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Application of Goldstein's Airscrew Theory to Design

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THE APPLICATION OF GOLDSTEIN'S THEORY TO THE PRACTICAL DESIGN OF AIRSCREWS.

By C. N. H. Lock, M.A.

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November, 1930.

Summary.—A recent paper on the Vortex theory of screw propellers by Dr. S. Goldstein in the Proceedings of the Royal Society, contains a solution of the problem of the potential flow past a body consisting of a finite number of coaxial helicoids of infinite length but finite radius moving through a fluid with constant velocity. The results are applied to the case of an ideal airscrew having a finite number of blades and a particular distribution of circulation along the blade for small values of the thrust.

The present paper contains a summary of Goldstein's results, which are then applied to the airscrew problem by a method which leads to formulae differing from the standard formulae of the "Vortex theory" by the addition of a factor to the formulae for the components of inflow; the value of this factor may be obtained from a chart embodying the results of Goldstein's calculations. The formulae of the Vortex theory are developed simultaneously from first principles by an analogous method which differs somewhat from the method used by its originator and brings out clearly the close analogy with the Prandtl theory of a monoplane wing; they also represent the limit of the Goldstein formulae for the case of an infinite number of blades.

The first application of the results is to show that for a screw of constant geometrical pitch (measured from the zero lift line of the section), a plan form (variation of chord with radius) may be calculated which gives Goldstein's value of the distribution of circulation with radius for all small values of the thrust; the screw is then analogous to the untwisted monoplane wing of elliptic plan form. For the two-bladed airscrew of experimental mean pitch ratio 1·57 the plan form of the family of airscrews approximates closely to the plan form required to satisfy Goldstein's conditions; as the pitch is decreased or the number of blades increased, the required plan form becomes progressively more blunt than that of the family, but the discrepancy between the performance calculated by Goldstein's theory and by the Vortex theory becomes smaller. The discrepancy for the 2 blader is 14 per cent. for a screw of pitch ratio 1·57 falling to 8 per cent. for pitch ratio 0·63, the value of the thrust according to the Vortex theory being too high.

It is suggested that the Goldstein formulae, which strictly apply only where there is a particular distribution of circulation along the blade, should be applied without this restriction in place of the Vortex theory, at any rate to airscrews of plan form similar to that of the family of airscrews, the extra labour involved being negligible. The modification may also be considered as being of the nature of an allowance for tip loss, since it may be shown that for a blade of constant chord the thrust grading tends to zero at the tip, whereas the Vortex theory gives a finite value of thrust grading at the tip.

It is proposed to compare calculations by the Goldstein formulae with the results of the new programme of experiments which has recently been sanctioned on airscrews of high pitch.

LIST OF SYMBOLS.

Goldstein's problem.

- w Velocity of helicoid.
- ε Angle of pitch at radius r.
- E Angle of pitch at tip radius R.
- u_z Axial component velocity; u_θ Circumferential component velocity; u_r Radial component velocity — Close to surface of helicoid relative to fluid at rest at ∞.
- $\mu = \cot \varepsilon$.
- $\mu_0 = \cot \text{E}$.

Airscrew.

- Γ Circulation round a blade element at radius r.
- V Forward; Ω Angular — velocity of airscrew.
- $x = r/\text{R}$.
- $\Lambda = \text{V}/\text{R}\,\Omega$.
- B Number of blades.
- c Chord length; ϑ Blade angle — at radius r.
- T Thrust.
- Q Torque.
- K Coefficient of circulation defined by §3, equation (7).
- F Axial inflow coefficient defined by §3, equation (28).

Blade element.

- W Magnitude; ϕ Angle with plane of airscrew; u Axial component; $(1-a_2)\,r\,\Omega$ Circumferential component — of resultant velocity of air relative to a blade element at radius r (neglecting radial component); (see Fig. 2).
- L Lift.
- a_0 Slope of lift curve (in §4).
- k_L Lift coefficient; k_D Drag coefficient; α Incidence — of blade section at radius r reduced to infinite aspect ratio.
- $\mu = \cot \phi$, $(\phi = \varepsilon)$.
- μ_0 = value of μ at airscrew tip.
- k_T Thrust coefficient.
- k_Q Torque coefficient.
- k_P Coefficient of power loss defined by §7, equation (1).

Efficiency.

$j = - \delta \Lambda$.

$h = - \delta V / \delta w$.

P = Power loss. P_1 Power loss due to inflow.
P_2 Power loss due to profile drag.

$P_c = P/\pi \varrho R^5 \Omega^3$.

$Q_c = Q/\pi \varrho R^5 \Omega^2$.

$\omega_1\ \omega_2$ Coefficients of power loss §5, equations (7), (8), (11).

$q_1\ q_2$ Coefficients of torque §5, equations (13), (14).

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1. *Introduction.*—It was shown in R. & M. 892* and R. & M. 1040† that there is good agreement between the results of calculations by the "Vortex theory" and observations of overall thrust and torque on the family of airscrews, except for the screws of the highest pitch diameter ratio, for which the torque is calculated too high over the whole working range, the thrust too high except near zero thrust, and the efficiency too low near maximum efficiency. It was suggested in R. & M. 1040 that the discrepancies might be due in part to the use in the Vortex theory of the assumption of an infinite number of blades; the error of this assumption would be expected to be greater for large values of J, i.e., for airscrews of high pitch.

An exact solution has recently been obtained by Goldstein‡ of the problem of an airscrew with a finite number of blades at small thrust and for a particular class of airscrew. The solution applies where the distribution of circulation along the blade is such that the flow in the wake is identical with the potential flow produced by the uniform axial motion of a set of equidistant coaxial helicoidal surfaces of finite radius. The solution is closely analogous to that for a monoplane aerofoil with elliptic distribution of circulation across the span, for which the wake is identical with the potential flow round a plane lamina of finite breadth (equal to the span of the aerofoil)

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* Family of Airscrews, Part III.—Lock and Bateman.

† The accuracy of the Vortex Theory of Airscrews.—Glauert and Lock.

‡ On the Vortex Theory of Screw Propellers. S. Goldstein, Ph.D. Proc. Royal Soc. A. Vol. 123, 1929.

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moving normal to itself ; the condition of minimum energy loss for given thrust or lift is satisfied in the two cases of airscrew and aerofoil, respectively.*

It appears that Goldstein's results represent a better approximation in certain directions than the Vortex theory of airscrews, in that the former takes account of the finiteness of the number of blades. It is, however, in its rigorous form limited to the case of small thrust and to a particular distribution of circulation with radius. The object of the present report is to show how Goldstein's results may be applied to improve the approximation of the Vortex theory in the general case ; at the same time it seemed worth while to exhibit the results of his paper in a form more familiar to those who are accustomed to the use of the Vortex theory in the practical design of airscrews.

It was found that the alterations introduced in the Vortex theory by this improved approximation, become of greatest importance near the airscrew tip, for a small number of blades, and at high values of J, i.e., for airscrews of high pitch. In the last respect, at any rate, it might be hoped that the new theory would help to account for the discrepancies between the Vortex theory and experiment. On investigation, however, it was found that the existing experimental data was not altogether sufficient or suitable for the purpose of deciding the relative merits of the old and new theories, especially on account of the uncertainties introduced by the large boss of the family of airscrews. The Aeronautical Research Committee have now sanctioned additional experiments on the airscrews of the family of the highest pitch, by a method which will eliminate boss effect ; the range of pitch will be extended to still higher values and the number of observations of thrust grading will be considerably increased. It is, therefore, considered that a detailed comparison of the results of the new theory with experiment should be postponed till the completion of the new experimental programme ; only a brief summary of the present position in this respect is included at the conclusion of the report. It is considered that the theoretical results are of sufficient practical importance to be put forward on their own merits.

2. *Summary of Goldstein's solution.*—The essentially original part of Goldstein's paper contains the solution of a particular problem in the pure hydrodynamics of a non-viscous fluid, the application to airscrew theory being to some extent secondary. The present paper contains first of all a statement of this problem, together with the nature of the solution obtained without going into details of the method of solution ; afterwards the results are applied to airscrew

* Betz, " Göttinger Nachr," pp. 193–213 (1919) ; reprinted in " Vier Ablandlungen zur Hydrodynamik und Aerodynamik," L. Prandtl und A. Betz, Göttingen, 1927.

theory by a method which while the same in essentials as that used by the author is somewhat different in form. At the same time the ordinary formulae of the Vortex theory are developed by a closely analogous method up to a point at which the formulae of the two theories can be considered side by side.

The problem of pure hydrodynamics considered by Goldstein is that of the potential flow of fluid past a rigid body of a certain form moving with a uniform velocity. The form of this body is a helicoidal surface, of infinite length but finite radius, moving parallel to its axis with uniform velocity w, or more generally any finite number of such surfaces equally spaced on the same axis and of the same radius, corresponding in number to the number of blades of the airscrew (Fig. 1). The results which we shall require to use are as follows.

Let ε be the angle of pitch of one of the helicoidal surfaces at radius r and let R be the outside radius of the helicoidal surface (see Fig. 1). The pitch length of the surface is

$$h = 2\pi\, r \tan \varepsilon,$$

which is independent of radius, and we may therefore write

$$r \tan \varepsilon = \mathrm{R} \tan \mathrm{E}$$

$$= \text{constant}, \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where E is the angle of the helicoid at its outer edge ; if the number of equally spaced surfaces is B, the axial distance between consecutive surfaces is $(2\,\pi\, r \tan \varepsilon)/\mathrm{B}$.

