $\nu \to 0$, only first-order terms in frequency are retained throughout the reduced-wing calculation, as is described in Ref. 2. The lift L and moment $\mathcal{M}$ are then obtained generally by retaining only first-order terms in frequency in the lift distribution l.

The derivatives referred to the pitching axis $x' = 0$ are defined by

$$\left.\begin{aligned} \frac{L}{\rho U^2 S} &= (l_z + i\nu l_{\dot z})\bar{z} + (l_\alpha + i\nu l_{\dot\alpha})\alpha \\ \frac{\mathcal{M}}{\rho U^2 Sc} &= (m_z + i\nu m_{\dot z})\bar{z} + (m_\alpha + i\nu m_{\dot\alpha})\alpha \end{aligned}\right\} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

The usual transformation formulae, which are given in the list of symbols in Ref. 3, are used to calculate the derivatives referred to an axis at $x' = hc$.

4. *Results.*—For the rectangular wing of aspect ratio 4, the derivatives are evaluated by the method outlined in sections 2 and 3, for a range of values of the frequency parameter ν and Mach numbers $M = \sqrt{(1 - \beta^2)}$, $\beta = 1$, $\frac{2}{3}$ and $\frac{1}{2}$. The derivatives for plunging and pitching oscillations, referred to axis positions at the leading edge and at $0{\cdot}445$ chord, are tabulated in Table 1.

The above results are obtained by considering reduced wings of aspect ratio $A_r = 4$, $8/3$ and 2 in incompressible flow. When the downwash matrix corresponding to a frequency parameter ω is known for the reduced wing, then it is relatively simple, by solving for a different boundary condition (6), to obtain the lift distribution and hence the derivatives for a wing of aspect ratio $A = A_r/\beta$ oscillating at a frequency $\nu = \omega\beta^2$ in subsonic flow M. Thus, plunging and pitching derivatives are calculated for the rectangular wings $A = 2$ and $8/3$ at $M = 0$, $A = 3$ and 6 at $M = 0{\cdot}745$ for several values of ν, and in the particular case $\nu \to 0$ for the wings $A = 16/3$ and 8 at M $= 0{\cdot}866$. The values for an axis positioned at $0{\cdot}445$ chord are given in Table 2.

5. *Results Used for Comparison.*—The difficulty of investigating the problem of a wing oscillating in subsonic flow, is reflected in the limited information available even for rectangular wings. For so simple a plan-form there is no routine method that is general in aspect ratio, Mach number and frequency. The special case of infinite aspect ratio has been solved; values of the derivatives are given in Ref. 8 for arbitrary Mach number and frequency (Jordan, 1953). In incompressible flow, wings of moderately low aspect ratio are considered by Lawrence and Gerber[9] (1952), and the calculations cover a wide range of ν. For low frequencies, derivatives may be calculated by the Multhopp-Garner[5] subsonic lifting-surface theory, as indicated in section 5.1.

So far as the author is aware, the only method applicable to arbitrary aspect ratio, subsonic Mach number and frequency for which computations have been carried out* is that proposed by Acum[4]. The calculation of the downwash involves a complex influence function of four variables for each term of the chordwise loading. When the necessary functions are programmed

* Richardson's method has recently been used by D. E. Williams and P. C. Birchall, R.A.E., to calculate derivatives for a rectangular wing of aspect ratio 2.

for a high-speed computing machine, the method of Ref. 4 can be reduced to a practical routine. Meanwhile, although the computation on desk-machines is very laborious, derivatives have been calculated for rectangular wings at $M = 0 \cdot 866$.

At present, experimental results are only available for low-frequency parameters. Values of the pitching-moment derivatives m_α and $m_{\dot\alpha}$ have been measured for the wing $A = 4$ oscillating about an axis at $0 \cdot 445$ chord, in tests made at the National Physical Laboratory[7] for a range of subsonic Mach number.

5.1. *Low-Frequency Theory.*—Values of the stability derivatives are calculated by using the Multhopp-Garner lifting-surface theory[5] for low-frequency pitching oscillations in subsonic flow. The derivatives for a wing A at Mach number M are expressed in section 5 of Ref. 5 in terms of the seven coefficients

$$(I_L)_1\,,\ (I_L)_2\,,\ (I_L)_3\,,\ I_L^* = -(I_M)_1\,,\ -(I_M)_2\,,\ -(I_M)_3\,,\ I_M^*\,,$$

which arise from the solution for a reduced wing of aspect ratio $A_r = \beta A$ in incompressible flow. Values of these coefficients for rectangular wings $A_r = 2 \cdot 13$ and $4 \cdot 13$ have been obtained by Garner, using an 'improved'† solution with $m = 15$, $N = 2$. This unpublished work is used here to obtain the pitching derivatives referred to an axis at $0 \cdot 445$ chord for the following cases:

TABLE A

A	M	l_α	$l_{\dot\alpha}$	$-m_\alpha$	$-m_{\dot\alpha}$
2·13	0	1·289	+0·781	−0·299	0·167
3·195	0·745	1·934	+0·410	−0·451	0·554
4	0·846	2·421	−0·459	−0·562	1·083
4·26	0·866	2·579	−0·873	−0·599	1·307
4·13	0	1·838	+0·394	−0·395	0·287
6·285	0·745	2·757	−1·620	−0·592	1·102
8·26	0·866	3·676	−6·288	−0·789	2·722

These values correlate satisfactorily with the results obtained for $\nu \to 0$ by the present method which are given in Tables 1 and 2.

5.2. *General Frequency Theory.*—Acum has applied linearised theory to the case of a finite wing oscillating at any frequency in compressible subsonic flow[4]. He has derived general expressions for the influence functions analogous to those used in the Multhopp lifting-surface theory by using the exact integral equation in a form suggested by Watkins, Runyan and Woolston[6] (1954) The method has been used with $m = 7$ spanwise and $N = 2$ chordwise variables to calculate the plunging and pitching derivatives for the wing $A = 4$ at $M = 0 \cdot 866$; the following solutions were evaluated on desk machines and involved about six months' computation. Values of the derivatives referred to the pitching axis at $0 \cdot 445$ chord are given in the following Table.

