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Summary.—The wing of infinite span oscillating harmonically in incompressible flow but having a vortex trail of finite length Sc is discussed theoretically. The 'incomplete circulation functions' C_S which arises in this case is tabulated. As an example, the damping moment due to slow pitching oscillations is shown for several values of S. The result is of interest as a wind-tunnel correction, in particular in that range of small frequencies which occurs in flight stability oscillations.

Agreement with an experiment in an open-jet wind tunnel is obtained. A contradiction between different experiments in closed-jet wind tunnels is mentioned.

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1. *Introduction.*—The problem of the wing oscillating harmonically in incompressible two-dimensional flow has been treated by numerous authors† on the assumption that the motion has existed for so long that the vortex trail can be treated as having infinite length and harmonic vortex distribution. The results of these investigations are familiar in the form of aerodynamic coefficients which have long been used in flutter calculations and recently in flight stability calculations.

In the present paper the case of a vortex trail, which has finite length Sc (c = wing chord) but has still a harmonic vortex distribution, is discussed. It is possible that such a case arises, for example, when a wing oscillates in a wind tunnel. However, in this case it arises within the framework of a rather complicated system of limitations to the air stream. In order to have simple propositions the following system of two wings in an unrestricted medium is considered in the present paper.

The first or main wing (*see* Fig. 1), is the given wing; it is supposed that it has been oscillating harmonically for a long time so that its vortex trail can be assumed to be harmonic. In this trail and at a fixed distance Sc from the first wing, the second or auxiliary wing oscillates; it oscillates in such a way as just to cancel the vorticity of the trail, so that no vortex trail exists behind the second wing. As a further simplification the chord δc of the auxiliary wing is supposed to be negligibly small; thus this wing is in effect nothing but a single vortex of harmonic intensity. It is obvious that the existence, in some form or other, of this vortex is necessary in order that the physical system be complete; to call this vortex a 'wing' is a matter of illustration only.

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† The first was Birnbaum[1] (1922). An independent approach was developed by Wagner[2] (1925). Küssner[3] (1929) continued Birnbaum's work; Glauert[4] (1929) continued Wagner's work (*see* also section 2).

We assume the downwash $w(x,t)$ at the main wing to be given and require the resulting forces on this wing. S is the parameter of our problem.

The investigation here presented was made two years ago in connection with an unsuccessful attempt[5] to reproduce, in an open-jet wind tunnel, the undamped pitching oscillations that had been predicted theoretically by Glauert[4]. It was found that the introduction of the parameter S yielded as good an agreement between theory and experiment as could be expected. It was felt at the time that, for reasons explained in section 5, this agreement should be tested by further experiments, varying in particular the parameter S. These further experiments could not then be made and are not likely to be made in the near future. Justification for communicating the existing results now is seen in :

(*a*) the practical importance of the fact that Glauert's theoretical prediction could not be corroborated by careful experiments in an open-jet wind tunnel and that this failure could be explained quantitatively by the finite length of the vortex trail in these experiments

(*b*) the statement made in a recent paper by Runyan[6] that Glauert flutter has been observed in a closed-jet wind tunnel (this occurrence is discussed in section 6)

(*c*) the recently renewed interest[7] in the circulation function C of unsteady flow. This function arises in the limit $S \to \infty$ of the present analysis. This analysis, by virtue of the finite parameter S, avoids certain mathematical difficulties which arose in some of the previous treatments of the case of an infinite vortex trail.

2. *Remark on the Circulation Function.*—We start our discussion of a system of two wings by recalling a few of the results concerning a single wing. This problem* has a general solution in closed form† : the lift distribution $p(x,t)$ due to an arbitrary downwash $w(x,t)$ is :

$$p(x,t) = \frac{\rho V}{\pi} \oint_{-1}^{+1} \left[\frac{c}{V} \frac{\partial w(x,t)}{\partial t} A(x,\xi) + w(x,t) \left\{ \frac{2}{\xi - x} + T(\tfrac{1}{2}\nu) - 1 \right\} \left(\frac{1-x}{1+x} \frac{1+\xi}{1-\xi} \right)^{0\cdot5} \right] d\xi \quad \ldots \quad (1)$$

For details of notation *see* section 3.1. The symbol $\oint$ denotes Cauchy's principal value.

Both in the general formula (1) and in the more familiar individual aerodynamic coefficients a certain transcendental function of the frequency parameter ν occurs. This function has been defined differently by different authors:

$$\tfrac{1}{2}[1 + T(\tfrac{1}{2}\nu)] \equiv C(\tfrac{1}{2}\nu) \equiv A(\nu) - iB(\nu)$$
$$= \frac{H_1^{(2)}(\tfrac{1}{2}\nu)}{iH_0^{(2)}(\tfrac{1}{2}\nu) + H_1^{(2)}(\tfrac{1}{2}\nu)} = \frac{K_1(\tfrac{1}{2}i\nu)}{K_0(\tfrac{1}{2}i\nu) + K_1(\tfrac{1}{2}i\nu)} \cdot \quad \ldots \quad (2)$$

The notation T was introduced by Küssner[10], the notation C by Theodorsen[11]. The notation $A - iB$ is used in British flutter practice. The $H_r^{(2)}$ are Hankel functions of the second kind, the K_r are modified Bessel functions.

The difference in the definitions of the two functions T and C : $1 + T = 2C$ instead of $T = C$, is not arbitrary as the two functions have different physical significances. We may split the flow around the wing, and consequently the lift distribution p, into a circulatory part and a non-circulatory part: the circulatory part has the factor $C(\tfrac{1}{2}\nu)$. On the other hand we may split $p(x,t)$ into a part which depends only upon the momentary values of w and $\partial w/\partial t$, the instantaneous part, and a part which depends upon the history of the motion and thus upon

* As defined already in section 1 ; single wing, oscillating harmonically in two-dimensional incompressible flow.

† Discovered by Soehngen[8] and by Schwarz[9].

the distribution of the vorticity along the vortex trail, the transient part. This transient part has the factor $T(\frac{1}{2}\nu)$ in (1); in fact (1) remains valid for any history of the motion if the T-function is replaced by the function* appropriate to the history.

Thus the two functions C and T, often called after Theodorsen and Küssner respectively, are the 'circulation function' and 'transient-lift function' of harmonic motion. The first of these terms is well established. The second is not. Otherwise there is no practical reason for preferring the one or the other in the present problem. As a matter of fact the circulation as such has less significance in unsteady flow than it has in steady flow, while the transient part of the flow has less significance in harmonic motion than it has in non-uniform motion. The analysis of the present paper follows closely that of Schwarz[9], who uses the T-function.

In the present paper we set out to investigate that function T_S which replaces the function T in (1) if $p(x,t)$ and $w(x,t)$ are respectively the lift distribution and downwash of the main wing in our system of two wings (*see* section 1 and Fig. 1). Obviously we must expect:

$$\lim_{S \to \infty} T_S = T \qquad (3)$$

We may further define (*see* (2)):

$$C_S = \tfrac{1}{2}(1 + T_S) \qquad (3a)$$

with the consequence:

$$\lim_{S \to \infty} C_S = C \qquad (3b)$$

We call T_S and C_S the 'incomplete T-function' and the 'incomplete circulation function' respectively.

