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The Laminar Boundary Layer on an Inclined Cone

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Summary.

A solution to the problem of incompressible conical flow past a cone is presented, in which the boundary-layer equations are solved by an implicit finite difference procedure. Comparisons are made with experiments by Rainbird, Crabbe and Jurewicz, and with calculations made by Crabbe who used an approximate method due to Cooke. The present calculations agree reasonably well with experiment and show that the approximate method gives a fair overall picture.

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1. *Introduction.*

General solutions of the three-dimensional boundary-layer equations require a high-speed computer with a large store, and even if this is available the organisation of the computation is very complex and beset with pitfalls. The only method of solution for problems with completely general boundary conditions known to the author is that due to Raetz[1]. This method seems rather difficult to follow and nobody else, so far as the author is aware, has yet attempted to use it. A difficulty in the method seems to be

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that it is necessary to ensure that the finite difference intervals in the streamwise direction are very much smaller than those normal to it. Raetz indeed finds that this limits the number of steps across the boundary layer to about 20 usually, though he has sometimes used 40. It does not seem possible to increase this number if one wishes to cover a reasonable area of surface in a reasonable time with acceptable accuracy. Again, the variables chosen by Raetz are somewhat unusual, especially the choice of the z-co-ordinate, where z is distance from the wall along the normal to the surface. Raetz replaces z by $\zeta = (1-u/u_e)^{\frac{1}{2}}$, where u is the velocity component which is approximately streamwise, and u_e is its value in the external flow. This may lead to difficulties if there is overshoot, that is, if there is some place where u is greater than u_e.

For these reasons it was felt that the development of another method ought to be attempted, and that it was advisable not to go the whole way at first but to try a problem which is only quasi three-dimensional to start with in order to gain experience. Incompressible conical flow was decided upon for this reason, and because some experiments and approximate computations for this case had been done by Rainbird, Crabbe and Jurewicz[2].

Details of the method used are given later in the Report and in the Appendices. The equations are non-linear and so recourse has to be made to an iterative process. The method of Smith and Clutter[3], suitably extended, seemed to represent one possible line of attack, but experience at Farnborough has suggested that it is better to integrate *across* the boundary layer by a matrix method, since the method of Smith and Clutter would lead to a difficult double interpolation to obtain the correct values of the first derivatives of the two velocity components at the wall. This is avoided here, and it makes it possible to tie in the solution of the problem to the values of the velocity components at both ends, instead of leaving one end free as is necessary if one integrates outwards from the wall. No difficulty as regards stability was found but the method broke down as separation was approached. Since the type of singularity encountered at separation is known from the work of Brown[4] it was simple to extrapolate the solution up to the separation point, which thus could be found with fair accuracy (usually to three significant figures).

So far only the incompressible case has been dealt with, but there would seem to be no particular difficulty in extending the method to the compressible case, though machine time and storage space will be considerably increased.

2. *The Boundary-layer Equations.*

We use an orthogonal curvilinear co-ordinate system for which the line element is given by

$$ds^2 = h_1^2\,d\xi^2 + h_2^2\,d\eta^2 + d\zeta^2$$

where the surface is denoted by $\zeta = 0$, and ζ measures distance along normals, whilst ξ = constant and η = constant are two families of co-ordinate curves on the surface. h_1 and h_2 are generally taken to be independent of ζ. The equations for incompressible flow in this system are

$$\frac{U}{h_1}\frac{\partial U}{\partial \xi} + \frac{V}{h_2}\frac{\partial U}{\partial \eta} + W\frac{\partial U}{\partial \zeta} - K_2\,UV + K_1\,V^2 = -\frac{1}{\rho h_1}\frac{\partial p}{\partial \xi} + \frac{\partial}{\partial \zeta}\left(\nu\frac{\partial U}{\partial \zeta}\right)$$

$$\frac{U}{h_1}\frac{\partial V}{\partial \xi} + \frac{V}{h_2}\frac{\partial V}{\partial \eta} + W\frac{\partial V}{\partial \zeta} - K_1\,UV + K_2\,U^2 = -\frac{1}{\rho h_2}\frac{\partial p}{\partial \eta} + \frac{\partial}{\partial \zeta}\left(\nu\frac{\partial V}{\partial \zeta}\right)$$

$$\frac{\partial}{\partial \xi}\left(h_2\,U\right) + \frac{\partial}{\partial \eta}\left(h_1\,V\right) + \frac{\partial}{\partial \zeta}\left(h_1\,h_2\,W\right) = 0$$

$$K_1 = -\frac{1}{h_1\,h_2}\frac{\partial h_2}{\partial \xi}, \qquad K_2 = -\frac{1}{h_1\,h_2}\frac{\partial h_1}{\partial \eta}.$$

For a cone (not necessarily right circular) we write $\xi = r$, $\eta = \theta$, $h_1 = 1$, $h_2 = r$, where r is distance from the apex and θ is the angle between any generator and a fixed generator measured in the plane into which the cone can be developed.

The equations now reduce to

$$U\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial \theta}+W\frac{\partial U}{\partial \zeta}-\frac{V^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\frac{\partial^2 U}{\partial \zeta^2}$$

$$U\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \theta}+W\frac{\partial V}{\partial \zeta}+\frac{UV}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta}+\nu\frac{\partial^2 V}{\partial \zeta^2}$$

$$\frac{\partial U}{\partial r}+\frac{U}{r}+\frac{1}{r}\frac{\partial V}{\partial \theta}+\frac{\partial W}{\partial \zeta} = 0.$$

We now suppose the external flow to be conical. Hence we have $\partial p/\partial r = 0$. In this case it is possible to find solutions of the equations in which U and V are functions of $\zeta/r^{\frac{1}{2}}$ and θ only. We shall in fact write

$$z = \frac{\zeta}{r}\left(\frac{U_e r}{\nu}\right)^{\frac{1}{2}}, \qquad \bar{w} = \frac{W}{U_e}\left(\frac{U_e r}{\nu}\right)^{\frac{1}{2}}$$

$$u = \frac{U}{U_e}, \qquad v = \frac{V}{V_e}.$$

In conical irrotational flow we have $V_e = dU_e/d\theta$ and we write

$$K = \frac{V_e}{U_e} = \frac{1}{U_e}\frac{dU_e}{d\theta}, \qquad M = \frac{1}{U_e}\frac{dV_e}{d\theta}$$

and the equations reduce to

$$u_{zz}-u_z\left(\bar{w}-\frac{1}{2}zu+\frac{1}{2}K^2 zv\right)-K^2 uv-K vu_\theta = -K^2 v^2,$$

$$v_{zz}-v_z\left(\bar{w}-\frac{1}{2}zu+\frac{1}{2}K^2 zv\right)-v(u+Mv)-Kv v_\theta = -1-M,$$