The results of Goldstein's theory, which are of importance for application to airscrews, relate to the velocity distribution in the immediate neighbourhood of the helicoids when they are moving through the fluid with axial velocity w. Let u_z, u_θ, u_r be component fluid velocities at any point, being respectively axial, circumferential, and radial components relative to cylindrical polar-co-ordinates co-axial with the helicoid and at rest relative to the fluid at an infinite distance in the radial direction.

The first of Goldstein's results is that the radial component u_r alone changes discontinuously through a helicoid, the values on the two sides being equal and opposite ; it follows that the surfaces may be replaced by a suitable distribution of helical vortex lines. Again, the axial and circumferential components u_z, u_θ close to a surface are functions of r only for a given helicoid, and their vectorial sum is equal to the component velocity of the surface normal to itself, which is of magnitude $w \cos \varepsilon$ and makes an angle ε with the axis of z. Hence

$$u_z = w \cos^2 \varepsilon, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

$$-\, u_\theta = w \cos \varepsilon \sin \varepsilon, \quad \ldots \quad \ldots \quad (3)$$

where ε may be considered as a function of r defined by equation (1).

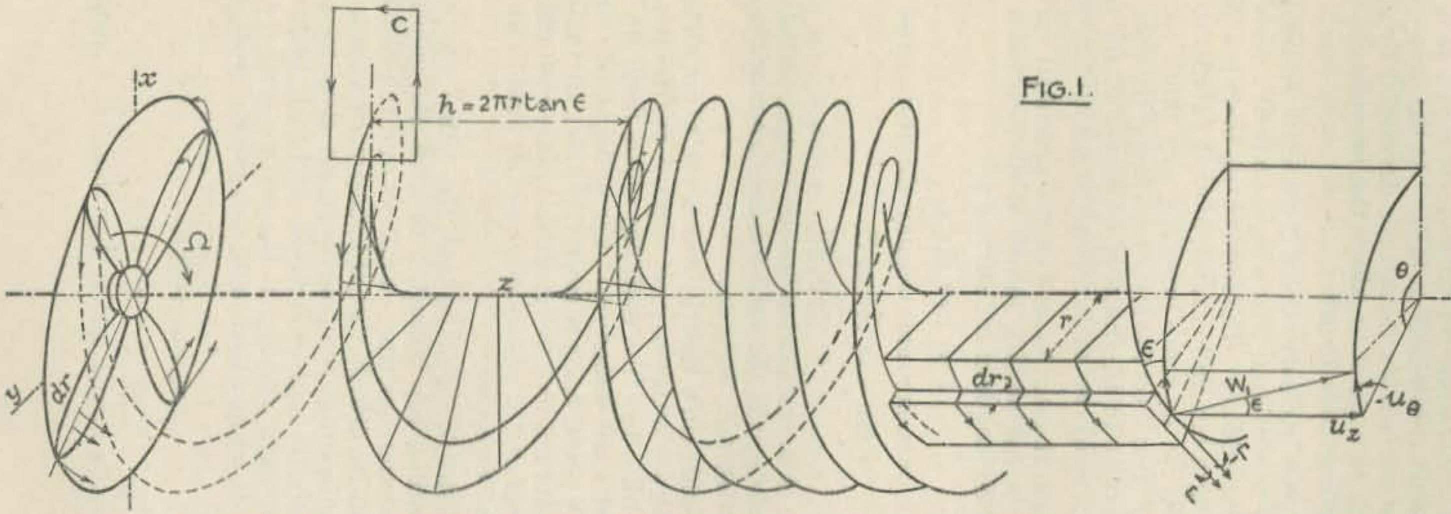

FIG. I.

On the extreme left is shown a four-bladed airscrew from one blade of which spring helical vortices of radius r and R respectively as shown by the dotted lines. After one complete turn, the complete helicoidal surface from one blade is shown, and after a further turn, the set of four helicoids springing from the four blades. Still further to the right is shown a section of this set of helicoids by circular cylinders of radius r and $r + dr$ respectively. These illustrate the approximations appropriate to the Vortex theory, in which the helices of radius r and $r + dr$ are replaced by continuous cylindrical vortex sheets corresponding to the case of an infinite number of blades.

The essential result of Goldstein's calculations is the determination of the strength of the equivalent helical vortex sheet as a function of radius. Let Γ be the circulation round a closed circuit (C, Fig. 1) cutting a helicoid at radius r and enclosing the part of the surface outside that radius. Then Γ is equal to the discontinuity of the velocity potential Φ across the helicoidal surface at radius r; $-d\,\Gamma/dr$ is, therefore, equal to the strength of the equivalent vortex sheet, or to the discontinuity in the radial component velocity u_r. It is obvious that Γ is proportional to w and that it is also a function of r/R, ε, and the number of surfaces B. Goldstein expresses Γ in terms of the non-dimensional coefficient K defined by

$$\Gamma = (Kw2\pi r \tan\varepsilon)/B, \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

so that K is a function of r/R, ε, and B only. For the case of two surfaces (two blades), K is tabulated in Table II on page 450 of Goldstein's paper for a series of values of the parameters μ and μ_0 defined by

$$\mu = \cot \varepsilon, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\mu_0 = \cot E. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

In Fig. 5 of the present report, values of $K/\cos^2 \varepsilon$ derived from this table are plotted against $\tan \varepsilon$ for a series of values of

$$x = r/R$$
$$= \mu/\mu_0.$$

For the case of four surfaces, Goldstein gives results in Table III, page 456, for the single value ($\mu_0 = 5$) and these are shown in Fig. 5 as isolated points. Results for other values of μ_0 could, however, be calculated without much labour from his auxiliary data.

The above results represent the whole of Goldstein's solution of his problem in pure hydrodynamics so far as it is required for application to airscrew theory.

3. *Comparison between the formulae of Goldstein's theory and of the Vortex theory.*—It is proposed to exhibit the "Vortex theory" of airscrews and Goldstein's theory on strictly parallel lines. With this object the fundamental basis of the Vortex theory on the one hand will be exhibited in a form as closely analogous as possible to Goldstein's theory; on the other hand, the final formulae of Goldstein's theory will be given in a form closely analogous to the standard formulae of the Vortex theory.

The fundamental conception in both theories is strictly analogous to the Prandtl conception of a monoplane wing. The airscrew blades are replaced by as many radial bound vortices of strength Γ (at radius r, Fig. 1) varying in general with r; from a blade element of length dr at radius r there springs a trailing vortex of strength $d\,\Gamma$. As in the theory of a monoplane wing, the effect of the induced velocities on the configuration of the vortices will at first be neglected;

these second order effects will be considered later. The trailing vortex, therefore, takes the form of a regular helix of radius r (Fig. 1) which is to the first order at rest in the fluid, and which coincides with the path of the blade element through the fluid ; i.e., its angle of pitch ε is given to the first order by the relation

$$\tan \varepsilon = V/r\ \Omega, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where V is the velocity of advance of the airscrew. The pitch length h of the helix is equal to $2\pi r \tan \varepsilon$, and the spacing in an axial direction between successive vortices is h/B where B is the number of blades.

Vortex theory.—We first proceed to determine a relation between Γ, r and $\tan \varepsilon$ on the basis of the assumption of the "Vortex theory" that the spacing between successive vortices is small or that $B/r \tan \varepsilon$ is large (infinite number of blades) so that the set of helical vortices of a particular radius from all the blades may be treated as a continuous tubular vortex sheet. This assumption is analogous to that commonly made in determining the field of an electric solenoid.

By analogy with the monoplane wing we adopt the artifice of assuming that a vortex of strength $-\Gamma$ leaves each blade at radius r (Fig. 1, extreme right), and a second vortex of equal and opposite strength Γ leaves each blade at radius $r + dr$, thus forming two concentric vortex solenoids of equal and opposite strength, enclosing a tubular element of fluid of thickness dr. Similarly, the next element from radius $r + dr$ to radius $r + 2dr$ is bounded by vortex solenoids of strength $-(\Gamma + d\Gamma)$ and $(\Gamma + d\Gamma)$ per blade. Thus the whole system is equivalent to the original distribution of strength $-d\Gamma/dr$ at radius r (or $r + dr$) ; the negative sign is consistent with the relation

$$\Gamma = \int_r^R -\frac{d\Gamma}{dr}\,dr$$

= circulation round the contour C (Fig. 1).

It is well known that the velocity field outside the outer, or inside the inner, member of a pair of vortex solenoids of equal and opposite strength, is zero. Since each blade element gives rise to a similar pair of solenoids, it follows that the velocity field between the solenoids is uninfluenced by the field outside ; the independence of neighbouring elements is, therefore, established to the first order on the assumption that $B/r \tan \varepsilon$ is large. To the same approximation the velocity (relative to air at infinity) between the solenoids at a large distance behind the airscrew is normal to the direction of the vortex lines and of magnitude equal to the strength of the vortex sheet equivalent to either solenoid.

The "spacing" in a direction normal to the surfaces of vorticity is

$$(h \cos \varepsilon)/B = (2\pi r \sin \varepsilon)/B,$$

and the velocity, relative to the air at an infinite distance, is in a direction making an angle ε with the Z axis, and is of magnitude W_1, equal to the strength of the equivalent vortex sheet, which is equal to Γ divided by the spacing, so that

$$W_1 = \Gamma B/(2 \pi r \sin \varepsilon).$$

Thus if u_z, u_θ, are the axial and circumferential component velocities (according to Goldstein's notation) it follows that

$$u_z = W_1 \cos \varepsilon$$
$$= \Gamma B/(2 \pi r \tan \varepsilon), \quad .. \quad .. \quad (2)$$
$$u_\theta = - W_1 \sin \varepsilon$$
$$= - \Gamma B/(2 \pi r). \quad .. \quad .. \quad .. \quad (3)$$

By an argument exactly similar to that used in the analogous case of a monoplane wing, it may be shown that the addition of a fictitious system of vortices, representing the reflection in the airscrew disc of the complete system of bound and trailing vortices, would give a system equivalent to a set of regular helical vortices extending to an infinite distance in both directions, since the bound vortices with their reflections would cancel each other. The same process would double the magnitude of the induced velocities at the airscrew disc, so that the actual induced velocities there would be one half the corresponding velocities due to the doubly infinite helical vortices and, therefore, equal to one half the actual induced velocities at a large distance behind the airscrew. Hence the components of "interference" velocity at the airscrew disc relative to air at rest at infinity are :—

$$\tfrac{1}{2}\, u_z,\ \tfrac{1}{2}\, u_\theta,$$

u_z, u_θ, being given by equations (2) and (3).