TABLE B

ν	l_z	$l_{\dot z}$	l_α	$l_{\dot\alpha}$	$-m_z$	$-m_{\dot z}$	$-m_\alpha$	$-m_{\dot\alpha}$
→0	0	2·479	2·479	−0·556	0	−0·589	−0·589	1·213
0·3	0·077	2·310	2·398	−0·136	−0·075	−0·479	−0·505	1·034
0·6	0·180	2·098	2·333	+0·027	−0·201	−0·314	−0·350	0·798

† See footnote on page 13 of Ref. 15.

By comparing these values with the results given in Table 1, it can be seen that there is considerable error in the present method when it is used for high values of both ν and M. This error is not unexpected, in view of the difference between the values at the centre point of the wing $A = 4$ when $\nu = 0 \cdot 6$ and $M = 0 \cdot 866$, of the downwash w_e obtained by the general theory and the simplified downwash w_s as computed by the present method (Ref. 4).

5.3. *Correction to the Present Method.*—To improve accuracy, a correction factor is now applied to the present results for the wing $A = 4$ at $M = 0 \cdot 866$, when $\nu = 0 \cdot 3$ and $0 \cdot 6$. It is assumed that the lift and moment are to a first approximation only dependent on the first term of the lift distribution $l\,e^{i\nu t}$ defined by equations (1) and (8), that is, on the loading $\Gamma = U\Gamma_0 A_1 C_{01}$. For a particular value of ν, the arbitrary coefficient C_{01} may be determined by satisfying the boundary condition $w = w_0$ at the centre point of the wing. Values of the simplified downwash w_s and the downwash w_e, which correspond to the loading $U\Gamma_0 A_1$, are given in Ref. 4. Then, analogous to the present method, $\Gamma \propto C_{01} = w_0/w$, whilst a more exact solution is $\Gamma \propto C_{01} = w_0/w_e$. It follows that the ratio w_s/w_e can be regarded as a simple correction to the lift distribution of the present method. Hence, the values of the lift L and moment $\mathscr{M}$ for $\nu = 0 \cdot 3$ and $0 \cdot 6$, as defined by equation (15) and the results given in Table 1, are corrected by multiplying by the complex factor w_s/w_e. Corrected values of the derivatives, referred to an axis at $0 \cdot 445$ chord, are then obtained for the wing $A = 4$ at $M = 0 \cdot 866$ as follows:

TABLE C

ν	l_z	$l_{\dot{z}}$	l_α	$l_{\dot{\alpha}}$	$-m_z$	$-m_{\dot{z}}$	$-m_\alpha$	$-m_{\dot{\alpha}}$
0·3	0·074	2·155	2·218	−0·101	−0·080	−0·464	−0·464	1·070
0·6	0·122	1·963	2·138	+0·209	−0·249	−0·336	−0·286	0·910

Comparison of these results with the values given in Table B shows that the accuracy of the derivatives is greatly improved by this simple correction.

6. *Discussion.*—The derivatives obtained by the present method for the wing $A = 4$ are compared in Figs. 1 to 5 with the wind-tunnel results and with the values calculated by other theories, as described in section 5. The effects of aspect ratio on the derivatives $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ are shown in Figs. 6 and 7. Comparison of the results for $\nu \to 0$ indicates that the present method is satisfactory for the calculation of stability derivatives for values of the Mach number as high as $M = 0 \cdot 866$.

6.1. *Derivatives for $A = 4$ Wing.*—In Fig. 1, the pitching-moment derivatives m_α and $m_{\dot{\alpha}}$ are plotted against the frequency parameter ν for several Mach numbers. For low frequencies the calculated derivatives for $M = 0 \cdot 866$ are seen to be in reasonable agreement with the wind-tunnel values, but for $M = 0 \cdot 745$ there is a marked difference between the calculated and measured values of the derivative $m_{\dot{\alpha}}$. The values show a similar variation due to frequency over the small range of ν considered in the tests.

The plunging and pitching derivatives for $M = 0 \cdot 866$, referred to an axis positioned at $0 \cdot 445$ chord, are shown in Figs. 2 and 3 respectively. The values of Table B which are obtained by Acum's method[4], are also plotted against the frequency parameter ν. It is evident that the present method gives values for certain of the derivatives, for this particular case, which are considerably in error even for small values of ν. The derivative most in error is $l_{\dot{\alpha}}$ and it is suggested that only for values of ν less than $0 \cdot 05$ is the error on $l_{\dot{\alpha}}$ likely to be small enough to be acceptable. It therefore seems that the present method gives sufficiently accurate values of all the derivatives when $M = 0 \cdot 866$, if the parameter $M\nu/(1 - M^2)$ does not exceed a value of $0 \cdot 175$. The values of Table C are shown as 'present method $\times$ correction' in Figs. 2 and 3; comparison of these results with the values of Table B shows that the simple correction described in section 5.3 greatly improves the accuracy of the derivatives for $M = 0 \cdot 866$.

It is difficult to assess the accuracy of the present method at $M = 0 \cdot 745$ without having values based on the exact theory[4], but some indication is given by the graphs of Fig. 4. The values of the derivatives $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ which are plotted against Mach number, for different values of ν, are obtained by cross-plotting with respect to ν the results given in Table 1. The graph of $l_{\dot{\alpha}}$ indicates again the rapid increase of the error in this derivative at $M = 0 \cdot 866$ as the parameter ν increases. The value for $\nu = 0 \cdot 6$ at $M = 0$ and the value from Table B for $\nu = 0 \cdot 6$ at $M = 0 \cdot 866$ suggest that the accurate curve for $\nu = 0 \cdot 6$ should follow the shape of the $\nu \to 0$ curve and it seems likely that accurate curves for lower values of ν would be similar. If this is so, then even for $M = 0 \cdot 745$ the present method will only give sufficiently accurate values of $l_{\dot{\alpha}}$ for a very limited range of ν. The graph of $m_{\dot{\alpha}}$ indicates that there is not such a large error in this derivative for the particular axis position of $0 \cdot 445$ chord.

The variation of $m_{\dot{\alpha}}$ with axis position and Mach number is considered in Fig. 5 for the frequency parameter values $\nu \to 0$, $0 \cdot 3$ and $0 \cdot 6$. The present method gives values of the right order for axis positions forward of the $0 \cdot 35$ chord for $\nu = 0 \cdot 3$ and $0 \cdot 6$ when $M = 0 \cdot 866$, but becomes considerably in error as the axis position moves towards the trailing edge. The values of $m_{\dot{\alpha}}$ obtained by correcting the results of the present method, as in section 5.3, are in surprisingly good agreement with the general theory values of Table B for all axis positions. It seems likely, from an inspection of the variation of $m_{\dot{\alpha}}$ with frequency when $M = 0$ and $M = 0 \cdot 866$, that the error in the present method at $M = 0 \cdot 745$ will be qualitatively similar to the error at $M = 0 \cdot 866$.