$$\bar{w}_z = \frac{1}{2}zu_z-u-Mv-K v_\theta-\frac{1}{2}K^2 zv_z.$$

Finally the equations are simplified by writing

$$\bar{w}-\frac{1}{2}zu+\frac{1}{2}K^2 zv = w$$

and they become

$$u_{zz}-wu_z-K^2 uv-K vu_\theta = -K^2 v^2 \tag{1}$$

$$v_{zz}-wv_z-v(u+Mv)-Kv v_\theta = -1-M \tag{2}$$

$$w_z = \frac{1}{2}K^2 v-\frac{3}{2}u-K v_\theta-Mv. \tag{3}$$

The boundary conditions are

$$w = u = v = 0 \qquad \text{at } z = 0\,,$$

$$u = v = 1 \qquad \text{at } z = \infty\,.$$

When difficulties arose in the computation a further transformation was made, namely writing $1-u$, $1-v$ instead of u, v. This did not resolve the difficulty, which in fact lay elsewhere. However, as a result, the equations actually solved were

$$u_{zz} - wu_z - K^2 u(1-v) - K\, u_\theta (1-v) = K^2(-v+v^2) \tag{1a}$$

$$v_{zz} - wv_z - v(1+2M-Mv-u) - K\, v_\theta (1-v) = u \tag{2a}$$

$$w_z = -\frac{1}{2}K^2(v-1) + \frac{3}{2}(u-1) + M(v-1) + K\, v_\theta\,, \tag{3a}$$

the boundary conditions now being

$$u = v = 1, \quad w = 0 \quad \text{for } z = 0\,,$$

$$u = v = 0 \quad \text{for } z = \infty\,.$$

3. *Numerical Solution.*

The method used is basically a Crank-Nicholson process and is described by Catherall and Mangler[5] and in more detail by Hall[6]. The attachment line is taken to be at $\theta = 0$ and the integration proceeds in steps of θ, advancing θ by an amount $\delta\theta$. The outer boundary is at $z = \infty$. We take it at $z = z_0$ where z_0 is sufficiently large for the accuracy required. $z_0 = 5$ was found to be adequate in the present problem. The range $z = 0$ to $z = z_0$ is divided into $N+1$ intervals, where $N+1 = z_0/\delta z$.

We shall denote $u(\theta, z) = u(m\,\delta\theta, n\delta z)$ by $u_{m,n}$; we note that $m = 0$ on the attachment line and $n = 0$ at the wall. We then use the central difference scheme:

$$\left(\frac{\partial u}{\partial \theta}\right)_{m+\frac{1}{2},n} = \frac{1}{\delta\theta}\,(u_{m+1,n} - u_{m,n}) \tag{4}$$

$$\left(\frac{\partial u}{\partial z}\right)_{m+\frac{1}{2},n} = \frac{1}{4\delta z}(u_{m+1,n+1} - u_{m+1,n-1} + u_{m,n+1} - u_{m,n-1}) \tag{5}$$

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{m+\frac{1}{2},n} = \frac{1}{2(\delta z)^2}(u_{m+1,n+1} - 2u_{m+1,n} + u_{m+1,n-1} + u_{m,n+1} - 2u_{m,n} + u_{m,n-1}) \tag{6}$$

$$u_{m+\frac{1}{2},n} = \frac{1}{2}(u_{m+1,n} + u_{m,n}) \tag{7}$$

$$u_{m+\frac{1}{2},n-\frac{1}{2}} = \frac{1}{4}(u_{m+1,n} + u_{m+1,n-1} + u_{m,n} + u_{m,n-1}) \tag{8}$$

$$\left(\frac{\partial w}{\partial z}\right)_{m+\frac{1}{2},n-\frac{1}{2}} = \frac{w_{m+\frac{1}{2},n} - w_{m+\frac{1}{2},n-1}}{\delta z}\,. \tag{9}$$

Equations (1) and (2) are written in finite difference form using the above scheme, being evaluated at the point $\theta = (m+\frac{1}{2})\,\delta\theta$, $z = n\delta z$. Equation (3) is evaluated at $[(m+\frac{1}{2})\,\delta\theta.\ (n-\frac{1}{2})\,\delta z]$. The difference equations are written out in full in Appendix A. Equations (1) and (2) thus become equations for u_{m+1} and v_{m+1} for all n. They involve $w_{m+\frac{1}{2}}$, which is found from equation (3). We suppose that all the u's and v's have been found up to the line $\theta = m\delta\theta$, and the problem is to find u_{m+1} and v_{m+1} for all n. As these quantities, not yet known, occur in equation (3), we put assumed or extrapolated values for them into equation (3) and thus $w_{m+\frac{1}{2}}$ is found for all n. Now equation (1) is linearized in u_{m+1}, again putting in assumed values of v_{m+1}. The solution is given in Appendix B. This gives a better estimate for u_{m+1}. This is used in equation (2) which is similarly linearized and solved, thus obtaining a better value for v_{m+1}. The iteration is repeated until sufficient accuracy is secured. The test for accuracy used is δ_1^*, defined by

$$\delta_1^* = \int_0^{z_0} (1-u)\,dz\,.$$

When this changes by less than some small amount the iteration is stopped and we proceed to the next step, advancing θ by an amount $\delta\theta$.

A separate and simpler procedure is required to find u and v on the stagnation line itself ($\theta = 0$) before the main step-by-step procedure begins.

4. *External Flow for the Right Circular Cone.*

If the semi-angle of the cone is denoted by θ_c and the incidence is α, then according to Crabbe we have, by slender body theory

$$\frac{U_e}{U_0} = \sec\theta_c \left[\cos\alpha - 2\alpha\tan\theta_c\cos\left(\frac{\theta}{\sin\theta_c}\right)\right].$$

$$\frac{V_e}{U_0} = 2\alpha\sin\left(\frac{\theta}{\sin\theta_c}\right),$$

where U_0 is the velocity at infinity. Slender body theory requires θ_c and α to be small and ignoring cubes and higher powers of these quantities we may write

$$\frac{U_e}{U_0} = 1 - \frac{1}{2}\alpha^2 + \frac{1}{2}\theta_c^2 - 2\alpha\,\theta_c\cos\left(\frac{\theta}{\theta_c}\right),$$

$$\frac{V_e}{U_0} = 2\alpha\sin\left(\frac{\theta}{\theta_c}\right).$$

Hence if we put $\lambda = \alpha/\theta_c$ we have

$$K = \frac{2\lambda\,\theta_c\sin(\theta/\theta_c)}{U_e/U_0},\ M = \frac{2\lambda\cos(\theta/\theta_c)}{U_e/U_0}$$

and so

$$K_{m+\frac{1}{2}} = \frac{2\lambda\,\theta_c}{(U_e/U_0)_{m+\frac{1}{2}}}\sin\left[\frac{(m+\frac{1}{2})\,\delta\theta}{\theta_c}\right],\quad M_{m+\frac{1}{2}} = \frac{2\lambda}{(U_e/U_0)_{m+\frac{1}{2}}}\cos\left[\frac{(m+\frac{1}{2})\,\delta\theta}{\theta_c}\right].$$

We note that at $\theta = 0$

$$K = 0, \quad M = 2\lambda\,U_0/U_e\,.$$

5. *Results.*

The first computations were done on a Mercury computer. Limitations of storage space were such that only 50 points across the boundary layer could be taken. The value of $\delta\theta$ was 0·02 or 0·03 initially, but it was found necessary to halve this interval as separation was approached, sometimes more than once. Even then the method finally broke down. The accuracy test for changes of δ_1^* per iteration was 0·0001. As a check, some of the computations were repeated on an Atlas computer with the intervals δz and $\delta\theta$ both halved and the accuracy test reduced to 0·00001. It was found that in no case was the change in any computed quantity more than one half of one per cent, and was usually less.