3. *Analysis*.—3.1. *Notation*.—

t		Time
ρ		Air density
$2\pi f$		Circular frequency
c		Chord of main wing
δc		Chord of auxiliary wing
V		Speed of air flow
v	$=$	$2V/c$ Reduced speed
ν	$=$	$2\pi fc/V$ Frequency parameter
ω	$=$	$i\nu/2 = 2\pi if/v$ Imaginary frequency parameter
x		Non-dimensional coordinate (*see* Fig. 1)
$sc/2$	$=$	$(S + 0 \cdot 5)c$ Distance between wings (*see* Fig. 1)
$w(x,t)$	$\equiv$	$w(x)V \exp(2\pi ift)$ Downwash produced by the two wings
$\varepsilon(x,t)$		Free vorticity
$\gamma(x,t)$	$\equiv$	$\gamma(x)V \exp(2\pi ift)$ Bound vorticity
$\varphi(x,t)$	$\equiv$	$\varphi(x)V \exp(2\pi ift)$ Total vorticity
$p(x,t)$		Local lift
$\mathcal{R}$		Real part
$\mathcal{I}$		Imaginary part
R		Reynolds number

* This function starts from zero as the motion starts from rest so that the adjective 'instantaneous' for the remaining part is adequate (Küssner[12]).

3.2. *Aerodynamic Propositions.*—The system of two wings oscillating harmonically in incompressible two-dimensional flow has already been described in section 1; it is shown in Fig. 1. The usual assumptions of linearised aerofoil theory are made: thin aerofoil, small amplitudes, potential flow.

Bound vorticity γ, free vorticity ε and total vorticity φ are interconnected by the relation:

$$\gamma(x,t) + \varepsilon(x,t) = \varphi(x,t) \,. \qquad (1)$$

Bound vorticity exists at the two wings only. Free vorticity exists at the two wings and also forms the vortex trail between them:

$$\begin{aligned} \varepsilon(x,t) \equiv \gamma(x,t) &\equiv 0 \text{ in } [-\infty, -1] \text{ and } [s + \delta, \infty] \\ \gamma(x,t) &\equiv 0 \text{ in } [1, s - \delta], \end{aligned} \qquad (1a)$$

the [ ] brackets referring to the x-coordinate.

Owing to (1a) the Kutta condition of smooth flow at the trailing edge need be stated for the main wing only:

$$|\gamma(1,t)| \neq \infty \,. \qquad (1b)$$

As vorticity is generated at the two wings only, the vorticity of the trail is subject to the condition:

$$\varepsilon(x,t) = \varepsilon\left(1,t - \frac{x-1}{v}\right) = \varepsilon\left(s - \delta, t + \frac{s - \delta - x}{v}\right) \; (1 \leqslant x \leqslant s - \delta) \,. \qquad (1c)$$

At a given point of either wing the bound vorticity γ and free vorticity ε are related by Helmholz's theorem, which yields:

$$\frac{\partial \gamma}{\partial t} + \frac{d\varepsilon}{dt} = \frac{\partial \varphi}{\partial t} + v\frac{\partial \varepsilon}{\partial x} = 0 \,. \qquad (2)$$

On integrating equation (2) and using equations (1a) and (1c) we obtain:

$$\begin{aligned} \varepsilon(x,t) &= -\frac{1}{v}\int_{-1}^{x} \frac{\partial}{\partial t}\varphi(\xi,t)\, d\xi \qquad && (-1 \leqslant x \leqslant 1) \\ &= -\frac{1}{v}\int_{-1}^{+1} \frac{\partial}{\partial t}\varphi\left(\xi,t - \frac{x-1}{v}\right) d\xi \\ &= +\frac{1}{v}\int_{s-\delta}^{s+\delta} \frac{\partial}{\partial t}\varphi\left(\xi,t + \frac{s-\delta-x}{v}\right) d\xi && \Bigg\} \; (1 \leqslant x \leqslant s - \delta) \\ &= +\frac{1}{v}\int_{x}^{s+\delta} \frac{\partial}{\partial t}\varphi(\xi,t)\, d\xi && (s - \delta \leqslant x \leqslant s + \delta) \,. \end{aligned} \qquad (3)$$

The total vorticity is related to the downwash $w(x,t)$ by the theorem of Biot and Savart:

$$\left(\oint_{-1}^{+1} + \oint_{s-\delta}^{s+\delta}\right) \frac{\varphi(x,t)}{x-\xi}\, d\xi + \int_{1}^{s-\delta} \frac{\varepsilon(\xi,t)}{x-\xi}\, d\xi = 2\pi w(x,t)$$

$$\begin{pmatrix} -1 \leqslant x \leqslant +1 \\ s - \delta \leqslant x \leqslant s + \delta \end{pmatrix} . \qquad (4)$$

The local lift p is related to the bound vorticity γ by the theorem of Kutta and Joukowski:

$$p(x,t) = \rho V \gamma(x,t) \,. \qquad (5)$$

3.3. *Harmonic Motion.*—We assume the instantaneous downwash w at the main wing to be given. The aerodynamic relations stated in the preceding section are not sufficient to determine the bound vorticity γ, and thus the local lift p (*see* section 3.2, equation (5)). We are still free to decide upon the history of the motion and also have a certain choice left for the downwash distribution over the auxiliary wing. These freedoms will now be eliminated by first introducing harmonic motion and then going to the limit $\delta \to 0$.

In the case of harmonic motion the free vorticity ε along the vortex trail can be written:

$$\varepsilon(x,t) = -2\pi\lambda V \exp\left[2\pi i f\left(t - \frac{x}{v}\right)\right], \quad \ldots \quad (1)$$

with an unknown constant λ. By means of (1), and using the non-dimensional notation defined in section 3.1, equation 3.2(4) becomes:

$$\frac{1}{2\pi}\left[\oint_{-1}^{+1} + \oint_{s-\delta}^{s+\delta}\right] \frac{\varphi(\xi)}{x - \xi}\, d\xi = w(x) + \lambda \int_{1}^{s-\delta} \frac{e^{-\omega\xi}}{x - \xi}\, d\xi . \quad \ldots \quad (2)$$

For λ from equation 3.2(3):

$$\lambda = \frac{\omega}{2\pi} e^{\omega} \int_{-1}^{+1} \varphi(\xi)\, d\xi = -\frac{\omega}{2\pi} e^{(s-\delta)\omega} \int_{s-\delta}^{s+\delta} \varphi(\xi)\, d\xi . \quad \ldots \quad (3)$$

Equation (3) states the physically obvious fact that the total vorticities of the two wings must be equal, apart from a phase difference.

3.4. *Integral Equation and Solution.*—By going to the limit $\delta \to 0$ in equations 3.3(2) and (3) the integral equation which governs our problem is obtained:

$$\frac{1}{2\pi}\oint_{-1}^{+1} \frac{\varphi(\xi)}{x - \xi}\, d\xi = w(x) + \lambda \left\{\int_{-1}^{s} \frac{e^{-\omega\xi}}{x - \xi}\, d\xi + \frac{e^{-\omega s}}{\omega(x - s)}\right\} \quad (-1 \leqslant x \leqslant 1) , \quad \ldots \quad (1)$$

with:

$$\lambda = \frac{\omega}{2\pi} e^{\omega} \int_{-1}^{+1} \varphi(\xi)\, d\xi . \quad \ldots \quad (1a)$$

Equation (1) is an integral equation of the first kind of the form:

$$\frac{1}{2\pi}\oint_{-1}^{+1} \frac{f(\xi)}{x - \xi}\, d\xi = g(x) ,$$

which has the solution, due to A. Betz:

$$f(x) = \frac{2}{\pi}\sqrt{\left(\frac{1 - x}{1 + x}\right)} \oint_{-1}^{+1} g(\xi) \sqrt{\left(\frac{1 + \xi}{1 - \xi}\right)} \frac{d\xi}{\xi - x} . \quad \ldots \quad (2)$$

Equation (2) is the only solution of equation (1*a*) which, if formed for equation (1), fulfils the condition equation 3.2 (1*b*).