6.2. *Effects of Aspect Ratio*.—Values of the derivatives $l_{\dot{\alpha}}$ and $m_{\dot{\alpha}}$ referred to the $0 \cdot 445$ chord axis, are plotted against the reciprocal of aspect ratio in Figs. 6 and 7 for different values of the Mach number, $M = 0$, $0 \cdot 745$ and $0 \cdot 866$, and the frequency parameter, $\nu \to 0$ and $\nu = 0 \cdot 6$. The values obtained by the present method for $\nu \to 0$ are in good agreement for all Mach numbers with the values of Table A calculated by the Multhopp-Garner method[5]. For incompressible flow, the present results for $\nu = 0 \cdot 6$ correlate satisfactorily with the values obtained for infinite aspect ratio[8] and low aspect ratio[9]. The effect of aspect ratio at high frequency appears to be similar for $M = 0$ and $M = 0 \cdot 866$. This is suggested in Figs. 6 and 7, by the dotted curves for $\nu = 0 \cdot 6$ which join the reliable theoretical points for $A = 4$ and $A = \infty$. The values obtained by the present method when $M = 0 \cdot 745$ and $\nu = 0 \cdot 6$, indicate the correct variation with aspect ratio, although the numerical values of the derivative $l_{\dot{\alpha}}$ in Fig. 6 show the wrong trend with change in Mach number, as already noted in the $l_{\dot{\alpha}}$ curves of Fig. 4.

7. *Concluding Remarks*.—The present method gives satisfactory values of the stability derivatives for rectangular wings at high subsonic Mach number. Values of the flutter derivatives are only likely to be of sufficient accuracy for a very limited range of the parameter $M\nu/(1 - M^2)$. For the wing of aspect ratio 4, the results based on exact theory[4] indicate that the present method considerably overestimates the frequency effect at high Mach number. In view of the large error apparent in the values of the derivative $l_{\dot{\alpha}}$ it is suggested that the upper limit of the frequency parameter ν may be about $0 \cdot 175(1 - M^2)/M$ for this particular wing.

Although frequency effects may not be so marked for wings of swept and tapered plan-form, the results of this note suggest that the present method will be reliable only for low-frequency parameters at high Mach number. The importance of developing a routine method for the frequency-Mach-number problem therefore remains. The method of Ref. 4 deals with the exact problem, but will be extremely laborious to apply to a wing of general plan-form unless extensive use is made of a high-speed computing machine.

REFERENCES

No.	Author	Title, etc.
1	W. P. Jones	Oscillating wings in compressible subsonic flow. R. & M. 2855. October, 1951.
2	D. E. Lehrian	Calculation of stability derivatives for oscillating wings. R. & M. 2922. February, 1953.
3	D. E. Lehrian	Calculation of flutter derivatives for wings of general plan-form. R. & M. 2961. January, 1954.
4	W. E. A. Acum	Theory of wings oscillating with general frequencies in subsonic flow (with application to rectangular plan-forms).—With Appendix by D. E. Lehrian. A.R.C. 17,824, August, 1955; A.R.C. 18,630, August 1956. (To be published.)
5	H. C. Garner	Multhopp's subsonic lifting surface theory of wings in slow pitching oscillations. R. & M. 2885. July, 1952.
6	C. E. Watkins, H. L. Runyan and D. S. Woolston	On the kernel function of the integral equation relating the lift and downwash distributions of oscillating finite wings in subsonic flow. N.A.C.A. Report 1234. 1955.
7	J. B. Bratt, W. G. Raymer and J. E. G. Townsend	Measurements of the direct pitching-moment derivatives for two-dimensional flow at subsonic and supersonic speeds and for a wing of aspect ratio 4 at subsonic speeds. (A.R.C. report to be issued.)
8	P. F. Jordan	Aerodynamic flutter coefficients for subsonic, sonic and supersonic flow (linear two-dimensional theory). R. & M. 2932. April, 1953.
9	H. R. Lawrence and E. H. Gerber	The aerodynamic forces on low-aspect-ratio wings oscillating in an incompressible flow. *J. Ae. Sci.* Vol. 19. No. 11. p. 769. November, 1952, and Vol. 20. No. 4. p. 296. April, 1953.

APPENDIX

The Basic Equations of R. & M. 2855[1]

A wing of aspect ratio A describing simple harmonic oscillations of frequency $f/2\pi$ is introduced into a steady air stream of uniform density ρ and uniform velocity U. The oscillating wing gives rise to a disturbance of velocity potential ϕ superimposed on the steady flow. The linearised equations for this problem are transformed in Ref. 1 to the plane of a wing of reduced aspect ratio $A_r = \beta A$. In the notation of the present note the co-ordinates (x', y', z') of the original problem are related to those of the reduced wing problem as follows

$$x = x', \quad y = \beta y', \quad z = \beta z', \qquad \ldots \qquad (16)$$

where the Mach number $M = \sqrt{(1 - \beta^2)}$. If ϕ_a and ϕ_b are the values of ϕ above and below the plane $z' = 0$ of the wing A and its wake, the discontinuity in velocity potential $\phi_a - \phi_b = k(x', y')\, e^{ift}$, and this is expressed as

$$k(x', y') = K(x, y) \exp\left(i\frac{fx}{U}\frac{M^2}{\beta^2}\right). \qquad \ldots \qquad (17)$$

The lift distribution $l\, e^{ift}$ over the wing A is denoted by

$$l(x', y') = \rho U \Gamma(x, y) \exp\left(i\frac{fx}{U}\frac{M^2}{\beta^2}\right). \qquad \ldots \qquad (18)$$

Furthermore, the tangential flow condition at a point on the wing A, that is

$$w\, e^{ift} = \frac{\partial \phi}{\partial z'},$$

transforms into a boundary condition at the surface of the reduced wing A_r such that

$$W(x, y) = \frac{w(x', y')}{\beta} \exp\left(-\, i \frac{fx'}{U} \frac{M^2}{\beta^2}\right). \qquad \ldots \qquad (19)$$