Goldstein's Theory.—We now proceed to consider Goldstein's assumptions, making use of the results of Section 2. He dispenses with the assumption that the spacing between successive helices is small. In place of this he makes the assumption that the velocity field of the whole of the vortex system *at a large distance behind the airscrew* is equivalent to the potential field of a series of rigid helicoidal surfaces of radius R; these occupy the position of the helicoidal vortex sheets formed by trailing vortices shed by the several blades, and move backwards with a velocity w small compared with $R\Omega$.* It will be noticed that this does not conflict with the assumption previously made that the trailing vortices are at rest to the first order.

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* It is helpful to notice that a precisely analogous assumption applies to a monoplane wing with elliptic loading. (See Introduction.)

Goldstein proceeds to determine the velocity field of the system of helicoidal surfaces completely by means of infinite series; we repeat the summary of his results given in the last section.

The discontinuity of velocity across the helicoidal surface is purely radial, thus verifying that the system is equivalent to a certain system of helical vortices. It follows that the axial and circumferential components close to the surface are definite (the same on both sides of the surface). The resultant velocity close to the surface is equal to the velocity of the surface normal to itself the value of which is

$$u_z \sec \varepsilon = - u_\theta \operatorname{cosec} \varepsilon = w \cos \varepsilon, \quad \ldots \quad \ldots \quad (4)$$

giving

$$u_z = w \cos^2 \varepsilon, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$u_\theta = - w \cos \varepsilon \sin \varepsilon. \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

The velocities in the wake are now no longer independent of axial or circumferential position, but the interference velocities close to the airscrew blades are still one half the corresponding components close to the helicoidal surface in the wake.

The circulation Γ defined in the previous section is (by Stokes' theorem) identical with the circulation round a blade of the airscrew at radius r. The strength $- d\,\Gamma/dr$ of the trailing vortex sheet from a blade element at radius r is equal to the discontinuity of radial component velocity across the helicoidal surface at the same radius; the circulation Γ round the blade is, therefore, equal to the corresponding discontinuity in the velocity potential Φ. This is obviously a function of r/R, ε, and number of blades B, and is a linear function of the velocity w of the helicoidal surfaces.

In Goldstein's tables it is expressed in the form of a coefficient K defined by the equation

$$\Gamma = \text{K} \cdot w \cdot 2\,\pi\, r \tan \varepsilon/\text{B}, \quad \ldots \quad \ldots \quad (7)$$

where K is a non-dimensional function of ε, r/R, and B only. Substituting for w from this equation in (5) and (6) we obtain

$$u_z = \frac{\cos^2 \varepsilon}{\text{K}} \cdot \frac{\Gamma\,\text{B}}{2\,\pi\, r \tan \varepsilon}, \quad \ldots \quad \ldots \quad (8)$$

$$- u_\theta = \frac{\cos^2 \varepsilon}{\text{K}} \cdot \frac{\Gamma\,\text{B}}{2\,\pi\, r}, \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

which differ from the formulae (2) and (3) of the Vortex theory by the factor $\cos^2 \varepsilon/\text{K}$ only, but with the proviso that the distribution of Γ along the blade must be such as to make w independent of r in accordance with equation (7). It follows from the previous argument that K must tend to the value $\cos^2 \varepsilon$ as a limit when the number of blades becomes infinite, but that it is then unnecessary to assume that w is independent of radius.

To the first order it is now a simple matter to apply either of the above formulae for interference velocities to strip theory calculations. At low rates of advance, however, second order effects become large and the exact method of applying the results becomes important.

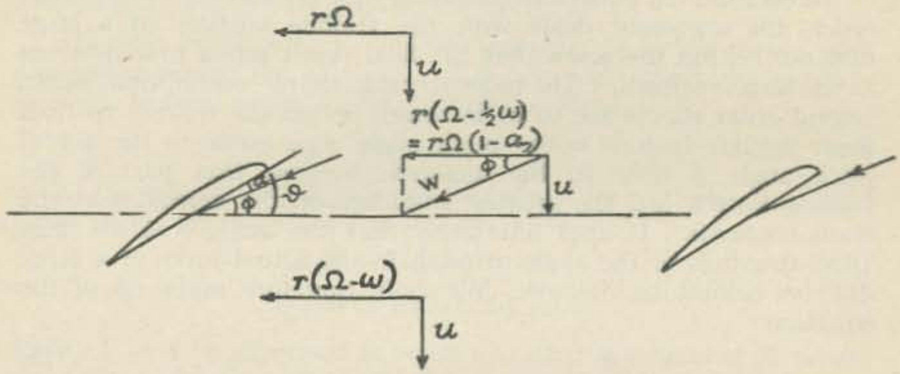

FIG. 2.

We adopt the usual notation of the Vortex theory (see Fig. 2). W is the total component in a plane normal to the blade of the effective velocity relative to a blade element and makes an angle ϕ with the plane of rotation. The axial and circumferential components of W are

$$W \sin \phi = u, \quad \ldots \quad (10)$$

$$W \cos \phi = (1 - a_2)\, r\, \Omega, \quad \ldots \quad (11)$$

so that

$$u = (1 - a_2)\, r\, \Omega \tan \phi, \quad \ldots \quad (12)$$

and $(u - V)$, $(- a_2 r\, \Omega)$, are the corresponding components of interference velocity relative to air at rest at infinity, and satisfy the relations

$$u - V = \tfrac{1}{2}\, u_z, \quad \ldots \quad (13)$$

$$a_2\, r\, \Omega = -\tfrac{1}{2}\, u_\theta. \quad \ldots \quad (14)$$

The following argument applies more especially to the results of Goldstein's assumptions, but should also apply to the Vortex theory as a limiting case.

Where account is taken of the velocity field of the vortices themselves, it is no longer true that the trailing vortices form regular helices, but both the pitch and radius will vary on proceeding backwards from the airscrew. It may still be assumed, however, that the fluid motion is steady relative to axes moving and rotating with the airscrew, and under these conditions it may be proved that the vortex lines coincide with the relative streamlines. In particular, the trailing vortex immediately it has left the blade coincides with the relative streamline at the same point, which (ignoring the angle of contraction of the slipstream) makes an angle ϕ with the airscrew disc.

In establishing equations (8) and (9) (or (2) and (3)) to the first order, the argument dealt with the trailing vortices at a large distance behind the screw, but the final result refers to conditions at the airscrew itself. The most accurate simple assumption, where second order effects are to be included, is that the trailing vortices form regular helices whose pitch angle ε is equal to the actual pitch angle ϕ close to the airscrew, because that part of the trailing vortex has the greatest influence on the velocities at the blade elements. It does not imply that the angle ϕ is the best approximation to the angle of pitch in the actual wake at a large distance behind the airscrew. We shall, therefore, make use of the equation

$$\varepsilon = \phi, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (15)$$

in the further development of both theories, an equation which is to supersede equation (1).

As in the analogous case of the monoplane wing, the individual velocities are normal to the direction of the trailing vortices, and it follows that a small change in direction of the trailing vortices (of the order of the ratios of u_x and u_θ to V and $r\,\Omega$) gives rise to changes in the components of induced velocity whose ratios to V and $r\,\Omega$ are of the second order in the above ratios.*

It will appear that the subsequent development of the Vortex theory equations leads to results identical with those commonly developed on the basis of considerations of energy and momentum, but it was an object of the present exposition to obtain all results on the basis of arguments strictly analogous to those used in the development of the Prandtl theory, of the monoplane wing. In the case of Goldstein's theory, considerations of energy and momentum could not be used since the velocity distribution, etc., is not independent of the angular co-ordinate.

We proceed to develop strip theory formulae on the basis of the Goldstein equations (8) and (9) ; the corresponding results for the Vortex theory may be obtained by putting $K = \cos^2 \varepsilon \; (= \cos^2 \phi)$ whenever it occurs ; it is to be remembered that w constant along the blade is a necessary condition in the former case but not in the latter.

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* The effect of substituting equation (1) for (15) is to substitute ε as defined by (1) for ϕ in (8), but to leave ϕ unaltered in (20). Thus (in the case of the Vortex theory) $u - V$ as given by (22) is multiplied by the factor $(\tan \phi \,/\tan \varepsilon) = u/V(1 - a_2)$, which changes the value of u/V by a quantity of the second order in $(u - V)/V$.

We notice first that equations (13), (12), (14), (5), (6) and (15) lead to the relation

$$V = r\ \Omega \tan\phi - \tfrac{1}{2} w, \qquad \ldots \qquad \ldots \qquad (16)$$

so that our assumption is identical with that which Goldstein attributes to Prandtl on page 460 of his paper.