This solution still contains the unknown λ which has to be eliminated by means of equation (1*a*). Having thus found the total vorticity $\varphi(x)$ we obtain the bound vorticity $\gamma(x)$ by means of equation 3.2 (1) and (3):

$$\gamma(x) = \varphi(x) + \omega \int_{-1}^{x} \varphi(\xi)\, d\xi . \quad \ldots \quad (3)$$

The lift distribution p is then obtained by means of equation 3.2(5):

$$p(x,t) = \rho V^2 \gamma(x) \exp(2\pi i f t) . \quad \ldots \quad (3a)$$

By letting s tend to infinity in our problem (1), (1*a*) and (3), we obtain the familiar problem of the wing with infinite vortex trail in the form discussed by Schwarz[9]. The final result obtained by Schwarz is equation 2(1); our aim is to find the alteration which occurs in equation 2(1) if s takes a finite value. Thus, in performing the integrations contained in equations (1*a*), (2) and (3) we need concern ourselves only with those terms which contain the parameter s. This part of the analysis is given in Appendix I; the result is as given below:

Equation 2(1) remains valid for our system of two wings if the T-function is replaced as follows:

$$T(\tfrac{1}{2}\nu) \rightarrow T_{S,x}(\tfrac{1}{2}\nu) \equiv T_S(\tfrac{1}{2}\nu) - \frac{1+x}{s-x} U_S(\tfrac{1}{2}\nu) \,, \qquad (4)$$

with

$$T_S(\tfrac{1}{2}\nu) = \frac{2H_{1S}(\tfrac{1}{2}\nu)}{iH_{0,S}(\tfrac{1}{2}\nu) + H_{1S}(\tfrac{1}{2}\nu)} - 1$$

$$U_S(\tfrac{1}{2}\nu) = -\frac{4}{\pi\omega} \frac{e^{-\omega s}}{iH_{0S}(\tfrac{1}{2}\nu) + H_{1S}(\tfrac{1}{2}\nu)} \frac{1}{\sqrt{(s^2-1)}} \,. \qquad (4a)$$

The functions H_{rS} are incomplete Hankel functions (*see* Appendix II). Owing to the relation:

$$\lim_{S \to \infty} H_{rS}(\tfrac{1}{2}\nu) = H_r^{(2)}(\tfrac{1}{2}\nu) \qquad (4b)$$

and equation 2(2), the condition 2(3) is fulfilled.

3.5. *Lift and Moment.*—Let L_x be the lift acting on the part of the chordwise section between the point x and the trailing edge, and let M_x be the moment about the mid-chord point $x = 0$ of L_x:

$$L_x = c\int_x^1 p(u,t)\,du$$

$$M_x = c^2\int_x^1 u\,p(u,t)\,du \,. \qquad (1)$$

Let L_x, M_x refer to the case of an infinite vortex trail. By applying equation (1) to equation 2(1) the familiar results are obtained; these depend upon the downwash $w(\xi)$ and contain the T-function (or the equivalent C-function).

Let L_{xS}, M_{xS} be the forces corresponding to L_x, M_x in our case of a finite vortex trail. These forces are obtained by again applying equation (1) to equation 2(1) but only after replacing the T-function by the T_{Sx}-function (*see* 3.4(4)). We find the formulae for L_x, M_x remain valid in the case of a finite vortex trail if we make the substitution:

$$T(\tfrac{1}{2}\nu) \rightarrow T_S(\tfrac{1}{2}\nu) - U_S(\tfrac{1}{2}\nu) \times \begin{cases} I_2/I_0 & \text{for } L_{xS} \\ (sI_2 - I_0 - I_1)/I_1 & \text{for } M_{xS} \end{cases} . \qquad (2)$$

The I_r are the following integrals:

$$I_0 = \int_x^1 \sqrt{\left(\frac{1-u}{1+u}\right)}\,du = \tfrac{1}{2}\pi - \sin^{-1}x - \sqrt{(1-x^2)}$$

$$I_1 = \int_x^1 u\sqrt{\left(\frac{1-u}{1+u}\right)}\,du = (1-\tfrac{1}{2}x)\sqrt{(1-x^2)} - \tfrac{1}{2}(\tfrac{1}{2}\pi - \sin^{-1}x)$$

$$I_2 = \int_x^1 \frac{1+u}{s-u}\sqrt{\left(\frac{1-u}{1+u}\right)}\,du = s(\tfrac{1}{2}\pi - \sin^{-1}x)$$

$$+ \sqrt{(1-x^2)} - 2\sqrt{(s^2-1)}\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\cdot\frac{s+1}{s-1}\right)} . \qquad (2a)$$

In the particular case $x = -1$, *i.e*, for total lift L and total moment M, we obtain for equation (2):

$$T(\tfrac{1}{2}\nu) \rightarrow T_S(\tfrac{1}{2}\nu) - U_S(\tfrac{1}{2}\nu) \times \begin{cases} s - \sqrt{(s^2-1)} = \dfrac{1}{2s} - \dfrac{1}{8s^3} \ldots \text{ for } L_S \\ 1 - 2s\{s - \sqrt{(s^2-1)}\} = -\dfrac{1}{4s^2} - \dfrac{1}{8s^4} \ldots \text{ for } M_S. \end{cases} \quad (2b)$$

3.6. *The Incomplete Circulation Function.*—The T-function is a function of the frequency parameter ν only; the function $T_{S,x}$ is a function of the three parameters ν, S and x. In order to simplify our result we now discard the dependence of $T_{S,x}$ upon x by neglecting the function U_S. Thus we use the function T_S of equation 3.4(4a) as the function T_S of equations 2(3) and (3a). From equations 3.4(4) and (4a) it can be seen that the error thus committed is of the order S^{-2} (*cf.* also Table 1 and Fig. 3).

From equations 2(2) and 3.4(4a) in conjunction with Appendix II(2) the following expression for the incomplete circulation function is obtained:

$$C_S(\tfrac{1}{2}\nu) = \frac{\displaystyle\int_1^s \frac{\eta}{\sqrt{(\eta^2-1)}} e^{-\omega\eta}\, d\eta + \frac{s}{\sqrt{(s^2-1)}} \frac{e^{-\omega s}}{\omega}}{\displaystyle\int_1^s \frac{\eta+1}{\sqrt{(\eta^2-1)}} e^{-\omega\eta}\, d\eta + \frac{s+1}{\sqrt{(s^2-1)}} \frac{e^{-\omega s}}{\omega}} \quad (1)$$

The limit $s \rightarrow \infty$ cannot formally be introduced in equation (1) as it stands but it can be thus introduced after transforming equation (1) to read:

$$C_S(\tfrac{1}{2}\nu) = \frac{\omega \displaystyle\int_1^s \left(\frac{\eta}{\sqrt{(\eta^2-1)}} - 1\right) e^{-\omega\eta}\, d\eta + \left(\frac{s}{\sqrt{(s^2-1)}} - 1\right) e^{-\omega s} + e^{-\omega}}{\omega \displaystyle\int_1^s \left\{\sqrt{\left(\frac{\eta+1}{\eta-1}\right)} - 1\right\} e^{-\omega\eta}\, d\eta + \left\{\sqrt{\left(\frac{s+1}{s-1}\right)} - 1\right\} e^{-\omega s} + e^{-\omega}}. \quad (1a)$$

Both numerator and denominator of equations (1), (1a) are nearly independent of s if s is large, the derivatives of both with respect to s being of the order s^{-2}. It will be noticed that this is due to the terms which represent the auxiliary wing, or, if for a moment we consider the case of a wing in free flight, to the terms which represent the initial vortex. By leaving these terms out we would obtain:

$$C_S(\tfrac{1}{2}\nu) = \frac{\displaystyle\int_1^s \frac{\eta}{\sqrt{(\eta^2-1)}} e^{-\omega\eta}\, d\eta}{\displaystyle\int_1^s \sqrt{\left(\frac{\eta+1}{\eta-1}\right)} e^{-\omega\eta}\, d\eta}. \quad (1b)$$

Equation (1b) becomes meaningless if s is replaced by ∞.