It is shown in Ref. 1 that over the reduced wing and wake

$$U\Gamma = i \frac{fK}{\beta^2} + U \frac{\partial K}{\partial x} \qquad \ldots \qquad (20)$$

with the condition $\Gamma = 0$ everywhere in the wake. Also, at any point $(x_1, y_1, 0)$ on the wing

$$4\pi W(x_1, y_1) = \iint\limits_{z_1 \to 0} K \frac{\partial^2}{\partial z_1{}^2} \left[\exp\left(-\, i \frac{M}{\beta^2} \frac{fr}{U}\right) \Big/ r\right] dx\, dy\,, \qquad \ldots \qquad (21)$$

where $r = \sqrt{\{(x - x_1)^2 + (y - y_1)^2 + z_1{}^2\}}$ and the integral is taken over the wing and wake. Equation (21) may be written as

$$W = W_s - I_0$$

where

$$W_s = \frac{1}{4\pi} \iint\limits_{z_1 \to 0} K \frac{\partial^2}{\partial z_1{}^2} \left(\frac{1}{r}\right) dx\, dy\,. \qquad \ldots \qquad (22)$$

It is shown in Ref. 1 that the integral I_0 for a wing of finite aspect ratio, is of order $\left(\frac{M\nu}{\beta^2}\right)^2 \log_e \left(\frac{M\nu}{\beta}\right)$ where the parameter $\nu = fc/U$. Hence to first order accuracy in the parameter $M\nu/\beta^2$

$$W = W_s\,. \qquad \ldots \qquad (23)$$

Then equations (20), (22), and (23) are identical to those which have to be satisfied in the problem of a wing of aspect ratio A_r which is oscillating at a frequency $f/2\pi\beta^2$ in incompressible flow.

TABLE 1

Derivatives for a Rectangular Wing of Aspect Ratio 4 describing Plunging and Pitching Oscillations

h	M	ν	l_z	$l_{\dot{z}}$	l_α	$l_{\dot{\alpha}}$	$-m_z$	$-m_{\dot{z}}$	$-m_\alpha$	$-m_{\dot{\alpha}}$
0	0	→0	0	1·791	1·791	+1·218	0	+0·417	+0·417	0·648
		0·2	−0·001	1·740	1·747	1·357	−0·007	0·406	0·403	0·680
		0·6	−0·106	1·579	1·563	1·507	−0·091	0·370	0·325	0·718
		1·2	−0·671	1·421	1·166	1·558	−0·424	0·340	+0·112	0·737
		2·4	−2·958	1·371	0·142	1·567	−1·815	0·352	−0·652	0·768
	0·745	→0	0	2·193	2·193	0·855	0	0·492	+0·492	0·870
		0·2	+0·009	2·047	2·066	1·358	−0·014	0·462	0·457	0·974
		0·6	−0·156	1·696	1·677	1·740	−0·170	0·387	0·312	1·034
		1·2	−0·878	1·353	1·049	1·608	−0·674	0·316	0·011	0·963
	0·866	→0	0	2·445	2·445	0·403	0	0·526	0·526	1·142
		0·05	0·003	2·411	2·414	0·725	−0·001	0·520	0·519	1·215
		0·3	−0·017	2·004	2·026	1·763	−0·062	0·433	0·411	1·357
		0·6	−0·256	1·580	1·503	1·925	−0·247	+0·320	+0·226	1·249
0·445	0	→0	0	1·791	1·791	0·421	0	−0·380	−0·380	0·275
		0·2	−0·001	1·740	1·747	0·582	−0·007	−0·369	−0·371	0·240
		0·6	−0·106	1·579	1·610	0·804	−0·044	−0·332	−0·351	0·196
		1·2	−0·671	1·421	1·465	0·926	−0·125	−0·292	−0·351	0·174
		2·4	−2·958	1·371	1·458	+0·958	−0·499	−0·258	−0·493	0·185
	0·745	→0	0	2·193	2·193	−0·121	0	−0·484	−0·484	0·705
		0·2	+0·009	2·047	2·062	+0·448	−0·018	−0·449	−0·454	0·570
		0·6	−0·156	1·696	1·747	0·986	−0·101	−0·367	−0·389	0·423
		1·2	−0·878	1·353	1·440	+1·005	−0·284	−0·286	−0·330	0·375
	0·866	→0	0	2·445	2·445	−0·685	0	−0·562	−0·562	1·213
		0·05	+0·003	2·411	2·413	−0·348	−0·002	−0·553	−0·554	1·138
		0·3	−0·017	2·004	2·033	+0·872	−0·054	−0·459	−0·467	0·776
		0·6	−0·256	1·580	1·617	+1·222	−0·133	−0·383	−0·384	0·562

The derivatives are referred to an axis at a position h on the chord.

TABLE 2

Derivatives for Rectangular Wings of Aspect Ratio A describing Plunging and Pitching Oscillations, referred to an Axis at 0·445 *Chord*

A	M	ν	l_z	$l_{\dot{z}}$	l_α	$l_{\dot{\alpha}}$	$-m_z$	$-m_{\dot{z}}$	$-m_\alpha$	$-m_{\dot{\alpha}}$
2	0	→0	0	1·223	1·223	+0·762	0	−0·281	−0·281	0·160
		0·2	−0·015	1·211	1·212	0·795	−0·003	−0·278	−0·279	0·154
		1·2	−0·682	1·102	1·106	0·857	−0·089	−0·247	−0·295	0·144
		2·4	−2·713	1·102	1·123	0·854	−0·407	−0·231	−0·441	0·158
8/3	0	→0	0	1·462	1·462	0·688	0	−0·323	−0·323	0·198
		0·45	−0·072	1·387	1·398	0·819	−0·020	−0·305	−0·314	0·170
		1·35	−0·887	1·238	1·257	0·902	−0·131	−0·263	−0·329	0·158
		2·70	−3·536	1·279	1·362	0·915	−0·610	−0·249	−0·543	0·175
3	0·745	→0	0	1·834	1·834	0·453	0	−0·421	−0·421	0·519
		0·5$\dot{3}$	−0·131	1·549	1·581	0·986	−0·070	−0·353	−0·369	0·377
		1·0$\dot{6}$	−0·681	1·290	1·343	+1·004	−0·205	−0·292	−0·328	0·339
6	0·745	→0	0	2·686	2·686	−1·460	0	−0·569	−0·569	1·052
		0·5$\dot{3}$	−0·065	1·952	2·019	+0·845	−0·104	−0·404	−0·427	0·510
		1·0$\dot{6}$	−0·703	1·552	1·652	+1·070	−0·267	−0·317	−0·358	0·416
16/3	0·866	→0	0	2·924	2·924	−2·313	0	−0·645	−0·645	1·701
8	0·866	→0	0	3·581	3·581	−5·852	0	−0·759	−0·759	2·597