Ignoring profile drag in the first instance, as in the theory of a monoplane wing, the lift on a blade element satisfies a relation identical in form to the Kutta Joukowski relation

$$d\,L = \varrho\, W\, \Gamma\, dr. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (17)$$

Using the strip theory relations for thrust and torque (ignoring profile drag) we have

$$dT/dr = B\,(dL/dr)\cos\phi, \qquad \ldots \qquad \ldots \qquad (18)$$

$$dQ/dr = Br\,(dL/dr)\sin\phi. \qquad \ldots \qquad \ldots \qquad (19)$$

Hence Γ may be expressed in terms of either the element of thrust or of torque by the equations

$$\Gamma = (\tan\phi/B\varrho u)\,(dT/dr) \qquad \ldots \qquad \ldots \qquad (20)$$

$$= (1/B\varrho r u)\,(dQ/dr). \qquad \ldots \qquad \ldots \qquad (21)$$

By substituting for Γ from these equations in (18) and (8), and (19) and (9), we can express the inflow factors in terms of either thrust or torque grading. When profile drag is included, equations (20) and (21) are, of course, no longer exactly consistent with the ordinary strip theory equations for the thrust and torque, viz., equations (26) and (27) below. The wake will now include thin regions of turbulence in the form of helicoidal surfaces representing the profile drag loss, in addition to the helical vortices corresponding to the lift; any application of the previous theory must, therefore, be only approximate at best. It is, therefore, legitimate to make a choice among equations (20), (21), (26) and (27), (18), (19), (23) and (24), and it seems reasonable to use equation (20), expressing Γ in terms of thrust, when calculating the axial component of interference velocity, and equation (21), expressing Γ in terms of torque, when calculating the rotational interference component. The results in the limiting case of an infinite number of blades will then be exactly consistent with the condition of conservation of angular momentum, and (less definitely) with the results deduced from considerations of axial momentum and energy. Substituting, therefore, from (20) in (13) and (8) and from (21) in (14) and (9) we have

$$u - V = \tfrac{1}{2} u_z = \frac{\cos^2\phi}{K} \cdot \frac{1}{4\pi\varrho r u} \cdot \frac{dT}{dr}, \quad \ldots \qquad (22)$$

$$a_2 r\ \Omega = -\tfrac{1}{2} u_\theta = \frac{\cos^2\phi}{K} \cdot \frac{1}{4\pi\varrho r^2 u} \cdot \frac{dQ}{dr}, \ldots \qquad (23)$$

and (using (7) and (20)),

$$\tfrac{1}{2}\, w = \frac{1}{4\,\pi\,\varrho\, r\, u\mathrm{K}} \cdot \frac{d\mathrm{T}}{dr} \quad \ldots \quad (24)$$

The ordinary strip theory considerations similar to those for a monoplane wing decide that aerofoil data for infinite aspect ratio must be used. If k_{L}, k_{D} are the two-dimensional lift and drag coefficients of the section at incidence α we have

$$\alpha = \vartheta - \phi, \quad \ldots \quad (25),$$

$$\frac{d\mathrm{T}}{dr} = \mathrm{B}\, c\, \varrho\, \mathrm{W}^2\, (k_{\mathrm{L}} \cos\phi - k_{\mathrm{D}} \sin\phi)$$

$$= \mathrm{B}\, c\, \varrho\, (1 - a_2)^2\, r^2\, \Omega^2\, (k_{\mathrm{L}} \cos\phi - k_{\mathrm{D}} \sin\phi) \sec^2\phi, \quad \ldots \quad (26),$$

using equation (11), and

$$\frac{d\mathrm{Q}}{dr} = \mathrm{B}\, c\, \varrho\, r\, \mathrm{W}^2\, (k_{\mathrm{D}} \cos\phi + k_{\mathrm{L}} \sin\phi)$$

$$= \mathrm{B}\, c\, \varrho\, r\, (1 - a_2)^2\, r^2\, \Omega^2\, (k_{\mathrm{D}} \cos\phi + k_{\mathrm{L}} \sin\phi) \sec^2\phi. \quad (27).$$

Substituting in (22) and (23) and using (12) we have

$$\mathrm{F} \equiv 1 - \frac{\mathrm{V}}{u} = \frac{\cos^2\phi}{\mathrm{K}} \cdot \frac{\mathrm{B}\,c}{4\,\pi\,r} \cdot \frac{k_{\mathrm{L}} \cos\phi - k_{\mathrm{D}} \sin\phi}{\sin^2\phi}, \quad \ldots \quad (28)$$

$$\frac{a_2}{1 - a_2} = \frac{\cos^2\phi}{\mathrm{K}} \cdot \frac{\mathrm{B}\,c}{4\,\pi\,r} \cdot \frac{k_{\mathrm{D}} \cos\phi + k_{\mathrm{L}} \sin\phi}{\sin\phi \cos\phi} \quad \ldots \quad (29)$$

Finally, using the definition of F in (28), equation (12) gives

$$\mathrm{V}/\mathrm{R}\,\Omega = x\, (1 - a_2)\, (1 - \mathrm{F}) \tan\phi \quad \ldots \quad (30)$$

On putting $\mathrm{K} = \cos^2\phi$, equations 25 to 30 become identical with the standard formulae of the Vortex theory as given, e.g., in R. & M. 892, page 6.

It is interesting to notice that although, when profile drag is included, the value of w obtained from ((7), (15), (20), (26)) differs from the value obtained from ((17), (15), (21), (27)), the following exact equation takes the place of (16),

$$\mathrm{V}/\mathrm{R}\,\Omega = x \tan\phi - (\mathrm{B}c/4\,\pi\,r)\, \{\, x\, (1 - a_2)/\mathrm{K}\, \}\, (k_{\mathrm{L}}/\sin\phi). \quad \ldots \quad (31)$$

This equation, which may be used in place of (30), does not involve k_{D} except through the factor a_2.

4. *Plan form of an airscrew which gives rise to a wake satisfying Goldstein's conditions.*—It has already been pointed out that Goldstein's solution requires that w should be independent of the radius of a blade element. Still bearing in mind the analogy of the monoplane wing with elliptic distribution of lift, it is obvious that for any given working condition it is possible to choose an infinite

variety of distributions of chord length and blade angle which make w independent of radius. With the same analogy in view we shall endeavour to solve the following problem. Neglecting profile drag, to find the (? unique) distribution of chord length and blade angle which makes w constant for all *small* values of the thrust for given solidity and experimental pitch ratio. The resulting blade will be analogous to the *non-twisted* monoplane wing with elliptic load distribution (having an elliptical plan form).

Neglecting profile drag, and assuming the same constant slope a_0 of the k_L curve for all sections and that the blade angle ϑ is measured from the zero lift line of the section, equation §3 (26) for thrust grading may be written

$$dT/dr = B\, c\, \varrho\, ur\, \Omega\, (1 - a_2)\, a_0\, \alpha/\sin\phi. \quad \ldots \quad \ldots \quad (1)$$

Again, equation (7) of §3 gives

$$\tfrac{1}{2} w = \frac{B}{4\pi r \tan\phi} \cdot \frac{\Gamma}{K} = \frac{1}{4\pi \varrho r u K} \cdot \frac{dT}{dr} \cdot \quad \ldots \quad (2)$$

Equation (16) of §3 may be written

$$\Lambda \equiv V/R\,\Omega = x \tan\phi - \tfrac{1}{2} w/R\,\Omega, \quad \ldots \quad \ldots \quad (3)$$

and equations (1) and (2) give

$$\tfrac{1}{2} \frac{w}{R\,\Omega} = \frac{B\,c}{4\pi r} \cdot \frac{x\,(1 - a_2)\, a_0\, \alpha}{K \sin\phi} \ldots \quad \ldots \quad \ldots \quad (4)$$

Equations (1), (3), (4) and §3, (12), and (25) are sufficient for the present problem; in (1), (4) and §3 (25), ϑ and α are supposed to be measured from the no lift line of the section. It follows from equation (3) that in order that w may be constant along a blade for a particular working condition, it is necessary that $x \tan\phi$ should be constant along the blade. To determine the condition that w should be constant along the blade for all small values of thrust (or α), we differentiate equations (3), (4) and §3 (25) and put $\alpha = 0$ after differentiation.

We use the symbol δ for $\delta\,\alpha$ and other quantities proportional to it to distinguish from differentiation with respect to r. The result after eliminating $\delta\,\phi$ is given by the two equations

$$-\delta\,\Lambda = x \sec^2\vartheta\,\delta\alpha + \tfrac{1}{2}\,\delta\,w/R\,\Omega, \quad \ldots \quad \ldots \quad \ldots \quad (5)$$

$$\tfrac{1}{2}\,\delta\,w/R\,\Omega = (B\,c/4\pi R)\, a_0\, \delta\alpha/K \sin\vartheta, \quad \ldots \quad \ldots \quad \ldots \quad (6)$$

where K is now a function of ϑ and x, with

$$x \tan\vartheta = \text{constant along the blade.} \quad \ldots \quad \ldots \quad \ldots \quad (7)$$

Eliminating $\delta\,\alpha$ between (5) and (6) gives

$$-\,\delta\,\Lambda = \tfrac{1}{2}\cdot\frac{\delta\,w}{\mathrm{R}\,\Omega}\left\{1+\frac{4\,\pi\,r\,\mathrm{K}\sin\vartheta}{\mathrm{B}\,c\,a_0\cos^2\vartheta}\right\}. \qquad .. \qquad (8)$$