Calculations were made for a cone angle of $\theta_c = 7\frac{1}{2}$ deg with $\lambda(= \alpha/\theta_c)$ having values 0·5, 1·0, 1·3 and 2·0. Fig. 1 shows the u profile on the stagnation line for $\lambda = 0{\cdot}5$ as calculated by the present method, compared with that by Moore's method[8] and Cooke's approximate method[9] as worked out by Crabbe[7]. The latter method of course *assumes* a profile and only the scale comes out of the calculations. Moore's profile show a rather surprising overshoot, but one would only expect Moore's results to be good when λ is small. They come from the sum of the first two terms in an expansion in powers of λ (the first term being the Blasius distribution) and here $\lambda = 0{\cdot}5$.

In Fig. 2 we show a comparison for $\lambda = 1{\cdot}3$ between the experiments of Rainbird *et al*[2], the approximate method[9] and the present method, in respect of surface flow angle, that is, the angle between the limiting streamlines and the generators of the cone. Rainbird *et al* believe that their experiments gave too low a surface angle because the injected dye they used was somewhat too thick and probably gave the direction of flow at a point very near to but not quite *on* the surface.

Figs. 3 and 4 give comparisons between the present solution and the approximate method in respect of streamwise and cross-wise skin friction for the case $\lambda = 1$. For the streamwise skin friction the approximate method gives values in general too low and for the cross-wise skin friction the opposite is the case. However, as separation is approached the approximate method comes closer to the correct answer but goes too far. Fig. 5 shows the surface flow angle (β) for all four cases as far as they could be computed. Expressed in terms of θ the curves all end quite near to separation since they are coming down towards $\beta = 0$ very rapidly. In Appendix C we show that near separation we might expect a curve of θ against $(\tan\beta)^2$ to approach a straight line and in Fig. 6a we see that this is indeed the case and separation can be estimated quite accurately by extrapolation of the lines to the θ axis, at any rate for the higher values of λ. At $\lambda = 0{\cdot}5$ a more refined procedure is necessary, as seen in Fig. 6b. This value of λ is indeed a critical value in that for $\lambda = 0{\cdot}5$ separation occurs almost exactly on the leeward generator of the cone and will not occur at all for lower values. The boundary layers going round either side of the cone encounter one another at the leeward generator and what occurs in this region does not yet seem to be fully understood. We show in Fig. 7 some cross-flow profiles at the point where $\theta = 0{\cdot}2$, that is, approximately 90 deg round the cone from the stagnation line $\theta = 0$. It was not possible to make a direct comparison between these results and those given by Crabbe[7] owing to lack of detailed information but we can compare the maximum values of V_c/Q_e and these we give in the table below:

λ	Max (V_c/Q_e)	
	Exact	Approx
0·5	0·032	0·034
1·0	0·052	0·061
1·5	0·078	0·083

In each case the approximate value is too large.

Finally in Fig. 8 we show the position of separation as calculated by the present method and as found by the experiments. The calculations by the approximate method are not shown. They form a curve below that of the experiments but not so low as the present method. One must remember that the external pressure distribution used in the present calculations were those based on slender-body theory. In a complete theory, the external pressure distribution would have to be calculated for a flow which includes the vortex sheets associated with the separation; possibly, the slenderness assumption would have to be dropped.

The actual pressure distribution was not measured in the experiments but if it had been one could have obtained a fairer comparison between the experiments and the theory. Rainbird *et al*[2] consider that although they could not be certain about the surface flow angle of their experiments their measured separation points are not in doubt. They did attempt to allow for a changed external pressure distribution in their approximate calculations and they found separation to be earlier than that obtained by the simple slender-body assumption. Such corrections have not been attempted in the present study.

6. *Conclusions.*

The problem was undertaken (1) to test the possibility of the method of approach used for solving the partial differential equations concerned, with a view to its extension to the more important compressible case, and (2) to provide a means of testing Cooke's approximate method against an exact solution.

It was found that the method was indeed capable of solving the incompressible problem studied to satisfactory accuracy with reasonable economy of machine time, and of providing a good estimate of the position of the separation line. The test of Cooke's approximate method of calculating general boundary layers in three dimensions shows that in some cases at least the approximate method is capable of giving a fair qualitative picture of the flow and of predicting the point of separation.

LIST OF SYMBOLS

A	Matrix in Appendix B
a_n, b_n, c_n, d_n	Coefficients in (A.5)
a'_n, b'_n, c'_n, d'_n	Coefficients in (A.6)
d	Matrix of d_n
h_1, h_2	Coefficients in line element
K	$V_e/U_e = (1/U_e)(d\,U_e/d\theta)$
K_1	$-(1/h_1\,h_2)(\partial h_2/\partial \xi)$
K_2	$-(1/h_1\,h_2)(\partial h_1/\partial \eta)$
L	Matrix in Appendix B
l_n	Terms in the matrix L
M	$(1/U_e)(d\,V_e/d\theta)$
$N+1$	Number of intervals normal to surface
p	Pressure
Q_e	Resultant external velocity
r	Distance from the apex
U,V,W	Velocity components
U_0	Velocity at infinity
U	Matrix in Appendix B
U_n	Terms in matrix U
u	U/U_e
$\bar{U}$	$u_{m+1,n}+u_{m,n}$
u	Matrix of u_n in Appendix B
$\bar{V}$	$v_{m+1,n}+v_{m,n}$
V_c	Cross-flow velocity component
v	V/V_e
$\bar{w}$	$\frac{W}{U_e}\left(\frac{U_e r}{\nu}\right)^{\frac{1}{2}}$
w	$\bar{w}-\frac{1}{2}zu+\frac{1}{2}K^2 zv$
y	Matrix Uu in Appendix B
y_n	Terms in matrix y
z	$\frac{\zeta}{r}\left(\frac{U_e r}{\nu}\right)^{\frac{1}{2}}$
α	Incidence of the cone
β	Surface flow angle, or angle between limiting streamlines and generators

δ_1^*	$\int_0^{z_0} (1-u)\,dz$
θ	Angle between generators in the developed plane
θ_s	Value of θ at separation
θ_c	Semi-angle of cone
λ	α/θ_c
μ	Coefficient of viscosity
ν	Kinematic viscosity
ξ, η	Curvilinear orthogonal co-ordinates on the surface
ζ	Distance normal to surface
ρ	Density
τ_{01}, τ_{02}	Streamwise and crosswise skin-friction components

Subscript e refers to values just outside the boundary layer

Subscripts m,n: $u_{m,n}$ means the value of u at the point $m\,\delta\theta$, $n\delta z$