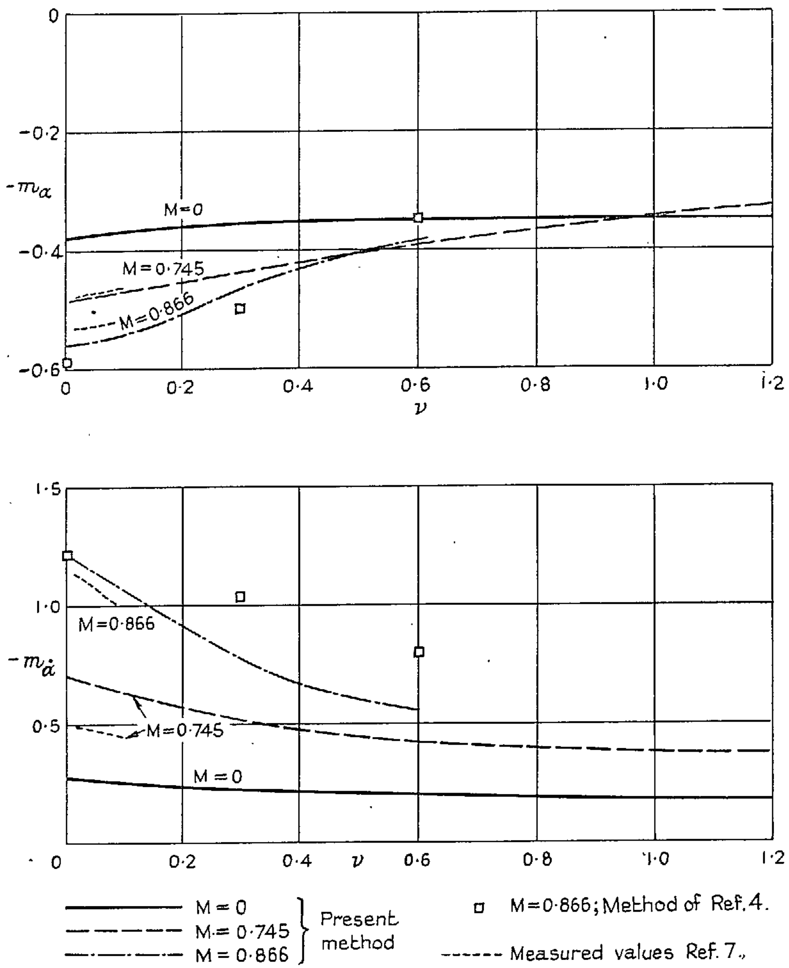

FIG. 1. Wing $A = 4$.—Variation of the pitching-moment derivatives, referred to an axis positioned at 0·445 chord, with Mach number M and frequency parameter $\nu = fc/U$.

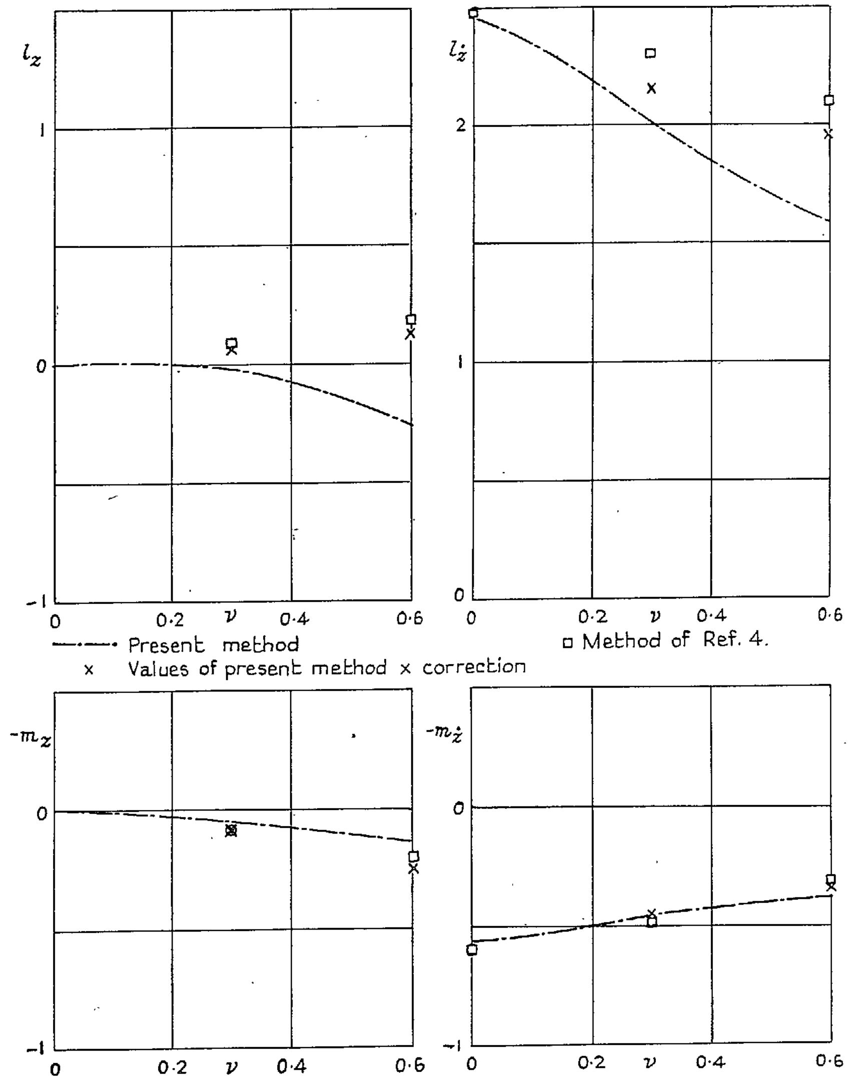

FIG. 2. Wing $A = 4$.—Derivatives for plunging oscillations, referred to an axis positioned at 0·445 chord, for Mach number $M = 0·866$.