From (7) it follows that the screw must be of constant geometrical pitch (measured from the zero lift line); equations (8) and (7) show that the variation of chord with radius must be chosen so as to make

$$\mathrm{K}/c\,\cos\vartheta = \text{constant} \;.. \qquad .. \qquad .. \qquad .. \qquad .. \qquad (9)$$

along the blade. The constant value of $\delta\,w/\delta\Lambda$ may be adjusted so as to give any chosen value to the chord length c at a standard radius. Next $\delta\alpha$ is determined as a function of radius so as to satisfy equation (6), the chord having been already determined. It is to be noticed that by equation (5),

$$\delta\alpha \text{ varies as } \cos^2\vartheta/x,$$

so that the present design of blade is open to the objection that the central sections would stall earlier than the outer sections. To the present approximation, equation (26) of §3 may be written

$$\delta\,d\mathrm{T}/dr = \mathrm{B}\,c\,\varrho\,r^2\,\Omega^2\,a_0\,\delta\alpha/\cos\vartheta, .. \qquad .. \qquad .. \qquad (10)$$

and eliminating $\delta\alpha$ between this equation and (6) gives

$$\delta\,d\mathrm{T}/dr = 2\,\pi\,\varrho\,r^2\,\Omega\,\mathrm{K}\tan\vartheta\;\delta w. \;.. \qquad .. \qquad .. \qquad (11)$$

Finally, the value of $-\,(\partial/\partial\Lambda)\,(d\mathrm{T}/dr)$, (the initial slope of the thrust grading against Λ at small thrust) is given by eliminating δw between this equation and (8). A similar result for the torque (still neglecting profile drag) is given by eliminating δw between (8) and the following equation

$$\begin{aligned}\delta\,d\mathrm{Q}/dr &= r\tan\vartheta\;\delta\,(d\mathrm{T}/dr)\\ &= (\mathrm{V}/\,\Omega)\;\delta\,(d\mathrm{T}/dr)\\ &= 2\,\pi\,\varrho\,r^2\,\mathrm{V}\,\mathrm{K}\tan\vartheta\;\delta\,w. \qquad .. \qquad .. \qquad (12)\end{aligned}$$

The slope of the thrust curve against Λ is then determined by graphical integration from equation (11).

Calculations by the above method were made for three of the cases worked out by Goldstein :—2 blades μ_0 ($\equiv \mathrm{R}\,\Omega/\mathrm{V}$) $= 2{\cdot}0$ and $5{\cdot}0$, and 4 blades, $\mu_0 = 5{\cdot}0$.

The constant value of $\delta w/\delta\Lambda$ was determined from (8) to make c equal to the chord of a blade of the family of airscrews at radius 0·7R. Since values of K for 4 blades, $\mu_0 = 2$, by Goldstein's method were not available it was decided to work out the 4 cases :— 2 and 4 blades, $\mu_0 = 2$ and 5, by the approximate formula for K given by Prandtl and quoted by Goldstein on pages 451 and 457 of his paper. This formula may be written in the form

$$\mathrm{K} = (2/\pi)\cos^2\phi\;\text{arc}\cos e^{-\mathrm{B}\lambda},$$

where

$$\left.\begin{aligned}\lambda &= \sin(\phi - \phi_0)/\sin\phi \sin 2\phi_0 \\ \tan\phi_0 &= V/R\,\Omega.\end{aligned}\right\} \quad .. \quad .. \quad (13)$$

For the same blade and the same values of ϑ and c, the calculations were then repeated on the basis of the assumption of the Vortex theory (i.e., by replacing K by $\cos^2\vartheta$); δw is then no longer constant along the blade.

Fig. 3 shows thrust grading curves in the form of the coefficient $t_1 = -(1/\pi\,\varrho\,R^2\,\Omega)\,(\partial/\partial V)\,(\partial T/\partial r)$, for small thrust ($\Omega$ constant), plotted against x for the particular case :—2 blades, $\mu_0 = 2$ (J = 1·57 at zero thrust) calculated by the Goldstein formulae and also with the same plan form by the Vortex theory for comparison. By integrating these and similar thrust grading curves, values at small thrust of the slope of the thrust coefficient curves were obtained in the form $-(1/\pi\,R^3\,\Omega)\,(\partial T/\partial V)$, ($\Omega$ constant), and are given in the following table :—

TABLE 1.

Values of $-\{1/\pi\,\varrho\,R^3\,\Omega\}\,(d\,T/d\,V)$ (Ω constant) $= -\frac{4}{\pi^2}\,(d\,k_T/d\,J)$,

for small thrust $\left(\frac{c}{R} = 0\cdot155 \text{ at } x = 0\cdot7\right)$.

J_0 (J at zero thrust.)	B (Number of blades.)	Goldstein *a.*	Vortex *b.*	Ratio *b/a.*	Prandtl *a.*	Vortex *b.*	Ratio *b/a.*
1·57	2	0·0653	0·0745	1·14	0·0672	0·0733	$1\cdot08_5$
1·57	4	—	—	—	0·121	0·129	1·06
0·628	2	0·0765	0·0822	1·08	0·0783	0·0825	$1\cdot05_5$
0·628	4	0·126	0·132	1·05	0·127	0·133	1·05

The largest discrepancy between the Goldstein and Vortex theories is 14 per cent. for the two blader ($J_0 = 1\cdot57$), and decreases with increase of the number of blades or decrease of pitch. The Prandtl approximation considerably underestimates the discrepancy in the extreme case. The true value of the discrepancy for a 4-blader $J_0 = 1\cdot57$ probably lies between 7 and 9 per cent.

The plan forms for the three cases of the Goldstein calculations are shown in Fig. 4 with the plan form of a blade of the family for comparison. The tip of the blade becomes more blunt as the number of blades increases or as the pitch decreases; for both 2-bladers the shapes are in fair agreement with a blade of the family. Since the blades of large pitch ratio of the family of airscrews (R. & M. 829) are of approximately constant geometrical pitch (the blade angle being measured from the zero lift line of the section) it follows that they also satisfy approximately the condition of minimum energy loss. A conclusion from Goldstein's theory would be that an airscrew blade should be made more blunt as the pitch decreases or as the number of blades increases. From the result, previously established, that for a blade designed as above, the incidence varies approximately inversely as the radius, it seems probable that a more efficient blade over a range of conditions would be obtained by reducing the twist and increasing the taper, in such a way as to maintain Goldstein's conditions for a single working condition at a fairly large thrust, whilst delaying the stalling of the blade sections of small radius.

5. *Method of calculating efficiency*.—The appearance of the velocity w in the fundamental equation of the present method, lends itself to the construction, on a definite physical basis, of simple formulae for the power wastage of an airscrew. It is easy to show that when profile drag is neglected, the total power wastage corresponding to an annular element dr of the airscrew disc is given by

$$\Omega\, d\mathrm{Q} - \mathrm{V}\, d\mathrm{T} = \tfrac{1}{2}\, w\, d\mathrm{T}, \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \qquad (1)$$

which, by use of §3 (24), may be written

$$\frac{1}{4\pi\varrho r u \mathrm{K}}\left(\frac{d\mathrm{T}}{dr}\right).\quad d\mathrm{T}\ . \qquad \ldots \quad \ldots \qquad (2)$$

In the special case in which w is constant along the blade the total power wastage is, therefore, equal to

$$\tfrac{1}{2}\, w\, \mathrm{T}^{*}\ldots \qquad \ldots \quad \ldots \quad \ldots \qquad (3)$$

These results have the advantage of combining into a single expression the components of power loss associated with both axial and rotational component velocities in the wake, as used in e.g., R. & M. 1034† and 1238.‡

When profile drag is included, the definition of w is no longer exactly unique, but if it is assumed as

$$\tfrac{1}{2}\ \frac{w}{r\Omega} = \frac{\mathrm{B}c}{4\pi r}\cdot\frac{k_{\mathrm{L}}(1-a_2)}{\mathrm{K}\sin\phi,} \quad \ldots \quad \ldots \quad \ldots \qquad (4)$$

§3 (31) may be written

$$\mathrm{V}/\mathrm{R}\,\Omega = x\tan\phi - \tfrac{1}{2}\, w/\mathrm{R}\,\Omega, \qquad \ldots \quad \ldots \qquad (5)$$

---

* This practically amounts to a proof that Goldstein's distribution corresponds to minimum energy loss.

† The efficiency of an airscrew.—H. Glauert.

‡ The effect of body interference on the efficiency of an airscrew.—Lock.

and the following equation for the total power loss may be shown to be exactly consistent with equations §(3), (26–31) :—

$$\Omega \frac{dQ}{dr} - V \frac{dT}{dr} = \tfrac{1}{2} w.\ Bc\, \varrho\, (1 - a_2)^2 r^2 \Omega^2 k_L \sec^2 \phi \left[1 + \left(\frac{k_D}{k_L}\right)^2\right]$$

$$+ Bc\, \varrho\, (1 - a_2)^3 r^3 \Omega^3 \sec^3 \phi\, k_D \ldots \quad \ldots \quad \ldots \quad (6)$$

The first term on the right-hand side here includes the whole of the power wasted in imparting non-turbulent motion to the air stream, while the second term represents the wastage associated with the profile drag of the blade elements.

Vortex theory underestimates power loss.—It may be shown generally that the power loss calculated by Goldstein's method will always be greater than that calculated by the Vortex theory. Using suffixes g and v to distinguish the two cases we have in general

$$K_g < K_v$$

$$(K_v = \cos^2 \phi),$$

and so from equation (24) of §3

$$w_g \geq w_v,$$

and so

$$P_g > P_v,$$

for given dT/dr.