Some previous authors have used the right-hand side of equation (1b), with ∞ for s, as the definition for the C-function. This is certainly incorrect from a formal point of view. However, it can be shown* that indeed:

$$C(\tfrac{1}{2}\nu) = \lim_{s \rightarrow \infty} \left\{ \frac{\displaystyle\int_1^s \frac{\eta}{\sqrt{(\eta^2-1)}} e^{-\omega\eta}\, d\eta}{\displaystyle\int_1^s \sqrt{\left(\frac{\eta+1}{\eta-1}\right)} e^{-\omega\eta}\, d\eta} \right\}, \quad (1c)$$

* For example, by considering unstable oscillations, i.e., $\omega = \mu + i\nu$, going to the limit $s \rightarrow \infty$, applying equation AII(5), and considering the limit $\mu \rightarrow 0$ after that.

i.e., that the limit of the quotient (1*b*) exists and has the correct value. Thus the terms which represent the initial vortex are not strictly necessary in the process $s \to \infty$, but they simplify this process.

For the purpose of numerical evaluation it is convenient to transform equation (1) by introducing $\nu = -2i\omega$ and $S = (s-1)/2$. The following symmetrical expression is obtained for the T_S function:

$$T_S(\tfrac{1}{2}\nu) = \frac{i\nu \int_0^S e^{-i\nu\sigma} \sqrt{\left(\frac{\sigma}{1+\sigma}\right)} d\sigma + e^{-i\nu S} \sqrt{\left(\frac{S}{S+1}\right)}}{i\nu \int_0^S e^{-i\nu\sigma} \sqrt{\left(\frac{1+\sigma}{\sigma}\right)} d\sigma + e^{-i\nu S} \sqrt{\left(\frac{S+1}{S}\right)}} . \qquad (2)$$

Note that:

$$T_S(0) = \frac{S}{S+1} , \qquad (2a)$$

and that:

$$\left(\frac{\partial T_S}{\partial \nu}\right)_{\nu=0} = \begin{cases} 0 - i\left[\log 4S - 1 - O\left(\frac{1}{S}\log S\right)\right] & (S < \infty) \\ \tfrac{1}{2}\pi - i\infty & (S = \infty) . \end{cases} \qquad (3)$$

Numerical values for the function T_S and also for the function U_S have been calculated by methods explained in Appendix III. The results are tabulated in Table 1. In Table 2 the incomplete circulation function is given in the form:

$$C_S(\tfrac{1}{2}\nu) \equiv A_S(\nu) - iB_S(\nu) . \qquad (4)$$

(*Cf.* 2(2)). T_S is shown graphically in Fig. 2.

4. *Damping Moment Due to Slow Pitching Oscillations—Comparison with Experiment.*—The case of a harmonically oscillating wing having a vortex trail of finite length arises in wind-tunnel tests. Such tests are made in order to obtain information regarding the wing with infinite vortex trail ($S = \infty$). In fact the theoretical effect of a finite parameter value S, as seen from Fig. 2, is not very important in such cases, generally speaking, as S is usually fairly large ($S \sim 10$, say).

There is an exception: the relative effect which S has on the imaginary part of the function T_S remains large even for large values of S if the frequency ν is sufficiently small. This can also be seen from equation 3.6(3): the function $\mathcal{I}\, T$, owing to the term $\nu \log \nu$ which it contains, has an infinite slope as ν tends to zero, while $\mathcal{I}\, T_S$ has a finite slope ($S \neq \infty$). This becomes important in connection with the single-degree-of-freedom pitching flutter which exists, theoretically, in a certain range of forward axis positions. This flutter was discovered by Glauert[4] and hence should be called 'Glauert flutter'.

Fig. 3 refers to this flutter. It shows, in the range $0 \leqslant \nu \leqslant 0 \cdot 12$ and for the pitching axis at $x_0 = -5/3$, the out-of-phase damping moment:

$$-\mathcal{I}\, M_\alpha = -\rho c^2 V^2 \nu m_{\dot\alpha} \exp(2\pi i f t) ,$$

which results for different values of S. Glauert flutter arises if $-\mathcal{I}\, M_\alpha < 0$. 'Exact theory' refers to equation 3.4(4); 'simplified theory' refers to equation 3.6(1). 'Quasi-steady assumption' refers to the assumption $T(\nu) \equiv 1$ which, until recently, was in general use in flight stability calculations.

The axis position $x_0 = -5/3$ was chosen as being well suited for an attempt to verify Glauert flutter experimentally. Let F be the force which, if applied at mid-chord, would produce the negative damping moment $\mathcal{I}\, M_\alpha$ about x_0. The axis position of maximum moment $\mathcal{I}\, M_\alpha$ is forward, the axis position of maximum force F is aft of $x_0 = -5/3$.

Glauert flutter does not arise (as is well known) if the quasi-steady assumption is used. It arises in unsteady theory proper if $S = \infty$ and $\nu < 0 \cdot 077$; it persists (as shown in Fig. 3) if $S = 50$ and still persists if $S = 20$, though in a reduced range and with much reduced intensity. It has disappeared when S comes down to 10.

Fig. 3 also shows an experimental result[5], obtained with the wind-tunnel arrangement sketched in Fig. 4. The parameter S takes the value $S = 11$ if the fan is supposed to represent the auxiliary wing. With $S = 11$ the experimental result agrees well with the theoretical results, as some allowance must be made for damping effects due to the finite thickness of the wing, to its finite effective span and to the existence of a boundary layer, and these effects are not allowed for in the theoretical curves of Fig. 3*.

5. *Application as a Wind-tunnel Correction.*—The difference between the two theoretical results for finite and infinite length of the vortex trail suggests itself as a wind-tunnel correction, and the good agreement between theoretical prediction and experimental result shown in Fig. 3 would seem to speak convincingly enough for the validity of this correction for arrangements of the kind shown in Fig. 4.

However, this suggestion needs some cautionary consideration. Our simple theoretical assumption of two wings in an infinitely extended medium is, after all, only a very rough approximation of the complicated arrangement shown in Fig. 4.

Consider first the finite cross-section of the actual jet. W. P. Jones[13], assuming fixed walls, has shown that the finite cross-section has a damping effect similar to that of the finite length S of the trail. Indeed, it is conceivable that the fixed walls restrict the effectiveness of the more distant parts of the vortex trail. However, the jet of Fig. 4 is not closed throughout but is open behind the wing; somewhat optimistically, we might claim that the effects of the two different kinds of boundaries can be expected roughly to cancel each other on the wing itself.

Another suspicion arises in respect of the effect, attributed to the fan, of cancelling the vortex trail: it is by no means proved that this effect does exist. Consider in particular the limiting case of steady flow. Here the wing produces not a vortex trail but a downwash trail which will hardly be affected much by the fan; on the other hand the flow would be straightened effectively by the collector which encloses the fan. The latter effect has been investigated theoretically† (for the case of steady flow, i.e., $\nu = 0$) and has been found to be of the order exp $(-S)$, while our theory (at $\nu = 0$), assuming a second wing having a lift on the first, yields an effect of order S^{-1} (*cf.* 3.6(2*a*)). This would indicate that the 'collector effect' decreases much more rapidly as S tends to infinity than does the 'fan effect', whereas it seems inconceivable, in the case of steady flow, that the collector should have less effect than the fan.