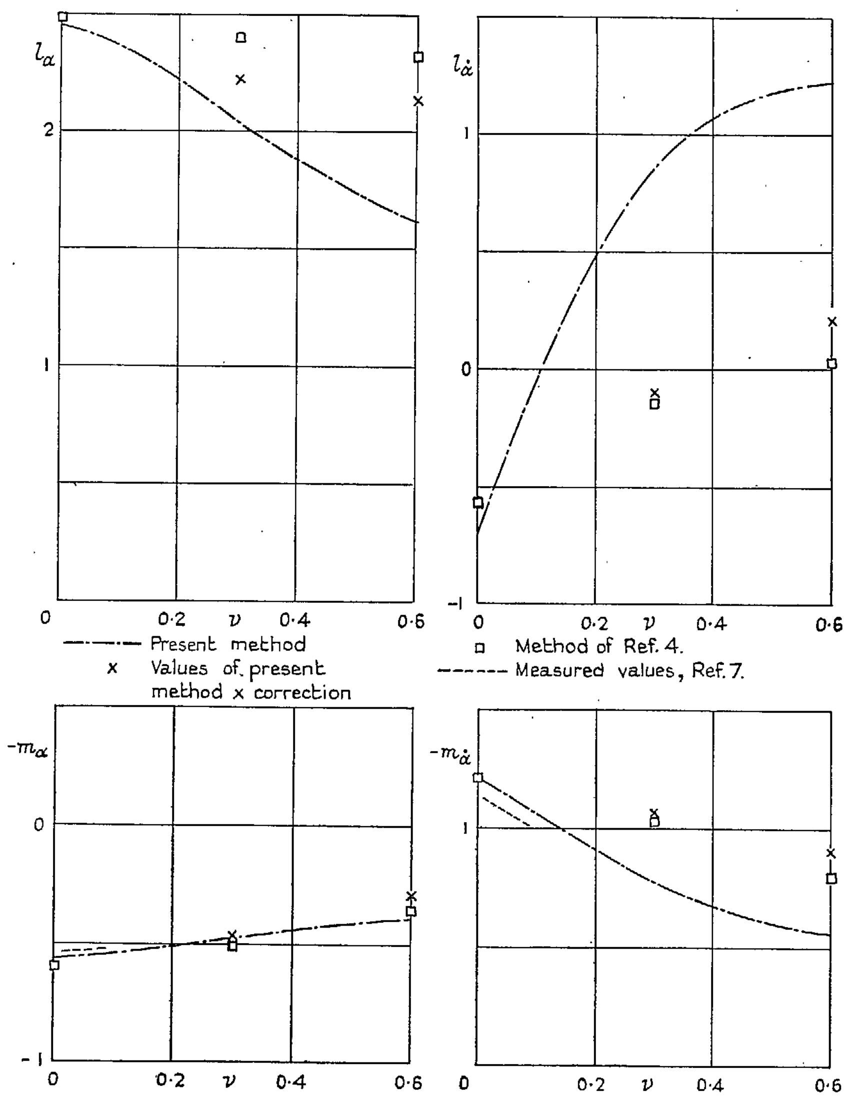

FIG. 3. Wing $A = 4$.—Derivatives for pitching oscillations, referred to an axis positioned at $0 \cdot 445$ chord, for a Mach number $M = 0 \cdot 866$.

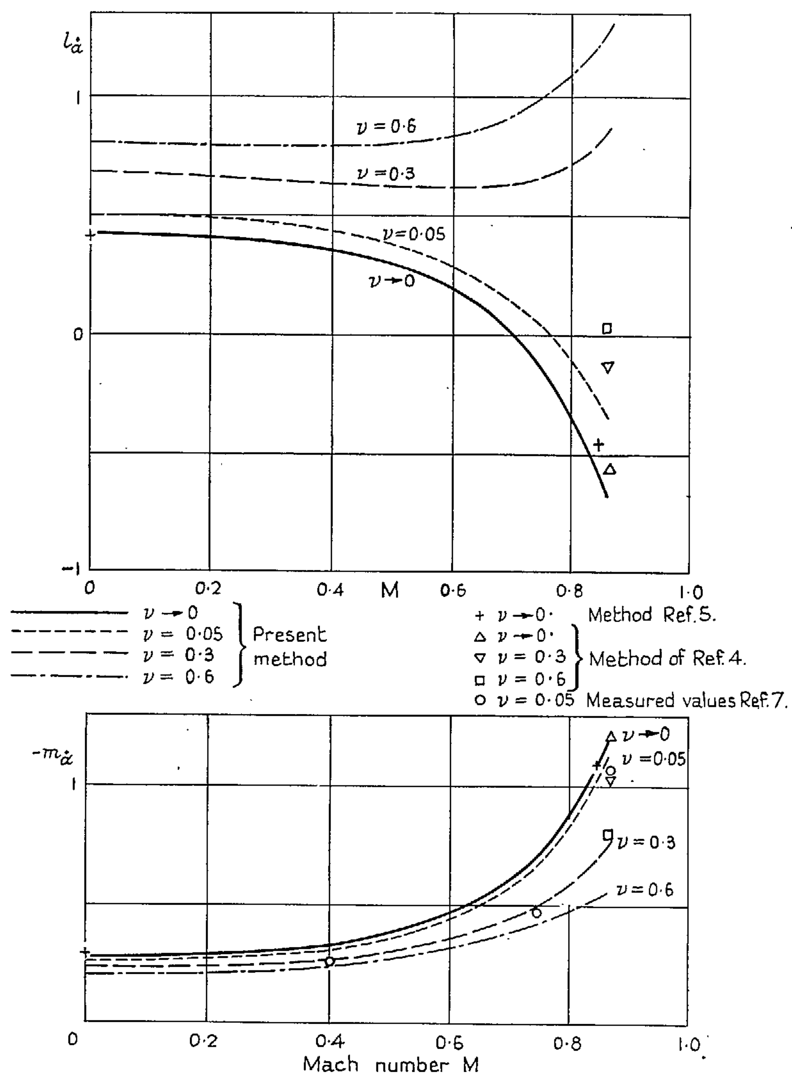

FIG. 4. Wing $A = 4$.—Damping derivatives for pitching oscillations, referred to an axis positioned at $0 \cdot 445$ chord, for frequency parameter values $\nu = fc/U$.