Value of maximum efficiency corresponding to the approximation of §4.—We proceed to apply the formulae for small thrust of §4 to the calculation of power loss and efficiency. Write j for $-\ \delta\Lambda$ where as before

$$\Lambda = V/R\,\Omega\,;$$

j and $\delta\Lambda$ vanish with δa and with thrust. For the coefficient of power loss, defined by

$$P_c \equiv P/\pi\, \varrho\, R^5\, \Omega^3,$$

it follows from equation (1) that when profile drag is neglected and P is expanded in powers of j, all terms after the first in the expansion being neglected as in §4,

$$P_c = \omega_1 j^2 \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (7),$$

where ω_1 is a non-dimensional quantity which is finite (i.e., independent of j or $\delta\Lambda$) and which can be expressed as an integral along the blade by means of the results of §4 in the form :—

$$\omega_1 = \int_0^1 (t_1/2h)\, dx \quad \ldots \quad \ldots \quad \ldots$$

where

$$t_1 = -\frac{1}{\pi \varrho R^2 \Omega} \cdot \frac{\delta}{\delta V} \cdot \frac{d T}{d r},$$

and

$$h = -\delta V/\delta w$$

$$= j R \Omega/\delta w.$$

On Goldstein's theory w is constant along the blade and so ω_1 may be written

$$\omega_1 = (1/2 h) \int_0^1 t_1 \, dx. \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (8)$$

In order to calculate maximum efficiency it is essential to take account of the effect of profile drag on the blade elements. It will be assumed that thrust and incidence may be treated as small quantities up to maximum efficiency (see footnote on next page) so that the profile drag coefficient k_D at a given radius can be treated as independent of j. Write P_1 for the power expended in producing the motion of the wake as given by formula (7) above and P_2 for the power lost in profile drag at the blade elements given by

$$dP_2/dr = \varrho B c (r \Omega \sec \vartheta)^3 k_D. \quad \ldots \quad \ldots \quad \ldots \quad (9)$$

The whole power loss coefficient is then given by

$$P_c = (P_1 + P_2)/\pi \varrho R^5 \Omega^3$$

$$= \omega_1 j^2 + \omega_2, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (10)$$

where ω_1, ω_2, are considered as constants, ω_2 being given by the equation

$$\omega_2 = \int_0^1 \frac{B c}{\pi R} (x \sec \vartheta)^3 k_D \, dx. \quad \ldots \quad \ldots \quad \ldots \quad (11)$$

Similarly, taking account to the same degree of approximation of the effect of profile drag on the torque we have

$$Q/\pi \varrho R^5 \Omega^2 = q_1 \cdot j + q_2, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (12)$$

where

$$q_1 = \Lambda_0 \int_0^1 t_1 \, dx, \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (13)$$

and

$$q_2 = \int_0^1 (B c/\pi R) x^3 k_D \sec \vartheta \, dx. \quad \ldots \quad \ldots \quad (14)$$

Hence the efficiency is given by

$$1 - \eta = P_c/Q_c = \frac{\omega_1 j^2 + \omega_2}{q_1 j + q_2}. \quad \ldots \quad \ldots \quad \ldots \quad (15)$$

The maximum value of η for variations of j may be determined in the usual way ;* it occurs when j is the positive root of the equation

$$j^2 + \frac{2q_2}{q_1} j - \frac{\omega_2}{\omega_1} = 0 , \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (16)$$

giving

$$j = - q_2/q_1 + \{ (q_2/q_1)^2 + \omega_2/\omega_1 \}^{1/2} , \quad \ldots \quad (17)$$

and

$$1 - \eta = \frac{2\,\omega_1}{q_1} j = \frac{2\,\omega_1{}^{1/2}\,\omega_2{}^{1/2}}{q_1} \left\{ \sqrt{1 + \frac{q_2^2\,\omega_1}{q_1^2\,\omega_2}} - \frac{q_2\,\omega_1{}^{1/2}}{q_1\,\omega_2{}^{1/2}} \right\} \quad \ldots \quad (18)$$

The calculation of maximum efficiency was carried out by this method for the case of the airscrew with plan form satisfying Goldstein's conditions :—2 blades, $R\,\Omega/V = 2\cdot0$. The values of k_D were taken from R. & M. 892, Table 3, to correspond to $\alpha = 0$ (measured from the chord) as corresponding to a suitable mean value of k_D over the range of α between zero thrust and maximum efficiency. The results of the calculation are given in the following Table.

TABLE.

	Goldstein.	Vortex.
t_1	0·065	0·0745 (as in Table 1).
ω_1	0·0162	0·0108
ω_2	0·00036	0·00036
q_1	0·0326	0·0372
$V/R\,\Omega$ (Max. eff.)	0·36	0·32
η Max.	0·86	0·90

Thus the calculation on Goldstein's theory gives a maximum efficiency 4 per cent. lower than the calculation by the Vortex theory. In support of the accuracy of the method, the value of maximum efficiency on the Vortex theory is in agreement with the value (0·897) calculated for 2-blader No. 5 of the family with zero allowance for boss drag.

6. *General method of applying Goldstein's results.*—It has been observed that the plan form appropriate to Goldstein's method for a high pitched two-blader agrees approximately with the plan form of the family. For lower pitch and larger number of blades the plan form becomes progressively more blunt (Fig. 4), but at the same time the discrepancies between Goldstein and Vortex

* The justification for assuming that j may be treated as small up to maximum efficiency is that ω_2 and q_2 contain k_D as a factor, and are therefore small in comparison with ω_1 and q_1.

theory become smaller. It seems reasonable, therefore, to apply Goldstein's method to airscrews of plan form similar to that of the family of airscrews by substituting equations §3 (28) and (29) for the standard equations of the Vortex theory without regard to the condition of w constant along the blade. For this purpose a chart has been prepared (Fig. 5) in which $K/\cos^2\phi$ is plotted against $\tan\phi$, for two-bladed airscrews, for a series of standard values of the radius x; the values of K being obtained by cross plotting from Goldstein's Table II.

It is now possible to explain in what sense the method constitutes an allowance for tip effect. It appears from the chart (Fig. 5) or from Goldstein's analysis that K or $K/\cos^2\phi$ considered as a function of r/R tends to zero on approaching the airscrew tip (i.e., as r/R tends to unity). It follows from §3 equation (22) for $u - V$, in which $u - V$ is equated to an expression containing $(1/K)\cdot(dT/dr)$ as a factor, that for any finite value of V or J, dT/dr must tend to zero at the airscrew tip. This would still be true even in the extreme case of a square tipped blade in which the chord is constant up to the tip, whereas according to the Vortex theory, the thrust grading would tend to a finite value at the tip.

The method of performing an actual design calculation is now identical with that commonly used in the Vortex theory (see R. & M. 892). Calculations are made in the first instance for a series of values of α and a series of radii. The lift and drag coefficients being known as functions of α, values of a_2 and F are determined from §3 equations (28) and (29), K being obtained from the chart. dT/dr and dQ/dr are then obtained by the formulae §3 (26, 27) and V/nD from §3 (30) or the equivalent (31). If desired the efficiency may be determined directly by means of formula §5 (6) for the power wastage.

The values of non-dimensional coefficients of dT/dr and dQ/dr are then plotted against V/nD, cross plotted against r/R and graphically integrated in the usual way to give the thrust and torque coefficients for the whole airscrew.

7. *Calculations and comparison with experiment.*—The method has so far been applied only to a single airscrew, the 2-blader of P/D 1·5 of the family of airscrews. The original calculations by the formulae of the Vortex theory were used as a basis, and to save labour only the axial inflow factor F was corrected by multiplying F by $\cos^2\phi/K$ in accordance with §3 equation (28). The results were then cross plotted against $J(= V/nD)$, cross plotted against x and integrated in the usual manner.

In order to increase the accuracy of the determination of efficiency a power loss coefficient defined by

$$\frac{dk_P}{dx} = \frac{dk_Q}{dx} - \frac{J}{2\pi}\frac{dk_T}{dx} \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

was plotted in addition to the torque grading, and in place of the thrust grading. If the curve of dk_P/dx is made to tend to zero as x tends to zero (as in Fig. 7) the results may be considered as a standard representing zero allowance for boss drag. In order to increase the accuracy of the comparison between the results of Goldstein's theory and the Vortex theory, the differences

$$\delta\frac{dk_Q}{dx}=\frac{dk_Q}{dx}\text{ (Vortex)}-\frac{dk_Q}{dx}\text{ (Goldstein)}, \quad .. \quad (2)$$

$$\delta\frac{dk_P}{dx}=\frac{dk_P}{dx}\text{ (Vortex)}-\frac{dk_P}{dx}\text{ (Goldstein)}, \quad .. \quad (3)$$

were plotted and integrated.

The thrust coefficient k_T (with zero allowance for boss) is then obtained from the inverse of equation (1)

$$k_T=(2\,\pi/J)\cdot(k_Q-k_P).$$

Values of k_Q and k_T obtained in this way are given in the following table, together with the ratio (Vortex/Goldstein).

TABLE 2*

2-Bladed Airscrew P/D 1·5.