REFERENCES

No.	*Author(s)*	*Title, etc.*
1	G. S. Raetz	A method of calculating three-dimensional laminar boundary layers of steady compressible flow. Northrop Aircraft Inc., Report No. NAI-58-73 (BLC-144), A.R.C. 23 634, December 1957.
2	W. J. Rainbird, R. S. Crabbe,.. L. S. Jurewicz	A water tunnel investigation of the flow separation about circular cones at incidence. NRC Aero Report LR-385, September 1963.
3	A. M. O. Smith, ... D. W. Clutter	A general method for solving the compressible laminar boundary layer equations. Douglas Aircraft Division. Paper 1699, September 1963.
4	Susan N. Brown	Singularities associated with separating boundary layers. *Phil. Trans. Roy. Soc.* A No. 1084, Vol. 257, p. 409, 1964.
5	D. Catherall, ... K. W. Mangler	An indirect method for the solution of the Navier-Stokes equations for laminar incompressible flow at large Reynolds numbers. RAE Report Aero 2683, ARC 25599, October 1963.
6	M. G. Hall	A numerical method for solving the equations for a vortex core. RAE Tech. Rep. No. 65106, ARC 27150, May 1965.
7	R. S. Crabbe	An application of Cooke's method to the laminar incompressible boundary layer on a right circular cone at incidence. NRC Aero Report LR-391, December 1963.
8	F. K. Moore	Laminar boundary layer on a circular cone in supersonic flow at a small angle of attack. NACA TN 2521, October 1951.
9	J. C. Cooke	Approximate calculation of three-dimensional laminar boundary layers. ARC R & M No. 3201, October 1959.

APPENDIX A

Finite Difference Equations.

We write down equations (1), (2) and (3), namely

$$u_{zz}-wu_z-K^2\,uv-Kv\,u_\theta = -K^2\,v^2 \tag{A.1}$$

$$v_{zz}-wv_z-v(u+Mv)-Kv\,v_\theta = -1-M \tag{A.2}$$

$$w_z = \frac{1}{2}K^2\,v-\frac{3}{2}u-K\,v_\theta-Mv \tag{A.3}$$

in finite difference form using (4), (5), (6) (7), (8) and (9). The boundary conditions are

$$u_{m,0} = v_{m,0} = w_{m+\frac{1}{2},0} = 0$$

$$u_{m,N+1} = v_{m,N+1} = 1\,.$$

Equation (A.3) is solved first. It may be written evaluating it at $[(m+\frac{1}{2})\delta\theta,\, (n-\frac{1}{2})\delta z]$

$$\begin{aligned} w_{m+\frac{1}{2},n} = w_{m+\frac{1}{2},n-1}+\delta z\left(\frac{1}{8}K^2-\frac{1}{4}M\right)&(v_{m+1,n}+v_{m+1,n-1}+v_{m,n}+v_{m,n-1})\\ &-\frac{3\delta z}{8}(u_{m+1,n}+u_{m+1,n-1}+u_{m,n}+u_{m,n-1})\\ &-\frac{K\delta z}{2\delta\theta}(v_{m+1,n}-v_{m,n}+v_{m+1,n-1}-v_{m,n-1})\,. \end{aligned} \tag{A.4}$$

In this and the succeeding equations we have written for brevity K for $K_{m+\frac{1}{2}}$ and M for $M_{m+\frac{1}{2}}$. We give n the values 1, 2, . . . , N in succession and thus if values are given for the u's and v's then $w_{n+\frac{1}{2}}$ can be obtained.

The solution of (A.2) requires some care in the method of linearizing. In the product vv_θ the term v_{m+1} occurs in each factor. One of these is supposed known by extrapolation and the other is taken as unknown and to be determined. It seems essential to decide correctly which to take as known and which unknown. It was found that it was necessary to take the $v_{m+1,n}$ which occurs in v_θ to be that which is unknown.

Equations (A.1) and (A.2) are evaluated at $[(m+\frac{1}{2})\delta\theta,\, n\,\delta z]$. (A.1) may be written

$$a_n\,u_{m+1,n+1}+b_n\,u_{m+1,n}+c_n\,u_{m+1,n-1} = d_n \tag{A.5}$$

where

$$a_n = \frac{1}{2}\left(\frac{1}{\delta z}\right)^2-\frac{1}{4\delta z}\,w_{m+\frac{1}{2},n}$$

$$b_n = -\left(\frac{1}{\delta z}\right)^2-\frac{1}{4}K^2\,\overline{V}-\frac{K}{2\delta\theta}\,\overline{V}$$

$$c_n = \frac{1}{2}\left(\frac{1}{\delta z}\right)^2+\frac{1}{4\delta z}\,w_{m+\frac{1}{2},n}$$

$$\begin{aligned} d_n = -\frac{1}{4}K^2\,\overline{V}^2-a_n\,u_{m,n+1}-b_n\,u_{m,n}-c_n\,u_{m,n-1}\\ -\frac{K}{\delta\theta}\,\overline{V}u_{m,n} \end{aligned}$$

where

$$\bar{V} = v_{m+1,n} + v_{m,n}, \qquad u_{m+1,N+1} = 1, \qquad u_{m+1,0} = 0 \, .$$

(A.2) may be written

$$a_n' v_{m+1,n+1} + b_n' v_{m+1,n} + c_n' v_{m+1,n-1} = d_n' \tag{A.6}$$

where

$$a_n' = a_n, \qquad c_n' = c_n$$

$$b_n' = -\left(\frac{1}{\delta z}\right)^2 - \frac{K}{2\delta\theta}\bar{V} - \frac{1}{4}[\bar{U} + M\bar{V}]$$

$$d_n' = -1 - M - a_n' v_{m,n+1} - b_n' v_{m,n} - c_n' v_{m,n-1}$$

$$-\frac{K}{\delta\theta}\bar{U} v_{m,n} \, ,$$

where

$$\bar{U} = u_{m+1,n} + u_{m,n}, \qquad v_{m+1,N+1} = 1, \qquad v_{m+1,0} = 0 \, .$$

For equations (1a), (2a) and (3a) the corresponding formulae have a_n, a_n', c_n, c_n' unchanged, but have

$$b_n = -\left(\frac{1}{\delta z}\right)^2 + \frac{1}{4}K^2(\bar{V}-2) + \frac{K}{2\delta\theta}(\bar{V}-2) \, ,$$

$$d_n = \frac{1}{4}K^2 \bar{V}^2 - \frac{1}{2}K^2 \bar{V} + \frac{K}{\delta\theta}(\bar{V}-2) u_{m,n}$$

$$- a_n u_{m,n+1} - b_n u_{m,n} - c_n u_{m,n-1} \, ,$$

$$b_n' = -\left(\frac{1}{\delta z}\right)^2 - \frac{1}{2} + \frac{1}{4}M(\bar{V}-4) + \frac{K}{2\delta\theta}(\bar{V}-2) + \frac{1}{4}\bar{U} \, ,$$

$$d_n' = \frac{K}{\delta\theta}(\bar{V}-2) v_{m,n} + \frac{1}{2}\bar{U}$$

$$- a_n' v_{m,n+1} - b_n' v_{m,n} - c_n' v_{m,n-1} \, .$$

The solution of (A.5) and (A.6) is described in Appendix B.