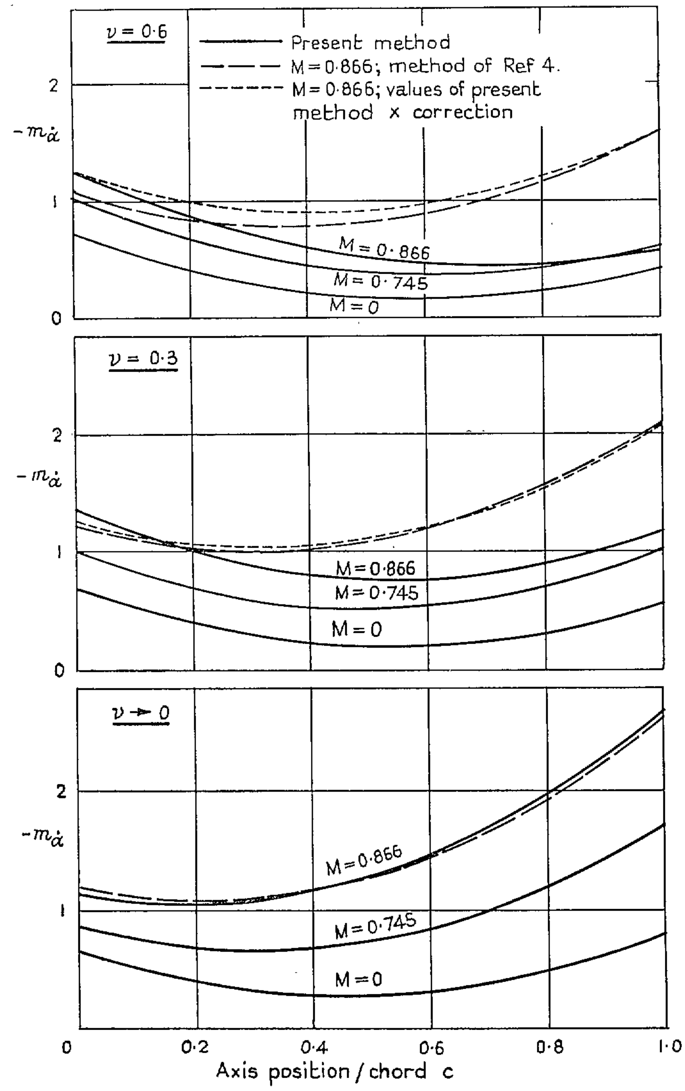

FIG. 5. Wing $A = 4$.—Variation of the pitching-moment damping derivative with axis position, Mach number M and frequency parameter $\nu = fc/U$.

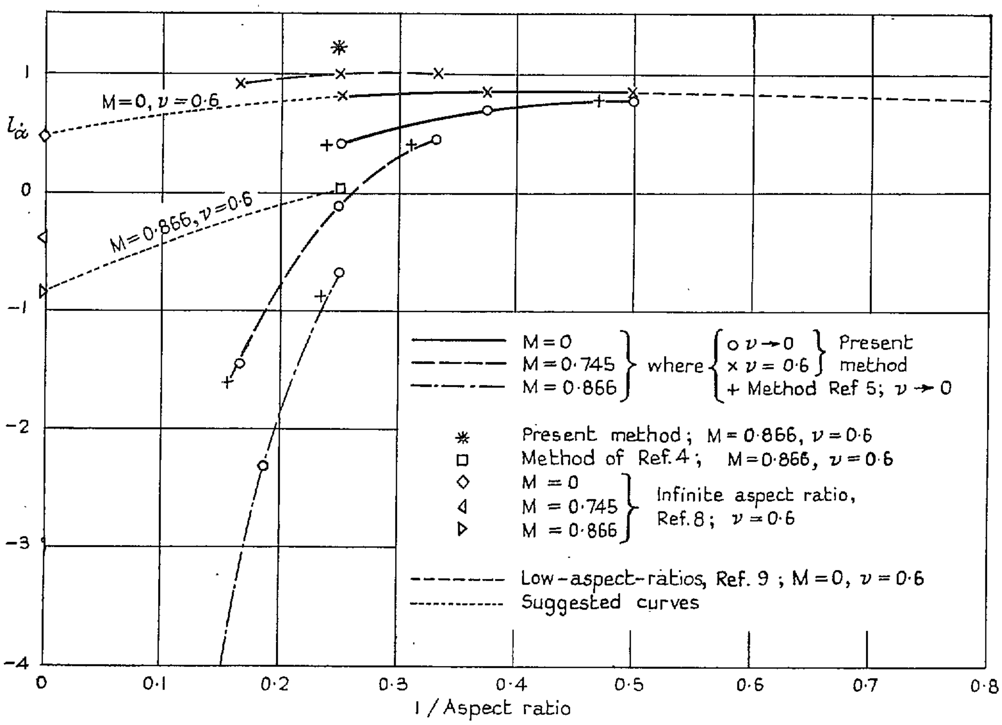

FIG. 6. Effect of aspect ratio on the derivative $l_{\dot{\alpha}}$, referred to an axis at 0·445 chord, for different values of Mach number and frequency.