J	k_Q.			k_T (zero boss effect).			Efficiency (zero boss effect).		
	Goldstein *a*.	Vortex *b*.	Ratio *b/a*.	Goldstein *a*.	Vortex *b*.	Ratio *b/a*.	Goldstein.	Vortex.	Difference.
0·8	0·0230	0·0226	0·98	0·132	0·136	1·03	0·73	$0{\cdot}76_5$	$0{\cdot}03_5$
0·9	0·0228	0·0231	1·01	0 123	0·130	$1{\cdot}05_5$	0·77	$0{\cdot}80_5$	$0{\cdot}03_5$
1·0	0·0222	0·0228	1·03	$0{\cdot}112_5$	0·121	$1{\cdot}07_5$	$0{\cdot}80_5$	0·84	$0{\cdot}03_5$
1·1	0·0212	0·0221	1·04	0·101	$0{\cdot}109_5$	$1{\cdot}08_5$	$0{\cdot}83_5$	$0{\cdot}86_5$	0·03
1·2	0·0198	0·0208	1·05	0·089	$0{\cdot}096_5$	$1{\cdot}08_5$	0·86	$0{\cdot}88_5$	0·02
1·3	0·0179	0·0189	$1{\cdot}05_5$	0 076	0 082	1·08	0·87	$0{\cdot}89_5$	$0{\cdot}02_5$
1·4	0·0154	0·0164	$1{\cdot}06_5$	0·060	0·066	1·10	$0{\cdot}87_5$	$0{\cdot}89_5$	0·02
1·5	0·0126	0·0134	$1{\cdot}06_5$	0·045	0·049	1·09	0·85	$0{\cdot}87_5$	$0{\cdot}02_5$
1·6	0·0090	0·0096	$1{\cdot}06_5$	0·027	0·030	1·11	0·77	$0{\cdot}79_5$	$0{\cdot}02_5$
1·7	0·0053	0·0054	1·02	0·010	0·010	1·00	0·51	$0{\cdot}50_5$	$0{\cdot}00_5$

The ratio of values of k_T is roughly constant over a considerable range (J = 1·1 to 1·6) in which its mean value is 1·09 : this may be compared with the result of the calculation by the method of §4 given in Table 1 as 1·14 for a two-blader with $J_0 = 1{\cdot}57$. The discrepancy is probably due to the neglect of the effect on rotational inflow in the latter calculations so that the value 1·14 is probably more nearly accurate than 1·09 for the two blader No. 5.

* The above results are based on aerofoil data reduced to infinite aspect ratio by formulae for the elliptic wing, and are therefore not exactly comparable with those of R. & M. 892 (see footnote p. 5, R. & M. 892).

The corresponding ratio for the torque will be somewhat smaller as indicated in the above table. For comparison the value of the ratio (calculated k_Q/observed k_Q) from R. & M. 829 and 892 is recorded below.

TABLE 3.

k_Q. (Vortex theory).

J.	Observed *a*.	Calculated *b*.	Ratio *b*/*a*.
0·8	0·0221	0·0234	1·06
0·9	0·0221	0·0235	$1{\cdot}06_5$
1·0	0·0216	0·0233	1·08
1·1	0·0205	0·0227	1·11
1·2	0·0187	0·0214	$1{\cdot}14_5$
1·3	0·0165	0·0195	1·18
1·5	0·0108	0·0137	1·27
1·7	0·0024	0·0055	2·29

It appears that the effect of Goldstein's theory is in the right direction to explain the discrepancy but is not of sufficient magnitude * ; a final decision on this point may well be reserved until the additional experiments on high pitch airscrews have been completed. The same may be said still more strongly of the thrust coefficient ; an attempt was made to estimate the possible uncertainties due to boss correction, but the results were too indefinite to be worth recording in view of the prospect of further experiments.

The maximum efficiency calculated by the present method according to Goldstein is 0·02 smaller than that calculated by the Vortex theory ; the difference calculated by the somewhat uncertain approximate method given in §5 is 0·04, the two values being 0·86 and 0·90 respectively. The latter value is in agreement with the value in Table 2 and the true value of the difference allowing for effect of the correction on rotational inflow is probably about 0·03. This difference is in the wrong direction to account for the observed discrepancy, but here again it appeared probable on investigation that errors in estimation of boss effect may account for the whole of the observed discrepancy.

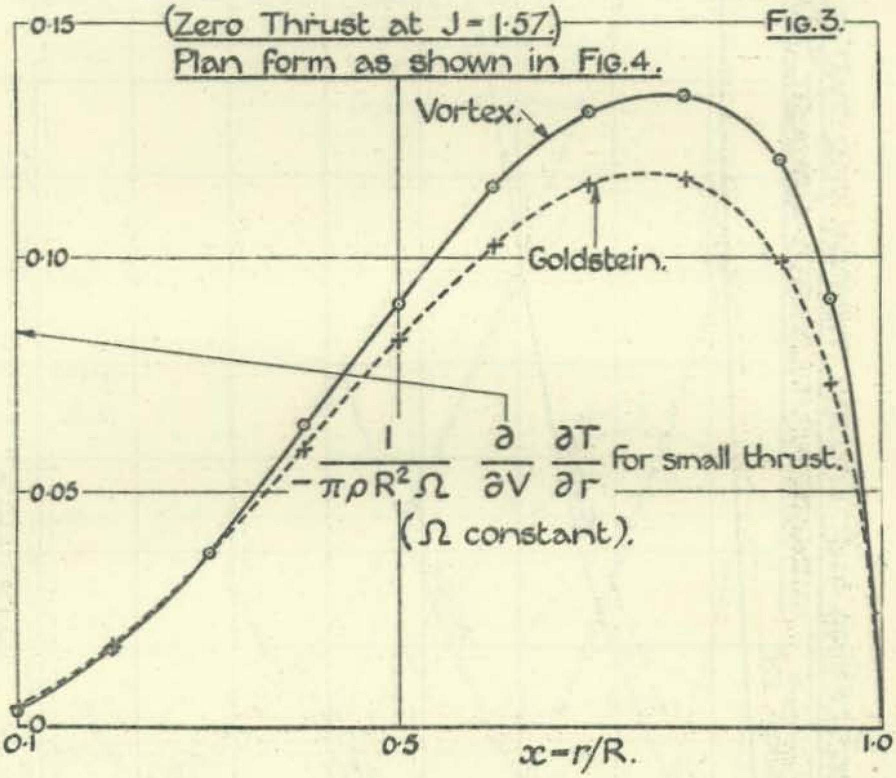
Finally it is worth while pointing out that the differences between the two thrust grading curves of Fig. 3 are qualitatively in agreement with the difference between observed and calculated thrust grading curves, in e.g., R. & M. 892, Fig. 2. Here again the absence of observations of thrust grading on the 2-blader P/D 1·5 makes it necessary to await the new programme of experiments for a quantitative comparison.

* The discrepancy of k_Q at zero thrust evidently represents a disagreement between the drag coefficient of the blade elements and the drag coefficient of a rectangular aerofoil at zero lift.

R.& M. 1377.

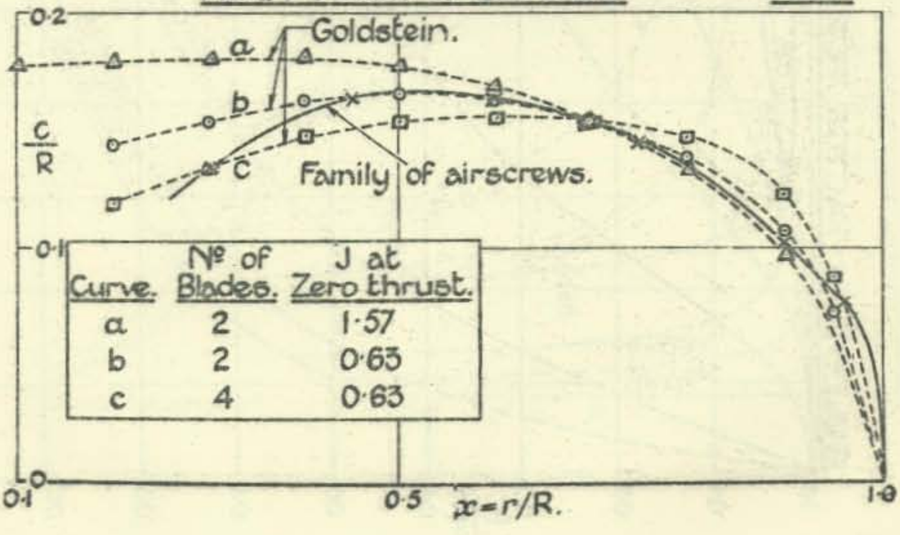
CALCULATED THRUST GRADING, 2 BLADED AIRSCREW.

Figs. 3 & 4.



PLAN FORM of BLADES.

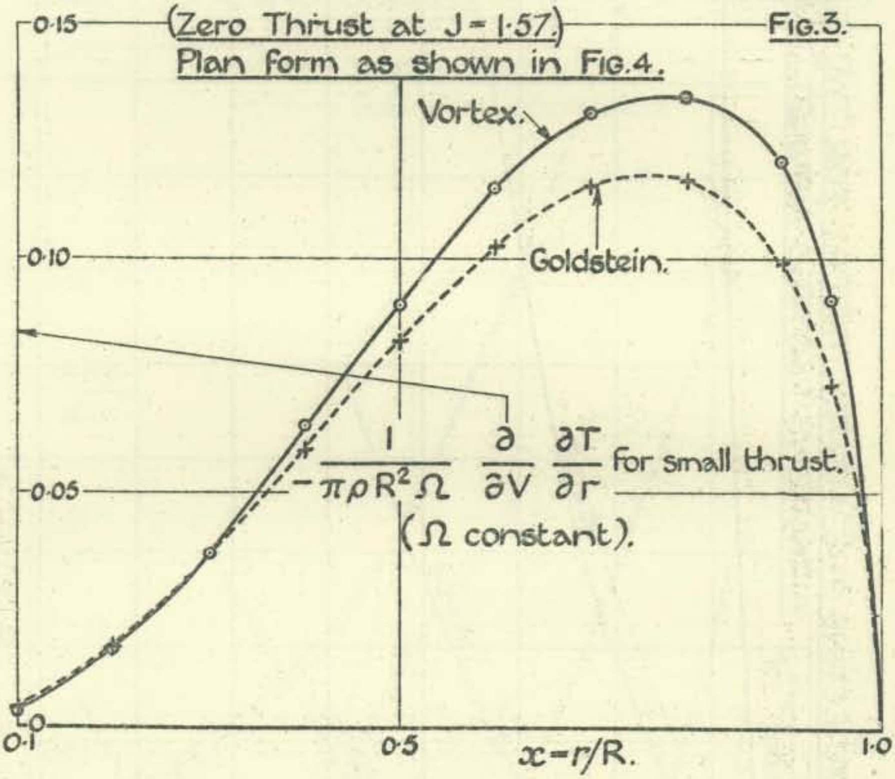
Fig. 4.