APPENDIX B

The Solution of the Linearized Finite Difference Equation.

We wish to solve an equation of the type

$$a_n\, u_{n+1} + b_n u_n + c_n u_{n-1} = d_n \qquad (1 \leq n \leq N)$$

where we have omitted the first subscript in the U's. It is $m+1$ throughout.

We give n the values 1, 2, . . . , N in succession and we find that we must solve the matrix equation

$$Au = d\,,$$

where A is a tri-diagonal matrix having the value

$$A = \left|\begin{array}{cccc|ccc} b_N & c_N & 0 & 0 & 0 & 0 & 0 \\ a_{N-1} & b_{N-1} & c_{N-1} & 0 & 0 & 0 & 0 \\ 0 & a_{N-2} & b_{N-2} & c_{N-2} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & a_1 & b_1 \end{array}\right|$$

and u and d are column matrices of the u's and d's except that we write $d_N - a_N = \bar{d}_N$ in place of d_N if $u_{N+1} = 1, u_0 = 0$ or $d_1 - c_1 = \bar{d}_1$ in place of d_1 if $u_{N+1} = 0, u_0 = 1$.

Now A can be split up into two triangular matrices, that is $A = LU$ or written in full

$$A = \left|\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_{N-1} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & l_{N-2} & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & l_2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & l_1 & 1 \end{array}\right| \quad \left|\begin{array}{ccc|ccc} U_N & c_N & 0 & 0 & 0 & 0 \\ 0 & U_{N-1} & c_{N-1} & 0 & 0 & 0 \\ 0 & 0 & U_{N-2} & c_{N-2} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & U_2 & c_2 \\ 0 & 0 & 0 & 0 & 0 & U_1 \end{array}\right|$$

On carrying out the multiplication we find

$$U_N = b_N$$

$$l_{n-1}\, U_n = a_{n-1}, \qquad l_{n-1}\, c_n + U_{n-1} = b_{n-1} \qquad (2 \leqslant n \leqslant N)$$

and hence we have

$$U_N = b_N$$

$$U_{n-1} = b_{n-1} - \frac{a_{n-1}\, c_n}{U_n}. \qquad (2 \leqslant n \leqslant N)\,.(\text{B.1})$$

Next we solve $LUu = d$ by writing it as $Uu = y$, $Ly = d$. The triangular matrix equations are easy to solve and lead to

$$y_N = d_N, \qquad y_{n-1} = d_{n-1} - \frac{a_{n-1}\, y_n}{U_n} \tag{B.2}$$

$$u_n = \frac{1}{U_n}(y_n - c_n\, u_{n-1}), \qquad u_1 = \frac{y_1}{U_1}. \tag{B.3}$$

Since the a's, b's and c's are known we can first find the U's by equations (B.1) starting at $n = N$ and going down to $n = 2$. Having found the U's we next find the y's by (B.2) again going down from $n = N$ to $n = 2$. Finally we find the u's by (B.3), going up from $n = 1$ to $n = N$.

Hence the solution is effected.

APPENDIX C

The Singularity at Separation.

The nature of the singularity at separation has been investigated by Brown[4]. In the present notation she writes

$$\xi = (\theta_s - \theta)^{\frac{1}{4}}, \qquad \eta = \left(\frac{1}{4(\theta_s - \theta)}\right)^{\frac{1}{4}} z'$$

$$\psi = 2^{3/2}\,\xi^3 f(\xi,\eta), \qquad s = 2^{-\frac{1}{2}}\,g(\xi,\eta),$$

where

$$u' = \frac{\partial s}{\partial z'}, \qquad v' = \frac{\partial \psi}{\partial z'}$$

and θ_s is the value of θ at separation.

She finds that

$$f(\xi,\eta) = \sum_{n=0}^{\infty} \xi^n f_n(\eta) + \text{terms involving powers of } \log \xi\,,$$

$$g(\xi,\eta) = \sum_{n=0}^{\infty} \xi^n g_n(\eta) + \text{terms involving powers of } \log \xi\,,$$

and that

$$f_0 = \frac{1}{6}\eta^3, \qquad g_0 = 0, \qquad f_1 = \alpha_1 \eta^2, \qquad g_1 = 0$$

$$f_2 = \alpha_2 \eta^2 - \frac{1}{15}\alpha_1^2 \eta^5\,, \qquad g_2 = B_2 \eta^2\,.$$

where α_1, α_2, B_2 are unknown constants, depending on the particular problem.

If we denote the surface flow angle by β we have

$$\tan\beta = \left(\frac{v'}{u'}\right)_{z'=0} = 4\xi^3 \left(\frac{f_\eta}{g_\eta}\right)_{\eta=0}$$

$$= \frac{4\alpha_1}{B_2}\left[(\theta_s-\theta)^{\frac{1}{2}} + \frac{\alpha_2}{\alpha_1}(\theta_s-\theta)^{\frac{3}{4}}\right] = A(\theta_s-\theta)^{\frac{1}{2}} + B(\theta_s-\theta)^{\frac{3}{4}} \quad \text{(say)}.$$

Hence we have for θ sufficiently near to θ_s

$$\theta_s - \theta \simeq (k \tan\beta)^2$$

where k is a constant.

Now we have $\beta = 0$ at separation and hence if we plot θ against $(\tan\beta)^2$ the above analysis suggests that we should obtain a straight line near to separation. We can extrapolate this line to $\beta = 0$ and hence determine the value of θ at separation. We do in fact find a line which is remarkably straight for the higher values of λ. (*See* Fig. 6a). This is not so for $\lambda = 0{\cdot}5$ and for this case it is necessary to take in the second term.