The last argument looks less conclusive in the case of unsteady flow, *i.e.*, in the case $\nu \neq 0$, where there is a vortex trail. Such a trail must be cancelled somehow at its end, and our theoretical assumption of a second wing would thus appear to be adequate. Indeed in our theory the length S has a decisive effect, even for very small values of ν, on the imaginary part of the T_S-function (*see* Fig. 3), and thus on the damping forces which disappear altogether as ν tends to zero. Perhaps the experimental result can be taken as indicating that our simplified theory is appropriate as regards the imaginary part of the function T_S, though it is probably inadequate as regards the real part of T_S if ν is small. However, it is felt that a more extensive set of experimental results should be available before a judgment of this kind can be given.

6. *Remark on Runyan's Experimental Result.*—In the preceding section the relation has been discussed between wind-tunnel experiments and the theoretical case of a wing with an infinite harmonical vortex trail in two-dimensional unrestricted flow. A marked difference was found. However, wind-tunnel tests are made mainly for application to actual flight, and here the

* Mechanical friction is eliminated from the experimental result in Fig 3.

† *See*, for example, Vandrey[14].

agreement can be expected to be better. The vortex trail of the actual wing in free flight has also a finite effective length, for it tends to die out owing to (*a*) its inherent instability, (*b*) the viscosity of the air, (*c*) three-dimensional effects.

This position would appear to be comforting. On the other hand the contradiction, stated below, between different wind-tunnel results is disquieting. The evidence available does not suffice to solve this contradiction. It is felt that the whole problem of wind-tunnel interference needs careful investigation in view of existing programmes for measuring low-frequency aerodynamic coefficients in various wind tunnels.

The contradiction arises as follows: Runyan[6] claims to have obtained Glauert flutter in the Langley 4·5-ft Closed-Jet Flutter Research Tunnel with a wing 8 in. × 47 in., pitching about the axis $x_0 = -1 \cdot 24$. Forces were not measured in these tests but the highest value $\nu_{\alpha\,\max}$ of the parameter $\nu_\alpha = 2\pi f_\alpha c/V$ at which sustained oscillations were observed was recorded for various relative densities of the wing*.

Runyan calculated theoretical values $\nu_{\alpha\,\max\,\mathrm{th}}$ by assuming unrestricted two-dimensional flow and $S = \infty$ but allowing for the mechanical friction of the model suspension. Ignoring for a moment the finite height of the wind tunnel (*i.e.*, its extension vertical to the wing) we must expect:

$$\nu_{\alpha\,\max} < \nu_{\alpha\,\max\,\mathrm{th}} \qquad \ldots \qquad (1)$$

for each of the following reasons:

(i) the finite value of the parameter S

(ii) finite thickness of wing, and boundary layer effects

(iii) gaps of 3·5 in. each between wing tips and tunnel walls.

However, Runyan found, in contradiction to (1):

$$\nu_{\alpha\,\max} > \nu_{\alpha\,\max\,\mathrm{th}} \qquad \ldots \qquad (1a)$$

for all wing densities.

Taking the evidence so far listed we would expect the explanation for the contradiction (1), (1*a*) to arise from the finite tunnel height in Runyan's experiment; namely, that this height had a negative damping effect in the case of a closed-jet tunnel. This conclusion, however, would contradict Jones's theory[13] (*see* section 5), and would thus contradict the experimental results[13] which were obtained by J. B. Bratt, again in a closed-jet tunnel, and which support Jones's theory†.

The suspicion of course arises that an unknown source of negative damping was present in Runyan's tests.

7. *Conclusions*.—The chordwise distribution of lift on a wing in incompressible flow which oscillates harmonically with an arbitrary downwash $w(x)$ and has a vortex trail of finite length Sc has been discussed (two-dimensional linearised problem). The familiar results for the wing with infinite vortex trail remain valid if in them Küssner's T-function for the transient lift, or Theodorsen's circulation function C, as the case may be, are replaced by corresponding incomplete functions T_S and C_S respectively. These functions are tabulated; they become functions of frequency parameter ν and length of trail S alone if terms of order S^{-2} are neglected. The term $i\nu \log \nu$, which occurs in the T- and C-functions, does not occur in the T_S- and C_S-functions if S is finite. As a consequence, Glauert flutter, which means single-degree-of-freedom pitching flutter, is not possible if S is smaller than about 18.

* f_α = natural frequency of pitching. The wind tunnel could be operated at any air pressure between atmospheric and 0·5 in. of mercury and the relative wing density could thus be varied.

† In Runyan's experiments the ratio of tunnel height to wing chord was 6·75. This value is just sufficiently small, according to Jones's theory, to exclude Glauert flutter without resort to mechanical friction or the other damping effect listed above ((i) to (iii)).

Glauert flutter occurs at small frequencies, corresponding to flight stability oscillations. Here the effect of S being finite is significant. On the other hand, this effect becomes negligible if S is not too small ($S > 10$, say) in the frequency range of ordinary flutter ($\nu > 0 \cdot 1$, say).

The theoretical results have been confirmed by tests in an open-jet wind tunnel but further tests are required in order to separate the present effect from wall interference effects.

A contradiction has been pointed out between tests in closed-jet wind tunnels made by Bratt and by Runyan. Bratt's tests support an interference theory by W. P. Jones, whereas Runyan's tests would seem to contradict it. It is felt that the problem of wind-tunnel corrections for slowly oscillating wings requires further investigation.

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APPENDIX I

Integrations

Applying equation 3.4(2) to equation 3.4(1) we obtain:

$$\varphi(x) = \frac{2}{\pi}\sqrt{\left(\frac{1-x}{1+x}\right)}\oint_{-1}^{+1} w(\xi)\sqrt{\left(\frac{1+\xi}{1-\xi}\right)}\frac{d\xi}{\xi - x} + \lambda\pi F(x) \quad \ldots \quad (1)$$

or

$$\varphi(x) = q(x) + \lambda\pi F(x)\,, \quad \ldots \quad (1a)$$

where $q(x)$ is independent of s and:

$$F(x) = \frac{2}{\pi^2}\sqrt{\left(\frac{1-x}{1+x}\right)}\oint_{-1}^{+1}\left\{\int_1^s \frac{e^{-\omega\eta}}{\xi-\eta}\,d\eta + \frac{e^{-\omega s}}{\omega(\xi - s)}\right\}\sqrt{\left(\frac{1+\xi}{1-\xi}\right)}\frac{d\xi}{\xi - x}$$

$$= \frac{2}{\pi}\sqrt{\left(\frac{1-x}{1+x}\right)}\left\{\int_1^s \frac{e^{-\omega\eta}}{x-\eta}\sqrt{\left(\frac{\eta+1}{\eta-1}\right)}\,d\eta + \frac{e^{-\omega s}}{\omega(x-s)}\sqrt{\left(\frac{s+1}{s-1}\right)}\right\}. \quad \ldots \quad (1b)$$

The last line is obtained by changing the order of integration and performing the then inner integration.