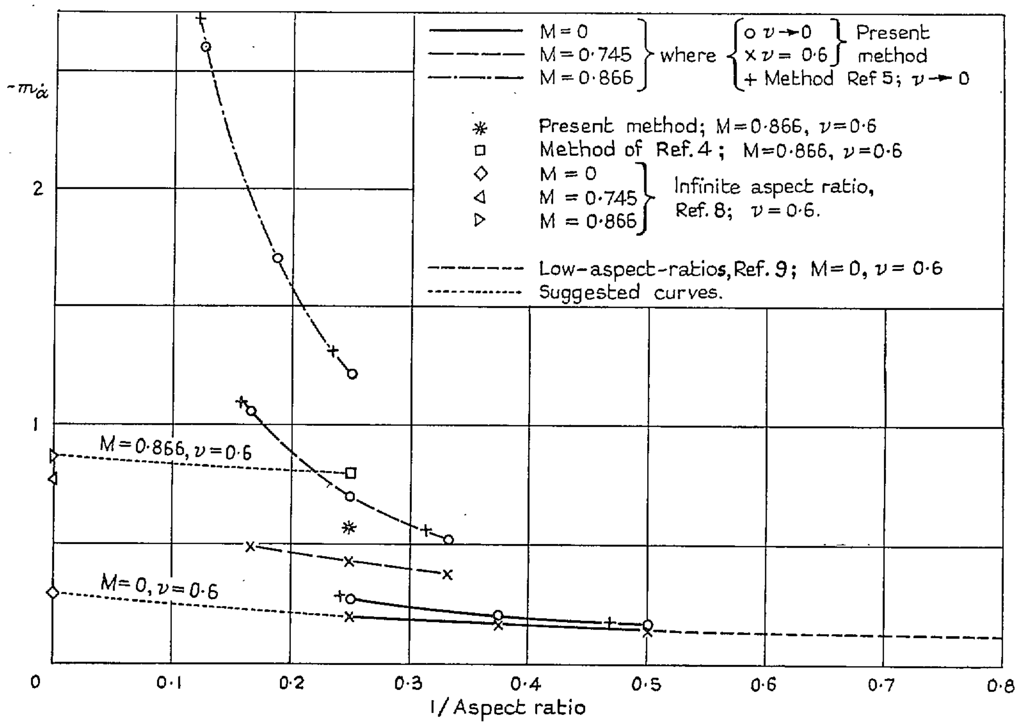

FIG. 7. Effect of aspect ratio on the derivative $-m_{\dot{\alpha}}$, referred to an axis at 0·445 chord, for different values of Mach number and frequency.

(71644) Wt. 52/8943 R.7 10/58 Hw.

PRINTED IN GREAT BRITAIN

Publications of the Aeronautical Research Council

ANNUAL TECHNICAL REPORTS OF THE AERONAUTICAL RESEARCH COUNCIL (BOUND VOLUMES)

1939 Vol. I. Aerodynamics General, Performance, Airscrews, Engines. 50*s*. (52*s*.).
Vol. II. Stability and Control, Flutter and Vibration, Instruments, Structures, Seaplanes, etc. 63*s*. (65*s*.)

1940 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Icing, Stability and Control, Structures, and a miscellaneous section. 50*s*. (52*s*.)

1941 Aero and Hydrodynamics, Aerofoils, Airscrews, Engines, Flutter, Stability and Control, Structures. 63*s*. (65*s*.)

1942 Vol. I. Aero and Hydrodynamics, Aerofoils, Airscrews, Engines. 75*s*. (77*s*.)
Vol. II. Noise, Parachutes, Stability and Control, Structures, Vibration, Wind Tunnels. 47*s*. 6*d*. (49*s*. 6*d*.)

1943 Vol. I. Aerodynamics, Aerofoils, Airscrews. 80*s*. (82*s*.)
Vol. II. Engines, Flutter, Materials, Parachutes, Performance, Stability and Control, Structures. 90*s*. (92*s*. 9*d*.)

1944 Vol. I. Aero and Hydrodynamics, Aerofoils, Aircraft, Airscrews, Controls. 84*s*. (86*s*. 6*d*.)
Vol. II. Flutter and Vibration, Materials, Miscellaneous, Navigation, Parachutes, Performance, Plates and Panels, Stability, Structures, Test Equipment, Wind Tunnels. 84*s*. (86*s*. 6*d*.)

1945 Vol. I. Aero and Hydrodynamics, Aerofoils. 130*s*. (132*s*. 9*d*.)
Vol. II. Aircraft, Airscrews, Controls. 130*s*. (132*s*. 9*d*.)
Vol. III. Flutter and Vibration, Instruments, Miscellaneous, Parachutes, Plates and Panels, Propulsion. 130*s*. (132*s*. 6*d*.)
Vol. IV. Stability, Structures, Wind Tunnels, Wind Tunnel Technique. 130*s*. (132*s*. 6*d*.)

Annual Reports of the Aeronautical Research Council—

1937 2*s*. (2*s*. 2*d*.) 1938 1*s*. 6*d*. (1*s*. 8*d*.) 1939–48 3*s*. (3*s*. 5*d*.)

Index to all Reports and Memoranda published in the Annual Technical Reports, and separately—

April, 1950 - - - - R. & M. 2600 2*s*. 6*d*. (2*s*. 10*d*.)

Author Index to all Reports and Memoranda of the Aeronautical Research Council—

1909—January, 1954 R. & M. No. 2570 15*s*. (15*s*. 8*d*.)

Indexes to the Technical Reports of the Aeronautical Research Council—

December 1, 1936—June 30, 1939	R. & M. No. 1850	1*s*. 3*d*. (1*s*. 5*d*.)
July 1, 1939—June 30, 1945	R. & M. No. 1950	1*s*. (1*s*. 2*d*.)
July 1, 1945—June 30, 1946	R. & M. No. 2050	1*s*. (1*s*. 2*d*.)
July 1, 1946—December 31, 1946	R. & M. No. 2150	1*s*. 3*d*. (1*s*. 5*d*.)
January 1, 1947—June 30, 1947	R. & M. No. 2250	1*s*. 3*d*. (1*s*. 5*d*.)

Published Reports and Memoranda of the Aeronautical Research Council—

Between Nos. 2251–2349	R. & M. No. 2350	1*s*. 9*d*. (1*s*. 11*d*.)
Between Nos. 2351–2449	R. & M. No. 2450	2*s*. (2*s*. 2*d*.)
Between Nos. 2451–2549	R. & M. No. 2550	2*s*. 6*d*. (2*s*. 10*d*.)
Between Nos. 2551–2649	R. & M. No. 2650	2*s*. 6*d*. (2*s*. 10*d*.)
Between Nos. 2651–2749	R. & M. No. 2750	2*s*. 6*d*. (2*s*. 10*d*.)

Prices in brackets include postage

HER MAJESTY'S STATIONERY OFFICE

York House, Kingsway, London W.C.2; 423 Oxford Street, London W.1; 13a Castle Street, Edinburgh 2; 39 King Street, Manchester 2; 2 Edmund Street, Birmingham 3; 109 St. Mary Street, Cardiff; Tower Lane, Bristol 1; 80 Chichester Street, Belfast, *or through any bookseller*.

S.O. Code No. 23-3068