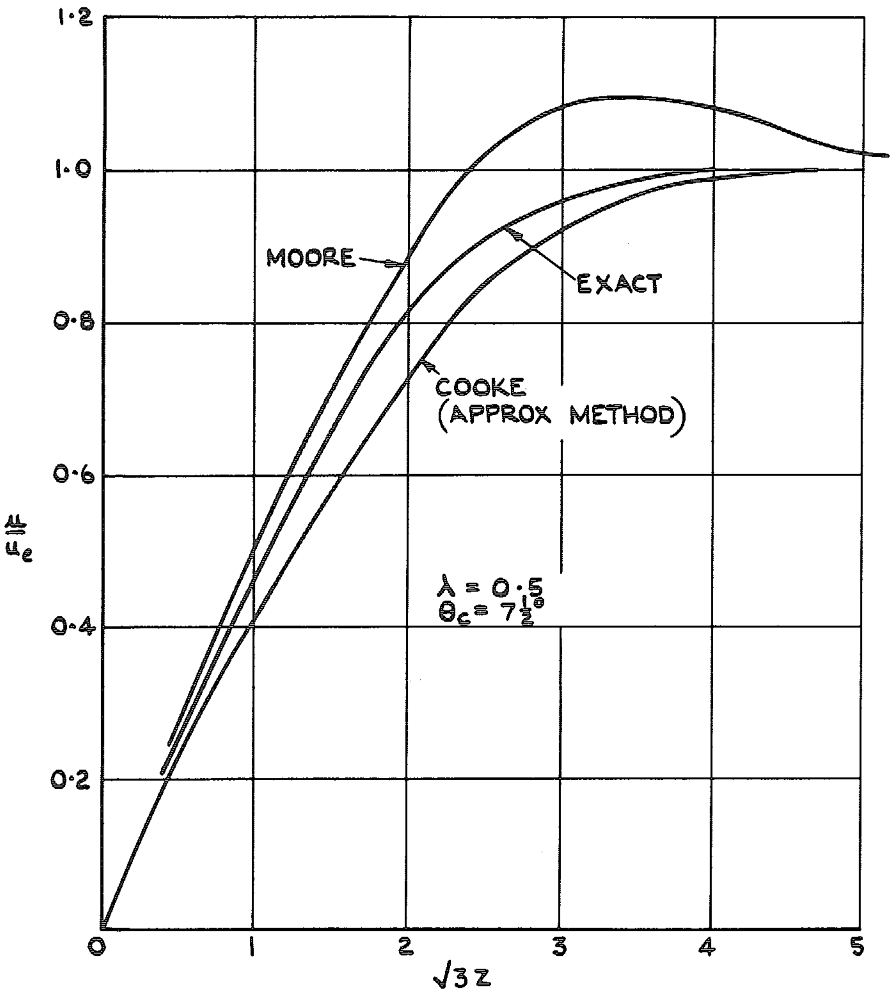

FIG. 1. Profile on stagnation line.

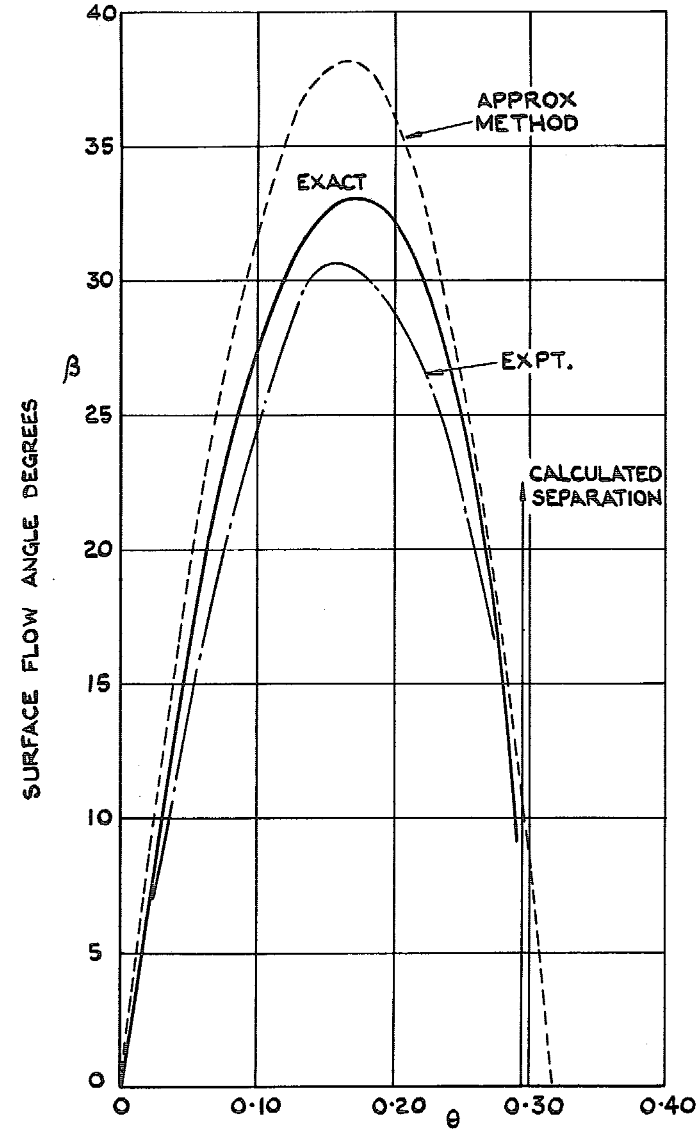

FIG. 2. Surface flow angle $\lambda = 1{\cdot}3$.

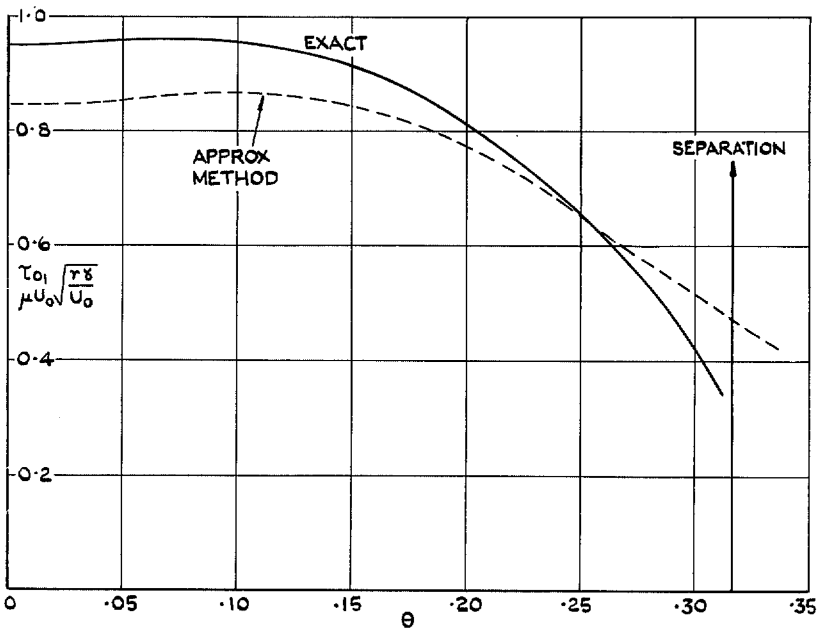

FIG. 3. Stream-wise skin friction $\lambda = 1$.