According to equations 3.4(1*a*) and 3.4(3) the function $\varphi(x)$ given by (1) and (1*b*) has to be integrated to give the quantities λ and $\gamma(x)$. The integral required for λ is a particular case of that required for $\gamma(x)$. We require only those terms which depend upon the parameter s, i.e., we require the integral of $F(x)$. After changing the order of integration the inner integration can be performed:

$$\int_{-1}^{x}\sqrt{\left(\frac{1-x}{1+x}\right)}\frac{dx}{\eta - x} = \tfrac{1}{2}\pi + \sin^{-1}x + 2\sqrt{\left(\frac{\eta-1}{\eta+1}\right)}\left(k(x,\eta) - \tfrac{1}{2}\pi\right), \quad \ldots \quad (2)$$

with

$$k(x,\eta) = \tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\frac{\eta+1}{\eta-1}\right)}. \quad \ldots \quad (2a)$$

Thus

$$\int_{-1}^{x} F(\xi)\,d\xi = -\frac{2}{\pi}\int_1^s e^{-\omega\eta}R(x,\eta)\sqrt{\left(\frac{\eta+1}{\eta-1}\right)}\,d\eta + \ldots, \quad \ldots \quad (3)$$

if $R(x,\eta)$ is the right-hand side of equation (2). It is readily verified that

$$R(x,\eta) = 0\left(\frac{1}{\eta}\right), \quad \ldots \quad (3a)$$

so that the integral (3) exists in the limit $s \to \infty$; but this is not true of the integrals over the individual terms of $R(x,\eta)$. Because of this, mathematical difficulties have arisen in previous treatments of the wing with infinite vortex trail. These difficulties are eliminated when we assume s to be finite. Then $R(x,n)$ can be split into its terms without hesitation; we obtain:

$$\int_{-1}^{x} F(\xi)\,d\xi = \{\tfrac{1}{2}\pi + \sin^{-1}x\}H_S + \frac{2}{\omega}e^{-\omega} - \frac{4}{\pi}\left\{\int_1^s e^{-\omega\eta}k(x,\eta)\,d\eta + \frac{e^{-\omega s}}{\omega}k(x,s)\right\}, \quad \ldots \quad (4)$$

with

$$H_S \equiv iH_{0S}(-i\omega) + H_{1S}(-i\omega)\,. \quad \ldots \quad (4a)$$

The functions H_{rS} are defined in the Appendix II.

We have $k(1,\eta) \equiv 0$ and hence:

$$\frac{1}{\pi}\int_{-1}^{+1}\varphi(\xi)\,d\xi = 2Q + \lambda\left(\pi H_S + \frac{2}{\omega}e^{-\omega}\right), \quad \ldots \quad (5)$$

where

$$2\pi Q = \int_{-1}^{+1} q(\xi)\,d\xi \quad \text{(independent of } s\text{)}\,.$$

From equations (5) and 3.4(1a):

$$\lambda = -\frac{2Q}{\pi H_S}. \qquad \ldots \qquad (6)$$

The total vorticity φ is obtained by inserting equations (1b) and (6) in equation (1).

In order to obtain the bound vorticity γ (*see* 3.4(3)), we have to transform the integral in equation (2):

$$\begin{aligned}
&\frac{2}{\pi}\left[\omega\int_1^s e^{-\omega\eta}\, k(x,\eta)\, d\eta + e^{-\omega s}\, k(x,s)\right] \\
&= e^{-\omega} + \frac{2}{\pi}\int_1^s e^{-\omega\eta}\frac{\partial}{\partial\eta}\, k(x,\eta)\, d\eta \\
&= e^{-\omega} + \frac{1}{\pi}\sqrt{\left(\frac{1-x}{1+x}\right)}\int_1^s \frac{e^{-\omega\eta}}{\sqrt{(\eta^2-1)}}\left(1-\frac{\eta+1}{\eta-x}\right) d\eta \\
&= e^{-\omega} + \tfrac{1}{2}\left[F(x) - \sqrt{\left(\frac{1-x}{1+x}\right)}\left(iH_{0S} - \frac{2}{\pi}\frac{e^{-\omega s}}{\omega\sqrt{(s^2-1)}}\frac{1+x}{s-x}\right)\right]. \qquad \ldots \qquad (7)
\end{aligned}$$

Equations (1) to (7), inserted in equations 3.4(3) and (3a), yield equation 3.4(4).

APPENDIX II

Definition of Incomplete Hankel Functions

Let:

$$r(\eta) = \begin{cases} \dfrac{-i}{\sqrt{(\eta^2-1)}} & r = 0 \\[2ex] \dfrac{\eta}{\sqrt{(\eta^2-1)}} & r = 1 \end{cases} \qquad \ldots \qquad (1)$$

and let:

$$H_{rS}(\tfrac{1}{2}\nu) = -\frac{2}{\pi}\left\{\int_1^s e^{-\omega\eta}\, r(\eta)\, d\eta + r(s)\frac{e^{-\omega s}}{\omega}\right\}. \qquad \ldots \qquad (2)$$

The functions H_{rS} occur in Appendix I, equation (4).

Consider the limit $S \to \infty$. As

$$\left(\frac{\partial H_{rS}}{\partial S}\right)_{S\to\infty} = \left(-\frac{4}{\pi}\, r'(s)\frac{e^{-\omega s}}{\omega}\right)_{s\to\infty} \equiv 0, \qquad \ldots \qquad (3)$$

this limit exists. In the case $r = 0$ we can formally let s tend to infinity in equation (2); in the case $r = 1$ we have first to transform the integral as follows:

$$\int_1^s e^{-\omega\eta}\frac{\eta}{\sqrt{(\eta^2-1)}}\, d\eta \equiv \int_1^s e^{-\omega\eta}\left(\frac{\eta}{\sqrt{(\eta^2-1)}} - 1\right) d\eta + \frac{1}{\omega}\left(e^{-\omega} - e^{-\omega s}\right). \qquad \ldots \qquad (4)$$

There is thus obtained:

$$\begin{aligned}
H_{0\infty}(\tfrac{1}{2}\nu) &= \frac{2i}{\pi}\int_1^\infty e^{-\omega\eta}\frac{d\eta}{\sqrt{(\eta^2-1)}} &&= H_0^{(2)}(\tfrac{1}{2}\nu) \\
H_{1\infty}(\tfrac{1}{2}\nu) &= -\frac{2}{\pi}\left\{\int_1^\infty e^{-\omega\eta}\left(\frac{\eta}{\sqrt{(\eta^2-1)}} - 1\right) d\eta + \frac{1}{\omega}e^{-\omega}\right\} &&= H_1^{(2)}(\tfrac{1}{2}\nu), \qquad \ldots \qquad (5)
\end{aligned}$$

that is, the functions H_{rS} become Hankel functions of the second kind in the limit $S \to \infty$. For this reason we call the functions H_{rS} the 'incomplete Hankel functions'.

APPENDIX III

Numerical Evaluation

Formula 3.6(2) for the incomplete T-function may be written:

$$T_S(\tfrac{1}{2}\nu) = \frac{T_1}{T_2}, \qquad (1)$$

with

$$\left.\begin{aligned} T_1 &= \mathrm{e}^{-i\nu S}\sqrt{\left(\frac{S}{S+1}\right)} + i\nu \sum_0^\infty \frac{(-i\nu)^n}{n!}\, a_n \\ T_2 &= \mathrm{e}^{-i\nu S}\sqrt{\left(\frac{S+1}{S}\right)} + i\nu \sum_0^\infty \frac{(-i\nu)^n}{n!}\, b_n \end{aligned}\right\}, \qquad (1a)$$

or alternatively:

$$\left.\begin{aligned} T_1 &= -\frac{\nu\pi}{4}\,\mathrm{e}^{i\nu/2}\,[H_0^{(2)}(\tfrac{1}{2}\nu) + iH_1^{(2)}(\tfrac{1}{2}\nu)] - \Delta_1 \\ T_2 &= +\frac{\nu\pi}{4}\,\mathrm{e}^{i\nu/2}\,[H_0^{(2)}(\tfrac{1}{2}\nu) - iH_1^{(2)}(\tfrac{1}{2}\nu)] + \Delta_2 \end{aligned}\right\}. \qquad (1b)$$