CALCULATED THRUST GRADING, 2 BLADED AIRSCREW.

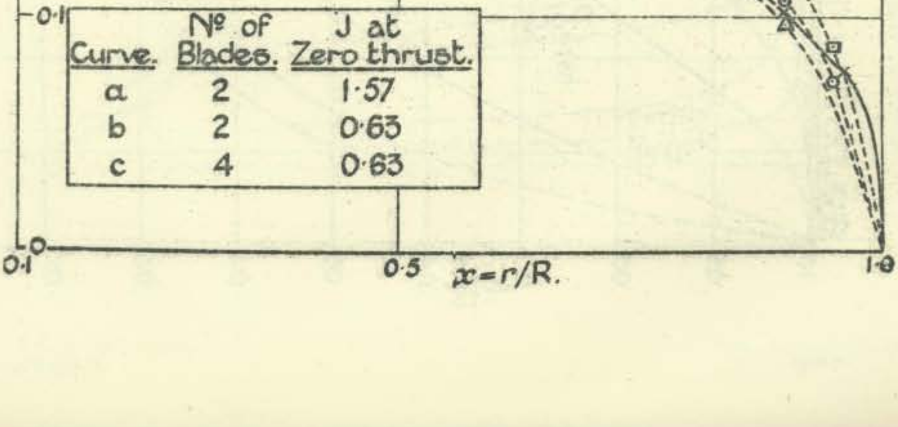
FIGS. 3 & 4.

FIG. 3.



PLAN FORM OF BLADES.

FIG. 4.



GOLDSTEIN FACTOR FOR CORRECTING THE VORTEX THEORY FOR TWO-BLADED AIRSCREWS PLOTTED AGAINST TAN ϕ

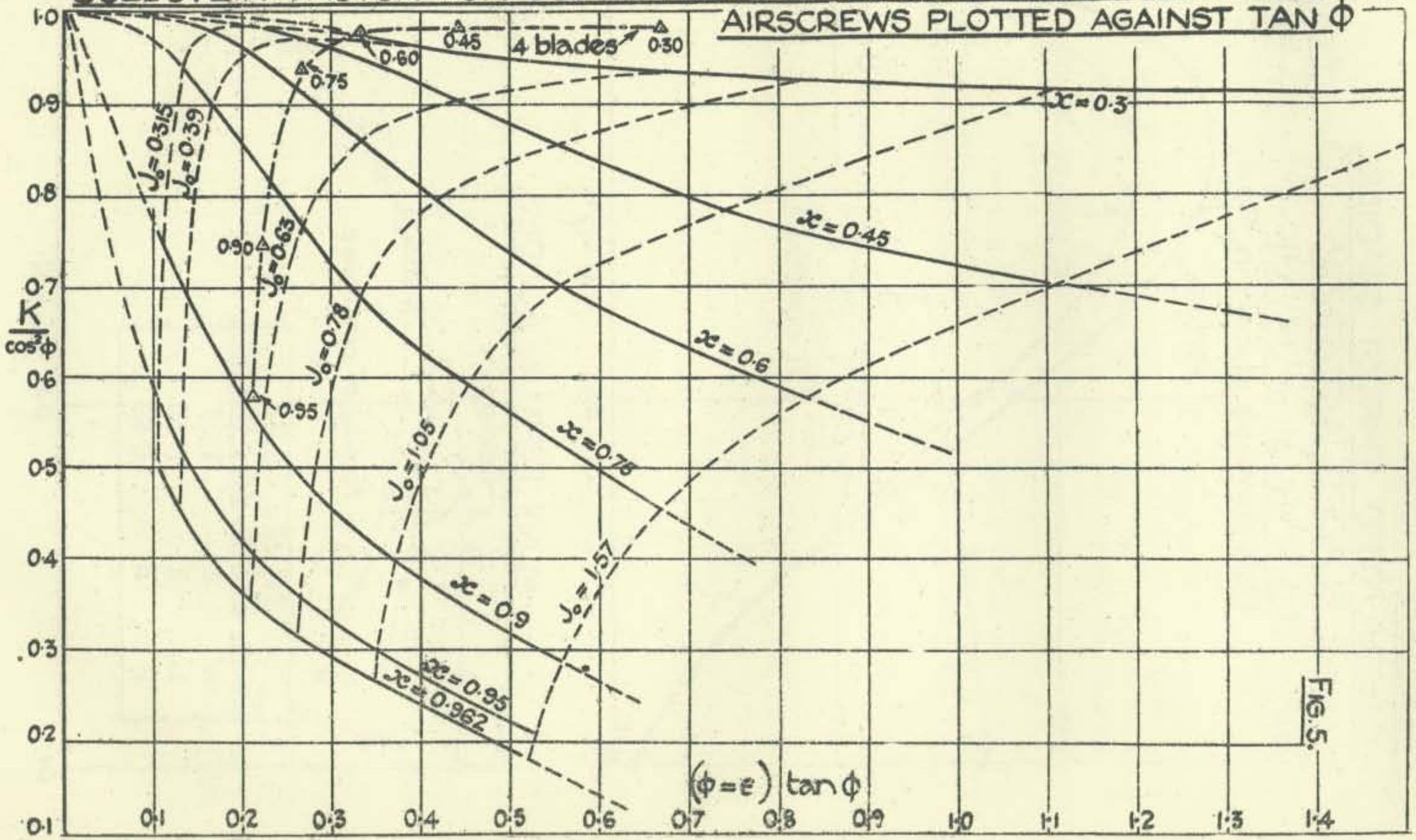


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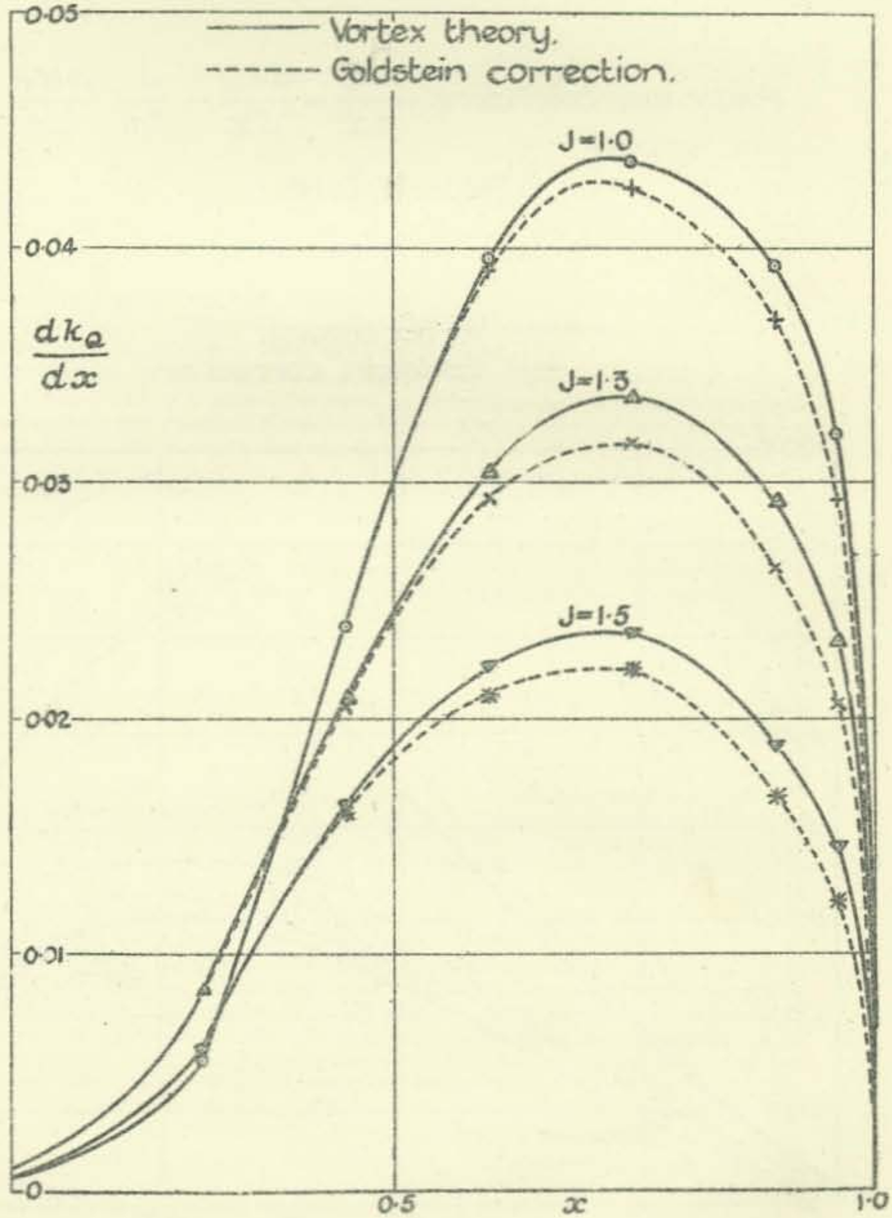
FIG. 6.

TORQUE GRADING CURVES. 2 BLADER P/D 1·5.



R.& M. 1377.

Fig. 7.

Power loss coefficient $\frac{dk_P}{dx} = \frac{dk_Q}{dx} - \frac{J}{2\pi}\frac{dk_T}{dx}$

2 Blader P/D. 1·5.