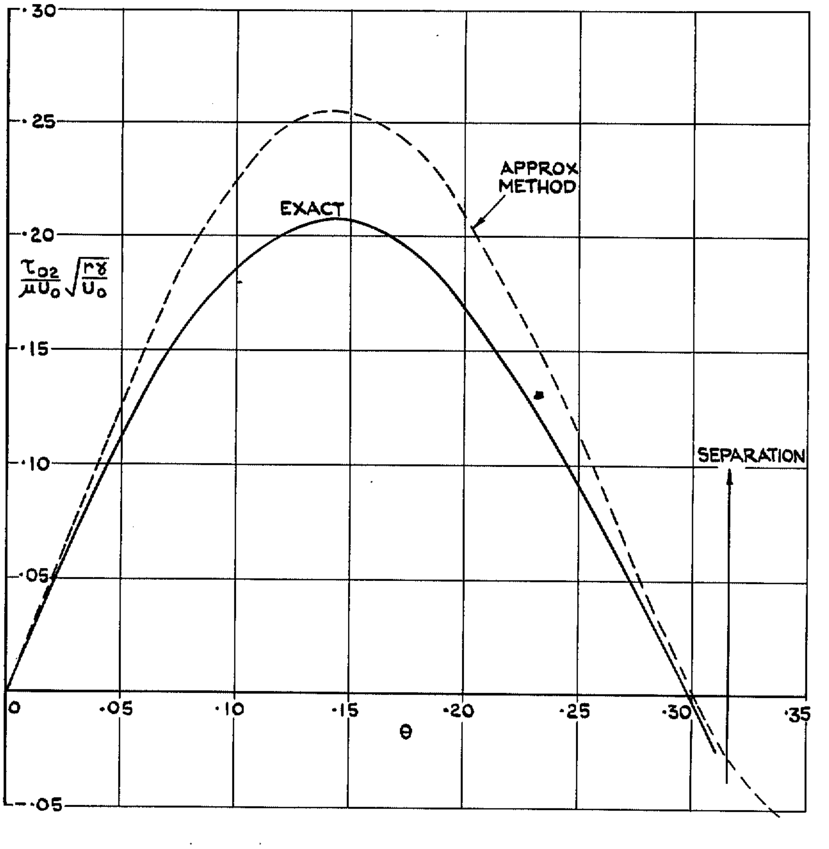

FIG. 4. Cross-flow skin friction, $\lambda = 1$.

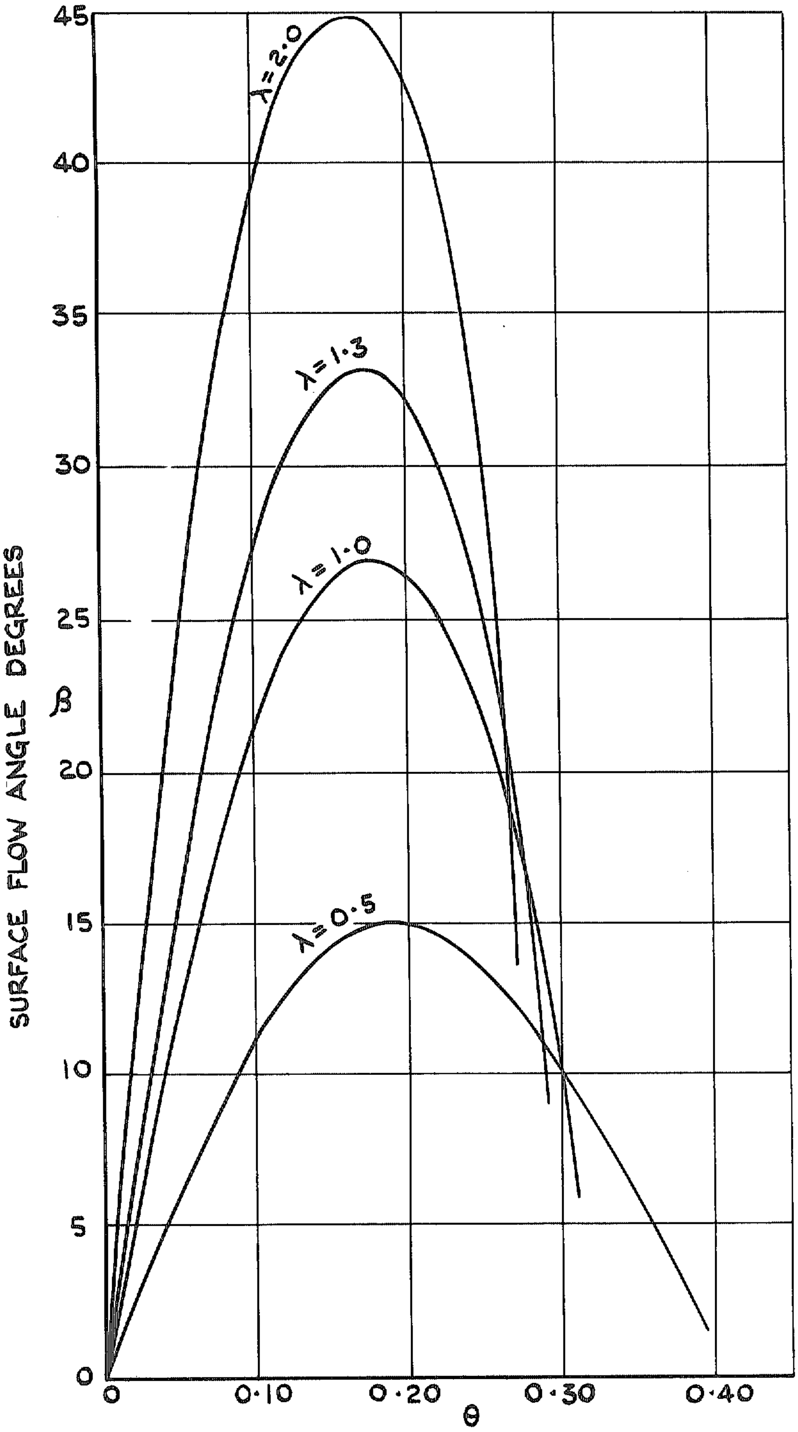

FIG. 5. Surface flow angle.

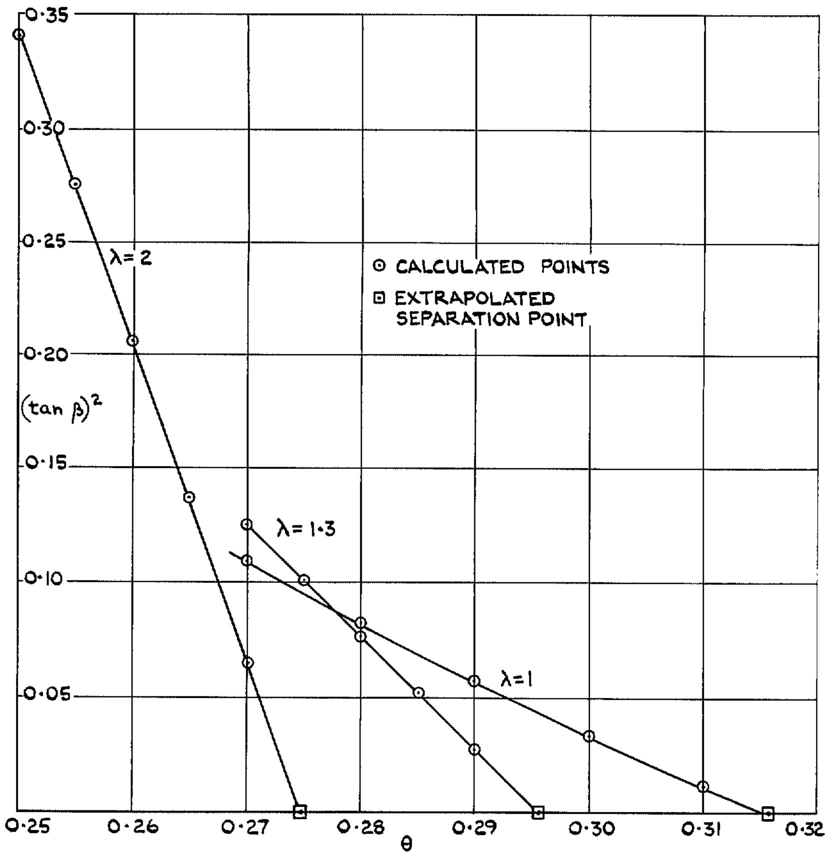

FIG. 6a. Determination of separation point, $\lambda = 1, 1{\cdot}3, 2$.