The two methods indicated by (1*a*) and (1*b*) have been used for calculating the numerical Tables 1 and 2 in overlapping ranges:

method (*a*) : $S = 1, 2$

method (*b*) : $S = 2, 5, 10, 20, 50$

Method (*a*). We have:

$$a_n = \int_0^S v^n \sqrt{\left(\frac{v}{v+1}\right)}\, dv \qquad (2)$$

and thus:

$$a_{-1} = \cosh^{-1}(2S+1). \qquad (2a)$$

Further:

$$\left.\begin{aligned} (n+1)a_n &= S^n\sqrt{\{S(S+1)\}} - (n+\tfrac{1}{2})a_{n-1} \\ b_n &= a_{n-1} + a_n \end{aligned}\right\}. \qquad (2b)$$

Method (*b*). Writing, compare equation 3.6(1*a*) with Appendix II, equations (2), (4) and (5):

$$\begin{aligned} \Delta_1 &= \left\{1 - \sqrt{\left(\frac{S}{S+1}\right)}\right\}\mathrm{e}^{-i\nu S} + \sum_1^\infty (-)^n\,\bar{n}(C_n + iD_n) \\ \Delta_2 &= \left\{\sqrt{\left(\frac{S+1}{S}\right)} - 1\right\}\mathrm{e}^{-i\nu S} + \sum_1^\infty (-)^n\,\frac{\bar{n}}{2n-1}\,(C_n + iD_n), \end{aligned} \qquad (3)$$

with

$$\bar{n} = \frac{1.3.5\ldots(2n-1)}{2.4.6.\ldots 2n},$$

we find:

$$C_n = \nu\int_S^\infty \frac{\sin \nu v}{v^n}\, dv\,;\; D_n = \int_S^\infty \frac{\cos \nu v}{v^n}\, dv. \qquad (3a)$$

Thus:

$$C_1 = \nu[\tfrac{1}{2}\pi - S_i(\nu S)]\,;\; D_1 = -\nu Ci(\nu S), \qquad (3b)$$

where $Si(x)$ and $Ci(x)$ denote sine-integral and cosine-integral.

Further:

$$C_{n+1} = \frac{\nu}{n}\left(\frac{\sin \nu S}{S^n} + D_n\right);\; D_{n+1} = \frac{\nu}{n}\left(\frac{\cos \nu S}{S^n} - C_n\right). \qquad (3c)$$

TABLE 1

Incomplete T-Function

ν \ S	1	2	5	10	20	50	∞
0	0·5000	0·6667	0·8333	0·9091	0·9524	0·9804	1
0·01	0·5000	0·6666	0·8329	0·9083	0·9509	0·9771	0·9830
0·02	0·4999	0·6662	0·8318	0·9059	0·9464	0·9674	0·9648
0·04	0·4994	0·6648	0·8273	0·8966	0·9292	0·9340	0·9275
0·07	0·4983	0·6611	0·8150	0·8721	0·8866	0·8708	0·8716
0·1	0·4965	0·6554	0·7968	0·8373	0·8322	0·8152	0·8180
0·2	0·4863	0·6236	0·7042	0·6859	0·6578	0·6628	0·6638
0·4	0·4489	0·5201	0·4832	0·4455	0·4577	0·4557	0·4552
0·7	0·3680	0·3493	0·2747	0·2904	0·2870	0·2856	0·2858
1	0·2833	0·2228	0·1896	0·1932	0·1966	0·1959	0·1959

$\mathscr{R}\, T_S$

ν	1	2	5	10	20	50	∞
0	0	0	0	0	0	0	
0·01	0·0044	0·0089	0·0175	0·0249	0·0324	0·0419	0·0533
0·02	0·0087	0·0178	0·0350	0·0496	0·0643	0·0819	0·0913
0·04	0·0174	0·0356	0·0695	0·0981	0·1251	0·1504	0·1504
0·07	0·0304	0·0620	0·1198	0·1661	0·2039	0·2190	0·2149
0·1	0·0432	0·0879	0·1673	0·2259	0·2636	0·2605	0·2613
0·2	0·0848	0·1678	0·2944	0·3526	0·3487	0·3454	0·3446
0·4	0·1579	0·2837	0·3948	0·3813	0·3790	0·3773	0·3772
0·7	0·2316	0·3454	0·3558	0·3432	0·3453	0·3447	0·3446
1	0·2637	0·3301	0·2944	0·3023	0·3017	0·3013	0·3014

$-\mathscr{I}\, T_S$

ν \ S	1	2	5	10	20	50	∞
0	0·5000	0·3333	0·1667	0·0909	0·0476	0·0196	0
0·01	0·5000	0·3333	0·1666	0·0909	0·0476	0·0196	0
0·04	0·4996	0·3327	0·1659	0·0902	0·0470	0·0191	0
0·1	0·4976	0·3296	0·1621	0·0866	0·0441	0·0177	0
0·4	0·4648	0·2845	0·1222	0·0620	0·0321	0·0130	0
1	0·3503	0·1838	0·0798	0·0418	0·0214	0·0087	0

$\mathscr{R}\, (U_S\, e^{i\nu S})$

ν	1	2	5	10	20	50	∞
0	0	0	0	0	0	0	0
0·01	−0·0031	−0·0031	+0·0023	0·0016	0·0010	0·0005	0
0·04	−0·0125	−0·0124	+0·0093	0·0064	0·0040	0·0019	0
0·1	−0·0310	−0·0307	+0·0225	0·0148	0·0085	0·0034	0
0·4	−0·1136	−0·1005	+0·0560	0·0283	0·0145	0·0059	0
1	−0·1944	−0·1260	+0·0522	0·0277	0·0142	0·0057	0

$-\mathscr{I}\, (U_S\, e^{i\nu S})$

TABLE 2

Incomplete Circulation Function C_S

S / ν	1	2	5	10	20	50	∞
0	0·7500	0·8333	0·9167	0·9545	0·9762	0·9902	1·0000
0·01	0·7500	0·8333	0·9165	0·9541	0·9754	0·9885	0·9915
0·02	0·7499	0·8331	0·9159	0·9530	0·9732	0·9837	0·9824
0·04	0·7497	0·8324	0·9136	0·9483	0·9646	0·9670	0·9637
0·07	0·7491	0·8306	0·9075	0·9361	0·9433	0·9354	0·9358
0·1	0·7483	0·8277	0·8984	0·9187	0·9161	0·9076	0·9090
0·2	0·7432	0·8118	0·8521	0·8430	0·8289	0·8314	0·8319
0·4	0·7245	0·7600	0·7416	0·7227	0·7288	0·7278	0·7276
0·7	0·6840	0·6746	0·6373	0·6452	0·6435	0·6428	0·6429
1	0·6416	0·6114	0·5948	0·5966	0·5983	0·5980	0·5979

$$A_S(\nu) = \mathscr{R}\, C_S(\tfrac{1}{2}\nu)$$

S / ν	1	2	5	10	20	50	∞
0	0	0	0	0	0	0	0
0·01	0·0022	0·0045	0·0088	0·0124	0·0162	0·0210	0·0266
0·02	0·0043	0·0089	0·0175	0·0248	0·0321	0·0410	0·0457
0·04	0·0087	0·0178	0·0348	0·0490	0·0626	0·0752	0·0752
0·07	0·0152	0·0310	0·0599	0·0830	0·1020	0·1095	0·1074
0·1	0·0216	0·0440	0·0836	0·1130	0·1318	0·1302	0·1306
0·2	0·0424	0·0839	0·1472	0·1763	0·1743	0·1727	0·1723
0·4	0·0789	0·1419	0·1974	0·1907	0·1895	0·1887	0·1886
0·7	0·1158	0·1727	0·1779	0·1716	0·1726	0·1724	0·1723
1	0·1319	0·1651	0·1472	0·1512	0·1508	0·1507	0·1507