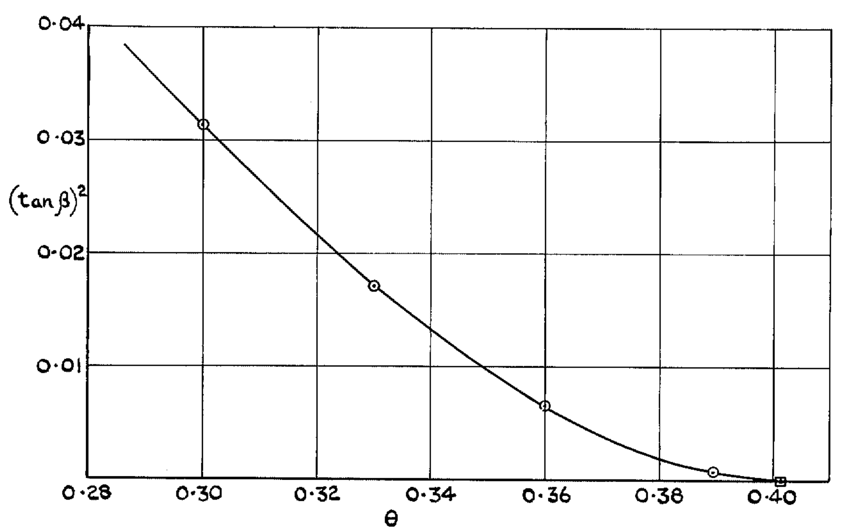

FIG. 6b. Determination of separation point, $\lambda = 0{\cdot}5$.

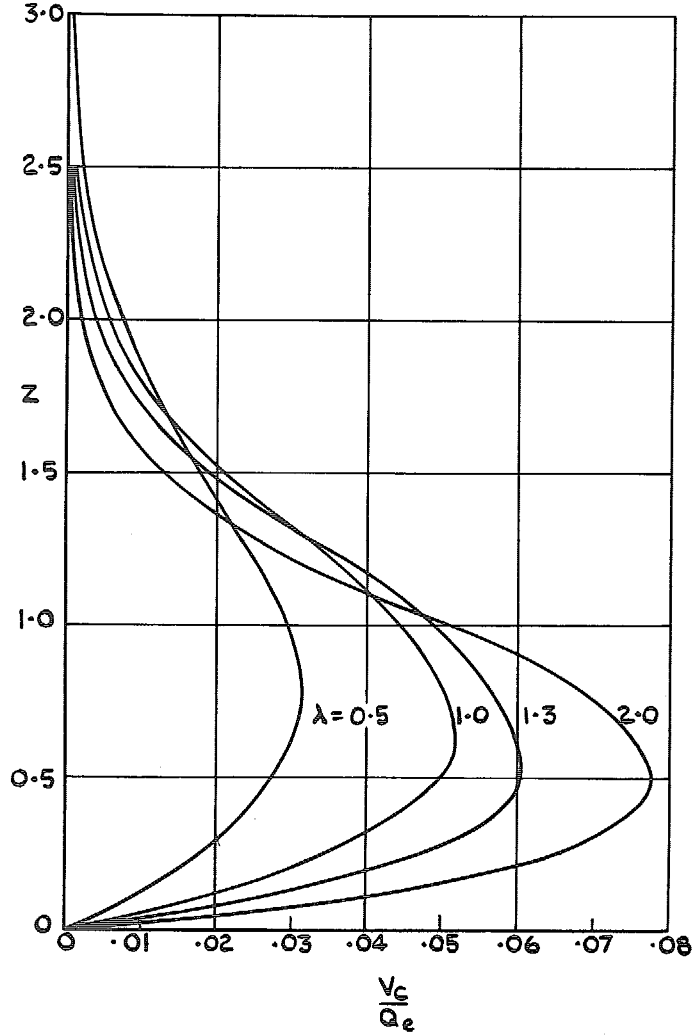

FIG. 7. Cross-flow velocity profiles at $\theta = 0{\cdot}2$.

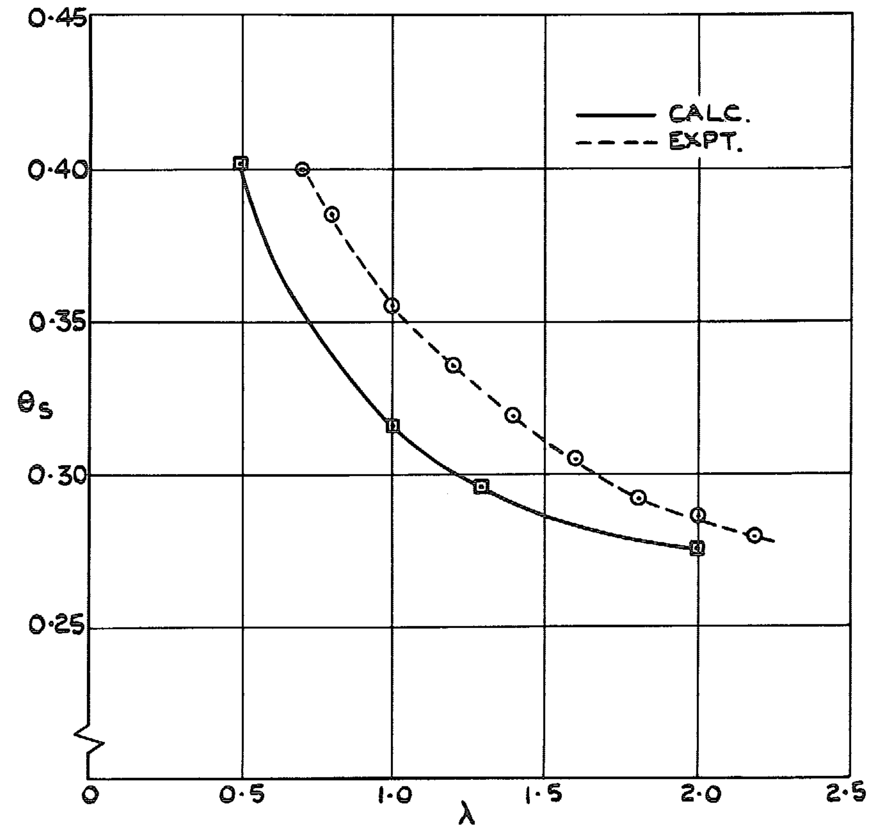

FIG. 8. Positions of separation.

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