$$B_S(\nu) = -\,\mathscr{I}\, C_S(\tfrac{1}{2}\nu)$$

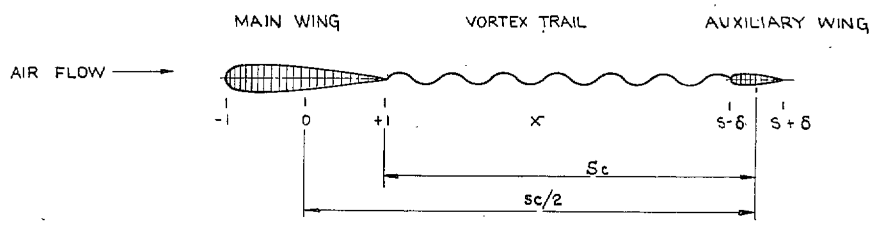

FIG. 1. System of two wings and notation.

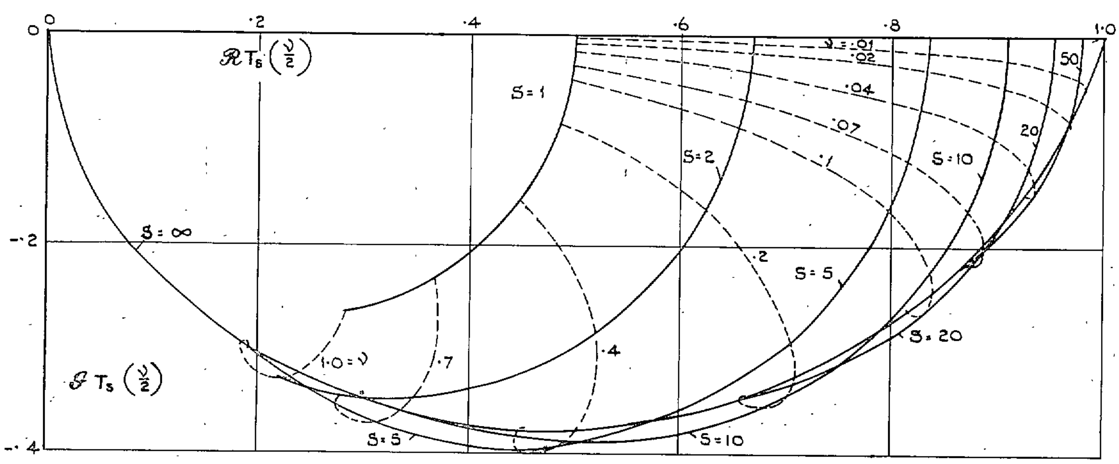

FIG. 2. T_s-function.

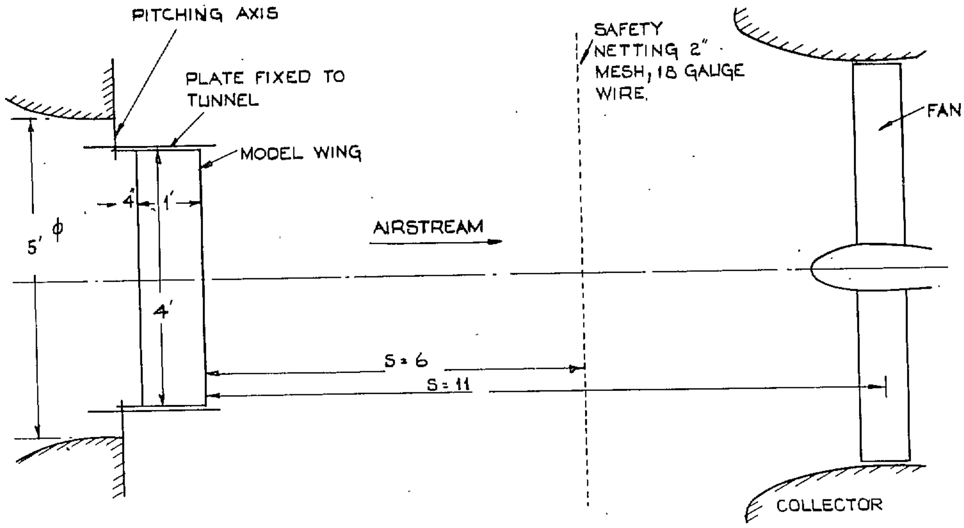

FIG. 4. Wind-tunnel arrangement.

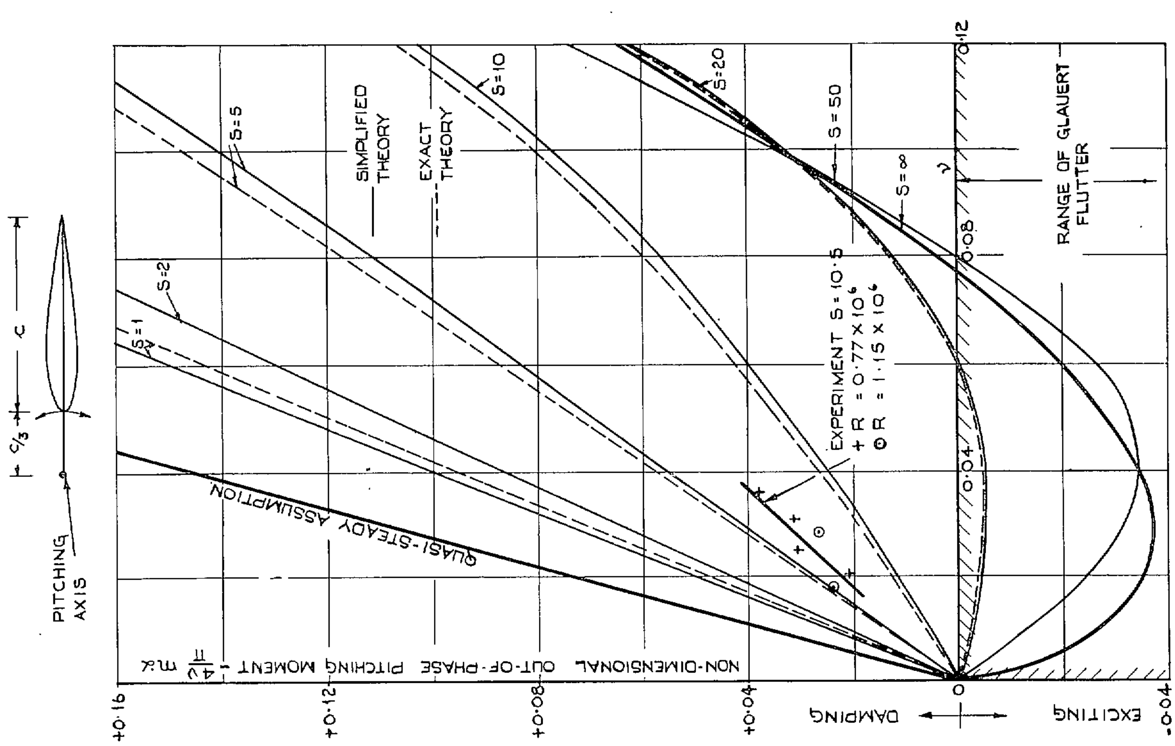

FIG. 3. Out-of-phase pitching moment. Theory and experiment.

(38983) Wt. 20/9036 K7 10/57 Hw.

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